Journal of Lie Theory
Volume **34** (2024) 1–16

Nonabelian Tensor Squares of Compact Groups via Quotients of Free Compact Groups

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Communicated by K.-H. Neeb

Abstract. We provide an explicit construction for the nonabelian tensor square of compact groups in terms of quotients of free compact groups. This has several consequences in terms of structural results and, just to mention two of them, one is a new upper bound for the weight of the nonabelian tensor square, another is the description of complements for the nonabelian tensor squares when we focus on the case of pro-p-groups.

Mathematics Subject Classification: 22C05, 20E18, 20J05, 20J06.

Key Words: Compact groups, nonabelian exterior square, Schur multiplier, varieties of topological groups, free groups.

1. Introduction and statement of the main results

In this paper we consider only topological groups, which satisfy the axiom of separation of Hausdorff. There are a series of properties of categorical nature that must be checked, when one wants to generalize the theory of finitely presented groups in the context of topological groups. [1] introduced the notion of finitely presented compact groups, and more recently his theory was successfully extended to locally compact groups by Cornulier and de la Harpe [9], introducing a geometric group theory for finitely presented locally compact groups. Totally disconnected compact groups, which are realized by projective limits of finite groups, are called profinite (see [17, Theorem 1.34]) and among these a special role is played by the projective limits of finite p-groups (p prime), which are described in terms of generators and relations in [42, Chapter 12]. For instance, the group of p-adic integers $\mathbb{Z}_p$ is a classical example of a pro-p-group, which is topologically generated by a single nontrivial element x, i.e.: $\mathbb{Z}_p = \overline{\langle x \rangle}$; here $\overline{\langle x \rangle}$ denotes the closure of the cyclic subgroup $\langle x \rangle$ in $\mathbb{Z}_p$ endowed with the p-adic topology. On the other hand, $\mathbb{Z}_p \neq \langle x \rangle$ so $\mathbb{Z}_p$ is topologically finitely generated with respect to the p-adic topology, but is no longer finitely generated if we consider the discrete topology on $\mathbb{Z}_p$.

Wilson [42, §12.1] uses the symbol $d(G)$ to denote the minimal number of topological generators of a pro-p-group G, following the agreement that $d(G) = \infty$ when G is not topologically finitely generated. Of course, $d(\mathbb{Z}_p^n) = n$, where $\mathbb{Z}_p^n$ denotes the direct sum of n-copies (n positive integer) of $\mathbb{Z}_p$. Following [42, Proposition 5.1.3'], if E is the free abstract group on the set X and I the set of all normal subgroups U of E such that E/U is a finite p-group and $X \setminus U$ is finite, then the

ISSN 0949–5932 / \$2.50

completion $FX = \lim_{U\in I} E/U$ of E with respect to I, together with the embedding $j : x \in X \to j(x) = (xU)_{U\in I} \in FX$ is the *free pro-p-group* on X. See details in [42, §1.4] concerning the well known notion of completion, when a given abstract group E and a filter I are given.

Definition 1.1. (See [42], p. 238) For a normal subgroup R of the free pro-p-group FX on a subset X, R is said to be topologically generated as a normal subgroup by X if R is the smallest closed normal subgroup of FX containing X. In this situation $d_{FX}(R)$ denotes the minimal number of topological generators of R with the agreement that $d_{FX}(R) = \infty$ if R is not topologically finitely generated as a normal subgroup. A presentation of a pro-p-group G is an epimorphism of pro-p-groups $\pi : FX \to G$, where $d(FX)$ coincides with the cardinality of X and the presentation is said to be finite if both $d(FX)$ and $d_{FX}(\ker \pi)$ are finite. ■

In Definition 1.1 it turns out that $G \simeq FX/\ker\pi$ and $G = \langle X \mid R\rangle$ means that we have a pro-p-presentation as per Definition 1.1 with $R = \ker\pi$. In this situation one should note that $\langle X\rangle$ is dense in G, that is, $G = \overline{\langle X\rangle}$, that is, G is *topologically finitely generated by* X. In order to formulate Definition 1.1 for larger classes of compact groups, one should look at [1, 9] and follow the ideas in these references. In fact profinite groups and pro-p-groups generalize well the usual finite groups, but when we consider compact groups it is possible to have many issues with connectivity which are absent for finite groups. Consequently, the notions will change when we want to describe generating sets.

Definition 1.2. Following [5, 6, 4], it is possible to introduce the *nonabelian tensor square* $G \otimes G$ of a finite group G by the following presentation:
for all $g_1, g_2, h_1, h_2 \in G$,

$$G\otimes G = \langle g_1\otimes h_1 \mid g_1g_2\otimes h_1 = ({g_2}^{g_1}\otimes {h_1}^{g_1})(g_1\otimes h_1),\ g_1\otimes h_1h_2 = (g_1\otimes h_1)({g_1}^{h_1}\otimes {h_2}^{h_1})\rangle,$$

where ${g_2}^{g_1} = g_1^{-1}g_2g_1$, ${h_1}^{g_1} = g_1^{-1}h_1g_1$, ${g_1}^{h_1} = h_1^{-1}g_1h_1$ and ${h_2}^{h_1} = h_1^{-1}h_2h_1$. ■

Note that G acts on itself by conjugation in Definition 1.2. Moreover the *nonabelian exterior square* of G is defined as the following quotient group

$$G \wedge G := \frac{G\otimes G}{\nabla(G)}, \quad \text{where} \quad \nabla(G) = \langle g\otimes g \mid g \in G\rangle$$

and $\nabla(G)$ turns out to be a central subgroup of $G \otimes G$.

