Journal of Lie Theory
Volume **33** (2023) 1177–1192

Characterization of Lipschitz Functions via the Commutators of Maximal Functions on Stratified Lie Groups

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Communicated by G. Ólafsson

Abstract. Our main aim is to consider the boundedness of the Hardy-Littlewood maximal commutator M_b and the nonlinear commutator $[b, M]$ on the Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$ when the symbols of the commutators belong to the Lipschitz space. Meanwhile, the corresponding end-point estimates on the Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$ are considered as well. As a result, some new characterizations of the Lipschitz spaces on Lie group via M_b and $[b, M]$ are given.

Mathematics Subject Classification: 42B35, 43A80, 43A15.

Key Words: Stratified Lie group, maximal function, Lipschitz function, commutator, Morrey space.

1. Introduction and main results

Stratified groups appear in quantum physics and many parts of mathematics, including several complex variables, Fourier analysis, geometry and topology et al. [10,31]. The geometry structure of stratified groups is so good that it inherits many analysis properties from the Euclidean spaces [11, 30]. Apart from this, the study of function spaces on stratified groups is more complicated since the difference between the geometry structures of stratified groups and Euclidean spaces. However, many harmonic analysis problems on stratified Lie groups deserve a further investigation since most results of the theory of Fourier transforms and distributions in Euclidean spaces cannot yet be duplicated.

Let T be a classical singular integral operator. The commutator $[b, T]$ generated by T and a suitable function b is defined by

$$[b, T]f = bT(f) - T(bf). \tag{1}$$

It is known that the commutators are intimately related to the regularity properties of the solutions of certain partial differential equations (PDE), see [4, 7, 28].

The first result for the commutator $[b, T]$ was established by Coifman et al. [6], and the authors proved that $b \in \mathrm{BMO}(\mathbb{R}^n)$ (bounded mean oscillation functions) if and only if the commutator (1) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978,

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ISSN 0949–5932 / \$2.50

Janson [18] generalized the results in [6] to functions belonging to a Lipschitz space and gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via commutator (1), and the author proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ if and only if $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also [27]).

In addition, Milman and Schonbek [24] used the real interpolation technique to establish a commutator result that applies to both the Hardy-Littlewood maximal function and a large class of nonlinear operators. In 2000, Bastero et al. [1] proved the necessary and sufficient conditions for the boundedness of the nonlinear commutator $[b,M]$ on L^p spaces, and the similar problems for $[b,M_\alpha]$ were also studied by Zhang and Wu [34]. In 2017, Zhang [33] considered some new characterizations of the Lipschitz spaces via the boundedness of maximal commutator M_b and the (nonlinear) commutator $[b,M]$ in Lebesgue spaces and Morrey spaces on Euclidean spaces. In 2018, Zhang et al. [35] gave necessary and sufficient conditions for the boundedness of the nonlinear commutator $[b,M_\alpha]$ on Orlicz spaces when the symbol b belongs to Lipschitz spaces, and obtained some new characterizations of non-negative Lipschitz functions. Recently, Guliyev [14, 15] gave necessary and sufficient conditions for the boundedness of some maximal commutators in the Orlicz spaces $L^\Phi(\mathbb{G})$ on stratified Lie group $\mathbb{G}$, where the symbol b belongs to $\mathrm{BMO}(\mathbb{G})$ spaces and Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{G})$ respectively, and obtained separately some new characterizations for certain subclasses of $\mathrm{BMO}(\mathbb{G})$ and $\dot{\Lambda}_\beta(\mathbb{G})$.

Inspired by the above literature, the purpose of this paper is to characterize the Lipschitz spaces $\dot{\Lambda}_\beta(\mathbb{G})$ in terms of the boundedness of the Hardy-Littlewood maximal commutator M_b and the nonlinear commutator $[b,M]$ in the context of the Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$, where the symbols belong to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{G})$. Moreover, the corresponding endpoint estimates on the Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$ are also established.

Let $f \in L^1_{\mathrm{loc}}(\mathbb{G})$, the Hardy–Littlewood maximal function M is given by

$$M(f)(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y)| \mathrm{d}y$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing x, and $|B|$ is the Haar measure of the $\mathbb{G}$-ball B. And the maximal commutator M_b generated by the operator M and a locally integrable function b is defined by

$$M_b(f)(x) = \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)| \mathrm{d}y.$$

On the other hand, similar to (1), the nonlinear commutator of the Hardy-Littlewood maximal function M with a locally integrable function b is defined by

$$[b,M](f)(x) = b(x)M(f)(x) - M(bf)(x).$$

Note that the operators M_b and $[b,M]$ essentially differ from each other. For example, M_b is positive and sublinear, but $[b,M]$ is neither positive nor sublinear.

The first part of this paper is to study the mapping properties of M_b on Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$ in the case when the symbol b belongs to a Lipschitz space. Therefore, some new characterizations of these Lipschitz spaces via such commutator are established.

Theorem 1.1. *Let b be a locally integrable function and $0 < \beta < 1$. Then the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{G})$.

(ii) *M_b is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for all p, q with $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$.*

(iii) *M_b is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for some p, q with $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$.*

(iv) *M_b satisfies the weak-type $(1, Q/(Q-\beta))$ estimates, namely, there exists a positive constant C such that*

$$\left|\{x \in \mathbb{G} : M_b(f)(x) > \lambda\}\right| \le C\left(\lambda^{-1}\|f\|_{L^1(\mathbb{G})}\right)^{Q/(Q-\beta)} \qquad (2)$$

holds for all $\lambda > 0$.

