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Full Projective Oscillator Representations of Special Linear Lie Algebras and Combinatorial Identities

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Abstract. Using the projective oscillator representation of $\mathfrak{sl}(n+1)$ and Shen's mixed product for Witt algebras, Y. Zhao and the second author [*Generalized projective representations for* $\mathfrak{sl}(n+1)$, J. Algebra 328 (2011) 132–154] constructed a new functor from $\mathfrak{sl}(n)$-**Mod** to $\mathfrak{sl}(n+1)$-**Mod**. In this paper, we start from $n = 2$ and use the functor successively to obtain a full projective oscillator realization of any finite-dimensional irreducible representation of $\mathfrak{sl}(n+1)$. The representation formulas of all the root vectors of $\mathfrak{sl}(n+1)$ are given in terms of first-order differential operators in $n(n+1)/2$ variables. One can use the result to study tensor decompositions of finite-dimensional simple modules by solving certain first-order linear partial differential equations, and thereby obtain the corresponding physically interested Clebsch-Gordan coefficients and exact solutions of Knizhnik-Zamolodchikov equation in WZW model of conformal field theory.

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1. Introduction

Sum-product type identities are important objects both in combinatorics and number theory. The Jacobi triple identity and quintuple product identity are such well-known examples. Macdonald [23] used affine analogues of the root systems of finite-dimensional simple Lie algebras to derive new type sum-product identities with the above two identities as special cases. Kac [17] found the character formula for the integrable modules of affine Kac-Moody algebras and showed that the Macdonald identities are exactly the denominator identities. Kang and Kim [19] found a number of interesting combinatorial identities from various expressions of the Witt partition functions in connection with the denominator identity of a certain graded Lie algebra. The denominator identities of finite-dimensional simple Lie algebras are Vandermonde determinant type identities, which do not produce sum-product identities of numbers. In this paper, we show that our full projective oscillator representations of special linear Lie algebras naturally give rise to certain sum-product type identities of finite type.

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Finite-dimensional irreducible representations of finite-dimensional simple Lie algebra over $\mathbb{C}$ were abstractly determined by Cartan and Weyl in early last century. However, explicit representation formulas of the root vectors in these simple algebras are in general very difficult to be given. In 1950, Gelfand and Tsetlin [8,9] used a sequence of corank-one subalgebras to obtain a basis whose elements were labeled by upside-down triangular data for finite-dimensional irreducible representations of general linear Lie algebras and orthogonal Lie algebras, respectively. Moreover, the corresponding matrix elements of the actions of simple root vectors (or Chevalley basis) were explicitly given. Based on the theory of Mickelsson algebras and the representation theory of the Yangians, Molev [24] constructed a weight basis for finite-dimensional irreducible representations of symplectic Lie algebras and obtained explicit formulas for the matrix elements of generators. He [25] has also done similar work for $o(2n+1)$. There are also many interesting works in this direction (e.g., cf. [6,7,12,13,22,24]). Knowing the representation formulas of simple roots is not enough to solve the general decomposition problem of the tensors of irreducible representations because they are not commuting operators.

In this paper, we present a first-order differential operator realization of any finite-dimensional irreducible representation of $\mathfrak{sl}(n+1)$ in $n(n+1)/2$ variables (cf. Theorem 2.3). Moreover, the explicit formulas of all the root vectors are given, which will be helpful in solving the general decomposition problem of the tensors of irreducible representations. In physics, the Clebsch-Gordan coefficients are numbers that arise in angular momentum coupling in quantum mechanics. They appeared as the expansion coefficients of total angular momentum eigenstates in an uncoupled tensor product basis (e.g., cf. [4,5]). We refer [1,2,21,26] for more applications and later developments. Mathematically, the numbers are those of explicitly determining the irreducible components in the tensor of two finite-dimensional simple modules in terms of orthonormal bases. The first fundamental step is to find explicit formulas for the highest-weight vectors of those irreducible components. Even that is in general a very difficult problem. Our result in this paper simplifies the problem to solving certain first-order linear partial differential equations. As a very special example, we find the explicit formulas of the highest-weight vectors of irreducible components in the tensor module of the simple module with highest weight $k\omega_1$ and any finite-dimensional simple module of special linear Lie algebras. The well-known $\mathfrak{sl}(n+1) \downarrow \mathfrak{sl}(n)$ branching rule and the multiplicity-one theorem of $\mathfrak{gl}(n+1) \downarrow \mathfrak{gl}(n)$ are direct consequences. Our representation formulas can also be used to find exact solutions of Knizhnik-Zamolodchikov equation in WZW model of conformal field theory (cf. [20,29,31]). An important feature of our representation is its connection with certain sum-product combinatorial identities of finite type. Below we give a more detailed introduction.

Let $n > 1$ be an integer. Denote by $GL(n+1,\mathbb{R})$ the group of $(n+1)\times(n+1)$ invertible matrices. A *projective transformation* on $\mathbb{R}^n$ is given by

$$u \mapsto \frac{Au+\vec{b}}{\vec{c}^{\,t}u+d} \qquad \text{for } u \in \mathbb{R}^n, \tag{1}$$

where all the vector in $\mathbb{R}^n$ are in column form and

$$\begin{pmatrix} A & \vec{b} \\ \vec{c}^{\,t} & d \end{pmatrix} \in GL(n+1,\mathbb{R}). \tag{2}$$

It is well-known that a transformation mapping straight lines to straight lines must be a projective transformation. The group of projective transformations is the fundamental group of n-dimensional projective geometry. Physically, the group with $n = 4$ consists of all the transformations of keeping free particles including light signals moving with constant velocities along straight lines (e.g., cf. [14,15]). Based on the embeddings of the Poincaré group and De Sitter group into the projective group with $n = 4$, Guo, Huang and Wu [14, 15] proposed three kinds of special relativity.

For simplicity, we assume that the base field is the field $\mathbb{C}$ of complex numbers in the rest of this paper. Let $E_{r,s}$ be the square matrix with 1 as its (r, s)-entry and 0 as the others. The special linear Lie algebra

$$\mathfrak{sl}(n+1) = \sum_{1\leq i<j\leq n+1} (\mathbb{C}E_{i,j} + \mathbb{C}E_{j,i}) + \sum_{r=1}^{n} \mathbb{C}(E_{r,r} - E_{r+1,r+1}). \tag{3}$$

Let $\mathcal{A} = \mathbb{C}[x_1, x_2, \cdots, x_n]$ be the polynomial algebra in n variables. Set

$$D = \sum_{s=1}^{n} x_s\partial_{x_s}, \quad I_n = \sum_{i=1}^{n} E_{i,i} \in \mathfrak{gl}(n). \tag{4}$$

Let M be an $\mathfrak{sl}(n)$-module. We fix $c \in \mathbb{C}$ and make M a $\mathfrak{gl}(n)$-module by letting $I_n|_M = c\,\mathrm{Id}_M$. Denote

$$\widehat{M} = \mathscr{A} \otimes_{\mathbb{C}} M. \tag{5}$$

For any two integers $p \leq q$, we denote $\overline{p,q} = \{p, p+1, \cdots, q\}$. Differentiating the transformations in (1), we get an inhomogeneous first-order differential operator representation of $\mathfrak{sl}(n+1)$. Using this representation and Shen's mixed product for Witt algebras in [28], Zhao and the second author [35] obtained the following representation of $\mathfrak{sl}(n+1)$ on $\widehat{M}$:

$$E_{i,j}|_{\widehat{M}} = x_i\partial_{x_j} \otimes \mathrm{Id}_M + \mathrm{Id}_{\mathscr{A}} \otimes E_{i,j}|_M, \tag{6}$$

$$(E_{i,i} - E_{j,j})|_{\widehat{M}} = (x_i\partial_{x_i} - x_j\partial_{x_j}) \otimes \mathrm{Id}_M + \mathrm{Id}_{\mathscr{A}} \otimes (E_{i,i} - E_{j,j})|_M, \tag{7}$$

$$E_{i,n+1}|_{\widehat{M}} = x_iD \otimes \mathrm{Id}_M + x_i \otimes I_n|_M + \sum_{r=1}^{n} x_r \otimes E_{i,r}|_M, \quad E_{n+1,i}|_{\widehat{M}} = -\partial_{x_i} \otimes \mathrm{Id}_M, \tag{8}$$

$$(E_{n,n} - E_{n+1,n+1})|_{\widehat{M}} = (D + x_n\partial_{x_n}) \otimes \mathrm{Id}_M + \mathrm{Id}_{\mathscr{A}} \otimes (I_n + E_{n,n})|_M \tag{9}$$

for $i, j \in \overline{1,n}$ with $i \neq j$. The first factors in the first terms of (6)–(9) are from differentiating (1).

Moreover,

$$\overline{M} = U(\mathfrak{sl}(n+1))(1 \otimes M) \tag{10}$$

forms an $\mathfrak{sl}(n+1)$-submodule. Denote by $\mathscr{A}_k$ the subspace of polynomials in $\mathscr{A}$ with degree k.

Set

$$\widehat{M}_k = \mathscr{A}_k \otimes_{\mathbb{C}} M, \quad \overline{M}_k = \overline{M} \bigcap \widehat{M}_k. \tag{11}$$

According to (6) and (7), $\widehat{M}_k$ is the tensor module of the $\mathfrak{sl}(n)$-modules $\mathscr{A}_k$ and M, and $\overline{M}_k$ is an $\mathfrak{sl}(n)$-submodule of $\widehat{M}_k$. If M is a simple $\mathfrak{sl}(n)$-module, then $\overline{M}$ is a simple $\mathfrak{sl}(n+1)$-module (cf. [34, 35]). When M is a finite-dimensional simple $\mathfrak{sl}(n)$-module, Zhao and the second author [35] found a sufficient condition for $\widehat{M}$ to be a simple $\mathfrak{sl}(n+1)$-module; equivalently $\widehat{M} = \overline{M}$.

When M is a highest-weight simple $\mathfrak{sl}(n)$-module, $\overline{M}$ is a highest-weight simple $\mathfrak{sl}(n+1)$-module for some highest weight λ. As we will see that λ can be any dominant integral weight of $\mathfrak{sl}(n+1)$. Starting from $n=2$ and a finite-dimensional irreducible first-order differential operator representation of $\mathfrak{sl}(2)$, we apply (5)–(9) inductively in this paper to obtain a first-order differential operator realization of any finite-dimensional irreducible representation of $\mathfrak{sl}(n+1)$ in $n(n+1)/2$ variables. Denote by $\mathbb{N}$ the set of nonnegative integers. Throughout the whole text, we use the notaion $V_n(\lambda)$ to stand for a highest-weight simple $\mathfrak{sl}(n+1)$-module for some highest weight λ.

Let $\{\omega_1,\cdots,\omega_n\}$ represent the fundamental dominat weights for $\mathfrak{sl}(n+1)$, suppose that $\lambda=\sum_{i=1}^n k_i\omega_i$ is a dominant integral weight of $\mathfrak{sl}(n+1)$. We find that the degree of $\overline{M}$ equals:

$$\max\{r\in\mathbb{N}\mid \overline{M}_r\neq\{0\}\}=\sum_{i=1}^n k_i=|\lambda|. \tag{12}$$

Moreover, we have the $\mathfrak{sl}(n)$-modules

$$\overline{M}_0=M=V_{n-1}(\sum_{i=1}^{n-1}k_{i+1}\omega_i),\;\;\overline{M}_{|\lambda|}=V_{n-1}(\sum_{i=1}^{n-1}k_i\omega_i). \tag{13}$$

Indeed, the $\mathfrak{sl}(n)$-modules $\overline{M}_r$ and $\overline{M}_{|\lambda|-r}$ have the same number of irreducible components. According to Weyl's character formula we have

$$d_n(\lambda)=\dim V_n(\lambda)=\prod_{1\leq i<j\leq n+1}\frac{\sum_{r=i}^{j-1}k_r+j-i}{j-i} \tag{14}$$

(e.g., cf. [16, 34]). If $\lambda=k\omega_i$, then the equation $\sum\limits_{r=0}^{|\lambda|}\dim\overline{M}_r=d_n(\lambda)$ is directly equivalent to the well-known classical combinatorial identity

$$\sum_{r=0}^{k}\binom{k-r+i-1}{i-1}\binom{r+n-i}{n-i}=\binom{k+n}{k}$$

(e.g., cf. p. 10 in [27]). As a byproduct, we prove

$$\sum_{s=0}^{n-1}(-1)^s\binom{\sum_{j=s+1}^{n}(k_j+1)}{n}d_{n-1}(\sum_{i=1}^{s}k_i\omega_i+\omega_s+\sum_{i=s}^{n-1}k_{i+1}\omega_i)=d_n(\sum_{\ell=1}^{n}k_\ell\omega_\ell).$$

When $k_1=k_2=\cdots=k_n=k$, $\overline{M}$ is a Steinberg module and the above equation is equivalent to another well-known classical combinatorial identity

$$\sum_{s=0}^{n-1}(-1)^s\binom{(n-s)(k+1)}{n}\binom{n}{s}=(k+1)^n$$

(e.g., cf. p. 51 in [27]).

In Section 2, we present the inductive construction of the differential-operator realization of any finite-dimensional representation of $\mathfrak{sl}(n+1)$. In Section 3, we determine a basis for any finite-dimensional representation of $\mathfrak{sl}(n+1)$ and its relation with the Gelfand-Tsetlin basis. Moreover, we find all the highest-weight vectors for the irreducible $\mathfrak{sl}(n)$-components in $\widehat{M}$ and $\overline{M} = V_n(\lambda)$. Section 4 is devoted to the detailed study on the special cases when the highest weights are $k_1\omega_1 + k_2\omega_2$, $k_1\omega_1 + k_2\omega_n$ and $k\omega_i$, respectively.

2. General construction

In this section, we start from $n = 2$ and repeatedly use (5)–(9) to construct the objective representation of $\mathfrak{sl}(n+1)$. We also use the fact that the tensor algebra of two polynomial algebras is isomorphic to the polynomial algebra in all involved variables.

Let some positive integer $n > 1$ be given. Recall the special linear Lie algebra $\mathfrak{sl}(n+1)$ in (3).

Set
$$h_i = E_{n+2-i,n+2-i} - E_{n+1-i,n+1-i}, \qquad i \in \overline{1,n}. \tag{15}$$

The subspace
$$H = \sum_{i=1}^{n} \mathbb{C}h_i \tag{16}$$

forms a Cartan subalgebra of $\mathfrak{sl}(n+1)$. We choose

$$\{E_{i,j} \mid 1 \leq j < i \leq n+1\} \text{ as positive root vectors.} \tag{17}$$

In particular, we have

$$\{E_{n+2-i,n+1-i} \mid i = 1, 2, \cdots, n\} \text{ as positive simple root vectors.} \tag{18}$$

Accordingly,
$$\{E_{i,j} \mid 1 \leq i < j \leq n+1\} \text{ are negative root vectors} \tag{19}$$

and we have

$$\{E_{n+1-i,n+2-i} \mid i = 1, 2, \cdots, n\} \text{ as negative simple root vectors.} \tag{20}$$

The representation formulas in [8] were given only for the elements in (15), (18) and (20). In particular,

$$\mathfrak{sl}(n+1)_- = \sum_{1\leq i<j\leq n+1} \mathbb{C}E_{i,j} \text{ and } \mathfrak{sl}(n+1)_+ = \sum_{1\leq i<j\leq n+1} \mathbb{C}E_{j,i} \tag{21}$$

are the nilpotent subalgebra of negative root vectors and the nilpotent subalgebra of positive root vectors, respectively. A *singular vector* of an $\mathfrak{sl}(n+1)$-module V is a weight vector annihilated by the elements all positive root vectors. The fundamental weights $\omega_i \in H^*$ are
$$\omega_i(h_r) = \delta_{i,r}. \tag{22}$$

Set
$$\mathscr{G}_0 = \mathfrak{sl}(n) + \mathbb{C}(E_{n,n} - E_{n+1,n+1}),\ \ \mathscr{G}_+ = \sum_{i=1}^{n} \mathbb{C}E_{n+1,i},\ \mathscr{G}_- = \sum_{j=1}^{n} \mathbb{C}E_{j,n+1}. \tag{23}$$

Then $\mathscr{G}_0$ is a Lie subalgebra of $\mathfrak{sl}(n+1)$ and $\mathscr{G}_\pm$ are abelian Lie subalgebras of $\mathfrak{sl}(n+1)$. Moreover,
$$\mathfrak{sl}(n+1) = \mathscr{G}_- \oplus \mathscr{G}_0 \oplus \mathscr{G}_+. \tag{24}$$

Let M be an $\mathfrak{sl}(n)$-module, recall the $\mathfrak{sl}(n+1)$-module $\widehat{M}$ and its submodule $\overline{M}$ defined in (4)–(11). According to (6)–(9),

$$\mathscr{G}_+(1\otimes M)=\{0\},\qquad U(\mathscr{G}_0)(1\otimes M)=1\otimes M. \tag{25}$$

By the PBW theorem,

$$U(\mathfrak{sl}(n+1))(1\otimes M)=U(\mathscr{G}_-)U(\mathscr{G}_0)U(\mathscr{G}_+)(1\otimes M)=U(\mathscr{G}_-)(1\otimes M). \tag{26}$$

Furthermore, (11) and the first equation in (8) imply

$$\overline{M}_r=\mathscr{G}_-^r(1\otimes M). \tag{27}$$

First we consider the case $n=2$. For $k\in\mathbb{N}$, we take

$$M=\sum_{i=0}^{k}\mathbb{C}y^i\subset\mathbb{C}[y], \tag{28}$$

be the k-dimensional $\mathfrak{sl}(2)$-module with the formulas given by

$$E_{1,2}|_M=y^2\partial_y-ky,\ E_{2,1}|_M=-\partial_y,\ (E_{1,1}-E_{2,2})|_M=2y\partial_y-k. \tag{29}$$