In the category $\mathbb{KTOP}$ of compact groups, the corresponding notion of nonabelian tensor square is more sophisticated to formulate. Especially when G and H are compact groups which are not totally disconnected. There are very few results in this direction; Brown and others [4] describe nonabelian tensor squares categorically, so we have a clear description of universal and functorial properties of nonabelian tensor squares of compact groups, but only Sahleh [37] begins a systematic study in $\mathbb{KTOP}$ of compact groups. Di Micco and van der Linden [10, 11] investigated the relations between nonabelian tensor squares and double categories, generalizing an intuition of Edaladtzadeh [14] to the context of semi-abelian varieties. Again this helps to understand the general behaviour of the structure of nonabelian tensor squares of compact groups, but concrete examples and relations with representation theory of compact groups appear to be absent at the moment in the literature. We shall also mention some recent works of Donadze and others [2, 12, 13] in the context of abstract groups, focusing on interactions with homological group theory and algebraic topology.

Definition 1.3. (See [25]) If G is a pro-p-group, we may introduce the *complete nonabelian exterior square* $G\widehat{\otimes}G$ as the pro-p-group *topologically generated* by the symbols $x\widehat{\otimes}y$ subject to the relations

$$xy\widehat{\otimes}z = (x^y\widehat{\otimes}z^y)(y\widehat{\otimes}z) \quad \text{and} \quad y\widehat{\otimes}tz = (y\widehat{\otimes}z)(y^z\widehat{\otimes}t^z)$$

for $x, y, z \in G$; here $x^y = y^{-1}xy$ denotes the conjugate of x with respect to y. ■

In Definition 1.3, the subgroup $\widehat{\nabla}(G)$ generated by elements of the form $x\widehat{\otimes}x$ is a closed normal subgroup of $G\widehat{\otimes}G$. Then we may consider the quotient group $(G\widehat{\otimes}G)/\widehat{\nabla}(G) = G\widehat{\wedge}G$, which is called the *complete nonabelian exterior square* of G. It is well known (see [17, Theorem 1.34]) that in a pro-p-group G the filter basis $\mathcal{P}(G)$ of all closed normal subgroups N of G such that G/N is finite p-group allows us to approximate G, that is, $G = \lim_{N\in\mathcal{P}(G)} G/N$. From [36, Theorem 1.1], one can also look at $G\widehat{\otimes}G$ as the pro-p-group $\lim_{N,M\in\mathcal{P}(G)}(G/N \otimes G/M)$, obtained as projective limit on $\mathcal{P}(G)$ of the (discrete) finite p-groups $G/N \otimes G/M$ introduced in Definition 1.2.

The general case of compact groups, which are neither profinite nor connected, requires different techniques. Since compact Lie groups are characterized to have a faithful finite dimensional orthogonal (or unitary) representation [17, Corollary 2.40] and since a compact group G possesses

$$\mathcal{N}(G) = \{N = \overline{N} \lhd G \mid G/N \text{ is a Lie group}\}$$

as filter basis for its topology, we may apply the Approximation Theorem for Compact Lie Groups [18, Corollary 2.43] and conclude that any compact group G may be written as a projective limit with respect to $\mathcal{N}(G)$, namely $G = \lim_{N\in\mathcal{N}(G)} G/N$.

One could introduce *the nonabelian tensor square* $G\overline{\otimes}G$ of a compact group G as projective limit of a projective system of compact Lie groups, but this needs an approach via category theory and some additional observations. For instance, we know that $G/N \otimes G/M$ is a finite p-group when so are G/N and G/M (see [13] for details) but when G/N and G/M are compact Lie groups it is not automatic that $G/N \otimes G/M$ is a compact Lie group. Secondly, we shall study some homological algebra of short exact sequences, in fact the theory of the nonabelian tensor square is strongly related to homological algebra by results in [5, 6, 4, 31]. Already these two issues create significant differences with the profinite case, therefore a new perspective may be useful.

Our first main result describes the nonabelian tensor square of compact groups in terms of a universal property and produces a prototype of construction, which allows us to skip the approach via finite presentations. This is possible, thanks to the existence of free compact groups in [17, Chapter 11]. In fact we can say more on the nonabelian tensor square of compact groups with the help of the representation theory of compact groups.

In particular $C_0(X, \mathrm{U}(n))$ denotes the set of all continuous base point preserving functions from a compact pointed topological space X into the unitary group $\mathrm{U}(n)$

and $\mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$ denotes the space of continuous functions from $C_0(X,\mathrm{U}(n))$ to $\mathrm{U}(n)$ endowed with the topology of the pointwise convergence. These function spaces and matrices play an important role in the representation theory of compact groups, as explained in [17, Chapters 2,3,4].

We shall also mention that the notion of *continuous crossed pairing* below is a generalization of the usual notion of bilinear map in the context of usual abelian tensor products of compact abelian groups, see [17, Propositions A1.44, A1.45, A1.46 and A1.48]. More details will be formalized in Definition 2.3. Let's also recall that the notion of *weight* for a compact group can be found in [17, Definition A4.7] and will be recalled formally later on.

Theorem 1.4. (Existence and Uniqueness of Nonabelian Tensor Squares of Compact Groups) *Let G be a compact group that acts on itself by conjugation. Then*

(i) *There exist a compact group $G\overline{\otimes}G$ and a continuous crossed pairing*

$$(g,h) \in G \times G \mapsto g\overline{\otimes}h \in G\overline{\otimes}G$$

such that for any continuous crossed pairing $f : G\times G \to A$ of compact groups, there is a unique continuous homomorphism $f' : G\times G \to A$ of compact groups making commutative the following diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{(g,h)\mapsto g\overline{\otimes}h} & G\overline{\otimes}G \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ A & & \end{array}$$

i.e., $f'(g\overline{\otimes}h) = f(g,h)$.