Theorem 1.2. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$, $0 < \lambda < Q - \beta p$.*

(i) *If $1/q = 1/p - \beta/(Q-\lambda)$. Then $b \in \dot{\Lambda}_\beta(\mathbb{G})$ if and only if M_b is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\lambda}(\mathbb{G})$.*

(ii) *If $1/q = 1/p - \beta/Q$ and $\lambda/p = \mu/q$. Then $b \in \dot{\Lambda}_\beta(\mathbb{G})$ if and only if M_b is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\mu}(\mathbb{G})$.*

The second part study the mapping properties of the nonlinear commutator $[b, M]$ on Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$ in the case when the symbol b belongs to some Lipschitz space. To state our results, we recall the definition of the maximal operator with respect to a ball. For a fixed ball B^*, the Hardy-Littlewood maximal function with respect to B^* of a locally integrable function f is given by

$$M_{B^*}(f)(x) = \sup_{\substack{B \ni x \\ B \subseteq B^*}} |B|^{-1} \int_B |f(y)| \mathrm{d}y,$$

where the supremum is taken over all balls B with $B \subseteq B^*$ and $x \in B$.

Theorem 1.3. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$. Then the following statements are equivalent:*

(i) *$b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \ge 0$.*

(ii) *$[b, M]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$.*

(iii) *There exists a constant $C > 0$ such that*

$$\sup_{B \ni x} |B|^{-\beta/Q} \left(|B|^{-1} \int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x\right)^{1/q} \le C. \qquad (3)$$

Theorem 1.4. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$, $0 < \lambda < Q - \beta p$ and $1/q = 1/p - \beta/(Q-\lambda)$. Then the following statements are equivalent:*

(i) *$b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \ge 0$.*

(ii) *$[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\lambda}(\mathbb{G})$.*

Theorem 1.5. *Let b be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$, $0 < \lambda < Q - \beta p$, $1/q = 1/p - \beta/Q$ and $\lambda/p = \mu/q$. Then the following statements are equivalent:*

(i) $b \in \dot{\Lambda}_\beta(\mathbb{G})$ *and* $b \geq 0$.

(ii) $[b, M]$ *is bounded from* $L^{p,\lambda}(\mathbb{G})$ *to* $L^{q,\mu}(\mathbb{G})$.

This paper is organized as follows. In the next section, we recall some basic definitions and known results. In Section 3, we will prove Theorems 1.1 and 1.2. Section 4 is devoted to proving the Theorems 1.3 to 1.5.

Throughout this paper, the letter C always stands for a constant independent of the main parameters involved and whose value may differ from line to line.

2. Preliminaries and lemmas

2.1. Lie group $\mathbb{G}$

To prove the main results, we first recall some necessary notions and remarks. Below we give some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to [3, 10, 30].

Definition 2.1. We say that a Lie algebra $\mathcal{G}$ is *stratified* if there is a direct sum vector space decomposition

$$\mathcal{G} = \oplus_{j=1}^m V_j = V_1 \oplus \cdots \oplus V_m \tag{4}$$

such that $\mathcal{G}$ is nilpotent of step m if m is the smallest integer for which all Lie brackets (or iterated commutators) of order $m+1$ are zero, that is, we have

$$[V_1, V_j] = \begin{cases} V_{j+1}, & 1 \leq j \leq m-1 \\ 0, & j \geq m \end{cases}.$$

It is not difficult to find that the above V_1 generates the whole of the Lie algebra $\mathcal{G}$ by taking Lie brackets.

Remark 2.2. (see [36]) Let $\mathcal{G} = \mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_{m+1} = \{0\}$ denote the lower central series of $\mathcal{G}$, and $\{X_1, \ldots, X_N\}$ be a basis for V_1 of $\mathcal{G}$.

(a) The direct sum decomposition (4) can be constructed by identifying each $\mathcal{G}_j$ as a vector subspace of $\mathcal{G}$ and setting $V_m = \mathcal{G}_m$ and $V_j = \mathcal{G}_j \setminus \mathcal{G}_{j+1}$ for $j = 1, \ldots, m-1$.

(b) The dimension of $\mathbb{G}$ at infinity as the integer Q is given by

$$Q = \sum_{j=1}^m j \dim(V_j) = \sum_{j=1}^m \dim(\mathcal{G}_j).$$

Definition 2.3. A Lie group $\mathbb{G}$ is said to be *stratified* when it is a connected simply-connected Lie group and its Lie algebra $\mathcal{G}$ is stratified. ■

If $\mathbb{G}$ is stratified, then its Lie algebra $\mathcal{G}$ admits a canonical family of dilations $\{\delta_r\}$, namely, for $r > 0,\ X_k \in V_k\ (k = 1, \dots, m)$,

$$\delta_r\Big(\sum_{k=1}^{m} X_k\Big) = \sum_{k=1}^{m} r^k X_k,$$

which are Lie algebra automorphisms. By the Baker-Campbell-Hausdorff formula for sufficiently small elements X and Y of $\mathcal{G}$ one has

$$\exp X \exp Y = \exp H(X, Y),$$

where $\exp : \mathcal{G} \to \mathbb{G}$ is the exponential map, $H(X,Y) = X + Y + \frac{1}{2}[X,Y] + \cdots$ is an infinite linear combination of X and Y and their Lie brackets, and the dots denote terms of order higher than two.

The following properties can be found in [29](see Proposition 1.1.1, or Proposition 2.1 in [32] or Proposition 1.2 in [10]).