Now we convert M into a simple $\mathfrak{gl}(2)$-module by letting

$$(E_{1,1}+E_{2,2})|_M=c\,\mathrm{Id}_M, \tag{30}$$

where $c\in\mathbb{C}$ is an arbitrary constant. Applying (5)–(9). Note

$$\widehat{M}=\mathbb{C}[x_1,x_2]\otimes_{\mathbb{C}}M=\mathrm{Span}\{x_1^{\alpha_1}x_2^{\alpha_2}y^\beta\mid\alpha_1,\alpha_2\in\mathbb{N},\ \beta\in\overline{0,k}\}\subset\mathbb{C}[x_1,x_2,y]. \tag{31}$$

Then $\overline{M}=U(\mathfrak{sl}(3)_-)(1\otimes M)$ is a simple $\mathfrak{sl}(3)$-submodule of $\widehat{M}$ with highest weight vector $1\otimes 1$ whose weight is

$$\lambda=-(\frac{3}{2}\,c+\frac{k}{2})\,\omega_1+k\,\omega_2. \tag{32}$$

Moreover, it is finite-dimensional if and only if we choose $c\in\frac{-2\mathbb{N}-k}{3}$. Denote

$$k_1=-(\frac{3}{2}\,c+\frac{k}{2}),\ k_2=k, \tag{33}$$

where k_1 to be any nonnegative integer. Fix k_1, we have the following full projective oscillator representation of $\mathfrak{sl}(3)$:

$$E_{1,2}|_{\widehat{M}}=x_1\partial_{x_2}+y^2\partial_y-k_2y,\ \ E_{2,1}|_{\widehat{M}}=x_2\partial_{x_1}-\partial_y, \tag{34}$$

$$E_{1,3}|_{\widehat{M}}=x_1(x_1\partial_{x_1}+x_2\partial_{x_2}-k_1)+(x_1+x_2y)(y\partial_y-k_2), \tag{35}$$

$$E_{2,3}|_{\widehat{M}}=x_2(x_1\partial_{x_1}+x_2\partial_{x_2}-k_1)-(x_1+x_2y)\partial_y, \tag{36}$$

$$E_{3,1}|_{\widehat{M}}=-\partial_{x_1},\qquad E_{3,2}|_{\widehat{M}}=-\partial_{x_2}, \tag{37}$$

$$(E_{1,1}-E_{2,2})|_{\widehat{M}}=x_1\partial_{x_1}-x_2\partial_{x_2}+2y\partial_y-k_2, \tag{38}$$

$$(E_{2,2}-E_{3,3})|_{\widehat{M}}=x_1\partial_{x_1}+2x_2\partial_{x_2}-y\partial_y-k_1. \tag{39}$$

Recall (11) and (12). We have:

Proposition 2.1. *Let* $0 \le r \le k_1$, $\overline{M}_r = \widehat{M}_r$. *When* $k_1 < r \le k_1 + k_2$,

$$\mathcal{B}_r = \{x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r-k_1} y^\beta | \alpha_1, \alpha_2 \in \mathbb{N},\ \alpha_1 + \alpha_2 = k_1; \beta \in \overline{0, k_1 + k_2 - r}\} \quad (40)$$

forms a basis of $\overline{M}_r$. *If* $r > k_1 + k_2$, $\overline{M}_r = \{0\}$. *In particular,*

$$\{x_1^{\alpha_1} x_2^{\alpha_2} y^\beta | \alpha_1, \alpha_2 \in \mathbb{N},\ \alpha_1 + \alpha_2 \le k_1;\ \beta \in \overline{0, k_2}\} \bigcup (\bigcup_{r=k_1+1}^{k_1+k_2} \mathcal{B}_r) \quad (41)$$

is a basis of the simple $\mathfrak{sl}(3)$*-module* $V_2(k_1\omega_1 + k_2\omega_2) = \overline{M}$.

Proof. We prove the theorem by induction on r. If $r = 0$, $\overline{M}_0 = 1 \otimes M = \widehat{M}_0$. Suppose $\overline{M}_r = \widehat{M}_r$ for $r < \ell \le k_1$. Consider $r = \ell$. For any $(\alpha_1, \alpha_2, \beta) \in \mathbb{N}^3$ with $\alpha_1 + \alpha_2 = \ell - 1$ and $\beta \leqslant k_2 - 1$, we have

$$x_1^{\alpha_1} x_2^{\alpha_2} y^\beta, x_1^{\alpha_1} x_2^{\alpha_2} y^{\beta+1} \in \overline{M}_{\ell-1}. \quad (42)$$

Moreover, (35) and (36) give

$$\begin{pmatrix} E_{13}(x_1^{\alpha_1} x_2^{\alpha_2} y^\beta) \\ E_{23}(x_1^{\alpha_1} x_2^{\alpha_2} y^{\beta+1}) \end{pmatrix} = \begin{pmatrix} (\ell + \beta - k_1 - k_2 - 1) & \beta - k_2 \\ -\beta - 1 & \ell - \beta - k_1 - 2 \end{pmatrix} \begin{pmatrix} x_1^{\alpha_1+1} x_2^{\alpha_2} y^\beta \\ x_1^{\alpha_1} x_2^{\alpha_2+1} y^{\beta+1} \end{pmatrix} \quad (43)$$

and
$$x_1^{\alpha_1+1} x_2^{\alpha_2} y^{k_2} = \frac{1}{\ell - k_1 - 1} E_{13}(x_1^{\alpha_1} x_2^{\alpha_2} y^{k_2}) \in \overline{M}_\ell, \quad (44)$$

$$x_1^{\alpha_1} x_2^{\alpha_2+1} = \frac{1}{\ell - k_1 - 1} E_{23}(x_1^{\alpha_1} x_2^{\alpha_2}) \in \overline{M}_\ell. \quad (45)$$

Since

$$\begin{vmatrix} (\ell + \beta - k_1 - k_2 - 1) & \beta - k_2 \\ -\beta - 1 & \ell - \beta - k_1 - 2 \end{vmatrix} = (\ell - k_1 - 1)(\ell - k_1 - k_2 - 2) \ne 0, \quad (46)$$

Solving (43) yields
$$x_1^{\alpha_1+1} x_2^{\alpha_2} y^\beta, x_1^{\alpha_1} x_2^{\alpha_2+1} y^{\beta+1} \in \overline{M}_\ell. \quad (47)$$

According to (44)–(47), $\overline{M}_\ell = \widehat{M}_\ell$. By induction, $\overline{M}_r = \widehat{M}_r$ for $r \in \overline{0, k_1}$.
For technical convenience, we allow $r = k_1$ in (40). Then

$$\mathcal{B}_{k_1} = \{x_1^{\alpha_1} x_2^{\alpha_2} y^\beta | \alpha_1, \alpha_2 \in \mathbb{N},\ \alpha_1 + \alpha_2 = k_1;\ \beta \in \overline{0, k_2}\} \text{ is a basis of } \overline{M}_{k_1} = \widehat{M}_{k_1}. \quad (48)$$

In this case, $\mathscr{G}_- = \mathbb{C}E_{1,3} + \mathbb{C}E_{2,3}$ (cf. (23)). For an element in (40), we have

$$\begin{aligned} & E_{13}(x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r-k_1} y^\beta) \\ = & (x_1(x_1\partial_{x_1} + x_2\partial_{x_2} - k_1) + (x_1 + x_2 y)(y\partial_y - k_2))(x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r-k_1} y^\beta) \\ = & (r + \beta - k_1 - k_2) x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r+1-k_1} y^\beta \end{aligned} \quad (49)$$

and

$$\begin{aligned} & E_{23}(x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r-k_1} y^\beta) \\ = & (x_2(x_1\partial_{x_1} + x_2\partial_{x_2} - k_1) - (x_1 + x_2 y)\partial_y)(x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r-k_1} y^\beta) \\ = & -\beta x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{r+1-k_1} y^\beta \end{aligned} \quad (50)$$

by (35) and (36). Note that if $\beta = k_1 + k_2 - r$, then (49) becomes 0. Thus if $\mathcal{B}_r$ is a basis of $\overline{M}_r$, (49) and (50) implies that $\mathcal{B}_{r+1}$ is a basis of $\overline{M}_{r+1}$ by (27). Starting from $r = k_1$ and (48), we obtain that $\mathcal{B}_r$ is a basis of $\overline{M}_r$ for any $r \in \overline{k_1 + 1, k_1 + k_2}$ by induction. Since

$$\mathcal{B}_{k_1+k_2} = \left\{ x_1^{\alpha_1} x_2^{\alpha_2} (x_1 + x_2 y)^{k_2} | \alpha_1, \alpha_2 \in \mathbb{N},\ \alpha_1 + \alpha_2 = k_1 \right\}, \tag{51}$$

(49) and (50) with $r = k_1 + k_2$ and $\beta = 0$ imply

$$E_{13}(\mathcal{B}_{k_1+k_2}) = E_{23}(\mathcal{B}_{k_1+k_2}) = 0. \tag{52}$$

By (27), $\overline{M}_r = \{0\}$ for $k_1 + k_2 < r \in \mathbb{N}$. ■

Note that
$$\dim \overline{M}_r = \begin{cases} (r+1)(k_2+1) & \text{if } 0 \leqslant r \leqslant k_1, \\ (k_1+1)(k_1+k_2+1-r) & \text{if } k_1 < r \leqslant k_1 + k_2. \end{cases} \tag{53}$$

Moreover, (14) gives

$$d_2(k_1\omega_1 + k_2\omega_2) = \frac{(k_1+1)(k_2+1)(k_1+k_2+2)}{2}. \tag{54}$$

Thus the equation $\sum_{r=0}^{k_1+k_2} \dim \overline{M}_r = \dim \overline{M} = \dim V_2(\lambda)$ becomes

$$\sum_{r=0}^{k_1} (r+1)(k_2+1) + \sum_{r=k_1+1}^{k_1+k_2} (k_1+1)(k_1+k_2+1-r) = \frac{(k_1+1)(k_2+1)(k_1+k_2+2)}{2}. \tag{55}$$

This completes the realization of the simple $\mathfrak{sl}(3)$-module $V_2(k_1\omega_1 + k_2\omega_2) = \overline{M}$ in the variables y, x_1, x_2 (cf. (32) and (33)). Let

$$\mathscr{C}_n = \mathbb{C}[x_{i,j} | 1 \leqslant j \leqslant i \leqslant n] \tag{56}$$

be the polynomial algebra in $n(n+1)/2$ variables. For convenience, we make the following convention

$$x_{i,j} = \begin{cases} \text{variable } x_{i,j}, & 1 \leqslant j \leqslant i, \\ -1, & j = i+1, \\ 0, & \text{otherwise}. \end{cases} \qquad i \in \overline{1, n}, \tag{57}$$

Moreover, we use the notation $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,i}) \in \mathbb{N}^i$ for $i \in \overline{1, n}$ and

$$\alpha = (\alpha_1, \alpha_2, .., \alpha_n) \in \mathbb{N}^{\frac{n(n+1)}{2}}.$$

Denote by $X_i^{\alpha_i}$ and X^α the monomials

$$X_i^{\alpha_i} = x_{i,1}^{\alpha_{i,1}} x_{i,2}^{\alpha_{i,2}} \cdots x_{i,i}^{\alpha_{i,i}} \quad \text{and} \quad X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}.$$

For $1 \leqslant j \leqslant i \leqslant n$ and $\Theta = (\theta_1, \theta_2, \cdots, \theta_{i-j+1}) \in \mathbb{N}^{i-j+1}$, we denote

$$d_{i,j}(\Theta) = \begin{vmatrix} x_{j,\theta_1} & x_{j,\theta_2} & \cdots & x_{j,\theta_{i-j+1}} \\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{i-j+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{i,\theta_1} & x_{i,\theta_2} & \cdots & x_{i,\theta_{i-j+1}} \end{vmatrix}. \tag{58}$$

In particular, we take

$$D_{i,j}(s) = d_{i,j}(s, j+1, j+2, \cdots, i)$$
$$= \begin{vmatrix} x_{j,s} & -1 & 0 & \cdots & 0 \\ x_{j+1,s} & x_{j+1,j+1} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i-1,s} & x_{i-1,j+1} & x_{i-1,j+2} & \cdots & -1 \\ x_{i,s} & x_{i,j+1} & x_{i,j+2} & \cdots & x_{i,i} \end{vmatrix}. \tag{59}$$

Lemma 2.2. *For $1 \leqslant s \leqslant j < i \leqslant n$, the following equations hold,*

(i) $D_{i,j}(s) = x_{i,s} + \sum\limits_{t=j+1}^{i} x_{i,t} D_{t-1,j}(s)$;

(ii) $D_{i,j}(s) = x_{i,s} + \sum\limits_{t=j}^{i-1} x_{t,s} D_{i,t+1}(t+1)$.

Proof. The algebraic cofactor of $x_{i,t}, (j+1 \leqslant t \leqslant i)$ in $D_{i,j}(s)$ is

$$A_{i,t} = (-1)^{i+t} \times$$
$$\begin{vmatrix} x_{j,s} & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ x_{j+1,s} & x_{j+1,j+1} & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{t-1,s} & x_{t-1,j+1} & x_{t-1,j+2} & \cdots & x_{t-1,t-1} & 0 & 0 & \cdots & 0 \\ x_{t,s} & x_{t,j+1} & x_{t,j+2} & \cdots & x_{t,t-1} & -1 & 0 & \cdots & 0 \\ x_{t+1,s} & x_{t+1,j+1} & x_{t+1,j+2} & \cdots & x_{t+1,t-1} & x_{t+1,t+1} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i-1,s} & x_{i-1,j+1} & x_{i-1,j+2} & \cdots & x_{i-1,t-1} & x_{i-1,t+1} & x_{i-1,t+2} & \cdots & -1 \end{vmatrix}$$
$$= \begin{vmatrix} x_{j,s} & -1 & 0 & \cdots & 0 \\ x_{j+1,s} & x_{j+1,j+1} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{t-1,s} & x_{t-1,j+1} & x_{t-1,j+2} & \cdots & x_{t-1,t-1} \end{vmatrix} = D_{t-1,j}(s).$$

Hence expanding the determinant (59) according to the last row, we get

$$D_{i,j}(s) = x_{i,s} + \sum_{t=j+1}^{i} x_{i,t} A_{i,t} = x_{i,s} + \sum_{t=j+1}^{i} x_{i,t} D_{t-1,j}(s).$$

which proves (i). Expanding the determinant (59) according to the first row, we obtain

$$\begin{aligned} D_{i,j}(s) &= x_{j,s} D_{i,j+1}(j+1) + D_{i,j+1}(s) \\ &= x_{j,s} D_{i,j+1}(j+1) + x_{j+1,s} D_{i,j+2}(j+2) + D_{i,j+2}(s) \\ &= \cdots = \sum_{t=j}^{i-1} x_{t,s} D_{i,t+1}(t+1) + D_{i,i}(s). \end{aligned}$$

Note that $D_{i,i}(s) = x_{i,s}$, therefore (ii) holds. ■

Recall our realization of the simple $\mathfrak{sl}(3)$-module $V_2(k_1\omega_1+k_2\omega_2)$ in (28)–(55). We redenote

$$x_{1,1}=y,\quad x_{2,1}=x_1,\quad x_{2,2}=x_2. \tag{60}$$

Thus we have used (4)–(12) to realize the $\mathfrak{sl}(3)$-module $V_2(k_1\omega_1+k_2\omega_2)$ as a $\mathfrak{sl}(3)$-submodule of the $\mathfrak{sl}(3)$-module $\mathscr{C}_2=\mathbb{C}[x_{1,1},x_{2,1},x_{2,2}]$, whose representation formulas are given in (34)–(39) with $\widehat{M}$ replaced by $\mathscr{C}_2$. Suppose now that we have used (4)–(12) successively to realize the simple $\mathfrak{sl}(n)$-module

$$M=V_{n-1}(k_2\omega_1+k_3\omega_2+\cdots+k_n\omega_{n-1})$$

as a simple submodule of the $\mathfrak{sl}(n)$-module $\mathscr{C}_{n-1}=\mathbb{C}[x_{i,j}|1\leqslant j\leqslant i\leqslant n-1]$.

Fix $c_n\in\mathbb{C}$ and impose
$$I_n|_M=c_n\mathrm{Id}_M. \tag{61}$$

Then M become a simple $\mathfrak{gl}(n)$-module. Use (4)–(12) with

$$x_1=x_{n,1},\quad x_2=x_{n,2},\quad \cdots,\quad x_n=x_{n,n}. \tag{62}$$

Now
$$\begin{aligned}\widehat{M}&=\mathbb{C}[x_{n,1},x_{n,2},\cdots,x_{n,n}]\otimes_{\mathbb{C}}M\\ &\subset \mathbb{C}[x_{n,1},x_{n,2},\cdots,x_{n,n}]\otimes_{\mathbb{C}}\mathscr{C}_{n-1}\\ &=\mathbb{C}[x_{n,1},x_{n,2},\cdots,x_{n,n}]\otimes_{\mathbb{C}}\mathbb{C}[x_{i,j}|1\leqslant j\leqslant i\leqslant n-1]\\ &=\mathbb{C}[x_{i,j}|1\leqslant j\leqslant i\leqslant n]=\mathscr{C}_n.\end{aligned} \tag{63}$$

Then $\overline{M}=V_n(\lambda)$ is a simple module with highest weight vector whose weight is

$$\lambda=-\frac{1}{n}((n+1)c_n+\sum_{i=1}^{n-1}(n-i)k_{i+1})\omega_1+k_2\omega_2+k_3\omega_3+\cdots+k_n\omega_n. \tag{64}$$

In particular, we can choose appropriate $c_n\in\mathbb{C}$ such that

$$k_1=-\frac{1}{n}((n+1)c_n+\sum_{i=1}^{n-1}(n-i)k_{i+1})\in\mathbb{N}. \tag{65}$$

Thus $\overline{M}$ is a finite-dimensional simple $\mathfrak{sl}(n+1)$-module. Indeed, k_1 can take any nonnegative integer.