(ii) *The compact group $G\overline{\otimes}G$ is topologically isomorphic to the compact group FX/K, where X is a compact pointed topological space,*

$$K = \overline{\langle \varepsilon(gz,h)\varepsilon(g,h)^{-1}\varepsilon(z^g,h^g)^{-1}, \varepsilon(g,ht)\varepsilon(g^h,t^h)^{-1}\varepsilon(g,h)^{-1} \mid g,z,h,t \in G\rangle}^{FX},$$

and $\varepsilon : x \in X \to \varepsilon(x) \in FX$ denotes the embedding of X in FX.

In particular, $G\overline{\otimes}G$ is a continuous homomorphic image of the cartesian product C of countably many copies of $\mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$, *where C has the product topology of* $\mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$.

(iii) *Any other compact group satisfying* (i) *above is topologically isomorphic to $G\overline{\otimes}G$, hence to FX/K in* (ii) *above.*

(iv) *The weight $w(G\overline{\otimes}G)$ of $G\overline{\otimes}G$ is upper bounded by $w(G\times G)^{\aleph_0}$.*

Our second main result describes more specifically the case of nonabelian tensor square of pro-p-groups. We shall report that there is more literature on the nonabelian tensor square of pro-p-groups (see for instance [25, 34, 36]), but results of structure are not available in the sense of what we prove below.

Theorem 1.5. *Consider the nonabelian tensor square $G\widehat{\otimes}G$ of a pro-p-group G.*

(i) *If $G = N \rtimes H$ is a semidirect product of pro-p-groups N and H, then*

$$G\widehat{\otimes}G = K \rtimes (H\widehat{\otimes}H)$$

for some closed normal subgroup K.

(ii) *If $\overline{G'}$ has procyclic complement C, then $G\widehat{\otimes}G = (G\widehat{\wedge}G) \times C$.*

The paper is organized in four sections. After the statement of the main results in Section 1, we report some facts from the literature on the nonabelian tensor squares in Section 2; here the reader can note that the tools for the proofs of Theorems 1.4 and 1.5 are naturally extended from the previous techniques in the literature, but now we focus on compact groups. Section 3 deals with some results on the representation theory of compact groups; these will play a fundamental role in the description of the structure of the free compact groups. A discussion is also present via varieties of topological groups, leaving some open questions for a possible extension of Theorems 1.4 and 1.5 to categories of topological groups which are broader than $\mathbb{KTOP}$. Finally, Section 4 contains the main proofs and some applications. Terminology and notations are standard and follow [1, 9, 16, 17, 42].

2. Topological presentations and nonabelian tensor squares

From Definition 1.2, it is clear that we find the usual abelian tensor square of an abelian group when the derived subgroup $[G,G] = G'$ is trivial. More precisely, $G/[G,G] \otimes G/[G,G]$ is the usual abelian tensor product by [6, Proposition 2.4]. In fact this is true also when G is a pro-p-group in Definition 1.3. The notion of nonabelian tensor product has indeed its origins in a generalization of the usual notion of abelian tensor products, see [5, 6, 4]. Let's begin with a concrete example of an abelian finitely presented pro-p-group, in order to stress the relevance of Definition 1.1 in the context of Definition 1.3.

Example 2.1. Consider $G = \mathbb{Z}(p)^m$ for a given positive integer m; this is an elementary abelian p-group of rank m and pro-p-presentations for G coincide with usual presentations in the present situation. Moreover Definition 1.2 and Definition 1.3 agree for $G = \mathbb{Z}(p)^m$, that is, $G \otimes G = G\widehat{\otimes}G$ and this is indeed the usual abelian tensor square of an elementary abelian p-group. Consider $X = \{(g,h) \mid g,h \in G\}$ and denote its cardinality by card X; of course it is finite and card $X \leq m^2$.

Invoking [42, Proposition 5.1.6], we note that the cartesian product $\mathbb{Z}(p)^X$ of card X copies of $\mathbb{Z}(p)$ is the free elementary abelian pro-p-group on X, that is, we have $FX = \mathbb{Z}(p)^X$. Note that the direct product of card X copies of $\mathbb{Z}(p)$ is denoted by $\mathbb{Z}(p)^{(X)}$, according to [16, 17]. Now consider the epimorphism of finite abelian p-groups

$$\pi : (u_1, \dots, u_{\text{card } X}) \in \mathbb{Z}(p)^{(X)} \subseteq FX \longmapsto (u_1V, \dots, u_{\text{card } X}V) \in \mathbb{Z}(p)^m \otimes \mathbb{Z}(p)^m,$$

where $\mathbb{Z}(p)^X$ possesses the following subgroup

$$V = \langle \varepsilon(g_1g_2,h_1)\varepsilon(g_2,h_1)^{-1}\varepsilon(g_1,h_1)^{-1}, \varepsilon(g_1,h_1h_2)\varepsilon(g_1,h_1)^{-1}\varepsilon(g_1,h_2)^{-1} \mid g_1,g_2,h_1,h_2 \in G\rangle.$$

The map $\varepsilon : (g_1, g_2) \in X \to \varepsilon(g_1, g_2) \in \mathbb{Z}(p)^X = FX$ is the evaluation map, that is, the embedding of X into FX and it allows us to identify elements of X as elements of $\mathbb{Z}(p)^X$, moreover

$$\ker \pi = \{(u_1, \ldots, u_{\text{card } X}) \mid (u_1, \ldots, u_{\text{card } X})V = V\} = V$$

and we find that $g_1 \otimes g_2 = \varepsilon(g_1, g_2)V$.