Proposition 2.4. *Let $\mathcal{G}$ be a nilpotent Lie algebra, and let $\mathbb{G}$ be the corresponding connected and simply-connected nilpotent Lie group. Then we have*

(i) *The exponential map* $\exp : \mathcal{G} \to \mathbb{G}$ *is a diffeomorphism. Furthermore, the group law $(x, y) \mapsto xy$ is a polynomial map if $\mathbb{G}$ is identified with $\mathcal{G}$ via* $\exp$.

(ii) *f λ is a Lebesgue measure on $\mathcal{G}$, then* $\exp \lambda$ *is a bi-invariant Haar measure on $\mathbb{G}$ (or a bi-invariant Haar measure* $\mathrm{d}x$ *on $\mathbb{G}$ is just the lift of Lebesgue measure on $\mathcal{G}$ via* $\exp$*).*

Notations:

- y^{-1} represents the inverse of $y \in \mathbb{G}$.
- $y^{-1}x$ stands for the group multiplication of y^{-1} by x.
- Let the group identity element of $\mathbb{G}$ be referred to as the origin denotes by e.
- χ_E denotes a characteristic function of a measurable set E of $\mathbb{G}$.
- L^p $(1 \le p \le \infty)$ denotes the standard L^p-space with respect to the Haar measure $\mathrm{d}x$, with the L^p-norm $\|\cdot\|_p$.

A homogenous norm $\rho :\ x \to \rho(x)$ defined on $\mathbb{G}$ is a continuous function from $\mathbb{G}$ to $[0, \infty)$, which is C^∞ on $\mathbb{G} \setminus \{0\}$ and satisfies

$$\begin{cases} \rho(x^{-1}) = \rho(x), \\ \rho(\delta_t x) = t\rho(x) \ \text{ for all } x \in \mathbb{G} \text{ and } t > 0, \\ \rho(e) = 0. \end{cases}$$

Moreover, there exists a constant $c_0 \ge 1$ such that $\rho(xy) \le c_0(\rho(x) + \rho(y))$ for all $x, y \in \mathbb{G}$.

With the norm above, we define the $\mathbb{G}$ ball centered at x with radius r by $B(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and by λB denote the ball $B(x, \lambda r)$ with $\lambda > 0$, let $B_r = B(e,r) = \{y \in \mathbb{G} : \rho(y) < r\}$ be the open ball centered at e with radius r, which is the image under δ_r of $B(e, 1)$.

And by ${}^{\complement}B(x,r) = \mathbb{G} \setminus B(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) \geq r\}$ denote the complement of $B(x,r)$. Let $|B(x,r)|$ be the Haar measure of the ball $B(x,r) \subset \mathbb{G}$, and there exists $c_1 = c_1(\mathbb{G})$ such that

$$|B(x,r)| = c_1 r^Q, \qquad x \in \mathbb{G}, r > 0.$$

The most basic partial differential operator in a stratified Lie group is the sub-Laplacian associated with $X = \{X_1, \ldots, X_n\}$, is the second-order partial differential operator on $\mathbb{G}$ given by

$$\mathcal{L} = \sum_{i=1}^{n} X_i^2.$$

In the context of a Lie groups $\mathbb{G}$, when Young function $\Phi(t) = t^p$ and its complementary function $\Psi(t) = t^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following results can be inferred from [14] by elementary calculations.

Lemma 2.5. (Hölder's inequality on $\mathbb{G}$) *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbb{G}$ be a measurable set and measurable functions $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then there exists a positive constant C such that*

$$\int_{\Omega} |f(x)g(x)| \mathrm{d}x \leq C \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Lemma 2.6. (Norms of characteristic functions) *Let $0 < p < \infty$ and $\Omega \subset \mathbb{G}$ be a measurable set with finite Haar measure. Then*

$$\|\chi_\Omega\|_{L^p(\mathbb{G})} = \|\chi_\Omega\|_{WL^p(\mathbb{G})} = |\Omega|^{1/p}.$$

2.2. Maximal function

Let $0 \leq \alpha < Q$ and $f : \mathbb{G} \to \mathbb{R}$ is a locally integrable function. The fractional maximal function is defined by

$$M_\alpha(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |f(y)| \mathrm{d}y,$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing x.

The fractional maximal function $M_\alpha(f)$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $M(f)(x) \equiv M_0(f)(x)$.

The following propositions can be found in [19].

Proposition 2.7. *Let $0 \leq \alpha < Q$ and $1 < p < \gamma^{-1} = \frac{Q}{\alpha}$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then the following two conditions are equivalent:*

(i) *There is a constant $C > 0$ such that for any $f \in L^p_\omega(\mathbb{G})$ we have the inequality*

$$\Big(\int_{\mathbb{G}} \big(M_\gamma(f\omega^\gamma)(x)\big)^q \omega(x) \mathrm{d}x \Big)^{1/q} \leq C \Big(\int_{\mathbb{G}} |f(x)|^p \omega(x) \mathrm{d}x \Big)^{1/p}.$$

(ii) $\omega \in A_{1+q/p'}(\mathbb{G})$, $p' = \dfrac{p}{p-1}$.

Proposition 2.8. *Let $0 < \alpha < Q$, $\gamma = \alpha/Q$, $q = (1-\gamma)^{-1}$, and $f \in L^q(\mathbb{G})$. Then the following two conditions are equivalent:*

$$\omega\{x \in \mathbb{G} : M_\gamma(f\omega^\gamma)(x) > \lambda\} \le C\lambda^{-q}\Big(\int_{\mathbb{G}} |f(x)|\mathrm{d}x\Big)^q$$

with a constant $C > 0$ independent of f and $\lambda > 0$.