For convenience, we extend the representation of $\mathfrak{sl}(n+1)$ on $\widehat{M}$ to the representation of $\mathfrak{sl}(n+1)$ on $\mathscr{C}_n$ by (4)–(9) with M replaced by $\mathscr{C}_{n-1}$ and $\widehat{M}$ by $\mathscr{C}_n$. Fix $c_{n+1}\in\mathbb{C}$. We make $\mathscr{C}_n$ as a $\mathfrak{gl}(n+1)$-module by imposing

$$I_{n+1}|_{\mathscr{C}_n}=c_{n+1}\mathrm{Id}_{\mathscr{C}_n}. \tag{66}$$

Theorem 2.3. *Take the convention in* (57). *The representation formulas of* $\mathfrak{gl}(n+1)$ *on* $\mathscr{C}_n$ *are as follows:*

$$E_{i,i}|_{\mathscr{C}_n}=\frac{1}{n+1}c_{n+1}-c_n+\sum_{r=i}^{n}(x_{r,i}\partial_{x_{r,i}}-k_{n-r+1})-\sum_{s=1}^{i-1}x_{i-1,s}\partial_{x_{i-1,s}}$$
$$\textit{for}\ \ i\in\overline{1,n+1}, \tag{67}$$

$$E_{i,j}|_{\mathscr{C}_n}=\sum_{r=i-1}^{n}x_{r,i}\partial_{x_{r,j}}\qquad \textit{for}\ \ 1\leqslant j<i\leqslant n+1, \tag{68}$$

and for $1 \leqslant i < j \leqslant n+1$:

$$E_{i,j}|_{\mathscr{C}_n} = \sum_{r=j}^{n} x_{r,i}\partial_{x_{r,j}} + \sum_{s=i-1}^{j-1}\sum_{t=1}^{s} D_{j-1,s}(t)x_{s,i}\partial_{x_{s,t}} - \sum_{s=i}^{j-1} D_{j-1,s}(i)k_{n+1-s}. \quad (69)$$

Proof. When $n = 2$, (67)–(69) become

$$E_{12}|_{\mathscr{C}_2} = x_{2,1}\partial_{x_{2,2}} + x_{1,1}^2\partial_{x_{1,1}} - k_2x_{1,1}, \ \ E_{21}|_{\mathscr{C}_2} = x_{2,2}\partial_{x_{2,1}} - \partial_{x_{1,1}}, \quad (70)$$

$$E_{13}|_{\mathscr{C}_2} = x_{2,1}(x_{2,1}\partial_{x_{2,1}} + x_{2,2}\partial_{x_{2,2}} - k_1) + (x_{2,1} + x_{2,2}x_{1,1})(x_{1,1}\partial_{x_{1,1}} - k_2), \quad (71)$$

$$E_{23}|_{\mathscr{C}_2} = x_{2,2}(x_{2,1}\partial_{x_{2,1}} + x_{2,1}\partial_{x_{2,1}} - k_1) - (x_{2,1} + x_{2,2}x_{1,1})\partial_{x_{1,1}}, \quad (72)$$

$$E_{31}|_{\mathscr{C}_2} = -\partial_{x_{2,1}}, \ \ E_{32}|_{\mathscr{C}_2} = -\partial_{x_{2,2}}, \quad (73)$$

$$E_{11}|_{\mathscr{C}_2} = x_{2,1}\partial_{x_{2,1}} + x_{1,1}\partial_{x_{1,1}} + \frac{1}{3}c_3 - c_2 - k_1 - k_2, \quad (74)$$

$$E_{22}|_{\mathscr{C}_2} = x_{2,2}\partial_{x_{2,2}} - x_{1,1}\partial_{x_{1,1}} + \frac{1}{3}c_3 - c_2 - k_1, \quad (75)$$

$$E_{33}|_{\mathscr{C}_2} = -x_{2,1}\partial_{x_{2,1}} - x_{2,2}\partial_{x_{2,2}} + \frac{1}{3}c_3 - c_2. \quad (76)$$

Under the identification (60), (70)–(73) are exactly (34)–(37). Moreover, (74) − (75) is (38) and (75) − (76) is (39). According to (33), the sum of (74)–(76) is exactly (66) with $n = 2$. So the theorem holds for the case of $\mathfrak{gl}(3)$.

Suppose that the theorem holds for $\mathfrak{sl}(n)$. In the rest of this paper, we make a convention that

$$\text{differential operators on } \mathscr{C}_{n-1} \text{ are also viewed as those on } \mathscr{C}_n. \quad (77)$$

Let $i \in \overline{1,n}$. First

$$(E_{i,i} - E_{n+1,n+1})|_{\mathscr{C}_n} = \sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}} + x_{n,i}\partial_{x_{n,i}} + (I_n + E_{i,i})|_{\mathscr{C}_{n-1}} \quad (78)$$

by (7) and (9). Note $M = V_{n-1}(k_2\omega_1 + k_3\omega_2 + \cdots + k_n\omega_{n-1})$. Applying (67) with n replaced by $n-1$, we get

$$(I_n + E_{i,i})|_{\mathscr{C}_{n-1}} = c_n + \frac{1}{n}c_n - c_{n-1} + \sum_{s=i}^{n-1}(x_{s,i}\partial_{x_{s,i}} - k_{n-s+1}) - \sum_{s=1}^{i-1} x_{i-1,s}\partial_{x_{i-1,s}}. \quad (79)$$

According to (65), $$c_n = -\frac{1}{n+1}(nk_1 + \sum_{i=1}^{n-1}(n-i)k_{i+1}). \quad (80)$$

By our inductive construction,

$$c_{n-1} = -\frac{1}{n}((n-1)k_2 + \sum_{i=1}^{n-2}(n-1-i)k_{i+2}). \quad (81)$$

Thus $$\frac{n+1}{n}c_n - c_{n-1} = -k_1. \quad (82)$$

The expressions (78)–(79) and (82) imply

$$(E_{i,i} - E_{n+1,n+1})_{\mathscr{C}_n} = \sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}} + \sum_{s=i}^{n}(x_{s,i}\partial_{x_{s,i}} - k_{n-s+1}) - \sum_{s=1}^{i-1} x_{i-1,s}\partial_{x_{i-1,s}}. \quad (83)$$

Hence
$$E_{n+1,n+1}|_{\mathscr{C}_n} = \frac{1}{n+1}(\sum_{r=1}^{n+1} E_{r,r} - \sum_{r=1}^{n}(E_{r,r} - E_{n+1,n+1}))|_{\mathscr{C}_n}$$
$$= \frac{1}{n+1}c_{n+1} - c_n - \sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}} \tag{84}$$
by (83), and
$$E_{i,i}|_{\mathscr{C}_n} = (E_{i,i} - E_{n+1,n+1} + E_{n+1,n+1})|_{\mathscr{C}_n}$$
$$= \frac{1}{n+1}c_{n+1} - c_n + \sum_{s=i}^{n}(x_{s,i}\partial_{x_{s,i}} - k_{n-s+1}) - \sum_{s=1}^{i-1} x_{i-1,s}\partial_{x_{i-1,s}}. \tag{85}$$
Assume $1 \leqslant j < i \leqslant n+1$. If $i = n+1$,
$$E_{n+1,j}|_{\mathscr{C}_n} = -\partial_{x_{n,j}} = x_{n,n+1}\partial_{x_{n,j}}, \tag{86}$$
where the last equation dues to the convention (57) that $x_{n,n+1} = -1$. When $i < n+1$, we have
$$E_{i,j}|_{\mathscr{C}_n} = x_{n,i}\partial_{x_{n,j}} + E_{i,j}|_{\mathscr{C}_{n-1}} = \sum_{r=i-1}^{n} x_{r,i}\partial_{x_{r,j}}, \tag{87}$$
where the last equation is from inductive assumption.

Let $1 \leqslant i < j \leqslant n+1$. We want to prove (69). Note that
$$\sum_{r=i+1}^{n}\sum_{s=i}^{r-1}\sum_{t=1}^{s} = \sum_{s=i}^{n-1}\sum_{t=1}^{s}\sum_{r=s+1}^{n}, \qquad \sum_{r=i+1}^{n}\sum_{s=i}^{r-1} = \sum_{s=i}^{n-1}\sum_{r=s+1}^{n}. \tag{88}$$
Consider $j = n+1$. By the inductive assumption and (88), we have
$$E_{i,n+1}|_{\mathscr{C}_n} = x_{n,i}\sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}} + x_{n,i}I_n|_{\mathscr{C}_{n-1}} + \sum_{r=1}^{n} x_{n,r}E_{i,r}|_{\mathscr{C}_{n-1}}$$
$$= x_{n,i}(c_n + \sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}}) + \sum_{r=1}^{n} x_{n,r}E_{i,r}|_{\mathscr{C}_{n-1}}$$
$$= x_{n,i}(c_n + \sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}}) + \sum_{r=1}^{i-1} x_{n,r}E_{i,r}|_{\mathscr{C}_{n-1}} + x_{n,i}E_{i,i}|_{\mathscr{C}_{n-1}} + \sum_{r=i+1}^{n} x_{n,r}E_{i,r}|_{\mathscr{C}_{n-1}}$$
$$= x_{n,i}(c_n + \sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}}) + \sum_{r=1}^{i-1}\sum_{s=i-1}^{n-1} x_{n,r}x_{s,i}\partial_{x_{s,r}}$$
$$+ x_{n,i}\left(\frac{1}{n}c_n - c_{n-1} + \sum_{s=i}^{n-1}(x_{s,i}\partial_{x_{s,i}} - k_{n-s+1}) - \sum_{s=1}^{i-1} x_{i-1,s}\partial_{x_{i-1,s}}\right)$$
$$+ \sum_{r=i+1}^{n} x_{n,r}\left(\sum_{s=r}^{n-1} x_{s,i}\partial_{x_{s,r}} + \sum_{s=i-1}^{r-1}\sum_{t=1}^{s} D_{r-1,s}(t)x_{s,i}\partial_{x_{s,t}} - \sum_{s=i}^{r-1} D_{r-1,s}(i)k_{n-s+1}\right)$$
$$= \sum_{s=i}^{n-1}\sum_{t=1}^{s}\left(x_{n,t} + \sum_{r=s+1}^{n} x_{n,r}D_{r-1,s}(t)\right)x_{s,i}\partial_{x_{s,t}} - \sum_{s=i}^{n-1}\left(x_{n,i} + \sum_{r=s+1}^{n} x_{n,r}D_{r-1,s}(i)\right)k_{n-s+1}$$
$$- \sum_{t=1}^{i-1}\left(x_{n,t} + x_{n,i}x_{i-1,t} + \sum_{r=i+1}^{n} x_{n,r}D_{r-1,i-1}(t)\right)\partial_{x_{i-1,t}} - k_1x_{n,i} + x_{n,i}\sum_{r=1}^{n} x_{n,r}\partial_{x_{n,r}}$$

$$=\sum_{s=i-1}^{n}\sum_{t=1}^{s}\left(x_{n,t}+\sum_{r=s+1}^{n}x_{n,r}D_{r-1,s}(t)\right)x_{s,i}\partial_{x_{s,t}}-\sum_{s=i}^{n}\left(x_{n,i}+\sum_{r=s+1}^{n}x_{n,r}D_{r-1,s}(i)\right)k_{n-s+1}$$

$$=\sum_{s=i-1}^{n}\sum_{t=1}^{s}D_{n,s}(t)x_{s,i}\partial_{x_{s,t}}-\sum_{s=i}^{n}D_{n,s}(i)k_{n-s+1}, \quad (89)$$

where in the last equality, we have used Lemma 2.1 (i). When $j<n+1$, inductive assumption and (69) with n replaced by $n-1$ imply

$$\begin{aligned} E_{i,j}|_{\mathscr{C}_n} &= x_{n,i}\partial_{x_{n,j}}+E_{i,j}|_{\mathscr{C}_{n-1}} \\ &= x_{n,i}\partial_{x_{n,j}}+\sum_{r=j}^{n-1}x_{r,i}\partial_{x_{r,j}}+\sum_{s=i-1}^{j-1}\sum_{t=1}^{s}D_{j-1,s}(t)x_{s,i}\partial_{x_{s,t}}-\sum_{s=i}^{j-1}D_{j-1,s}(i)k_{n+1-s} \\ &= \sum_{r=j}^{n}x_{r,i}\partial_{x_{r,j}}+\sum_{s=i-1}^{j-1}\sum_{t=1}^{s}D_{j-1,s}(t)x_{s,i}\partial_{x_{s,t}}-\sum_{s=i}^{j-1}D_{j-1,s}(i)k_{n+1-s}. \quad \blacksquare \quad (90) \end{aligned}$$

For $1\leqslant j\leqslant i\leqslant n$, we set

$$\mathbb{J}_{i,j}=\{\Theta=(\theta_1,\theta_2,\cdots,\theta_{i-j+1})\mid \theta_r\in\overline{1,n};\ d_{i,j}(\Theta)\neq 0\}. \quad (91)$$

Recall that $\lambda=\sum_{i=1}^{n}k_i\omega_i$ is a dominant integral weight of $\mathfrak{sl}(n+1)$. Denote

$$S_n(\lambda)=\left\{\prod_{1\leqslant j\leqslant i\leqslant n}\prod_{\Theta\in\mathbb{J}_{i,j}}d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)}\ \middle|\ \begin{array}{l}\alpha_{i,j}(\Theta)\in\mathbb{N};\ \text{for } j\in\overline{1,n}:\\ \sum_{i=j}^{n}\sum_{\Theta\in\mathbb{J}_{i,j}}\alpha_{i,j}(\Theta)\leqslant k_{n-j+1}\end{array}\right\}\subset\mathscr{C}_n. \quad (92)$$

Moreover, the homogeneous subset of $S_n(\lambda)$ with degree r in $\{x_{n,1},x_{n,2},...,x_{n,n}\}$ is

$$(S_n(\lambda))_r=\left\{\prod_{1\leqslant j\leqslant i\leqslant n}\prod_{\Theta\in\mathbb{J}_{i,j}}d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)}\in S_n(\lambda)\ \middle|\ \sum_{j=1}^{n}\sum_{\Theta\in\mathbb{J}_{n,j}}\alpha_{n,j}(\Theta)=r\right\}. \quad (93)$$

Recall $|\lambda|=\sum_{i=1}^{n}k_i$. Then $S_n(\lambda)=\bigcup_{r=0}^{|\lambda|}(S_n(\lambda))_r$.

According to (10)–(12) and (63), the simple $\mathfrak{sl}(n+1)$-module we have

$$V_n(\lambda)=\overline{M}\subset\widehat{M}\subset\mathscr{C}_n.$$

Theorem 2.4. *As a vector space, $V_n(\lambda)$ is spanned by $S_n(\lambda)$. Moreover, the homogeneous subspace $(V_n(\lambda))_r=\overline{M}_r$ is spanned by $(S_n(\lambda))_r$.*

Proof. See Appendix A. ∎

3. Bases and singular vectors

In this section, we determine a basis for the $\mathfrak{sl}(n+1)$-module $V_n(\lambda)$ realized in Theorem 2.4 and a connection with Gelfand-Tsetlin basis (cf. [8]). Moreover, we calculate all the $\mathfrak{sl}(n)$-singular vectors in $\widehat{M}$. In particular, the results restricted to $\widehat{M}_r$ and $V_n(\lambda)=\overline{M}$ yield to certain sum-product type combinatorial identities.