We get the presentation $G \otimes G = G\widehat{\otimes}G = \langle X \mid R \rangle$ with $V = R$ according to Definition 1.1 and this presentation describes Definition 1.2, because

$$\varepsilon(g_1 g_2, h_1)V = \varepsilon(g_2, h_1)V \ \varepsilon(g_1, h_1)V \Longleftrightarrow g_1 g_2 \otimes h_1 = (g_2 \otimes h_1)\ (g_1 \otimes h_1)$$

$$\varepsilon(g_1, h_1 h_2)V = \varepsilon(g_1, h_1)V \ \varepsilon(g_1, h_2)V \Longleftrightarrow g_1 \otimes h_1 h_2 = (g_1 \otimes h_1)(g_1 \otimes h_2)$$

Using the above information on π and ε, we also get $G \otimes G \simeq \mathbb{Z}(p)^X/V$ and one can see that the relations in Definition 1.2 are obtained by V, which is the kernel of the epimorphism in Definition 1.1. ■

It is useful to recall that for a finite group G (not necessarily a finite p-group), Brown and others [5, 6] show that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \xrightarrow{\kappa} & [G,G] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\
0 & \longrightarrow & H_2(G,\mathbb{Z}) & \longrightarrow & G \wedge G & \xrightarrow{\kappa'} & [G,G] & \longrightarrow & 0,
\end{array}$$

is a commutative diagram, where $H_2(G, \mathbb{Z})$ denotes the second group of integral homology over G (called *Schur multiplier* of G, according to [35, §11.4] and [20]) and the following maps

$$\kappa : x \otimes y \in G \otimes G \longmapsto [x, y] = x^{-1}y^{-1}xy \in [G, G],$$

$$\kappa' : x \wedge y \in G \wedge G \longmapsto [x, y] = x^{-1}y^{-1}xy \in [G, G],$$

are epimorphisms of finite groups. Moreover $\ker \kappa' \simeq H_2(G, \mathbb{Z})$ and the following example can help to understand the computational aspects of the nonabelian tensor square of finite groups.

Example 2.2. When G is a perfect finite group, $G \otimes G$ coincides with the covering group of G by [5, Corollary 1]. Recall from [20, p.16] that G^* is a covering group of G, if G^* possesses a subgroup A such that $A \subseteq Z(G^*) \cap [G^*, G^*]$, $A \simeq H_2(G, \mathbb{Z})$ and $G \simeq G^*/A$. Schur [20, Theorem 2.1.4] showed that G^* exists for any possible choice of the finite group G. In particular, the computation of A_n^* is known for alternating groups A_n by [20, Theorem 2.12.5]. Furthermore, if $H_2(G, \mathbb{Z})$ is finite, then $G \wedge G \cong [G, G]$ by [5, Corollary 2]. Now we report some facts from [5, pp.196-197] for convenience of the reader:

(i) For all $n > 4$, [20, Theorem 2.12.5] shows that

$$A_n = [A_n, A_n] \simeq A_n \wedge A_n \quad \text{and} \quad A_n \otimes A_n \simeq A_n^*.$$

(ii) In particular, for $n = 4$ we have

$$A_4 \otimes A_4 = \mathbb{Z}(3) \times Q_8,$$

where Q_8 is the quaternion group of order 8. See [5, p.183] for details. This is an example of nonabelian tensor square which is not abelian. ■

Following [25, 33, 36], in case of a pro-p-group G, one should introduce the notion of *compatible continuous action*: G acts compatibly and continuously on itself by conjugation, if the action is continuous and the compatibility relations

$$x^{(y^z)} = x^{z^{-1}yz} \quad \text{and} \quad t^{(z^y)} = t^{y^{-1}zy}$$

are satisfied for all $x, y, z, t \in G$. However, it is also possible by [5, Remark 3] and [4] to introduce $G \otimes G$ using a universal property and then prove its existence which will implicitly yield the generators and relations. In order to follow this approach, we need to replace the bilinear maps which we usually have in the abelian context.

Definition 2.3. (Continuous crossed pairings of compact groups) Let A be a compact group and G a compact group acting compatibly and continuously on itself by conjugation. A function $f : G \times G \to A$ is called a *continuous crossed pairing* if for all $g, , h, t, z \in G$ we have

$$f(gz, h) = f(z^g, h^g)\, f(g, h) \quad \text{and} \quad f(g, ht) = f(g, h)\, f\left(g^h, t^h\right)$$ ■

Of course, Definition 2.3 works when G is a pro-p-group, see [33], and we find for discrete abelian compact groups the usual notion of bilinear maps, since Definition 2.3 specializes to

$$f(gz, h) = f(g, h)f(z, h) \text{ and } f(g, ht) = f(g, h)f(g, t).$$

We introduce the following universal property which specializes to the notion of nonabelian tensor product of abstract groups.