(ii) $\omega \in A_1(\mathbb{G})$.

The following strong and weak-type boundedness of M_α can be obtained from Propositions 2.7 and 2.8 when the weight $\omega = 1$, see Kokilashvili and Kufner [19] for more details. And the first part can also be obtained from Bernardis and Salinas [2].

Lemma 2.9. *Let $0 < \alpha < Q$, $1 \le p \le Q/\alpha$ with $1/q = 1/p - \alpha/Q$, and $f \in L^p(\mathbb{G})$.*

(i) *If $1 < p < Q/\alpha$, then there exists a positive constant C such that*

$$\|M_\alpha(f)\|_{L^q(\mathbb{G})} \le C\|f\|_{L^p(\mathbb{G})}$$

(ii) *If $p = 1$, then there exists a positive constant C such that*

$$|\{x \in \mathbb{G} : M_\alpha(f)(x) > \lambda\}| \le C\big(\lambda^{-1}\|f\|_{L^1(\mathbb{G})}\big)^{Q/(Q-\alpha)}$$

holds for all $\lambda > 0$.

Finally, similar to Lemma 2.3 in [34] (or see [1]), we give the pointwise relations of maximal function M, which can be deduced by elementary calculations.

Lemma 2.10. *Let $B \subset \mathbb{G}$ be a ball, and f be a locally integrable function. Then, the following results*

$$M(f\chi_B)(x) = M_B(f)(x) \text{ and } M(\chi_B)(x) = M_B(\chi_B)(x) = \chi_B(x)$$

are valid for all $x \in B$.

2.3. Lipschitz spaces on $\mathbb{G}$

Next we give the definition of the Lipschitz spaces on $\mathbb{G}$, and state some basic properties and useful lemmas.

Definition 2.11. (Lipschitz-type spaces on $\mathbb{G}$)

(i) Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{G})$ if there exists a constant $C > 0$ such that for all $x, y \in \mathbb{G}$,

$$|b(x) - b(y)| \le C(\rho(y^{-1}x))^\beta, \tag{5}$$

where ρ is the homogenous norm. The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})}$.

(ii) (see Macías and Segovia [22]) Let $0 < \beta < 1$ and $1 \le p < \infty$. The space $\mathrm{Lip}_{\beta,p}(\mathbb{G})$ is defined to be the set of all locally integrable functions b, i.e., there exists a positive constant C, such that

$$\sup_{B \ni x} \frac{1}{|B|^{\beta/Q}}\Big(\frac{1}{|B|}\int_B |b(x) - b_B|^p \mathrm{d}x\Big)^{1/p} \le C,$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing x and satsfying $b_B = \frac{1}{|B|}\int_B b(x)\mathrm{d}x$. The least constant C satisfying the conditions above shall be denoted by $\|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})}$.

Remark 2.12. (a) Similar to the definition of Lipschitz space $\dot{\Lambda}_\beta(\mathbb{G})$ in Definition 2.11(i), we also have the definition form as following (see [5, 9, 20] et al.)

$$\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} = \sup_{\substack{x,y\in\mathbb{G}\\ x\neq y}} \frac{|b(x)-b(y)|}{(\rho(y^{-1}x))^\beta}.$$

And $\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} = 0$ if and only if b is constant.

(b) In Definition 2.11(ii), when $p = 1$, we have

$$\|b\|_{\mathrm{Lip}_{\beta,1}(\mathbb{G})} = \sup_{B\ni x} \frac{1}{|B|^{\beta/Q}}\Big(\frac{1}{|B|}\int_B |b(x)-b_B|\mathrm{d}x\Big) := \|b\|_{\mathrm{Lip}_\beta(\mathbb{G})}.$$

(c) There are two basically different approaches to Lipschitz classes on the n-dimensional Euclidean space. Lipschitz classes can be defined via Poisson (or Weierstrass) integrals of L^p-functions, or, equivalently, by means of higher order difference operators (see Meda and Pini [23]).

Lemma 2.13. (see [5,21,22]) *Let $0 < \beta < 1$ and b be a locally integrable function on $\mathbb{G}$.*

(i) *When $1 \le p < \infty$, then $\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} = \|b\|_{\mathrm{Lip}_\beta(\mathbb{G})} \approx \|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})}$.*

(ii) *Let balls $B_1 \subset B_2 \subset \mathbb{G}$ and $b \in \mathrm{Lip}_{\beta,p}(\mathbb{G})$ with $p \in [1,\infty)$. Then there exists a constant C depends on B_1 and B_2 only, such that*

$$|b_{B_1} - b_{B_2}| \le C\|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})}|B_2|^{\beta/Q}.$$

(iii) *When $1 \le p < \infty$, then there exists a constant C depends on β and p only, such that*

$$|b(x) - b(y)| \le C\|b\|_{\mathrm{Lip}_{\beta,p}(\mathbb{G})}|B|^{\beta/Q}$$

holds for any ball B containing x and y.

2.4. Morrey spaces on $\mathbb{G}$

Morrey spaces were originally introduced by Morrey in [25] to study the local behavior of solutions to second-order elliptic partial differential equations.