First we define a lexicographic degree

$$\text{lexdeg}(X^\alpha) = (\alpha_{1,1}, \alpha_{2,1}, \alpha_{2,2}, \cdots, \alpha_{j,1}, \alpha_{j,2} \cdots, \alpha_{j,j}, \cdots, \alpha_{n,n})$$

for monomials $\mathscr{C}_n$ according to the order:

$$(1,1) < (2,1) < (2,2) \cdots < (j,1) < (j,2) \cdots < (j,j) \cdots < (n,n). \tag{94}$$

We say that $\text{lexdeg}(X^\alpha) > \text{lexdeg}(X^{\alpha'})$ if there exists (s,t) in (94) such that $\alpha_{i,j} = \alpha'_{i,j}$ for all indexes $(i,j) < (s,t)$ and $\alpha_{s,t} > \alpha'_{s,t}$. This gives a total order. For any polynomial $f \in \mathscr{C}_n$, we define $\text{lexdeg}(f)$ to be the maximal lexicographic degree of monomial appears in f. Define a relation in $(V_n(\lambda))_r$ $(0 \leqslant r \leqslant |\lambda|)$ as follows:

$$\forall u, v \in (V_n(\lambda))_r,\ u \sim v \quad \text{if} \ \ \text{lexdeg}(u) = \text{lexdeg}(v). \tag{95}$$

This is an equivalence relation. Let

$$\mathcal{B}(\lambda)_r \quad \text{be a set of representatives of the elements in } (S_n(\lambda))_r/\sim \tag{96}$$

(cf. (92) and (93)), and $\mathcal{Q}(\lambda)_r$ be the set of monomials with maximal lexicographic degree of the elements in $\mathcal{B}(\lambda)_r$. Suppose

$$f = \prod_{1 \leqslant j \leqslant i \leqslant n} \prod_{\Theta \in \mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)} \in (S_n(\lambda))_r. \tag{97}$$

Without loss of generality, we can rearrange the index $\Theta = (\theta_1, ..., \theta_{i-j+1}) \in \mathbb{J}_{i,j}$ (cf. (2.84)) in each factor $d_{i,j}(\Theta)$ of f such that $\theta_1 < \cdots < \theta_{i-j+1}$. Thus the monomial with maximal lexcoghraphic degree in $d_{i,j}(\Theta)$ is $\prod_{s=1}^{i-j+1} x_{j+s-1,\theta_s}$. Therefore, the the monomial with maximal lexcoghraphic degree in f is

$$\prod_{1 \leqslant j \leqslant i \leqslant n} \prod_{\substack{\Theta \in \mathbb{J}_{i,j} \\ \theta_1 < \cdots < \theta_{i-j+1}}} \prod_{s=1}^{i-j+1} x_{j+s-1,\theta_s}^{\alpha_{i,j}(\Theta)}. \tag{98}$$

By the definition of $S_n(\lambda)$ in (92) and (98), we conclude that $\mathcal{Q}(\lambda)_r$ is the set of monomials $X^\alpha \in \mathscr{C}_n$ satisfying the following conditions:

$$\sum_{j=1}^{n} \alpha_{n,j} = r, \quad \text{and} \tag{99}$$

$$\begin{aligned}
&0 \leqslant \alpha_{n,1} \leqslant k_1, \\
&0 \leqslant \alpha_{n,1} + \alpha_{n,2} \leqslant \alpha_{n-1,1} + k_1 \leqslant k_1 + k_2, \\
&\cdots \\
&0 \leqslant \sum_{j=1}^{l} \alpha_{n,j} \leqslant \sum_{j=1}^{l-1} \alpha_{n-1,j} + k_1 \leqslant \cdots \leqslant \sum_{j=1}^{l-s} \alpha_{n-s,j} + \sum_{i=1}^{s} k_i \leqslant \cdots \leqslant \sum_{i=1}^{l} k_i, \\
&\cdots \\
&0 \leqslant \sum_{j=1}^{n} \alpha_{n,j} \leqslant \sum_{j=1}^{n-1} \alpha_{n-1,j} + k_1 \leqslant \cdots \leqslant \alpha_{1,1} + \sum_{i=1}^{n-1} k_i \leqslant \sum_{i=1}^{n} k_i.
\end{aligned} \tag{100}$$

By Theorem 2.4,

$$\mathcal{B}(\lambda) = \bigcup_r \mathcal{B}(\lambda)_r \tag{101}$$

is the polynomial basis of $V_n(\lambda)$.

Set
$$\mathcal{Q}(\lambda) = \bigcup_r \mathcal{Q}(\lambda)_r. \tag{102}$$

Next we associate $X^\alpha \in \mathcal{Q}(\lambda)$ to a Gelfand-Tsetlin pattern of shape λ. By Theorem 2.3, let

$$\tau_{n+1,i} = E_{n+2-i,n+2-i}(1) = \frac{1}{n+1}c_{n+1} - c_n - \sum_{s=1}^{i-1} k_s \quad \text{for } i \in \overline{2, n+1}, \tag{103}$$

and set
$$\tau_{n+1,1} = \frac{1}{n+1}c_{n+1} - c_n. \tag{104}$$

Put
$$\varepsilon_i = (0, \cdots, 0, \overset{i}{1}, 0, \cdots 0). \tag{105}$$

Then we can extend the weight λ of $\mathfrak{sl}(n+1)$ to that of $\mathfrak{gl}(n+1)$

$$\lambda = \tau_{n+1,1}\varepsilon_1 + \tau_{n+1,2}\varepsilon_2 + \cdots + \tau_{n+1,n+1}\varepsilon_{n+1} \tag{106}$$

(cf. (15), (22), (80) and (81)). Define

$$\tau_{i,j} = \tau_{i+1,j+1} + \alpha_{j-i+n,n-i+1} \quad \text{for } 1 \leqslant j \leqslant i \leqslant n \tag{107}$$

inductively. Expression (100) yields

$$\tau_{i+1,j} \geqslant \tau_{i,j} \geqslant \tau_{i+1,j+1} \qquad \text{for } 1 \leqslant j \leqslant i \leqslant n. \tag{108}$$

Therefore, we obtain a Gelfand-Tsetlin pattern of shape λ:

$$\begin{array}{ccccccc} \tau_1 & & \tau_2 & & \cdots & & \tau_{n+1} \\ & \tau_{n,1} & & \cdots & & \tau_{n,n} & \\ & & \cdots & \cdots & \cdots & & \\ & & \tau_{2,1} & & \tau_{2,2} & & \\ & & & \tau_{1,1} & & & \end{array} \tag{109}$$

So we have a one-to-one correspondence between $\mathcal{Q}(\lambda)$ and the Gelfand-Tsetlin basis. Note that there is one-to-one correspondence between $\mathcal{B}(\lambda)$ and $\mathcal{Q}(\lambda)$. In summary, we have:

Theorem 3.1. *The set $\mathcal{B}(\lambda)$ is a basis of $V_n(\lambda)$. Moreover, there is a bijection between $\mathcal{B}(\lambda)$ and the Gelfand-Tsetlin basis.*

Recall that a singular vector is a weight vector annihilated by positive root vectors, next we want to find all the $\mathfrak{sl}(n)$-singular vectors in $\widehat{M}$ and $\overline{M} = V_n(\lambda)$ by the method of determinants given in [32].

Lemma 3.2. *A rational function in $\mathcal{X} = \{x_{i,j} | 1 \leqslant j \leqslant i \leqslant n\}$ annihilated by positive root vectors $\{E_{s,t} \mid 1 \leq t < s \leq n\}$ must be a rational function in $\{D_{n,1}(1), D_{n,2}(2), \cdots, D_{n,n}(n)\}$.*

Proof. Set
$$y_{i,j} = \begin{cases} D_{n,i}(i), & \text{if } i = j, \\ x_{i,j}, & \text{if } 1 \leqslant j < i \leqslant n \end{cases} \tag{110}$$

Indeed,

$$
\begin{aligned}
y_{k,k} &= \begin{vmatrix} x_{k,k} & -1 & 0 & \cdots & 0 \\ x_{k+1,k} & x_{k+1,k+1} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1,k} & x_{n-1,k+1} & x_{n-1,k+2} & \cdots & -1 \\ x_{n,k} & x_{n,k+1} & x_{n,k+2} & \cdots & x_{n,n} \end{vmatrix} \\
&= x_{k,k}y_{k+1,k+1} + \begin{vmatrix} x_{k+1,k} & -1 & 0 & \cdots & 0 \\ x_{k+2,k} & x_{k+2,k+2} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1,k} & x_{n-1,k+2} & x_{n-1,k+3} & \cdots & -1 \\ x_{n,k} & x_{n,k+2} & x_{n.k+3} & \cdots & x_{n,n} \end{vmatrix}. \qquad (111)
\end{aligned}
$$

Thus

$$x_{k,k} = y_{k+1,k+1}^{-1}y_{k,k} + \sum_{l=k+1}^{n} g_l y_{l,k}, \qquad (112)$$

where

$$g_l\text{'s are rational functions in } \{y_{i,j}|k+1 \leqslant j \leqslant i \leqslant n\}. \qquad (113)$$

So the sets of variables $\mathcal{X}$ and

$$\mathcal{Y} = \{y_{i,j}|1 \leqslant j \leqslant i \leqslant n\}$$

are functionally equivalent. Let f be a rational function in $\mathcal{X}$, which is annihilated by positive root vectors $\{E_{s,t} \mid 1 \leq t < s \leq n\}$. Then it can also be written as a rational function in $\mathcal{Y}$. We prove that f is independent of $\{y_{i,j}|1 \leqslant j < i \leqslant n\}$ by backward induction. By (68) and (111),

$$E_{s,t}(y_{k,k}) = 0 \qquad \text{for any } 1 \leqslant k \leqslant n \text{ and } 1 \leqslant t < s \leqslant n. \qquad (114)$$

In particular,

$$E_{n,n-1}(f) = \sum_{1\leqslant j<i\leqslant n} (x_{n,n}\partial_{n,n-1} - \partial_{x_{n-1,n-1}})(y_{i,j})\frac{\partial f}{\partial y_{i,j}} = y_{n,n}\frac{\partial f}{\partial y_{n,n-1}} = 0 \qquad (115)$$

by (68) and (110). Thus f is independent of $y_{n,n-1}$. Assume that f is independent of $\{y_{i,j}|k \leqslant j < i \leqslant n\}$ for some $1 < k \leqslant n-1$. By (68), (112) and (114),

$$
\begin{aligned}
E_{k,k-1}(f) &= \sum_{1\leqslant j<i\leqslant n} E_{k,k-1}(y_{i,j})\frac{\partial f}{\partial y_{i,j}} = \sum_{l=k+1}^{n} y_{l,k}\frac{\partial f}{\partial y_{l,k-1}} + x_{k,k}\frac{\partial f}{\partial y_{k,k-1}} \\
&= \sum_{l=k+1}^{n} y_{l,k}\Big(\frac{\partial f}{\partial y_{l,k-1}} + g_l\frac{\partial f}{\partial y_{k,k-1}}\Big) + y_{k+1,k+1}^{-1}y_{k,k}\frac{\partial f}{\partial y_{k,k-1}} = 0. \qquad (116)
\end{aligned}
$$

By inductive assumption, f is independent of $\{y_{l,k}|k+1 \leqslant l \leqslant n\}$. Thus the above equation yields

$$y_{k+1,k+1}^{-1}y_{k,k}\frac{\partial f}{\partial y_{k,k-1}} = 0. \qquad (117)$$

So f is independent of $y_{k,k-1}$. Then (116) implies

$$\sum_{l=k+1}^{n} y_{l,k}\frac{\partial f}{\partial y_{l,k-1}} = 0 \Longrightarrow \frac{\partial f}{\partial y_{k+1,k-1}} = \frac{\partial f}{\partial y_{k+1,k-1}} = \cdots = \frac{\partial f}{\partial y_{n,k-1}} = 0, \qquad (118)$$

which implies that f is independent of $\{y_{i,j}|k-1 \leqslant j < i \leqslant n\}$.

By induction, f is independent of $\{y_{i,j}|1 \leqslant j < i \leqslant n\}$. Namely, f is a rational function in $\{y_{1,1}, y_{2,2}, \cdots, y_{n,n}\}$. ∎

Suppose a rational function in $\{D_{n,i}(i)|1 \leqslant i \leqslant n\}$:

$$\frac{F(D_{n,1}(1), \cdots, D_{n,n}(n))}{G(D_{n,1}(1), \cdots, D_{n,n}(n))} = H \in \mathscr{C}_n, \tag{119}$$

where F and G are polynomials in n variables. Write

$$F(D_{n,1}(1), \cdots, D_{n,n}(n)), G(D_{n,1}(1), \cdots, D_{n,n}(n)) \text{ and } H \tag{120}$$

as polynomials in $\{x_{i,j} \mid 1 \leqslant j \leqslant i \leqslant n-1\}$ with coefficients in $\mathbb{C}[x_{n,1}, \cdots, x_{n,n}]$. Denote the "constant terms" by f, g and h respectively. Then

$$f(x_{n,1}, \cdots, x_{n,n}) = g(x_{n,1}, \cdots, x_{n,n})h(x_{n,1}, \cdots, x_{n,n}). \tag{121}$$

Note that the "constant term" in $D_{n,k}(k)$ is just $x_{n,k}$. Thus we have

$$f(x_{n,1}, \cdots, x_{n,n}) = F(x_{n,1}, \cdots, x_{n,n}), \;\; g(x_{n,1}, \cdots, x_{n,n}) = G(x_{n,1}, \cdots, x_{n,n}). \tag{122}$$

So (119) becomes

$$F(x_{n,1}, \cdots, x_{n,n}) = G(x_{n,1}, \cdots, x_{n,n})h(x_{n,1}, \cdots, x_{n,n}). \tag{123}$$

Substituting $D_{n,k}(k)$ into $x_{n,k}$ in the above equation, we get

$$\frac{F(D_{n,1}(1), \cdots, D_{n,n}(n)}{G(D_{n,1}(1), \cdots, D_{n,n}(n))} = h(D_{n,1}(1), \cdots, D_{n,n}(n)) \tag{124}$$

is a polynomial of $\{D_{n,i}(i)|1 \leqslant i \leqslant n\}$. Therefore, every polynomial in $\mathscr{C}_n$ annihilated by $\{E_{s,t} \mid 1 \leqslant t < s \leqslant n\}$ must be a polynomial of $\{D_{n,i}(i)|1 \leqslant i \leqslant n\}$.

For $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{N}^n$, we set

$$\lambda(\beta) = \sum_{i=1}^{n-1}(k_{i+1} + \beta_i - \beta_{i+1})\omega_i \tag{125}$$

and

$$\xi(\beta) = (D_{n,n}(n))^{\beta_1}(D_{n,n-1}(n-1))^{\beta_2} \cdots (D_{n,1}(1))^{\beta_n}. \tag{126}$$

Moreover, we calculate that

$$\text{the weight of } \xi(\beta) \text{ is } \lambda(\beta) \tag{127}$$

by (15), (22) and (67).

Let $|\beta|$ be the length of the multi-index β, namely $|\beta| = \sum_{i=1}^n \beta_i$. Set

$$\Omega_r = \big\{\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{N}^n \mid |\beta| = r, \; 0 \leqslant \beta_i \leqslant k_i, \; i \in \overline{1,n}\big\}, \tag{128}$$

$$\Omega = \bigcup_{r=0}^{|\lambda|} \Omega_r, \quad \text{and} \tag{129}$$

$$\widehat{\Omega}_r = \big\{\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{N}^n \mid |\beta| = r, \; 0 \leqslant \beta_i \leqslant k_i, \; i \in \overline{2,n}\big\}. \tag{130}$$

Then $\Omega_r \subset \widehat{\Omega}_r$. In the following, we identify a singular vector with its nonzero constant multiple. Denote by $\mathscr{A}_r$ the subspace of homogeneous polynomials with degree r in $\mathbb{C}[x_{n,1}, x_{n,2}, \cdots, x_{n,n}]$. By (91), (92), (126) and Lemma 3.2, we have:

Theorem 3.3. *For $r \in \mathbb{N}$, the $\mathfrak{sl}(n)$ singular vectors in $\widehat{M_r}$ are $\{\xi(\beta) \mid \beta \in \widehat{\Omega}_r\}$, which generate all the $\mathfrak{sl}(n)$ irreducible components in*

$$\widehat{M_r} = \mathscr{A}_r \otimes_{\mathbb{C}} V_{n-1}\Big(\sum_{i=1}^{n-1} k_{i+1}\omega_i\Big). \tag{131}$$

Numerically, it implies the sum-product identity (cf. (14))

$$\sum_{\beta\in\widehat{\Omega}_r} d_{n-1}(\lambda(\beta)) = \binom{n+r-1}{n-1} d_{n-1}\Big(\sum_{i=1}^{n-1} k_{i+1}\omega_i\Big). \tag{132}$$

For $r \in \overline{0, |\lambda|}$, the $\mathfrak{sl}(n)$ singular vectors in $(V_n(\lambda))_r = \overline{M}_r$ are $\{\xi(\beta) \mid \beta \in \Omega_r\}$. Moreover, $\{\xi(\beta) \mid \beta \in \Omega\}$ are the highest-weight vectors of all the $\mathfrak{sl}(n)$ irreducible components in $V_n(\lambda) = \overline{M}$. The numeric $\mathfrak{sl}(n+1) \downarrow \mathfrak{sl}(n)$ branching rule

$$\sum_{\beta\in\Omega} d_{n-1}(\lambda(\beta)) = d_n(\lambda) \tag{133}$$

naturally follows (cf. (14)). If $r \leq k_1$, then $\widehat{\Omega}_r = \Omega_r$, which yields $(V_n(\lambda))_r = \widehat{M_r}$.

By (132) we can derive a formula for $(V_n(\lambda))_r$ and $V_n(\lambda)$ as follows with the proof being presented in Appendix B:

Proposition 3.4. *We have* $\dim(V_n(\lambda))_r =$

$$= \sum_{s=0}^{n-1}(-1)^s \binom{n+r-1-\sum_{j=1}^{s}(k_j+1)}{n-1} d_{n-1}\Big(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i\Big) \tag{134}$$

for $r \in \overline{0, |\lambda|}$. Moreover,

$$\sum_{s=0}^{n-1}(-1)^s \binom{\sum_{j=s+1}^{n}(k_j+1)}{n} d_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i) = d_n(\sum_{\ell=1}^{n} k_\ell\omega_\ell). \tag{135}$$

Proof. See Appendix B. ∎

Consider the case $k_1 = k_2 = \cdots = k_n = k$. The $\mathfrak{sl}(n+1)$-module $V_n(k\sum_{i=1}^{n}\omega_i)$ is called a *Steinberg module*. According to (14), its dimension

$$d_n(k\sum_{i=1}^{n}\omega_i) = (k+1)^{\frac{n(n+1)}{2}}. \tag{136}$$

Moreover,

$$d_{n-1}\Big(\sum_{i=1}^{s} k\omega_i + \omega_s + \sum_{i=s}^{n-1} k\omega_i\Big) = \binom{n}{s}(k+1)^{\frac{n(n-1)}{2}} \tag{137}$$

By (134), the dimension of homogeneous space of degree $r \in \overline{0, nk}$ is

$$\begin{aligned}\dim(V_n(\lambda))_r &= \sum_{s=0}^{n-1}(-1)^s \binom{n+r-1-s(k+1)}{n-1} d_{n-1}\Big(\sum_{i=1}^{s} k\omega_i + \omega_s + \sum_{i=s}^{n-1} k\omega_i\Big) \\ &= \sum_{s=0}^{n-1}(-1)^s \binom{n+r-1-s(k+1)}{n-1}\binom{n}{s}(k+1)^{\frac{n(n-1)}{2}} \end{aligned} \tag{138}$$

and (135) becomes

$$\begin{aligned}(k+1)^{\frac{n(n+1)}{2}} &= \sum_{s=0}^{n-1}(-1)^s \binom{(n-s)(k+1)}{n} d_{n-1}\Big(\sum_{i=1}^{s} k\omega_i + \omega_s + \sum_{i=s}^{n-1} k\omega_i\Big) \\ &= \sum_{s=0}^{n-1}(-1)^s \binom{(n-s)(k+1)}{n}\binom{n}{s}(k+1)^{\frac{n(n-1)}{2}}, \end{aligned} \tag{139}$$

which is equivalent to the well-known classical combinatorial identity:

$$\sum_{s=0}^{n}(-1)^s \binom{(n-s)(k+1)}{n}\binom{n}{s} = (k+1)^n \tag{140}$$

(e.g., cf. p. 51 in [27]).