Definition 2.4. (Universal property for nonabelian tensor squares of compact groups) In the category $\mathbb{KTOP}$ of compact groups consider a compact group A, and a compact group G acting compatibly and continuously on itself by conjugation. The nonabelian tensor square of the compact group G is the compact group $G\overline{\otimes}G$ together with a continuous crossed pairing $(g, h) \in G \times G \mapsto g\overline{\otimes}h \in G\overline{\otimes}G$ such that for any continuous crossed pairing $f : G \times G \to A$ there is a unique homomorphism $f' : G\overline{\otimes}G \to A$ of compact groups making commutative the following diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{(g,h)\mapsto g\overline{\otimes}h} & G\overline{\otimes}G \\ {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle f'} & \\ A & & \end{array}$$

i.e., $f'(g\overline{\otimes}h) = f(g, h)$ for all $g, h \in G$. ■

The reader can see easily that Definition 2.4 generalizes the universal property of abelian tensor products of discrete groups in the category $\mathbb{AB}$ of abelian groups, where no topology is present (or equivalently one looks at groups with the discrete topology only). We end this section with a property of extension of continuous crossed pairings that will be used in the proof of Theorem 1.5 later on.

Lemma 2.5. *Let G be a compact group acting compatibly and continuously on itself by conjugation. Assume that B is another compact group in the same conditions. Consider two continuous homomorphisms $\theta : G \to B$ and $\phi : G \to B$ of compact groups satisfying the compatibility conditions $\phi(h^g) = \phi(h)^{\theta(g)}$ and $\theta(g^h) = \theta(g)^{\phi(h)}$ for all $g, h \in G$. Then there exists a continuous homomorphism of compact groups*

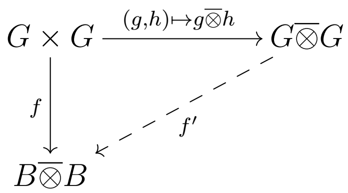
$$\theta\overline{\otimes}\phi : g\overline{\otimes}h \in G\overline{\otimes}G \longmapsto \theta\overline{\otimes}\phi\ (g\overline{\otimes}h) = \theta(g)\overline{\otimes}\phi(h) \in B\overline{\otimes}B.$$

Furthermore, if θ and ϕ are surjective, then so is $\theta\overline{\otimes}\phi$.

Proof. The universal property of $G\overline{\otimes}G$ is expressed by the commutative diagram in Definition 2.4 and can be realized replacing A with $B\overline{\otimes}B$ and choosing

$$f : (g, h) \in G \times G \mapsto f(g, h) = \theta(g)\overline{\otimes}\phi(h) \in G\overline{\otimes}G,$$

which is a continuous crossed pairing because of the compatibility conditions and of the continuity of θ and ϕ. Therefore the following diagram is commutative



with continuous homomorphism $f' = \theta\overline{\otimes}\phi : G\overline{\otimes}G \to B\overline{\otimes}B$ such that $\theta\overline{\otimes}\phi\ (g\overline{\otimes}h) = \theta(g)\overline{\otimes}\phi(h)$. Finally, we note that the compact group $B\overline{\otimes}B$ is topologically generated by the elements of the form $a\overline{\otimes}b$ with $a, b \in B$. Therefore, if θ, ϕ are surjective, then $\theta\overline{\otimes}\phi$ is surjective. ■

3. Varieties of topological groups and corresponding free groups

In the present section, we recall some notions concerning varieties of topological groups, which are used frequently in [18, 32] but due originally to Morris [3, 26, 27, 28] for topological groups. Following [18], we define the operations $\mathbf{S}, \overline{\mathbf{S}}, \overline{\mathbf{Q}}, \mathbf{C}$ on a class of topological groups to be, respectively, the operations of taking arbitrary subgroups, closed subgroups, quotients with respect to closed subgroups, and arbitrary cartesian products.

Definition 3.1. (See [18], p. 65) A class of Hausforff topological groups Ω is called a *variety of Hausdorff groups*, if the following conditions are satisfied:

(i) if $G \in \Omega$ and $H \leq G$, then $H \in \Omega$;

(ii) if N is a closed normal subgroup of G and $G \in \Omega$, then $G/N \in \Omega$;

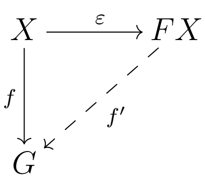
(iii) if $\{G_{i\in I}\}$ is a collection of groups with $G_i \in \Omega$ for all $i \in I$, then $\prod_{i\in I} G_i \in \Omega$. ■

The condition (i) means that $\mathbf{S}\Omega = \Omega$, (ii) means $\overline{\mathbf{Q}}\Omega = \Omega$, and (iii) means $\mathbf{C}\Omega = \Omega$. The smallest variety of Hausdorff groups containing Ω is denoted by $\mathfrak{V}(\Omega)$.

We remark that Definition 3.1 does not correspond to the original formulation in [3, 26, 27, 28] for arbitrary classes of topological groups, but it is designed for Hausdorff topological groups. This has been already noted in [18]. The point is that for a Hausdorff group G, and a normal subgroup H of G, the quotient group G/H is Hausdorff if and only if $H = \overline{H}$. This explains why we take $\overline{\mathbf{Q}}$ instead of $\mathbf{Q}$, where $\mathbf{Q}$ denotes the operator of taking quotients with respect to normal subgroups which are not necessarily closed. With the help of $\mathbf{S}$, $\overline{\mathbf{Q}}$ and $\mathbf{C}$, it is possible to construct arbitrary varieties of Hausdorff groups in the following way: firstly, we choose a property of groups, then we define the corresponding variety of Hausdorff groups, requiring that the new class contains all groups with the prescribed property, hence the closure with respect to $\mathbf{S}$, $\overline{\mathbf{Q}}$ and $\mathbf{C}$.