Definition 2.14. (Morrey-type spaces on $\mathbb{G}$ [8])

(i) Let $1 \le p < \infty$ and $0 \le \lambda \le Q$. The Morrey-type spaces $L^{p,\lambda}(\mathbb{G})$ is defined by $L^{p,\lambda}(\mathbb{G}) = \{f \in L^p_{\mathrm{loc}}(\mathbb{G}) : \|f\|_{L^{p,\lambda}(\mathbb{G})} < \infty\}$ with

$$\|f\|_{L^{p,\lambda}(\mathbb{G})} = \sup_{\substack{B\ni x\\ B\subset\mathbb{G}}} \Big(\frac{1}{|B|^{\lambda/Q}}\int_B |f(y)|^p\mathrm{d}y\Big)^{1/p},$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing x.

(ii) Let $1 \le p < \infty$ and $\varphi(x,r)$ be a positive measurable function on $\mathbb{G}\times(0,\infty)$. The generalized Morrey space $\mathcal{L}^{p,\varphi}(\mathbb{G})$ is defined for all functions $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ by the finite norm

$$\|f\|_{\mathcal{L}^{p,\varphi}(\mathbb{G})} = \sup_{\substack{B\ni x\\ B\subset\mathbb{G}}} \frac{1}{\varphi(x,r)}\Big(\frac{1}{|B|}\int_B |f(y)|^p\mathrm{d}y\Big)^{1/p},$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing x. And the weak generalized Morrey space $W\mathcal{L}^{p,\varphi}(\mathbb{G})$ is defined for all functions $f \in L^p_{\mathrm{loc}}(\mathbb{G})$ by the finite norm

$$\|f\|_{W\mathcal{L}^{p,\varphi}(\mathbb{G})} = \sup_{\substack{B \ni x \\ B \subset \mathbb{G}}} \frac{r^{-Q/p}}{\varphi(x,r)} \|f\|_{WL^p(B)},$$

where WL^p denotes the weak L^p space of measurable functions f.

Remark 2.15. (Guliyev [13])

(a) It is well known that if $1 \le p < \infty$ then

$$L^{p,\lambda}(\mathbb{G}) = \begin{cases} L^p(\mathbb{G}) & \text{if } \lambda = 0, \\ L^\infty(\mathbb{G}) & \text{if } \lambda = Q, \\ \Theta & \text{if } \lambda < 0 \text{ or } \lambda > Q, \end{cases}$$

where Θ is the set of all functions equivalent to 0 on $\mathbb{G}$.

(b) In Definition 2.14(ii), when $1 \le p < \infty$ and $0 \le \lambda \le Q$, we have $\mathcal{L}^{p,\varphi}(\mathbb{G}) = L^{p,\lambda}(\mathbb{G})$ if $\varphi(x,r) = |B|^{(\lambda/Q-1)/p}$, where $B \subset \mathbb{G}$ denotes the ball with radius r containing x. ■

We now recall the results on the boundedness of the fractional maximal operator in the generalised Morrey spaces, which can be found in [16] (Theorems 3.2 and 3.3, or see [12, 17, 26]).

Proposition 2.16. (Spanne-Guliyev type) *Let $1 \le p < \infty$, $0 \le \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$ and (φ_1, φ_2) satisfy the condition*

$$\sup_{r<t<\infty} t^{\alpha - Q/p} \operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{Q/p} \le C \varphi_2(x,r),$$

where $C > 0$ does not depend on r and $x \in \mathbb{G}$.

(i) *Then, for $1 < p < \infty$ and any $f \in \mathcal{L}^{p,\varphi_1}(\mathbb{G})$, there exists some positive constant C such that $\|M_\alpha f\|_{\mathcal{L}^{q,\varphi_2}(\mathbb{G})} \le C \|f\|_{\mathcal{L}^{p,\varphi_1}(\mathbb{G})}$.*

(ii) *Then, for $p = 1$ and any $f \in \mathcal{L}^{1,\varphi_1}(\mathbb{G})$, there exists some positive constant C such that $\|M_\alpha f\|_{W\mathcal{L}^{q,\varphi_2}(\mathbb{G})} \le C \|f\|_{\mathcal{L}^{1,\varphi_1}(\mathbb{G})}$.*

In the case $\alpha = 0$ and $p = q$, the conclusions of Proposition 2.16 are also valid.

Proposition 2.17. (Adams-Guliyev type) *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, and let $\varphi(x,\tau)$ satisfy the conditions*

$$\sup_{r<t<\infty} t^{-Q} \operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s) s^{Q} \le C \varphi(x,r)$$

and

$$\sup_{r<t<\infty} t^{\alpha} \varphi(x,\tau)^{1/p} \le C r^{-\alpha p/(q-p)},$$

where $C > 0$ does not depend on r and $x \in \mathbb{G}$.

(i) *Then, for $1 < p < \infty$ and any $f \in \mathcal{L}^{p,\varphi^{1/p}}(\mathbb{G})$, there exists some positive constant C such that $\|M_\alpha f\|_{\mathcal{L}^{q,\varphi^{1/q}}(\mathbb{G})} \le C \|f\|_{\mathcal{L}^{p,\varphi^{1/p}}(\mathbb{G})}$.*

(ii) *Then, for $p = 1$ and any $f \in \mathcal{L}^{1,\varphi^{1/p}}(\mathbb{G})$, there exists some positive constant C such that $\|M_\alpha f\|_{W\mathcal{L}^{q,\varphi^{1/q}}(\mathbb{G})} \le C \|f\|_{\mathcal{L}^{1,\varphi}(\mathbb{G})}$.*

In case we have

$\varphi_1(x,r) = |B|^{(\lambda/Q-1)/p} = c_1 r^{Q(\lambda/Q-1)/p}$ and $\varphi_2(x,r) = |B|^{(\mu/Q-1)/q} = c_2 r^{Q(\mu/Q-1)/q}$,

we can summarize the results as follows from Propositions 2.16 and 2.17 (see also Corollary 3.3 in [16]).