Remark 3.5. By (67) with $i=1$, $\{\lambda(\beta) \mid \beta \in \Omega\}$ are distinct weights of $\mathfrak{gl}(n)$. Thus the multiplicities of the $\mathfrak{gl}(n)$ irreducible components in $V_n(\lambda)$ are one; that is, the multiplicity-one theorem of $\mathfrak{gl}(n+1) \downarrow \mathfrak{gl}(n)$ holds. ■

By (128), $|\Omega_r| = |\Omega_{|\lambda|-r}|$. So $\mathfrak{sl}(n)$-modules $\overline{M}_r$ and $\overline{M}_{|\lambda|-r}$ have the same number of irreducible components. Moreover, (125), (128) and (130) give

$$\{\lambda(\beta) \mid \beta \in \widehat{\Omega}_{k_1+1} \setminus \Omega_{k_1+1}\} = \{(k_1+1)\omega_1 + \sum_{i=1}^{n-1} k_{i+1}\omega_i\}. \tag{141}$$

Thus
$$\widehat{M}_{k_1+1}/\overline{M}_{k_1+1} = V_{n-1}((k_1+1)\omega_1 + \sum_{i=1}^{n-1} k_{i+1}\omega_i). \tag{142}$$

According to (125) and (128),

$$\{\lambda(\beta) \mid \beta \in \Omega_{|\lambda|}\} = \Big\{\sum_{i=1}^{n-1} k_i\omega_i\Big\}. \tag{143}$$

Hence
$$\overline{M}_{|\lambda|} = V_{n-1}\Big(\sum_{i=1}^{n-1} k_i\omega_i\Big) \tag{144}$$

is a simple $\mathfrak{sl}(n)$-module.

Note that as an $\mathfrak{sl}(n)$-module, $\mathscr{A}_r = V_{n-1}(r\omega_1)$ in (131). By using the $\mathfrak{sl}(n)$ singular vectors $\{\xi(\beta) \mid \beta \in \widehat{\Omega}_r\}$ in

$$\widehat{M}_r = V_{n-1}(r\omega_1) \otimes V_{n-1}\Big(\sum_{i=1}^{n-1} k_{i+1}\omega_i\Big), \tag{145}$$

one can derive the corresponding Clebsch-Gordan coefficients. Moreover, they can also be used to construct $\mathfrak{sl}(n)$-module homomorphisms

$$\text{from } V_{n-1}(r\omega_1)\otimes_{\mathbb{C}} V_{n-1}\Big(\sum_{i=1}^{n-1} k_{i+1}\omega_i\Big) \text{ to } V_{n-1}(\mu) \tag{146}$$

for any dominant $\mathfrak{sl}(n)$ integral weight μ, which are fundamental in constructing corresponding intertwining operators of the vertex operator algebra associated with the affine Kac-Moody Lie algebra $A^{(1)}$ (e.g., cf. ([31])). Using these intertwining operators, one can obtain the corresponding exact solutions of Knizhnik-Zamolodchikov equation in WZW model of conformal field theory (cf. [20, 29, 31]).

4. Special cases

In this section, we give more detailed description on the special cases when the highest weights are $k_1\omega_1+k_2\omega_2, k_1\omega_1+k_n\omega_n$ and $k\omega_i$, respectively.

4.1. $\lambda = k_1\omega_1+k_2\omega_2$, $n\geqslant 3$

In this case, (92) becomes

$$S_n(\lambda)=\left\{\prod_{n-1\leqslant j\leqslant i\leqslant n}\prod_{\Theta\in\mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)}\;\middle|\; \begin{array}{l}\alpha_{i,j}(\Theta)\in\mathbb{N};\text{ for } j\in\overline{n-1,n}:\\ \displaystyle\sum_{i=j}^{n}\sum_{\Theta\in\mathbb{J}_{i,j}}\alpha_{i,j}(\Theta)\leqslant k_{n-j+1}\end{array}\right\} \tag{147}$$

because $k_3=k_4=\cdots=k_n=0$. According to (58), (147) only involves variables $\{x_{n-1,i},x_{n,j}\mid i\in\overline{1,n-1}, j\in\overline{1,n}\}$. Since $V_n(\lambda)$ is spanned by $S_n(\lambda)$, we have

$$V_n(\lambda)\subset\mathbb{C}[x_{n-1,i},x_{n,j}\mid i\in\overline{1,n-1}, j\in\overline{1,n}].$$

For simplicity, we redenote

$$y_i=x_{n-1,i},\quad x_j=x_{n,j}\qquad\text{for } i\in\overline{1,n-1}, j\in\overline{1,n}.$$

We also take the convention $y_n=-1$. Then the set

$$\left\{\prod_{i=1}^{n}x_i^{\alpha_i}\prod_{1\leqslant s<t\leqslant n}(x_sy_t-x_ty_s)^{\gamma_{s,t}}\prod_{j=1}^{n-1}y_j^{\beta_j}\;\middle|\;\begin{array}{c}\alpha_i,\beta_j,\gamma_{s,t}\in\mathbb{N};\ \displaystyle\sum_{i=1}^{n}\alpha_i\le k_1;\\ \displaystyle\sum_{1\leqslant s<t\leqslant n}\gamma_{s,t}+\sum_{j=1}^{n-1}\beta_j\le k_2\end{array}\right\}$$

spans $V_n(\lambda)$ by (147).

According (67)–(69), we have the following representation formulas of $\mathfrak{sl}(n+1)$:

$$E_{i,j}|_{V_n(\lambda)}=x_i\partial_{x_j}+y_i\partial_{y_j},\quad E_{i,n}|_{V_n(\lambda)}=x_i\partial_{x_n}+y_i\Big(\sum_{s=1}^{n-1}y_s\partial_{y_s}-k_2\Big),$$

$$E_{i,n+1}|_{V_n(\lambda)}=x_i\Big(\sum_{s=1}^{n}x_s\partial_{x_s}-k_1\Big)+y_i\Big(\sum_{s=1}^{n-1}(x_s+x_ny_s)\partial_{y_s}\Big)-k_2(x_i+x_ny_i),$$

$$E_{n,j}|_{V_n(\lambda)} = x_n\partial_{x_j} - \partial_{y_j}, \quad E_{n+1,j}|_{V_n(\lambda)} = -\partial_{x_j},$$

$$(E_{i,i} - E_{n+1,n+1})|_{V_n(\lambda)} = \sum_{s=1}^{n} x_s\partial_{x_s} + x_i\partial_{x_i} + y_i\partial_{y_i} - k_1 - k_2,$$

$$(E_{n,n} - E_{n+1,n+1})|_{V_n(\lambda)} = \sum_{s=1}^{n} x_s\partial_{x_s} + x_n\partial_{x_n} - \sum_{s=1}^{n-1} y_s\partial_{y_s} - k_1.$$

$$E_{n,n+1}|_{V_n(\lambda)} = x_n(\sum_{s=1}^{n} x_s\partial_{x_s} - k_1) - \sum_{s=1}^{n-1}(x_s + x_n y_s)\partial_{y_s}, \quad E_{n+1,n}|_{V_n(\lambda)} = -\partial_{x_n}$$

for $i, j \in \overline{1, n-1}$. By Theorem 3.3, $\mathfrak{sl}(n)$ singular vectors in $\widehat{M}_r$ are

$$\{x_n^s(x_{n-1} + x_n y_{n-1})^t \mid s, t \in \mathbb{N}; t \leq k_2; s + t = r\},$$

whose weights are

$$\{(k_2 + s - t)\omega_1 + t\omega_2 \mid s, t \in \mathbb{N}; t \leq k_2; s + t = r\}$$

for $r \in \mathbb{N}$. According to (14),

$$d_{n-1}(\ell_1\omega_1 + \ell_2\omega_2) = \frac{\ell_1 + 1}{n-1}\binom{\ell_1 + \ell_2 + n - 1}{n-2}\binom{\ell_2 + n - 2}{n-2}.$$

So (132) becomes

$$\sum_{s=\max\{0, r-k_2\}}^{r} \frac{k_2 + 2s - r + 1}{n-1}\binom{k_2 + s + n - 1}{n-2}\binom{r - s + n - 2}{n-2}$$
$$= \binom{n + r - 1}{n-1}\binom{n + k_2 - 1}{n-1}.$$

Let $k_3 = \cdots = k_n = 0$ in Corollary 3.4. Since $r \leqslant k_1 + k_2 < k_1 + k_2 + 2$,

$$\binom{n + r - 1 - \sum_{j=1}^{s}(k_j + 1)}{n-1} = 0, \quad \binom{\sum_{j=s+1}^{n}(k_j + 1)}{n} = \binom{n - s}{n} = 0$$

when $s > 1$. Thus (134) and (135) become

$$\begin{aligned}\dim(V_n(\lambda))_r &= \binom{n + r - 1}{n-1} d_{n-1}(k_2\omega_1) - \binom{n + r - 2 - k_1}{n-1} d_{n-1}((k_1 + k_2 + 1)\omega_1) \\ &= \binom{n + r - 1}{n-1}\binom{k_2 + n - 1}{n-1} - \binom{n + r - 2 - k_1}{n-1}\binom{k_1 + k_2 + n}{n-1}\end{aligned}$$

and

$$\begin{aligned}\dim V_n(\lambda) &= \binom{k_1 + k_2 + n}{n} d_{n-1}(k_2\omega_1) - \binom{k_2 + n - 1}{n} d_{n-1}((k_1 + k_2 + 1)\omega_1) \\ &= \binom{k_1 + k_2 + n}{n}\binom{k_2 + n - 1}{n-1} - \binom{k_2 + n - 1}{n}\binom{k_1 + k_2 + n}{n-1}.\end{aligned}$$

4.2. $\lambda = k_1\omega_1 + k_n\omega_n$

In this case, the representation of $\mathfrak{sl}(n+1)$ is given in the formulas (67)–(69) with $k_2 = \cdots = k_{n-1} = 0$. Recall that

$$D_{j,1}(1) = \begin{vmatrix} x_{1,1} & -1 & 0 & \cdots & 0 \\ x_{2,1} & x_{2,2} & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{j-1,1} & x_{j-1,2} & x_{j-1,3} & \cdots & -1 \\ x_{j,1} & x_{j,2} & x_{j,3} & \cdots & x_{j,j} \end{vmatrix}. \tag{148}$$

Now the $\mathfrak{sl}(n+1)$-module $V_n(\lambda)$ is spanned by

$$S_n(\lambda) = \left\{ \prod_{i=1}^{n} x_{n,i}^{\alpha_i} \prod_{j=1}^{n} (D_{j,1}(1))^{\beta_j} \;\middle|\; \alpha, \beta \in \mathbb{N}^n;\ |\alpha| \le k_1,\ |\beta| \le k_n \right\}.$$

By Theorem 3.3, $\mathfrak{sl}(n)$ singular vectors in $\widehat{M}_r$ are

$$\{x_n^s (D_{n,1}(1))^t \mid s, t \in \mathbb{N}; t \le k_n; s + t = r\},$$

whose weights are $\{s\omega_1 + (k_n - t)\omega_{n-1} \mid s, t \in \mathbb{N}; t \le k_n; s + t = r\}$ for $r \in \mathbb{N}$. So (132) becomes

$$\sum_{s=\max\{0, r-k_n\}}^{r} \frac{k_n + 2s - r + n - 1}{n-1} \binom{s+n-2}{n-2} \binom{k_n + s - r + n - 2}{n-2}$$
$$= \binom{n+r-1}{n-1} \binom{n+k_n-1}{n-1}.$$

Suppose $k_1 + k_n \ge r \ge k_1 + 1$. The set

$$\{ \prod_{i=1}^{n} x_{n,i}^{\alpha_i} \prod_{j=1}^{n} (D_{j,1}(1))^{\beta_j} \mid \alpha, \beta \in \mathbb{N}^n; |\alpha| = k_1; \beta_n = r - k_1; |\beta| - \beta_n \le k_1 + k_n - r\}$$

forms a basis of $(V_n(\lambda))_r$ by (148). Thus

$$\dim (V_n(\lambda))_r = \binom{k_1 + n - 1}{n-1} \binom{k_1 + k_n - r + n - 1}{n-1}.$$

By Theorem 3.3, $\mathfrak{sl}(n)$ singular vectors in $(V_n(\lambda))_r$ are

$$\{x_n^s (D_{n,1}(1))^t \mid s, t \in \mathbb{N}; s \le k_1; t \le k_n; s + t = r\},$$

whose weights are

$$\{s\omega_1 + (k_n - t)\omega_{n-1} \mid s, t \in \mathbb{N}; s \le k_1; t \le k_n; s + t = r\}.$$

So we have

$$\sum_{s=\max\{0, r-k_n\}}^{\min\{k_1, r\}} \frac{k_n + 2s - r + n - 1}{n-1} \binom{s+n-2}{n-2} \binom{k_n + s - r + n - 2}{n-2}$$
$$= \binom{k_1 + n - 1}{n-1} \binom{k_1 + k_n - r + n - 1}{n-1}.$$

4.3. $\lambda = k\omega_i,\ 2 \leqslant i \leqslant n-1$

For simplicity, we denote

$$d_j(\Theta) = d_j(\theta_1, \theta_2 \cdots, \theta_{i+j-n}) = \begin{vmatrix} x_{n-i+1,\theta_1} & x_{n-i+1,\theta_2} & \cdots & x_{n-i+1,\theta_{i+j-1}} \\ x_{n-i+2,\theta_1} & x_{n-i+2,\theta_2} & \cdots & x_{n-i+2,\theta_{i+j-1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{j,\theta_1} & x_{j,\theta_2} & \cdots & x_{j,\theta_{i+j-1}} \end{vmatrix}$$

for $j \in \overline{n-i+1, n}$ and $\Theta = (\theta_1, \cdots, \theta_{i+j-1}) \in \mathbb{J}_{i,j}$. Set

$$\mathbb{I}_{i,j} = \left\{(\theta_1, \cdots, \theta_{i+j-1}) \in \mathbb{N}^{i+j-1} \mid \theta_1 \leqslant n-i+1,\ \theta_1 < \theta_2 < \cdots . < \theta_{i+j-n} \leqslant j\right\}.$$

By Theorem 3.3, the homogeneous subspace $(V_n(\lambda))_r$ is spanned by

$$\left\{ \prod_{j=n-i+1}^{n} \prod_{\Theta\in\mathbb{I}_{i,j}} d_j(\Theta)^{\alpha_j(\Theta)} \mid \alpha_j(\Theta) \in \mathbb{N};\ \sum_{j=n-i+1}^{n} \sum_{\Theta\in\mathbb{I}_{i,j}} \alpha_j(\Theta) \leqslant k;\ \sum_{\Theta\in\mathbb{I}_{i,n}} \alpha_n(\Theta) = r\right\}.$$

So $V_n(\lambda)$ does not involve the variables $\{x_{r,s} \mid 1 \leq s \leq r \leq n-i\}$. The representation of $\mathfrak{sl}(n+1)$ is given in the formulas (67)–(69) with $k_r = 0$ for $i \neq r \in \overline{1,n}$, $|_{\mathscr{C}_n}$ replaced by $|_{V_n(\lambda)}$ and all the ingredients containing $x_{r,s}$ deleted for $1 \leq s \leq r \leq n-i$.