From [17, Definition A1.4, Proposition A1.6] we know that abstract free abelian groups exist and are of the form $\mathbb{Z}^{(X)}$ up to isomorphism. A similar situation applies to compact groups: free compact groups over a topological space X may be defined by the universal property and explicit constructions are possible:

Definition 3.2. [See [17], Definition 11.1] A free compact group over a pointed topological space (X, x_0) is a compact group FX together with a continuous function $\varepsilon : X \to FX$ mapping x_0 to the identity of FX such that for every basepoint preserving continuous function $f : X \to G$ into a compact group G (with 1 as its base point) there is a unique continuous homomorphism of groups $f' : FX \to G$ making commutative the following diagram



The problem of the existence of free topological groups is not elementary.

Remark 3.3. Comfort [8], Kakutani [19], Graev [15], Markov [22] and Morris [7, 21] contributed to the description of algebraic and topological properties of free topological groups. Successively, many authors studied free topological groups and Sipacheva [38] indicates the different techniques and methods of investigation in the recent literature. We shall also mention that Tkachenko [40, 41] worked on free topological groups via an approach with uniformities. Sipacheva [39, 38] provides also information on the constructions of topologies of free topological groups.

We are going to recall some details for free compact groups from [17, Chapter 11]. Note first of all that compact groups are isomorphic to closed subgroups of cartesian products of unitary groups from [17, Corollary 2.29]. Secondly, in a topological space X with topology $\mathcal{O}$ the set $\{\mathrm{card}(\mathcal{B}) \mid \mathcal{B} \text{ is a basis for } \mathcal{O}\}$ of all cardinalities of bases for $\mathcal{O}$ is well ordered and thus has a minimal element, called the *weight* of X and denoted by $w(X)$ (see [17, Definition A4.7]).

Proposition 3.4. (See [17], Proposition 11.6) *Let X be a compact pointed topological space.*

(i) *Let P be a compact group and $e : X \to P$ a continuous function of pointed spaces such that for every continuous function of pointed spaces $f : X \to \mathrm{U}(n)$ into a unitary group there is a function $f' : FX \to \mathrm{U}(n)$ such that $f = f' \circ e$. Then $FX = \overline{\langle e(X) \rangle}$ together with the corestriction $e_X : X \to FX$ is the free compact group on X.*

(ii) *Let $C_0(X, \mathrm{U}(n))$ be the set of all continuous base point preserving functions from X into $\mathrm{U}(n)$ and $\mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$ the space of the continuous functions from $C_0(X, \mathrm{U}(n))$ to $\mathrm{U}(n)$ with the topology of the pointwise convergence. Then*

$$G = \prod_{n=1}^{\infty} \mathrm{U}(n)^{C_0(X,\mathrm{U}(n))}$$

is a compact group satisfying the universal property (i) *above with*

$$e_n(x) : f \in C_0(X, \mathrm{U}(n)) \longmapsto e_n(x)(f) = f(x) \in \mathrm{U}(n)$$

and $e : x \in X \mapsto e(x) = (e_n(x))_{n\in\mathbb{N}} \in G$.

(iii) *The weight $w(FX)$ of the free compact group FX on X is upper bounded by the weight $w(X)^{\aleph_0}$.*

In particular, Proposition 3.4(i) shows that free compact groups exists and Proposition 3.4(ii) provides an explicit construction for a free compact group. Moreover Proposition 3.4 applies to free profinite groups and free pro-p-groups.

Remark 3.5. Wilson [42, Chapter 5] describes the properties of free profinite groups, originally studied by Mel'nikov [23, 24], considering an arbitrary class $\mathcal{C}$ of finite groups (containing nontrivial groups), closed under $\mathbf{S}$, $\overline{\mathbf{Q}}$ and $\mathbf{D}$. Here $\mathbf{D}$ is a special case of $\mathbf{C}$, that is, the operation of taking arbitrary direct products (instead of cartesian products). ■

If we consider the following filter basis for a compact group G

$$\mathcal{F}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a finite group}\},$$

then G is profinite whenever $G = \lim_{N\in\mathcal{F}(G)} G/N$ in the first case, but a pro-p-groups whenever $G = \lim_{N\in\mathcal{P}(G)} G/N$. The following example helps to understand that specializations of Proposition 3.4 are not immediate for profinite groups.

Example 3.6. Let G be a compact group. We have $\mathcal{P}(G) \subseteq \mathcal{F}(G) \subseteq \mathcal{N}(G)$. Moreover, if the identity component G_0 of G is trivial, then $\mathcal{N}(G) = \mathcal{F}(G)$. See [18, Theorem 1.34, Exercise E1.12 (iii)] for details. We can take $\mathcal{C}$ to be the class of all finite groups and in this case pro-$\mathcal{C}$-groups are exactly the profinite groups. Instead if we choose $\mathcal{C}$ to be the class of all finite p-groups, then pro-$\mathcal{C}$-groups are exactly pro-p-groups.

Let's see a more specific description of free groups in these two situations. From [42, Propositions 5.1.3 and 5.1.3', Theorem 5.5.4], if $r \geq 1$ is an integer, F_r the free abstract group on the set of generators $X = \{x_1, \ldots, x_r\}$ and

$$\mathcal{C}(F_r) = \{N = \overline{N} \triangleleft F_r \mid F_r/N \in \mathcal{C}\},$$

then the projective limit $FX = \widehat{F_{\mathcal{C}}} = \lim_{N \in \mathcal{C}(F_r)} F_r/N$ turns out to be the *free pro-$\mathcal{C}$ group on r-generators.* In particular, $\widehat{F_{\mathcal{C}}} = \overline{\langle X \rangle}$ and Proposition 3.4 applies. More specifically, [42, Propositions 5.1.3'] shows that $\widehat{F_{\mathcal{C}}}$ satisfies the universal property in Proposition 3.4(i). On the other hand, $\widehat{F_{\mathcal{C}}}$ is explictly constructed as a projective limit, and projective limits are closed subgroups of cartesian products (see [17, Lemma 1.26]), hence $\widehat{F_{\mathcal{C}}}$ specializes the construction in Proposition 3.4(ii) in the present circumstances.