Lemma 2.18. *Let $0 < \alpha < Q$, $1 < p < Q/\alpha$ and $0 < \lambda < Q - \alpha p$.*

(i) *If $1/q = 1/p - \alpha/(Q-\lambda)$, then there exists a positive constant C such that $\|M_\alpha f\|_{L^{q,\lambda}(\mathbb{G})} \le C\|f\|_{L^{p,\lambda}(\mathbb{G})}$ for every $f \in L^{p,\lambda}(\mathbb{G})$.*

(ii) *If $1/q = 1/p - \alpha/Q$ and $\lambda/p = \mu/q$. Then there exists a positive constant C such that $\|M_\alpha f\|_{L^{q,\mu}(\mathbb{G})} \le C\|f\|_{L^{p,\lambda}(\mathbb{G})}$ for every $f \in L^{p,\lambda}(\mathbb{G})$.*

3. Proofs of the Theorems 1.1 and 1.2

We now prove the mapping properties of M_b on Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$, where the symbol b belongs to a Lipschitz space. First, we prove Theorem 1.1.

Proof of Theorem 1.1. If $b \in \dot{\Lambda}_\beta(\mathbb{G})$, then, using Definition 2.11(i), we have

$$\begin{aligned} M_b(f)(x) &= \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)|\mathrm{d}y \\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} \sup_{B \ni x} |B|^{-1} \int_B |\rho(y^{-1}x)|^\beta |f(y)|\mathrm{d}y \\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} \sup_{B \ni x} \frac{1}{|B|^{1-\beta/Q}} \int_B |f(y)|\mathrm{d}y \\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})} M_\beta(f)(x). \end{aligned} \tag{6}$$

Therefore, (ii), (iii) and (iv) follow from Lemma 2.9 and above estimate.

(iii) $\Rightarrow$ (i): Suppose M_b is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for some p, q with $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$. Noting that $1/p + 1/q' = 1 + \beta/Q$, for any ball $B \subset \mathbb{G}$, using Lemma 2.5 (Hölder's inequality) and Lemma 2.6, one obtains

$$\begin{aligned} \frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - b_B|\mathrm{d}x &\le \frac{1}{|B|^{1+\beta/Q}} \int_B \Big(\frac{1}{|B|} \int_B |b(x) - b(y)|\mathrm{d}y\Big)\mathrm{d}x \\ &= \frac{1}{|B|^{1+\beta/Q}} \int_B \Big(\frac{1}{|B|} \int_B |b(x) - b(y)|\chi_B(y)\mathrm{d}y\Big)\mathrm{d}x \\ &\le \frac{1}{|B|^{1+\beta/Q}} \int_B M_b(\chi_B)(x)\mathrm{d}x \\ &\le \frac{1}{|B|^{1+\beta/Q}} \Big(\int_B \big(M_b(\chi_B)(x)\big)^q \mathrm{d}x\Big)^{1/q} \Big(\int_B \chi_B(x)\mathrm{d}x\Big)^{1/q'} \\ &\le \frac{C}{|B|^{1+\beta/Q}} \|\chi_B\|_{L^p(\mathbb{G})} \|\chi_B\|_{L^{q'}(\mathbb{G})} \le C. \end{aligned}$$

This together with Lemma 2.13 gives $b \in \dot{\Lambda}_\beta(\mathbb{G})$.

(iv) $\Rightarrow$ (i): Assume that M_b satisfies the weak-type $(1, Q/(Q-\beta))$ estimates and (2) is true. In order to verify $b \in \dot{\Lambda}_\beta(\mathbb{G})$, for any fixed ball $B^* \subset \mathbb{G}$, we have

$$|b(x) - b_{B^*}| \leq \frac{1}{|B^*|}\int_{B^*} |b(x) - b(y)|\mathrm{d}y$$

for all $x \in B^*$. On the other hand, we get

$$\begin{aligned} M_b(\chi_{B^*})(x) &= \sup_{B \ni x} \frac{1}{|B|}\int_B |b(x) - b(y)|\chi_{B^*}(y)\mathrm{d}y \\ &\geq \frac{1}{|B^*|}\int_{B^*} |b(x) - b(y)|\chi_{B^*}(y)\mathrm{d}y \\ &= \frac{1}{|B^*|}\int_{B^*} |b(x) - b(y)|\mathrm{d}y \\ &\geq |b(x) - b_{B^*}| \end{aligned}$$

for all $x \in B^*$. Thus, combined with (2), we obtain

$$\begin{aligned} &\left|\{x \in B^* : |b(x) - b_{B^*}| > \lambda\}\right| \leq \left|\{x \in B^* : M_b(\chi_{B^*})(x) > \lambda\}\right| \\ &\leq C\left(\lambda^{-1}\|\chi_{B^*}\|_{L^1(\mathbb{G})}\right)^{Q/(Q-\beta)} \leq C\left(\lambda^{-1}|B^*|\right)^{Q/(Q-\beta)}. \end{aligned}$$