By Theorem 3.3, $\mathfrak{sl}(n)$ singular vectors in $\widehat{M_r}$ are

$$\{x_n^s(D_{n,n-i+1}(n-i+1))^t \mid s,t \in \mathbb{N}; t \leq k; s+t = r\},$$

whose weights are

$$\{s\omega_1 + (k-t)\omega_{i-1} + t\omega_i \mid s,t \in \mathbb{N}; t \leq k; s+t = r\} \tag{149}$$

for $r \in \mathbb{N}$. Moreover,

$$d_{n-1}(k\omega_{i-1}) = \prod_{j=1}^{i-1} \frac{\binom{k+n-i+j}{k}}{\binom{k+j-1}{k}}.$$

So (132) becomes

$$\sum_{s=\max\{0,r-k\}}^{r} d_{n-1}(s\omega_1 + (k+s-r)\omega_{i-1} + (r-s)\omega_i)$$

$$= \binom{n+r-1}{n-1} \prod_{j=1}^{i-1} \frac{\binom{k+n-i+j}{k}}{\binom{k+j-1}{k}}.$$

Suppose $r \in \overline{0,k}$. By Theorem 3.3, $\mathfrak{sl}(n)$-module $(V_n(\lambda))_r$ is an simple module with highest-weight vector $(D_{n,n-i+1}(n-i+1))^r$, whose weight is $(k-r)\omega_{i-1} + r\omega_i$. Thus

$$\dim(V_n(\lambda))_r = \binom{k-r+i-1}{i-1}\binom{r+n-i}{n-i} \prod_{j=1}^{i-1} \frac{\binom{k+n-i+j}{k}}{\binom{k+j}{k}}.$$

Hence (133) is directly equivalent to

$$\sum_{r=0}^{k} \binom{k-r+i-1}{i-1}\binom{r+n-i}{n-i} = \binom{k+n}{k}.$$

A. Proof of Theorem 2.4

We will prove Theorem 2.4 in this section with two lemmas. Recall again that $x_{s,i}=0$ if $s<i-1$ by the convention (57), and $D_{n,s}(i)=0$ when $s<i\le n$ by (59). Thus by the last equality in (89), we can write

$$E_{i,n+1}|_{\mathscr{C}_n}=\Phi_i-\Psi_i$$

with
$$\Phi_i=\sum_{1\leqslant t\leqslant s\leqslant n}D_{n,s}(t)x_{s,i}\partial_{x_{s,t}},\quad \Psi_i=\sum_{s=1}^{n}D_{n,s}(i)k_{n-s+1}. \tag{150}$$

Lemma A.1. *For $i\in\overline{1,n}$, $1\leqslant t\leqslant s\leqslant n$ and $\Theta\in\mathbb{J}_{s,t}$, we have*

$$\Phi_i(d_{s,t}(\Theta))=D_{n,t}(i)d_{s,t}(\Theta)-d_{n,t}(\tilde{\Theta}_s) \tag{151}$$

where
$$\tilde{\Theta}_s=\begin{cases}(0,\cdots,0) & \text{if } s=n,\\ (\theta_1,\theta_2,\cdots,\theta_{s-t+1},i) & \text{if } s=n-1,\\ (\theta_1,\theta_2,\cdots,\theta_{s-t+1},i,s+2,\cdots,n) & \text{if } s\leqslant n-2.\end{cases} \tag{152}$$

Proof. We calculate: $\Phi_i(d_{s,t}(\Theta))=$

$$=\Phi_i\left(\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}}\\ x_{t+1,\theta_1} & x_{t+1,\theta_2} & \cdots & x_{t+1,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}}\end{vmatrix}\right)$$

$$=\sum_{j=t}^{s}x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}}\\ D_{n,j}(\theta_1) & D_{n,j}(\theta_2) & .. & D_{n,j}(\theta_{s-t+1})\\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}}\end{vmatrix}$$

$$=\sum_{j=t}^{s}x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}}\\ \sum\limits_{r=j}^{n}x_{r,\theta_1}D_{n,r+1}(r+1) & \sum\limits_{r=j}^{n}x_{r,\theta_2}D_{n,r+1}(r+1) & \cdots & \sum\limits_{r=j}^{n}x_{r,\theta_{s-t+1}}D_{n,r+1}(r+1)\\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}}\end{vmatrix}$$

$$=\sum_{j=t}^{s}x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}}\\ x_{j,\theta_1}D_{n,j+1}(j+1) & x_{j,\theta_2}D_{n,j+1}(j+1) & \cdots & x_{j,\theta_{s-t+1}}D_{n,j+1}(j+1)\\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}}\end{vmatrix}$$

$$+\sum_{j=t}^{s} x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}} \\ \sum\limits_{r=s+1}^{n} x_{r,\theta_1} D_{n,r+1}(r+1) & \sum\limits_{r=s+1}^{n} x_{r,\theta_2} D_{n,r+1}(r+1) & \cdots & \sum\limits_{r=s+1}^{n} x_{r,\theta_{s-t+1}} D_{n,r+1}(r+1) \\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}} \end{vmatrix}$$

$$= \sum_{j=t}^{s} x_{j,i} D_{n,j+1}(j+1) d_{s,t}(\Theta)$$

$$+\sum_{j=t}^{s} x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}} \\ D_{n,s+1}(\theta_1) & D_{n,s+1}(\theta_2) & \cdots & D_{n,s+1}(\theta_{s-t+1}) \\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}} \end{vmatrix} \qquad (153)$$

by Lemma 2.2(ii). Here we make the convention $D_{n,n+1}(n+1) = 1$ and $D_{n,n+1}(\theta) = 0$ if $1 \leqslant \theta \leqslant n$.

In the following, putting a bracket on a row in a determinant means removing that row. If $s = n$,

$$(153) = \sum_{j=t}^{n} x_{j,i} D_{n,j+1}(j+1) d_{n,t}(\Theta) = D_{n,t}(i) d_{n,t}(\Theta). \qquad (154)$$

Suppose $s = n-1$. Then we have

$$(153) \quad = \quad \sum_{j=t}^{n-1} x_{j,i} D_{n,j+1}(j+1) d_{n-1,t}(\Theta) + \sum_{j=t}^{n-1} x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{n-t}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{n-t}} \\ x_{n,\theta_1} & x_{n,\theta_2} & .. & x_{n,\theta_{n-t}} \\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{n-t+}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,\theta_1} & x_{n-1,\theta_2} & \cdots & x_{n-1,\theta_{n-t}} \end{vmatrix}$$

$$= \quad D_{n,t}(i) d_{s,t}(\Theta) - x_{n,i} d_{s,t}(\Theta) + \sum_{j=t}^{s} (-1)^{n-j-1} x_{j,i}\begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{n-t}} \\ \cdots & \cdots & \cdots & \cdots \\ (x_{j,\theta_1} & x_{j,\theta_2} & \cdots & x_{j,\theta_{n-t}}) \\ \cdots & \cdots & \cdots & \cdots \\ x_{n,\theta_1} & x_{n,\theta_2} & \cdots & x_{n,\theta_{n-t}} \end{vmatrix}$$

$$= \quad D_{n,t}(i) d_{s,t}(\Theta) + (-1)^{n-t-1} d_{n,t}(i, \theta_1, \cdots, \theta_{n-t}). \qquad (155)$$

Assume $s \leqslant n-2$.

Now we obtain

$$
\begin{aligned}
(153) = {} & \sum_{j=t}^{s} x_{j,i} D_{n,j+1}(j+1) d_{s,t}(\Theta) \\
& + \sum_{j=t}^{s} x_{j,i} \begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}} \\ D_{n,s+1}(\theta_1) & D_{n,s+1}(\theta_2) & .. & D_{n,s+1}(\theta_{s-t+1}) \\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}} \end{vmatrix} \\
= {} & D_{n,t}(i) d_{s,t}(\Theta) - D_{n,s+1}(i) d_{s,t}(\Theta) \\
& + \sum_{j=t}^{s} x_{j,i} \begin{vmatrix} x_{t,\theta_1} & x_{t,\theta_2} & \cdots & x_{t,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{j-1,\theta_1} & x_{j-1,\theta_2} & \cdots & x_{j-1,\theta_{s-t+1}} \\ D_{n,s+1}(\theta_1) & D_{n,s+1}(\theta_2) & .. & D_{n,s+1}(\theta_{s-t+1}) \\ x_{j+1,\theta_1} & x_{j+1,\theta_2} & \cdots & x_{j+1,\theta_{s-t+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{s,\theta_1} & x_{s,\theta_2} & \cdots & x_{s,\theta_{s-t+1}} \end{vmatrix} \\
= {} & \sum_{j=s+1}^{n} (-1)^{s-j} x_{j,i} \begin{vmatrix} x_{t,\theta_1} & \cdots & x_{t,\theta_{s-t+1}} & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{s,\theta_1} & \cdots & x_{s,\theta_{s-t+1}} & 0 & 0 & \cdots & 0 \\ x_{s+1,\theta_1} & \cdots & x_{s+1,\theta_{s-t+1}} & x_{s+1,s+1} & x_{s+1,s+2} & \cdots & x_{s+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (x_{j,\theta_1} & \cdots & x_{j,\theta_{s-t+1}} & x_{j,s+1} & x_{j,s+2} & \cdots & x_{j,n}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n,\theta_1} & \cdots & x_{n,\theta_{s-t+1}} & x_{n,s+1} & x_{n,s+2} & \cdots & x_{n,n} \end{vmatrix} \\
& + D_{n,t}(i) d_{s,t}(\Theta) + \sum_{j=t}^{s} (-1)^{s-j} \begin{vmatrix} x_{t,\theta_1} & \cdots & x_{t,\theta_{s-t+1}} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (x_{j,\theta_1} & \cdots & x_{j,\theta_{s-t+1}} & 0 & \cdots & 0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{s,\theta_1} & \cdots & x_{s,\theta_{s-t+1}} & 0 & \cdots & 0 \\ x_{s+1,\theta_1} & \cdots & x_{s+1,\theta_{s-t+1}} & x_{s+1,s+2} & \cdots & x_{s+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n,\theta_1} & \cdots & x_{n,\theta_{s-t+1}} & x_{n,s+2} & \cdots & x_{n,n} \end{vmatrix} \\
= {} & D_{n,t}(i) d_{s,t}(\Theta) + (-1)^{s-t} \begin{vmatrix} x_{t,i} & x_{t,\theta_1} & \cdots & x_{t,\theta_{s-t+1}} & x_{t,s+2} & \cdots & x_{t,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{s,i} & x_{s,\theta_1} & \cdots & x_{s,\theta_{s-t+1}} & x_{s,s+2} & \cdots & x_{s,n} \\ x_{s+1,i} & x_{s+1,\theta_1} & \cdots & x_{s+1,\theta_{s-t+1}} & x_{s+1,s+2} & \cdots & x_{s+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n,i} & x_{n,\theta_1} & \cdots & x_{n,\theta_{s-t+1}} & x_{n,s+2} & \cdots & x_{n,n} \end{vmatrix} \\
= {} & D_{n,t}(i) d_{s,t}(\Theta) + (-1)^{s-t} d_{n,t}(i, \theta_1, \cdots, \theta_{s-t+1}, s+2, \cdots, n). \qquad (156)
\end{aligned}
$$

Expressions (154)–(156) imply

$$\Phi_i(d_{s,t}(\Theta)) = D_{n,t}(i)d_{s,t}(\Theta) - d_{n,t}(\tilde{\Theta}_s) \tag{157}$$

where
$$\tilde{\Theta}_s = \begin{cases} (0,\cdots,0) & \text{if } s=n,\\ (\theta_1,\theta_2,\cdots,\theta_{s-t+1},i) & \text{if } s=n-1,\\ (\theta_1,\theta_2,\cdots,\theta_{s-t+1},i,s+2,\cdots,n) & \text{if } s\leqslant n-2.\end{cases} \tag{158}$$

This completes the proof. ■

For $1\leqslant i_1<i_2<\cdots<i_{n-m+1}\leqslant n$, $1\leqslant m\leqslant n-1$ and $f\in\mathscr{C}_n$, we define

$$\begin{bmatrix} E_{i_1,n+1} & E_{i_2,n+1} & \cdots & E_{i_{n-m+1},n+1}\\ x_{m,i_1} & x_{m,i_2} & \cdots & x_{m,i_{m-n+1}}\\ x_{m+1,i_1} & x_{m+1,i_2} & \cdots & x_{m+1,i_{m-n+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n-1,i_1} & x_{n-1,i_2} & \cdots & x_{n-1,i_{m-n+1}}\end{bmatrix}(f) = \sum_{j=1}^{n-m+1} E_{i_j,n+1}(a_{i_j}f), \tag{159}$$

where
$$a_{i_j} = (-1)^{1+j}d_{n-1,m}(i_1,\cdots,i_{j-1},i_{j+1},\cdots,i_{n-m+1}) \tag{160}$$

is the algebraic cofactor of $E_{i_j,n+1}$ in the matrix (cf. (58)).

Lemma A.2. *These first-order differential operators defined in* (159) *are*

$$\begin{aligned}&\begin{bmatrix} E_{i_1,n+1} & E_{i_2,n+1} & \cdots & E_{i_{n-m+1},n+1}\\ x_{m,i_1} & x_{m,i_2} & \cdots & x_{m,i_{m-n+1}}\\ x_{m+1,i_1} & x_{m+1,i_2} & \cdots & x_{m+1,i_{m-n+1}}\\ \cdots & \cdots & \cdots & \cdots\\ x_{n-1,i_1} & x_{n-1,i_2} & \cdots & x_{n-1,i_{m-n+1}}\end{bmatrix}\\ &= (-1)^{n-m}d_{n,m}(i_1,\cdots,i_{n-m+1})\left(m-n+\sum_{t=1}^{n} x_{n,t}\partial_{x_{n,t}} - \sum_{s=1}^{n-m+1} k_s\right)\\ &\quad + \sum_{1\leqslant t\leqslant s<m}\sum_{j=1}^{n-m+1} a_{i_j}D_{n,s}(t)x_{s,i_j}\partial_{x_{s,t}} - \sum_{s=1}^{m-1}\sum_{j=1}^{n-m+1} k_{n-s+1}D_{n,s}(i_j)a_{i_j}.\end{aligned} \tag{161}$$

Proof. By (91), (159) is equal to

$$\sum_{j=1}^{n-m+1}\big(\Phi_{i_j}(a_{i_j}f) - \Psi_{i_j}(a_{i_j}f)\big) = \sum_{j=1}^{n-m+1}\big(\Phi_{i_j}(a_{i_j})f + a_{i_j}\Phi_{i_j}(f) - \Psi_{i_j}(a_{i_j}f)\big) \tag{162}$$

According to (151),

$$\begin{aligned}&\sum_{j=1}^{n-m+1}\Phi_{i_j}(a_{i_j})\\ &= \sum_{j=1}^{n-m+1}\big(D_{n,m}(i_j)a_{i_j} - (-1)^{n-m}d_{n,m}(i_1,\cdots,i_{n-m+1})\big)\\ &= \sum_{j=1}^{n-m+1}\left(x_{n,i_j}a_{i_j} + \sum_{t=m}^{n-1} x_{t,i_j}D_{n,t+1}(t+1)a_{i_j}\right)\\ &\qquad - (-1)^{n-m}(n-m+1)d_{n,m}(i_1,\cdots,i_{n-m+1})\end{aligned}$$

$$= \sum_{j=1}^{n-m+1} x_{n,i_j} a_{i_j} + \sum_{t=m}^{n-1} \left(D_{n,t+1}(t+1) \sum_{j=1}^{n-m+1} x_{t,i_j} a_{i_j} \right)$$
$$- (-1)^{n-m}(n-m+1) d_{n,m}(i_1, \cdots, i_{n-m+1})$$
$$= (-1)^{n-m} d_{n,m}(i_1, \cdots, i_{n-m+1}) - (-1)^{n-m}(n-m+1) d_{n,m}(i_1, \cdots, i_{n-m+1})$$
$$= (m-n)(-1)^{n-m} d_{n,m}(i_1, \cdots, i_{n-m+1}), \tag{163}$$

where the second equality was obtained by Lemma 2.2 (ii). Moreover,

$$\sum_{j=1}^{n-m+1} a_{i_j} \Phi_{i_j} = \sum_{1 \leqslant t \leqslant s \leqslant n} \sum_{j=1}^{n-m+1} a_{i_j} D_{n,s}(t) x_{s,i_j} \partial_{x_{s,t}}$$
$$= \sum_{1 \leqslant t \leqslant s \leqslant n} D_{n,s}(t) \begin{vmatrix} x_{s,i_1} & x_{s,i_2} & \cdots & x_{s,i_{n-m+1}} \\ x_{m,i_1} & x_{m,i_2} & \cdots & x_{m,i_{n-m+1}} \\ x_{m+1,i_1} & x_{m+1,i_2} & \cdots & x_{m+1,i_{n-m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,i_1} & x_{n-1,i_2} & \cdots & x_{n-1,i_{n-m+1}} \end{vmatrix} \partial_{x_{s,t}}$$
$$= (-1)^{n-m} d_{n,m}(i_1, \cdots, i_{n-m+1}) \sum_{t=1}^{n} x_{n,t} \partial_{x_{n,t}}$$
$$+ \sum_{1 \leqslant t \leqslant s < m} D_{n,s}(t) \begin{vmatrix} x_{s,i_1} & x_{s,i_2} & \cdots & x_{s,i_{n-m+1}} \\ x_{m,i_1} & x_{m,i_2} & \cdots & x_{m,i_{n-m+1}} \\ x_{m+1,i_1} & x_{m+1,i_2} & \cdots & x_{m+1,i_{n-m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,i_1} & x_{n-1,i_2} & \cdots & x_{n-1,i_{n-m+1}} \end{vmatrix} \partial_{x_{s,t}}. \tag{164}$$

Furthermore,

$$\sum_{j=1}^{n-m+1} \Psi_{i_j} a_{i_j} = \sum_{j=1}^{n-m+1} \sum_{s=1}^{n} k_{n-s+1} D_{n,s}(i_j) a_{i_j}$$
$$= \sum_{s=m}^{n} \sum_{j=1}^{n-m+1} k_{n-s+1} D_{n,s}(i_j) a_{i_j} + \sum_{s=1}^{m-1} \sum_{j=1}^{n-m+1} k_{n-s+1} D_{n,s}(i_j) a_{i_j}$$
$$= \sum_{s=m}^{n} (-1)^{n-m} k_{n-s+1} d_{n,m}(i_1, \cdots, i_{n-m+1})$$
$$+ \sum_{s=1}^{m-1} \sum_{j=1}^{n-m+1} k_{n-s+1} D_{n,s}(i_j) a_{i_j}. \tag{165}$$

In summary, we obtain (161). ∎

Now we are going to prove the main theorem of Section 2.