Note in particular that in case $\mathcal{C}$ is the class of all abelian p-groups, then $\widehat{F_{\mathcal{C}}} = \mathbb{Z}_p^X$ is the free abelian pro-p-group on X. We could also choose $\mathcal{C}$ as the class of all elementary abelian p-groups, then $\widehat{F_{\mathcal{C}}} = \mathbb{Z}(p)^X$ is the free elementary abelian pro-p-group on X. See details in [42, Proposition 5.1.6]. Finally, regarding the specialization of Proposition 3.4(iii), one can refer to [42, Corollaries 5.5.8, 5.5.9].

We want to mention a few more results of structure of free compact groups.

Proposition 3.7. (See [17], Theorem 11.7) *Let X be a a compact pointed topological space, and F_0X the identity component of the free compact group FX on X. Then there is a totally disconnected subgroup D of FX such that $FX = (F_0X)D$. Moreover D contains a totally disconnected closed subgroup T such that $FX = (F_0X) \rtimes T$, that is, F_0X is always a semidirect factor of FX.*

We focus our attention on the center $Z(FX)$ of FX:

Proposition 3.8. (See [17], Theorem 11.14) *If X is a pointed compact topological space of cardinality* $\operatorname{card} X \geq 3$*, then $Z(FX) \subseteq F_0X$. In particular, if* $\operatorname{card} X = 2$*, then $Z(FX) = FX$ is the universal compact monothethic group, that is, $Z(FX) = FX \simeq \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}^{2^{\aleph_0}}$.*

From Proposition 3.8, if X is a compact pointed space with two points, then FX is a compact abelian group and coincides with its abelianization, that is,

$$F_{\mathrm{ab}}X = FX/\overline{[FX, FX]} = FX/\overline{[F'X]}.$$

Note also that in case FX is abelian, then automatically

$$FX = Z(FX) \subseteq F_0X,$$

that is, a free compact group that is abelian should be necessarily connected. Lastly, we look at the commutator subgroup $F'X$ of FX and note that it can split on FX:

Proposition 3.9. (See [17], Theorem 11.19, Corollary 11.20) *For a compact connected pointed topological space X, we have $FX \simeq F_{ab}X \times F'X$ as compact group if and only if the first integral cohomology group $H^1(X, \mathbb{Z})$ is divisible. In particular, if X is contractible, then $FX \simeq F_{ab}X \times F'X$.*

In the theory of varieties of abstract groups, one can adapt the construction and the universal property of free abstract groups, showing that any variety of abstract groups possesses free groups, see [35, 2.3.6]. Something similar can be done in the context of compact groups. For instance, [17, Definition 11.22 (iii)] introduces the *free semisimple compact connected group* $F_{\mathrm{ss}}X$ on a pointed topological space X

as a compact connected semisimple group together with a base point preserving continuous function $i_X : X \to F_{ss}X$ such that for every base point preserving essential continuous function $f : X \to S$ into a compact semisimple group S there is a unique morphism $f' : F_{ss}X \to S$ such that $f = f' \circ i_X$.

Proposition 3.10. (See [17], Proposition 11.23) *For a compact connected pointed topological space X, there exists $F_{\mathrm{ss}}X$. Moreover $F_{\mathrm{ss}}X = F'X/(Z_0(FX) \cap F'X)$.*

Varieties of topological groups (in the sense of Definition 3.1) containing compact groups have been considered in [29]. We shall also mention that the category of pro-Lie groups generalizes very well the category of compact groups, see [18, 30]. On the other hand, the following problem remains open in its generality:

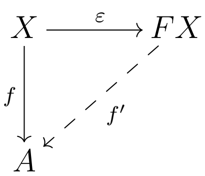
Problem 3.11. Provide results of existence and uniqueness of nonabelian tensor squares of topological groups via quotients of free topological groups, beginning from varieties of topological groups which contain compact groups.

4. Proofs of the main results

We are now able to prove our first main theorem:

Proof. (i). Let $X = G \times G$ be a pointed space with base point $x_0 = (1_G, 1_G)$, and let (FX, ε) be the free compact group over (X, x_0). Assume that $f : X \to A$ is a continuous homomorphism of compact groups, and in addition that f is a crossed pairing. By Definition 3.2, there is a continuous homomorphism $f' : FX \to A$, which is unique in making commutative the diagram:

(D1)



Let K be the smallest closed normal subgroup of FX that is topologically generated by the elements

$$\varepsilon(gz,h)\varepsilon(g,h)^{-1}\varepsilon(z^g,h^g)^{-1} \text{ and } \varepsilon(g,ht)\varepsilon(g^h,t^h)^{-1}\varepsilon(g,h)^{-1}$$

for all $g, z, h, t \in G$. Since f is, in addition, a crossed pairing, we have

$$f'\left(\varepsilon(gz,h)\varepsilon(g,h)^{-1}\varepsilon(z^g,h^g)^{-1}\right) = f'\left(\varepsilon(gz,h)\right) f'\left(\varepsilon(g,h)^{-1}\right) f'\left(\varepsilon(z^g,h^g)^{-1}\right) \quad (1)$$
$$= f(gz,h)f(g,h)^{-1}f(z^g,h^g)^{-1} = f(z^g,h^g)f(g,h)f(g,h)^{-1}f(z^g,h^g)^{-1} = 1.$$