Let $t > 0$ be a constant to be determined later; applying Fubini's theorem, one gets

$$\begin{aligned} &\int_{B^*} |b(x) - b_{B^*}|\mathrm{d}x = \int_0^\infty \left|\{x \in B^* : |b(x) - b_{B^*}| > \lambda\}\right|\mathrm{d}\lambda \\ &= \int_0^t \left|\{x \in B^* : |b(x) - b_{B^*}| > \lambda\}\right|\mathrm{d}\lambda + \int_t^\infty \left|\{x \in B^* : |b(x) - b_{B^*}| > \lambda\}\right|\mathrm{d}\lambda \\ &\leq t|B^*| + C\int_t^\infty \left(\lambda^{-1}|B^*|\right)^{Q/(Q-\beta)}\mathrm{d}\lambda \leq t|B^*| + C|B^*|^{Q/(Q-\beta)}\int_t^\infty \lambda^{-Q/(Q-\beta)}\mathrm{d}\lambda \\ &\leq C\left(t|B^*| + |B^*|^{Q/(Q-\beta)}t^{1-Q/(Q-\beta)}\right). \end{aligned}$$

Let $t = |B^*|^{\beta/Q}$ in the above estimate, we get

$$\int_{B^*} |b(x) - b_{B^*}|\mathrm{d}x \leq C|B^*|^{1+\beta/Q}.$$

It follows from Lemma 2.13 that $b \in \dot{\Lambda}_\beta(\mathbb{G})$ since B^* is an arbitrary ball in $\mathbb{G}$.

The proof of Theorem 1.1 is complete since (ii) $\Rightarrow$ (i) follows from (iii) $\Rightarrow$ (i). ∎

Proof of Theorem 1.2. (i): We first prove that the necessary condition. Assume $b \in \dot{\Lambda}_\beta(\mathbb{G})$, using (6) and Lemma 2.18, we obtain

$$\|M_b(f)\|_{L^{q,\lambda}(\mathbb{G})} \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})}\|M_\beta f\|_{L^{q,\lambda}(\mathbb{G})} \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{G})}\|f\|_{L^{p,\lambda}(\mathbb{G})}.$$

We now prove that the sufficient condition. M_b is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\lambda}(\mathbb{G})$, then for any ball $B \subset \mathbb{G}$,

$$\begin{aligned} &|B|^{-\beta/Q}\Big(|B|^{-1}\int_B |b(x) - b_B|^q\mathrm{d}x\Big)^{1/q} \leq |B|^{-\beta/Q}\Big(|B|^{-1}\int_B \left(M_b(\chi_B)(x)\right)^q\mathrm{d}x\Big)^{1/q} \\ &\leq |B|^{-\beta/Q-1/q+\lambda/(Qq)}\|M_b(\chi_B)\|_{L^{q,\lambda}(\mathbb{G})} \leq C|B|^{-\beta/Q-1/q+\lambda/(Qq)}\|\chi_B\|_{L^{p,\lambda}(\mathbb{G})} \leq C, \end{aligned}$$

where in the last step we have used $1/q = 1/p - \beta/(Q-\lambda)$ and the fact

$$\|\chi_B\|_{L^{p,\lambda}(\mathbb{G})} \le |B|^{(1-\lambda/Q)/p}. \tag{7}$$

It follows from Lemma 2.13 that $b \in \dot{\Lambda}_\beta(\mathbb{G})$. This completes the proof.

(ii): By a similar proof to (i) in Theorem 1.2, we can obtain the desired result. ■

4. Proofs of Theorems 1.3 to 1.5

Now, we prove the mapping properties of the nonlinear commutator $[b, M]$ on Lebesgue spaces and Morrey spaces over some stratified Lie group $\mathbb{G}$ when the symbol b belongs to some Lipschitz space. The following first proves Theorem 1.3.

Proof of Theorem 1.3. (i) ⇒ (ii): For any fixed $x \in \mathbb{G}$ such that $M(f)(x) < \infty$, since $b \ge 0$ then

$$\begin{aligned} |[b, M](f)(x)| &= |b(x)M(f)(x) - M(bf)(x)| \\ &\le \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)|\mathrm{d}y \\ &= M_b(f)(x). \end{aligned} \tag{8}$$

It follows from Theorem 1.1 that $[b, M]$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ since $b \in \dot{\Lambda}_\beta(\mathbb{G})$.

(ii) ⇒ (iii): For any fixed ball $B \subset \mathbb{G}$ and all $x \in B$, it follows from Lemma 2.10 that the pointwise relations

$$M(\chi_B)(x) = \chi_B(x) \quad \text{and} \quad M(b\chi_B)(x) = M_B(b)(x).$$

Then

$$\begin{aligned} &|B|^{-\beta/Q}\Big(|B|^{-1}\int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x\Big)^{1/q} \\ &= |B|^{-\beta/Q}\Big(|B|^{-1}\int_B |[b, M](\chi_B)(x)|^q \mathrm{d}x\Big)^{1/q} \\ &\le |B|^{-\beta/Q - 1/q}\|[b, M](\chi_B)\|_{L^q(\mathbb{G})} \\ &\le C|B|^{-\beta/Q-1/q}\|\chi_B\|_{L^p(\mathbb{G})} \le C, \end{aligned} \tag{9}$$

which implies (iii) since the ball $B \subset \mathbb{G}$ is arbitrary.