Theorem 2.4 *As a vector space, $V_n(\lambda)$ is spanned by $S_n(\lambda)$. Moreover, the homogeneous subspace $(V_n(\lambda))_r = \overline{M}_r$ is spanned by $(S_n(\lambda))_r$.*

Proof. We prove the theorem by induction on n. If $n = 2$, the sets

$$\mathbb{J}_{1,1} = \mathbb{J}_{2,2} = \{(1), (2)\}, \qquad \mathbb{J}_{2,1} = \{(1,2), (2,1)\}. \tag{166}$$

Moreover, $$d_{1,1}(1)=x_{1,1},\;\; d_{1,1}(2)=-1,\;\; d_{2,2}(1)=x_{2,1},\;\; d_{2,2}(2)=x_{2,2}, \tag{167}$$

$$d_{2,1}(1,2)=-d_{2,1}(2,1)=\begin{vmatrix} x_{1,1} & -1 \\ x_{2,1} & x_{2,2} \end{vmatrix}=x_{1,1}x_{2,2}+x_{2,1}. \tag{168}$$

Under the identification (60), the basis (41) is contained in the set

$$S_2(\lambda)\subset\left\{\pm x_{2,1}^{\alpha_{2,1}}x_{2,2}^{\alpha_{2,2}}x_{1,1}^{\alpha_{1,1}}(x_{2,1}+x_{2,2}x_{1,1})^{\alpha'_{1,1}}\;\middle|\;\begin{array}{l}0\leqslant\alpha_{2,1}+\alpha_{2,2}\leqslant k_1\\ \text{and}\\ 0\leqslant\alpha_{1,1}+\alpha'_{1,1}\leqslant k_2\end{array}\right\}\subset V(\lambda) \tag{169}$$

(cf. (32) and (33)). In consequence we obtain $V_2(\lambda)=\overline{M}=\text{Span}\,S_2(\lambda)$ and $(V_2(\lambda))_r=\overline{M}_r=\text{Span}\,(S_2(\lambda))_r$. So the theorem holds for $n=2$. Set

$$W(\lambda)_r=\text{Span}\,(S_n(\lambda))_r. \tag{170}$$

Suppose that the theorem holds for $n-1$, which is equivalent to

$$(V_n(\lambda))_0=V_{n-1}(k_2\omega_1+k_3\omega_2+\cdots+k_n\omega_{n-1})=W(\lambda)_0. \tag{171}$$

Assume $W(\lambda)_r=(V_n(\lambda))_r$ for some $0\leqslant r<|\lambda|=\sum_{i=1}^{n}k_i$. Take

$$f=\prod_{1\leqslant j\leqslant i\leqslant n}\prod_{\Theta\in\mathbb{J}_{i,j}}d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)}\in(S_n(\lambda))_r,$$

by Lemma A.1,

$$\begin{aligned}
E_{s,n+1}(f) &= (\Phi_s-\Psi_s)\left(\prod_{1\leqslant j\leqslant i\leqslant n}\prod_{\Theta\in\mathbb{J}_{i,j}}d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)}\right)\\
&= f\sum_{1\leqslant j\leqslant i\leqslant n}\sum_{\Theta\in\mathbb{J}_{i,j}}\alpha_{i,j}(\Theta)d_{i,j}(\Theta)^{-1}\left(D_{n,j}(s)d_{i,j}(\Theta)-d_{n,j}(\tilde{\Theta}_s)\right)\\
&\quad -f\sum_{j=s}^{n}k_{n-j+1}D_{n,j}(s)\\
&= \sum_{j=s}^{n}\left(\sum_{i=j}^{n}\sum_{\Theta\in\mathbb{J}_{i,j}}\alpha_{i,j}(\Theta)-k_{n-j+1}\right)D_{n,j}(s)f\\
&\quad -\sum_{1\leqslant j\leqslant i\leqslant n}\sum_{\Theta\in\mathbb{J}_{i,j}}\alpha_{i,j}(\Theta)d_{i,j}(\Theta)^{-1}d_{n,j}(\tilde{\Theta}_s)f\in W(\lambda)_{r+1},
\end{aligned} \tag{172}$$

where we have used the fact $D_{n,j}(s)=0$ when $j<s$, and $D_{n,j}(s)f\in(S_n(\lambda))_{r+1}$ if

$$\sum_{i=j}^{n}\sum_{\Theta\in\mathbb{J}_{i,j}}\alpha_{i,j}(\Theta)<k_{n-j+1}.$$

Therefore we get by (23) and (27)

$$(V_n(\lambda))_{r+1}=\sum_{s=1}^{n}E_{s,n+1}((V_n(\lambda))_r)\subseteq W(\lambda)_{r+1}. \tag{173}$$

On the other hand, we set

$$P^m_{r+1} = (S_n(\lambda))_{r+1} \bigcap \Big\{ \prod_{1\leqslant j\leqslant i\leqslant n} \prod_{\Theta\in\mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)} \ \Big| \ \sum_{j=1}^{m} \sum_{\Theta\in\mathbb{J}_{n,j}} \alpha_{n,j}(\Theta) > 0 \Big\} \tag{174}$$

for $1 \leqslant m \leqslant n$. It is obvious that

$$P^1_{r+1} \subseteq P^2_{r+1} \subseteq \cdots \subseteq P^n_{r+1} = (S_n(\lambda))_{r+1}. \tag{175}$$

We may assume $\sum_{s=1}^{n'-1} k_s \leqslant r < \sum_{s=1}^{n'} k_s$ with some $n' \in \overline{1,n}$. We claim

$$P^{n-n'+1}_{r+1} = (S_n(\lambda))_{r+1}. \tag{176}$$

Assume that it is not true. Take any

$$\prod_{1\leqslant j\leqslant i\leqslant n} \prod_{\Theta\in\mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\alpha_{i,j}(\Theta)} \in (S_n(\lambda))_{r+1} \setminus P^{n-n'+1}_{r+1}. \tag{177}$$

Then

$$\sum_{j=1}^{n-n'+1} \sum_{\Theta\in\mathbb{J}_{n,j}} \alpha_{n,j}(\Theta) = 0, \tag{178}$$

and

$$\sum_{j=n-n'+2}^{n} \sum_{\Theta\in\mathbb{J}_{n,j}} \alpha_{n,j}(\Theta) = r+1 > \sum_{s=1}^{n'-1} k_s. \tag{179}$$

However, by (92),

$$\sum_{j=n-n'+2}^{n} \sum_{\Theta\in\mathbb{J}_{n,j}} \alpha_{n,j}(\Theta) \leqslant \sum_{j=n-n'+2}^{n} \sum_{i=j}^{n} \sum_{\Theta\in\mathbb{J}_{i,j}} \alpha_{i,j}(\Theta) \leqslant \sum_{j=n-n'+2}^{n} k_{n-j+1} = \sum_{s=1}^{n'-1} k_s. \tag{180}$$

which contradicts (179). Thus

$$P^1_{r+1} \subseteq P^2_{r+1} \subseteq \cdots \subseteq P^{n-n'+1}_{r+1} = (S_n(\lambda))_{r+1}. \tag{181}$$

Take any $g \in P^1_{r+1}$. It can be written as $g = g'D_{n,1}(1)$ for some $g' \in (S_n(\lambda))_r$. By the definition of $S_n(\lambda)$,

$$g'd_{n-1,1}(\Theta) \in (S_n(\lambda))_r \subseteq (V_n(\lambda))_r \tag{182}$$

for any $\Theta \in \mathbb{J}_{n-1,1}$. According to (153), (154), (159) and (169),

$$\begin{bmatrix} E_{1,n+1} & E_{2,n+1} & \cdots & E_{n,n+1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n} \end{bmatrix} (g') = (-1)^{n-1} \left(r - \sum_{i=1}^{n} k_i - n + 1 \right) g. \tag{183}$$

Since the coefficient $r - \sum_{i=1}^{n} k_i - n + 1 < -n + 1$ is nonzero, $g \in (V_n(\lambda))_{r+1}$. Hence $P^1_{r+1} \subseteq (V_n(\lambda))_{r+1}$.

Next we assume that $P_{r+1}^{m-1} \subseteq (V_n(\lambda))_{r+1}$ for some $m \in \overline{2, n-n'+1}$ by (181). Take any $h \in P_{r+1}^m \backslash P_{r+1}^{m-1}$. Without loss of generality we assume $h = d_{n,m}(\vartheta_1, ..., \vartheta_{n-m+1})h'$ with $h' \in (S_n(\lambda))_r$ and $1 \leqslant \vartheta_1 < \cdots < \vartheta_{n-m+1} \leqslant n$. According to (93),

$$h' d_{n-1,m}(\Theta) \in (S_n(\lambda))_r \qquad \text{for any } \Theta \in \mathbb{J}_{n-1,m}. \tag{184}$$

Suppose

$$h' = \prod_{1 \leqslant j \leqslant i \leqslant n} \prod_{\Theta \in \mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\beta_{i,j}(\Theta)}. \tag{185}$$

We write $h' = h'_1 h'_2 h'_3$ according to the range of the index j as follows:

$$h'_1 = \prod_{j=1}^{m-1} \prod_{i=j}^{n-1} \prod_{\Theta \in \mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\beta_{i,j}(\Theta)}, \quad h'_2 = \prod_{j=m}^{n-1} \prod_{i=j}^{n-1} \prod_{\Theta \in \mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\beta_{i,j}(\Theta)}, \tag{186}$$

$$h'_3 = \prod_{j=m}^{n} \prod_{\Theta \in \mathbb{J}_{n,j}} d_{n,j}(\Theta)^{\beta_{n,j}(\Theta)} \tag{187}$$

based on (174) and $h \notin P_{r+1}^{m-1}$. We apply the operator defined in (159). By Lemma A.2,

$$\begin{aligned}
&\begin{bmatrix} E_{\vartheta_1,n+1} & E_{\vartheta_2,n+1} & \cdots & E_{n-m+1,n+1} \\ x_{m,\vartheta_1} & x_{m,\vartheta_2} & \cdots & x_{m,\vartheta_{m-n+1}} \\ x_{m+1,\vartheta_1} & x_{m+1,\vartheta_2} & \cdots & x_{m+1,\vartheta_{m-n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,\vartheta_1} & x_{n-1,\vartheta_2} & \cdots & x_{n-1,\vartheta_{m-n+1}} \end{bmatrix} (h') \\
= \;& (-1)^{n-m}(m-n-\sum_{s=1}^{n-m+1} k_s) h' d_{n,m}(\vartheta_1, \cdots, \vartheta_{n-m+1}) \\
& + \sum_{l=1}^{n-m+1} a_{\vartheta_l} \Phi_{\vartheta_l}(h') - \sum_{l=1}^{n-m+1} \sum_{s=1}^{m-1} k_{n-s+1} D_{n,s}(\vartheta_l) a_{\vartheta_l} h' \\
= \;& (-1)^{n-m}(m-n-\sum_{s=1}^{n-m+1} k_s) h - \sum_{l=1}^{n-m+1} \sum_{s=1}^{m-1} k_{n-s+1} D_{n,s}(\vartheta_l) a_{\vartheta_l} h' \\
& + \sum_{l=1}^{n-m+1} a_{\vartheta_l} \left(\Phi_{\vartheta_l}(h'_1) h'_2 h'_3 + \Phi_{\vartheta_l}(h'_2) h'_1 h'_3 + \Phi_{\vartheta_l}(h'_3) h'_1 h'_2 \right).
\end{aligned} \tag{188}$$

The first term is an integral multiple of h. Let us figure out what the remnant is. As (171), we have

$$\begin{aligned}
&\sum_{l=1}^{n-m+1} a_{\vartheta_l} \Phi_{\vartheta_l}(h'_1) h'_2 h'_3 - \sum_{l=1}^{n-m+1} \sum_{s=1}^{m-1} k_{n-s+1} D_{n,s}(\vartheta_l) a_{\vartheta_l} h' \\
&= \sum_{l=1}^{n-m+1} a_{\vartheta_l} h'_2 h'_3 \left(\Phi_{\vartheta_l}(h'_1) - \sum_{s=1}^{m-1} k_{n-s+1} D_{n,s}(\vartheta_l) h'_1 \right) \\
&= \sum_{l=1}^{n-m+1} a_{\vartheta_l} h'_2 h'_3 \Big[\sum_{j=1}^{m-1} \sum_{i=1}^{n} \sum_{\Theta \in \mathbb{J}_{i,j}} \beta_{i,j}(\Theta) d_{i,j}(\Theta)^{-1} \big(D_{n,j}(\vartheta_l) d_{i,j}(\Theta) - d_{n,j}(\tilde{\Theta}_{\vartheta_l}) \big) h'_1 \\
&\qquad - \sum_{s=1}^{m-1} k_{n-s+1} D_{n,s}(\vartheta_l) h'_1 \Big]
\end{aligned}$$

$$= \sum_{l=1}^{n-m+1} \sum_{j=1}^{m-1} \Big[\Big(\sum_{i=1}^{n} \sum_{\Theta\in\mathbb{J}_{i,j}} \beta_{i,j}(\Theta) - k_{n-j+1}\Big) D_{n,j}(\vartheta_l) h' a_{\vartheta_l} - \sum_{i=1}^{n} \sum_{\Theta\in\mathbb{J}_{i,j}} \beta_{i,j}(\Theta) d_{i,j}(\Theta)^{-1} d_{n,j}(\tilde{\Theta}_{\vartheta_l}) h' a_{\vartheta_l}\Big]. \tag{189}$$

Since $j \leqslant m-1$, $\beta_{i,j}(\Theta) \neq 0$ implies

$$d_{i,j}(\Theta)^{-1} d_{n,j}(\tilde{\Theta}_{\vartheta_l}) h' a_{\vartheta_l} \in P_{r+1}^{m-1}. \tag{190}$$

If $\sum_{i=1}^{n} \sum_{\Theta\in\mathbb{J}_{i,j}} \beta_{i,j}(\Theta) - k_{n-j+1} \neq 0$, (92) yields $D_{n,j}(\vartheta_l) h' a_{\vartheta_l} \in P_{r+1}^{m-1}$. Thus (189) is a polynomial in Span P_{r+1}^{m-1}.

Note that h_2' is a polynomial in $\{x_{i,j} \mid m \leqslant j \leqslant i \leqslant n-1\}$. By (164)

$$\sum_{l=1}^{n-m+1} a_{\vartheta_l} \Phi_{\vartheta_l}(h_2') h_1' h_3' = (-1)^{n-m} h_1' h_3' d_{n,m}(\vartheta_1, \cdots, \vartheta_{n-m+1}) \sum_{t=1}^{n} x_{n,t} \partial_{x_{n,t}}(h_2') + h_1' h_3' \sum_{1\leqslant t\leqslant s<m} D_{n,s}(t) \begin{vmatrix} x_{s,\vartheta_1} & x_{s,\vartheta_2} & \cdots & x_{s,\vartheta_{n-m+1}} \\ x_{m,\vartheta_1} & x_{m,\vartheta_2} & \cdots & x_{m,\vartheta_{n-m+1}} \\ x_{m+1,\vartheta_1} & x_{m+1,\vartheta_2} & \cdots & x_{m+1,\vartheta_{n-m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,\vartheta_1} & x_{n-1,\vartheta_2} & \cdots & x_{n-1,\vartheta_{n-m+1}} \end{vmatrix} \partial_{x_{s,t}}(h_2') = 0. \tag{191}$$

There remains one term in (188). Since h_3' is a polynomial of $\{x_{i,j} \mid m \leqslant j \leqslant i \leqslant n\}$ and is homogeneous in X_n with degree r, we have

$$\begin{aligned} &\sum_{l=1}^{n-m+1} a_{\vartheta_l} \Phi_{\vartheta_l}(h_3') h_1' h_2' \\ &= (-1)^{n-m} h_1' h_2' d_{n,m}(\vartheta_1, \cdots, \vartheta_{n-m+1}) \sum_{t=1}^{n} x_{n,t} \partial_{x_{n,t}}(h_3') \\ &\quad + h_1' h_2' \sum_{1\leqslant t\leqslant s<m} D_{n,s}(t) \begin{vmatrix} x_{s,\vartheta_1} & x_{s,\vartheta_2} & \cdots & x_{s,\vartheta_{n-m+1}} \\ x_{m,\vartheta_1} & x_{m,\vartheta_2} & \cdots & x_{m,\vartheta_{n-m+1}} \\ x_{m+1,\vartheta_1} & x_{m+1,\vartheta_2} & \cdots & x_{m+1,\vartheta_{n-m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n-1,\vartheta_1} & x_{n-1,\vartheta_2} & \cdots & x_{n-1,\vartheta_{n-m+1}} \end{vmatrix} \partial_{x_{s,t}}(h_3') \\ &= (-1)^{n-m} r h_1' h_2' h_3' d_{n,m}(\vartheta_1, \cdots, \vartheta_{n-m+1}) \\ &= (-1)^{n-m} r h. \end{aligned} \tag{192}$$

Therefore, by (189)–(192),

$$(188) = (-1)^{n-m}\Big(r + m - n - \sum_{s=1}^{n-m+1} k_s\Big) h + h'', \tag{193}$$

with some $h'' \in$ Span P_{r+1}^{m}.

Since $m \leqslant n - n' + 1$, we have

$$r + m - n \leqslant r < \sum_{s=1}^{n'} k_s \leqslant \sum_{s=1}^{n-m+1} k_s$$

and so the coefficient $(-1)^{n-m}(r + m - n - \sum_{s=1}^{n-m+1} k_s)$ is nonzero. By (176), h lies in $V(\lambda)_{r+1}$. Based on induction on m, we have $P_{r+1}^m \subseteq (V_n(\lambda))_{r+1}$ for all $1 \leqslant m \leqslant n - n' + 1$. Hence $(S_n(\lambda))_{r+1} \subseteq (V_n(\lambda))_{r+1}$ and $(V_n(\lambda))_{r+1} = W(\lambda)_{r+1}$ (cf. (170)).

It remains to show that $(V_n(\lambda))_r = \{0\}$ when $r > \sum_{i=1}^n k_i$. Recall $|\lambda| = \sum_{i=1}^n k_i$.

Let $f' = \prod_{1 \leqslant j \leqslant i \leqslant n} \prod_{\Theta \in \mathbb{J}_{i,j}} d_{i,j}(\Theta)^{\gamma_{i,j}(\Theta)}$ be some nonzero polynomial in $(S_n(\lambda))_{|\lambda|}$.

Since $\sum_{j=1}^n \sum_{\Theta \in \mathbb{J}_{n,j}} \gamma_{n,j}(\Theta) = \sum_{i=1}^n k_i$, (92) shows

$$f' = \prod_{j=1}^n \prod_{\Theta \in \mathbb{J}_{n,j}} d_{n,j}(\Theta)^{\gamma_{n,j}(\Theta)} \tag{194}$$

with $\sum_{\Theta \in \mathbb{J}_{n,j}} \gamma_{n,j}(\Theta) = k_{n-j+1}$ for $1 \leqslant j \leqslant n$. As (172), we have

$$E_{s,n+1}(f) = \sum_{j=s}^n \Big(\sum_{\Theta \in \mathbb{J}_{n,j}} \gamma_{n,j}(\Theta) - k_{n-j+1} \Big) D_{n,j}(s) f' = 0 \tag{195}$$

for $1 \leqslant s \leqslant n$. Hence $(V_n(\lambda))_{|\lambda|+1} = \{0\}$. Thereby, $(V_n(\lambda))_r = \{0\}$ for any $r > |\lambda|$ by (27). ∎

B. Proof of Proposition 3.4

In this section, we use (132) to prove

Proposition 3.4 *We have* $\dim(V_n(\lambda))_r =$

$$= \sum_{s=0}^{n-1} (-1)^s \binom{n + r - 1 - \sum_{j=1}^s (k_j + 1)}{n-1} d_{n-1}\Big(\sum_{i=1}^s k_i \omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1} \omega_i\Big) \tag{196}$$

for $r \in \overline{0, |\lambda|}$. *Moreover,*

$$\sum_{s=0}^{n-1} (-1)^s \binom{\sum_{j=s+1}^n (k_j + 1)}{n} d_{n-1}\Big(\sum_{i=1}^s k_i \omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1} \omega_i\Big) = d_n\Big(\sum_{\ell=1}^n k_\ell \omega_\ell\Big). \tag{197}$$

Proof. For $s \in \overline{1, n}$, let $r_s = r - \sum_{j=1}^s (k_j + 1)$. Set

$$\widehat{\Omega}_{r_s}^s = \{\beta \in \mathbb{N}^n \mid |\beta| = r_s;\ \beta_i \leqslant k_{i-1} \text{ for } i \in \overline{2, s};\ \beta_i \leqslant k_i \text{ for } i \in \overline{s+1, n}\},$$

$$\widetilde{\Omega}_{r_s}^s = \left\{ \beta \in \mathbb{N}^n \;\middle|\; \begin{array}{l} |\beta| = r_s;\ \beta_i \leqslant k_{i-1} \text{ for } i \in \overline{2, s}; \\ \beta_{s+1} \leqslant k_s + k_{s+1} + 1;\ \beta_i \leqslant k_i \text{ for } i \in \overline{s+2, n} \end{array} \right\}.$$

Since $r \leqslant \sum\limits_{i=1}^{n} k_i$, we have $r_n < 0$. So $\widehat{\Omega}^n_{r_n} = \widetilde{\Omega}^n_{r_n} = \emptyset$. Obviously $\widehat{\Omega}^s_{r_s} \subseteq \widetilde{\Omega}^s_{r_s}$ and

$$\widetilde{\Omega}^s_{r_s} \setminus \widehat{\Omega}^s_{r_s} = (0, \cdots, \overset{s+1}{k_{s+1}+1}, \cdots, 0) + \widehat{\Omega}^{s+1}_{r_{s+1}}.$$

By (131) and (132) with $\sum\limits_{i=1}^{n-1} k_{i+1}\omega_i$ replaced by $\sum\limits_{i=1}^{s} k_i\omega_i + \omega_s + \sum\limits_{i=s}^{n-1} k_{i+1}\omega_i$ and r replaced by r_s, note that $\mathscr{A}_{r_s} = \emptyset$ and $\widetilde{\Omega}^s_{r_s} = \emptyset$ if $r_s < 0$. We have

$$\begin{aligned}&\dim\big(\mathscr{A}_{r_s} \otimes V_{n-1}\big(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i\big)\big)\\ &= \sum_{\beta\in\widetilde{\Omega}^s_{r_s}} d_{n-1}\big(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i\big).\end{aligned} \tag{198}$$

Hence, by (137) and (138),

$$\begin{aligned}&\sum_{\beta\in\widetilde{\Omega}^s_{r_s}\setminus\widehat{\Omega}^s_{r_s}} d_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i)\\ &= \sum_{\beta\in\widehat{\Omega}^{s+1}_{r_{s+1}}} d_{n-1}\big(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\omega_i + (k_{s+1}+1)(\omega_{s+1} - \omega_s)\big)\\ &= \sum_{\beta\in\widehat{\Omega}^{s+1}_{r_{s+1}}} d_{n-1}\big(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i\big)\\ &= \big(\sum_{\beta\in\widetilde{\Omega}^{s+1}_{r_{s+1}}} - \sum_{\beta\in\widetilde{\Omega}^{s+1}_{r_{s+1}}\setminus\widehat{\Omega}^{s+1}_{r_{s+1}}}\big) d_{n-1}\big(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i\big)\\ &= \dim(\mathscr{A}_{r_{s+1}} \otimes V_{n-1}(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i))\\ &\quad - \sum_{\beta\in\widetilde{\Omega}^{s+1}_{r_{s+1}}\setminus\widehat{\Omega}^{s+1}_{r_{s+1}}} d_{n-1}(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i).\end{aligned} \tag{199}$$

Moving items, we get

$$\begin{aligned}&\dim(\mathscr{A}_{r_{s+1}} \otimes V_{n-1}(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i))\\ &= \sum_{\beta\in\widetilde{\Omega}^s_{r_s}\setminus\widehat{\Omega}^s_{r_s}} d_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i)\\ &\quad + \sum_{\beta\in\widetilde{\Omega}^{s+1}_{r_{s+1}}\setminus\widehat{\Omega}^{s+1}_{r_{s+1}}} d_{n-1}(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i - \beta_{i+1})\,\omega_i).\end{aligned} \tag{200}$$

Note that $\widetilde{\Omega}^{n-1}_{r_{n-1}} \setminus \widehat{\Omega}^{n-1}_{r_{n-1}} = (0,\cdots,0,(k_n+1)) + \widehat{\Omega}^n_{r_n} = \varnothing$, multiply (200) by $(-1)^s$ and sum over $1 \leqslant s \leqslant n-2$,

$$\begin{aligned}
&\sum_{s=1}^{n-2}(-1)^s \dim(\mathscr{A}_{r_{s+1}} \otimes V_{n-1}(\sum_{i=1}^{s+1} k_i\omega_i + \omega_{s+1} + \sum_{i=s+1}^{n-1} k_{i+1}\omega_i)) \\
&= -\sum_{\beta\in\widetilde{\Omega}^1_{r_1}\setminus\widehat{\Omega}^1_{r_1}} d_{n-1}((k_1+k_2+1)\omega_1 + \sum_{i=2}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i-\beta_{i+1})\omega_i) \\
&\quad + (-1)^{n-2}\sum_{\beta\in\widetilde{\Omega}^{n-1}_{r_{n-1}}\setminus\widehat{\Omega}^{n-1}_{r_{n-1}}} d_{n-1}(\sum_{i=1}^{n-2} k_i\omega_i + (k_{n-1}+k_n+1)\omega_{n-1} + \sum_{i=1}^{n-1}(\beta_i-\beta_{i+1})\omega_i) \\
&= -\sum_{\beta\in\widetilde{\Omega}^1_{r_1}\setminus\widehat{\Omega}^1_{r_1}} d_{n-1}((k_1+k_2+1)\omega_1 + \sum_{i=2}^{n-1} k_{i+1}\omega_i + \sum_{i=1}^{n-1}(\beta_i-\beta_{i+1})\omega_i). \qquad (201)
\end{aligned}$$

Since $\widehat{\Omega}_r \setminus \Omega_r = (k_1+1,0,\cdots,0) + \widehat{\Omega}^1_{r_1}$, as a conclusion, by (198) and (201),

$$\begin{aligned}
&\dim(\widehat{M}_r/(V_n(\lambda))_r) \\
&= \sum_{\beta\in\widehat{\Omega}^1_{r_1}} d_{n-1}((k_1+k_2+1+\beta_1-\beta_2)\omega_1 + \sum_{i=1}^{n-1}(k_{i+1}+\beta_i-\beta_{i+1})\omega_i) \\
&= \Big(\sum_{\beta\in\widetilde{\Omega}^1_{r_1}} - \sum_{\beta\in\widetilde{\Omega}^1_{r_1}\setminus\widehat{\Omega}^1_{r_1}}\Big) d_{n-1}((k_1+k_2+1+\beta_1-\beta_2)\omega_1 + \sum_{i=1}^{n-1}(k_{i+1}+\beta_i-\beta_{i+1})\omega_i) \\
&= \sum_{s=1}^{n-1}(-1)^{s-1}\dim(\mathscr{A}_{r_s} \otimes V_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i)) \\
&= \sum_{s=1}^{n-1}(-1)^{s-1}\binom{n+r-1-\sum_{j=1}^{s}(k_j+1)}{n-1} d_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i)
\end{aligned}$$

Here we make the convention that for $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!(p-q)!}, & \text{if } p \geqslant q, \\ 0, & \text{if } p < q. \end{cases}$$

Moreover, we have

$$\begin{aligned}
&\dim(V_n(\lambda))_r = \dim \widehat{M}_r - \dim(\widehat{M}_r/(V_n(\lambda))_r) \\
&= \binom{n+r-1}{n-1} d_{n-1}(\sum_{i=1}^{n-1} k_{i+1}\omega_i) \\
&\quad + \sum_{s=1}^{n-1}(-1)^s\binom{n+r-1-\sum_{j=1}^{s}(k_j+1)}{n-1} d_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i) \\
&= \sum_{s=0}^{n-1}(-1)^s\binom{n+r-1-\sum_{j=1}^{s}(k_j+1)}{n-1} d_{n-1}(\sum_{i=1}^{s} k_i\omega_i + \omega_s + \sum_{i=s}^{n-1} k_{i+1}\omega_i)
\end{aligned}$$

and

$$\begin{aligned}\dim V_n(\lambda) &= \sum_{r=1}^{|\lambda|}\dim(V_n(\lambda))_r\\ &= \sum_{r=1}^{|\lambda|}\sum_{s=0}^{n-1}(-1)^s\binom{n+r-1-\sum_{j=1}^{s}(k_j+1)}{n-1}d_{n-1}(\sum_{i=1}^{s}k_i\omega_i+\omega_s+\sum_{i=s}^{n-1}k_{i+1}\omega_i)\\ &= \sum_{s=0}^{n-1}(-1)^s\binom{n+|\lambda|-\sum_{j=1}^{s}(k_j+1)}{n}d_{n-1}(\sum_{i=1}^{s}k_i\omega_i+\omega_s+\sum_{i=s}^{n-1}k_{i+1}\omega_i)\\ &= \sum_{s=0}^{n-1}(-1)^s\binom{\sum_{j=s+1}^{n}(k_j+1)}{n}d_{n-1}(\sum_{i=1}^{s}k_i\omega_i+\omega_s+\sum_{i=s}^{n-1}k_{i+1}\omega_i),\end{aligned}$$

where we just treat $\omega_0 = 0$.

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References

[1] A. Alex, M. Kalus, A. Huckleberry, T. Alan, J. von Delft: *A numerical algorithm for the explicit calculation of $SU(N)$ and $SL(N,\mathbb{C})$ Clebsch-Gordan coefficients*, J. Math. Phys. 52/2 (2011), art. no. 023507, 21 pp.

[2] C. Bohning, H. Graf von Bothmer: *A Clebsch-Gordan formula for $SL_3(\mathbb{C})$ and applications to rationality*, Adv. Math. 224/1 (2010) 246–259.

[3] I. V. Cherednik: *A new interepretation of Gelfand-Tsetlin bases*, Duke Math. J. 54 (1987) 563–577.

[4] J. J. de Swart: *The octet model and its Clebsch-Gordan coefficients*, Rev. Mod. Phys. 35/4 (1963) 916–939.

[5] A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, Princeton University Press, Princeton (1957).

[6] V. M. Futorny, D. Grantcharov, L. E. Ramirez: *Singular Gelfand-Tsetlin modules of* $\mathfrak{gl}(n)$, Adv. Math. 290 (2016) 453–482.

[7] V. M. Futorny, D. Grantcharov, L. E. Ramirez: *New Singular Gelfand-Tsetlin $\mathfrak{gl}(n)$-modules of index 2*, Commun. Math. Phys. 355 (2017) 1209–1241.

[8] I. M. Gelfand, M. L. Tsetlin: *Finite-dimensional representations of the group of unimodular matrices*, Dokl. Akad. Nauk SSSR 71 (1950) 825–828.

[9] I. M. Gelfand, M. L. Tsetlin: *Finite-dimensional representations of the group of orthogonal matrices*, Dokl. Akad. Nauk SSSR 71 (1950) 1017–1020.

[10] M. D. Gould: *On the matrix elements of the $U(n)$ generators*, J. Math. Phys. 22 (1981) 15–22.

[11] M. D. Gould: *Wigner coefficients for a semisimple Lie group and the matrix elements of the $O(n)$ generators*, J. Math. Phys. 22 (1981) 2376–2388.

[12] M. D. Gould: *Representation theory of the symplectic groups. I*, J. Math. Phys. 30/6 (1989) 1205–1218.

[13] M. D. Gould, E. G. Kalnins: *A projection-based solution to the $Sp(2n)$ state labeling problem*, J. Math. Phys. 26 (1985) 1446–1457.

[14] H. Guo, C. Huang, H. Wu: *Yang's model as triply special relativity and the Snyder's model de Sitter special relativity duality*, Phys. Lett. B 663 (2008) 270–274.

[15] H. Guo, C. Huang, H. Wu: *The principle of relativity and the special relativity triple*, Phys. Letters B 670 (2009) 437–441.

[16] J. E. Humphreys: *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics 9, Springer, Berlin (1972).

[17] V. Kac: *Infinite Dimensional Lie Algebras*, Birkhäuser, Boston (1983).

[18] T. Kaeding: *Tables of $SU(3)$ isoscalar factors*, Atomic Data and Nuclear Data Tables 61/2 (1995) 233–288.

[19] S. Kang, M. Kim: *Free Lie algebras, generalized Witt formula and denominator identity*, J. Algebra 183/2 (1996) 560–594.

[20] V. G. Knizhnik, A. B. Zamolodchikov: *Current algebra and Wess-Zumino models in two dimensions*, Nuclear Phys. B 247 (1984) 83–103.

[21] X. Li, J. Paldus: *Unitary group tensor operator algebras for many-electron systems. I: Clebsch-Gordan and Racah coefficients*, J. Math. Chem. 4/1-1 (1990) 295–253.

[22] P. Littelmann: *An algorithm to compute bases and representation matrices for sl_{n+1} representations*, J. Pure Appl. Algebra 117/118 (1997) 447–468.

[23] I. G. Macdonald: *Affine root systems and Dedekind's η-function*, Invent. Math. 15 (1972) 91–143.

[24] A. I. Molev: *A basis for representations of symplectic Lie algebras*, Commun. Math. Phys. 201 (1999) 591–618.

[25] A. I. Molev: *Weight bases of Gelfand-Tsetlin type for representations of classical Lie algebras*, J. Phys. A: Math. Gen. 33 (2000) 4143–4158.

[26] J. Ramos: *Differential operators in terms of Clebsch-Gordan coefficients and the wave equation of massless tensor fields*, Gen. Ralativity Gravitation 38 (2006) 773–783.

[27] J. Riordan: *Combinatorial Identities*, R. E. Krieger Publishing Company, Malabar (1979).

[28] G. Shen: *Graded modules of graded Lie algebras of Cartan type (I) – mixed product of modules*, Science in China A 29 (1986) 570–581.

[29] A. Tsuchiya, Y. Kanie: *Vertex operators in conformal field theory on $\mathbf{P}^1$ and monodromy representations of braid group*, Adv. Stud. Pure Math. 16 (1988) 297–372.

[30] M. Wang, H. Yeh: *The most degenerate irreducible representations of the symplectic group*, J. Math. Phys. 21 (1980) 630–635.

[31] X. Xu: *Introduction to Vertex Operator Superalgebras and their Modules*, Kluwer Academic Publishers, Dordrecht (1998).

[32] X. Xu: *Differential invariants of classical groups*, Duke J. Math. 94/3 (1998) 543–572.

[33] X. Xu: *Differential operator representation of S_n and singular vectors in Verma modules*, Algebr. Represent. Theory 15 (2012) 211–231.

[34] X. Xu: *Representations of Lie Algebras and Partial Differential Equations*, Springer, Singapore (2017).

[35] Y. Zhao, X. Xu: *Generalized projective representations for* $\mathfrak{sl}(n+1)$, J. Algebra 328 (2011) 132–154.

[36] D. P. Zhelobenko: *The classical groups: spectral analysis of their finite-dimensional representations*, Russ. Math. Surveys 17 (1962) 1–94.

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