(iii) ⇒ (i): To prove $b \in \dot{\Lambda}_\beta(\mathbb{G})$, by Lemma 2.13, it suffices to verify that there is a constant $C > 0$ such that for all balls $B \subset \mathbb{G}$, one get

$$|B|^{-1-\beta/Q}\int_B |b(x) - b_B|\mathrm{d}x \le C. \tag{10}$$

For any fixed ball $B \subset \mathbb{G}$, let $E = \{x \in B : b(x) \le b_B\}$ and $F = \{x \in B : b(x) > b_B\}$. The following equality is trivially true (modifying the argument in [1], p. 3331):

$$\int_E |b(x) - b_B|\mathrm{d}x = \int_F |b(x) - b_B|\mathrm{d}x.$$

Since for any $x \in E$ we have $b(x) \le b_B \le M_B(b)(x)$, then for any $x \in E$,

$$|b(x) - b_B| \le |b(x) - M_B(b)(x)|.$$

Therefore

$$\begin{aligned} &\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - b_B| \mathrm{d}x = \frac{1}{|B|^{1+\beta/Q}} \int_{E\cup F} |b(x) - b_B| \mathrm{d}x \\ &= \frac{2}{|B|^{1+\beta/Q}} \int_E |b(x) - b_B| \mathrm{d}x \le \frac{2}{|B|^{1+\beta/Q}} \int_E |b(x) - M_B(b)(x)| \mathrm{d}x \\ &\le \frac{2}{|B|^{1+\beta/Q}} \int_B |b(x) - M_B(b)(x)| \mathrm{d}x. \end{aligned} \tag{11}$$

On the other hand, it follows from Lemma 2.5 (Hölder's inequality) and (3) that

$$\begin{aligned} &\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - M_B(b)(x)| \mathrm{d}x \\ &\le \frac{1}{|B|^{1+\beta/Q}} \left(\int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x \right)^{1/q} |B|^{1/q'} \\ &\le \frac{1}{|B|^{\beta/Q}} \left(|B|^{-1} \int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x \right)^{1/q} \le C. \end{aligned}$$

This together with (11) gives (10), and so we achieve $b \in \dot{\Lambda}_\beta(\mathbb{G})$.

In order to prove $b \ge 0$, it suffices to show $b^- = 0$, where $b^- = -\min\{b, 0\}$. Let $b^+ = |b| - b^-$, then $b = b^+ - b^-$. For any fixed ball $B \subset \mathbb{G}$, observe that

$$0 \le b^+(x) \le |b(x)| \le M_B(b)(x)$$

for $x \in B$ and thus we have that, for $x \in B$,

$$0 \le b^-(x) \le M_B(b)(x) - b^+(x) \le M_B(b)(x) - b^+(x) + b^-(x) = M_B(b)(x) - b(x).$$

Then, it follows from (3) that, for any ball $B \subset \mathbb{G}$,

$$\begin{aligned} \frac{1}{|B|} \int_B b^-(x) \mathrm{d}x &\le \frac{1}{|B|} \int_B |M_B(b)(x) - b(x)| \mathrm{d}x \\ &\le \left(\frac{1}{|B|} \int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x \right)^{1/q} \\ &= |B|^{\beta/Q} \left(\frac{1}{|B|^{\beta/Q}} \left(\frac{1}{|B|} \int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x \right)^{1/q} \right) \le C|B|^{\beta/Q}. \end{aligned}$$

Thus, $b^- = 0$ follows from Lebesgue's differentiation theorem.

The proof of Theorem 1.3 is completed. ■

Proof of Theorem 1.4. (i) $\Rightarrow$ (ii): We first prove that the necessary condition. Assume $b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \ge 0$. Using (8) and ((i)) in Theorem 1.2, it is not difficult to find that $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\lambda}(\mathbb{G})$.

(ii) $\Rightarrow$ (i): We now prove that the sufficient condition. Assume that $[b, M]$ is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\lambda}(\mathbb{G})$. Similarly to (9), for any ball $B \subset \mathbb{G}$, we obtain

$$\begin{aligned}
&|B|^{-\beta/Q}\Big(|B|^{-1}\int_B |b(x) - M_B(b)(x)|^q \mathrm{d}x\Big)^{1/q} \\
&= |B|^{-\beta/Q}\Big(|B|^{-1}\int_B |[b, M](\chi_B)(x)|^q \mathrm{d}x\Big)^{1/q} \\
&\le |B|^{\lambda/(Qq)-\beta/Q-1/q}\|[b, M](\chi_B)\|_{L^{q,\lambda}(\mathbb{G})} \\
&\le C|B|^{\lambda/(Qq)-\beta/Q-1/q}\|\chi_B\|_{L^{p,\lambda}(\mathbb{G})} \le C,
\end{aligned}$$

where in the last step we have used $1/q = 1/p - \beta/(Q - \lambda)$ and (7).

Using Theorem 1.3, we can obtain that $b \in \dot{\Lambda}_\beta(\mathbb{G})$ and $b \ge 0$. ■

Proof of Theorem 1.5. This proof can be done along the proof of Theorem 1.4. We omit the details. ■

Acknowledgments. The authors cordially thank the anonymous referees who gave valuable suggestions and useful comments which have lead to the improvement of this paper. The authors contributed equally to this work. Zhao is financially supported by the Scientific Research Fund of AHPU (No.S022022177). Wu is supported in parts by the Science and Technology Fund of Heilongjiang (No.2019-KYYWF-0909), the NNSF of China (No.11571160), the Reform and Development Foundation for Local Colleges and Universities of the Central Government(No.2020YQ07) and the Scientific Research Fund of MNU (No.D211220637).

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Received August 8, 2022
and in final form July 12, 2023