In (1) above, the first equality follows from the commutativity of (D1), the second equality is due to the fact that f is homomorphism, the final equality is due to the fact that f is a crossed pairing. We also have

$$f'\left(\varepsilon(g,ht)\varepsilon(g^h,t^h)^{-1}\varepsilon(g,h)^{-1}\right) = f'(\varepsilon(g,ht)f'(\varepsilon(g^h,t^h)^{-1})f'(\varepsilon(g,h)^{-1})$$

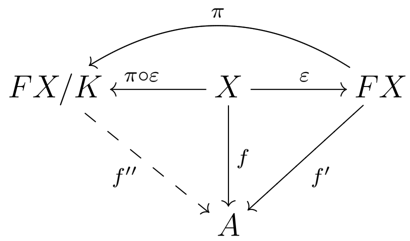
$$= f(g,ht)f(g^h,t^h)^{-1}f(g,h)^{-1} = f(g,h)f(g^h,t^h)f(g^h,t^h)^{-1}f(g,h)^{-1} = 1.$$

Again here it is used in the final equalities that f is a crossed pairing.

Therefore, f' vanishes on the following closed normal subgroup of FX

$$K = \overline{\langle \varepsilon(gz,h)\varepsilon(g,h)^{-1}\varepsilon(z^g,h^g)^{-1}, \varepsilon(g,ht)\varepsilon(g^h,t^h)^{-1}\varepsilon(g,h)^{-1} \mid g,z,h,t \in G\rangle}$$
$$= \overline{\langle \varepsilon(gz,h)\varepsilon(g,h)^{-1}\varepsilon(z^g,h^g)^{-1}, \varepsilon(g,ht)\varepsilon(g^h,t^h)^{-1}\varepsilon(g,h)^{-1} \mid g,z,h,t \in G\rangle^{FX}},$$

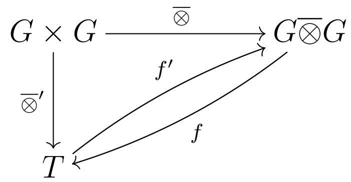
because it is a continuous homomorphism that vanishes on the topological generators of K. Hence, taking $\pi : FX \to FX/K$ to be the quotient homomorphism and its composition $\pi \circ \varepsilon : X \to FX/K$, then there is a continuous homomorphism $f'' : FX/K \to A$, which is unique in making commutative the following diagram



Putting $G\overline{\otimes}G = FX/K$, and $g\overline{\otimes}h = \pi(\varepsilon(g,h))$, we have checked that Definition 2.4 is satisfied by the right portion of the diagram above. Then (i) follows.

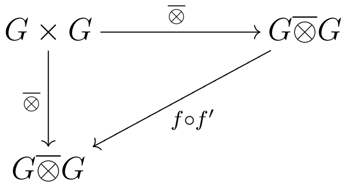
(ii) It follows from (i) that FX/K is topologically isomorphic to $G\overline{\otimes}G$.

(iii) If T is any other compact group together with a universal continuous crossed pairing $\overline{\otimes}'$ satisfying (i), by applying (i) to both $G\overline{\otimes}G$ and T, we obtain a commutative diagram:



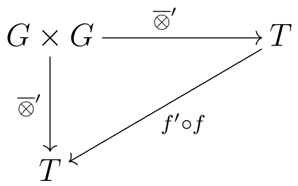
This in turn yields the commutative diagrams:

(D2)



and

(D3)



Furthermore, by uniqueness of $1_{G\overline{\otimes}G}$, respectively, 1_T, in making (D2), respectively (D3), commutative, we have $f \circ f' = 1_{G\overline{\otimes}G}$ and $f' \circ f = 1_T$, showing that T and $G\overline{\otimes}G$ are isomorphic as compact groups.

(iv) From Proposition 3.4, we know that the weight $w(FX)$ of the free compact group over $X = G \times G$ is upper bounded by $w(X)^{\aleph_0}$. Now $G\overline{\otimes}G \cong FX/K$ implies that the weight $w(G\overline{\otimes}G)$ of $G\overline{\otimes}G$ is upper bounded by $w(FX/K)$ which in turn is upper bounded by $w(FX)$, and (iv) follows. ■

Now we prove our second main theorem:

Proof. (i) By assumption, we have a short exact sequence of pro-p-groups:

$$1 \longrightarrow N \longrightarrow G \overset{\pi}{\longrightarrow} H \longrightarrow 1$$

where $\pi : G \to H$ is the natural projection. By Lemma 2.5, there is a continuous homomorphism $\pi \widehat{\otimes} \pi : G \widehat{\otimes} G \to H \widehat{\otimes} H$, giving rise to a short exact sequence of pro-p-groups

$$1 \longrightarrow \ker(\pi \widehat{\otimes} \pi) \longrightarrow G \widehat{\otimes} G \longrightarrow H \widehat{\otimes} H \longrightarrow 1.$$

By assumption, there is a continuous homomorphism $\alpha : H \to G$ such that $\pi \circ \alpha = 1_H$. By Lemma 2.5, there is a continuous homomorphism

$$\alpha \widehat{\otimes} \alpha : a \widehat{\otimes} b \in H \widehat{\otimes} H \mapsto \alpha \widehat{\otimes} \alpha \left(a \widehat{\otimes} b\right) = \alpha\left(a\right) \ \widehat{\otimes} \ \alpha\left(b\right) \in G \widehat{\otimes} G$$

Now $(\pi \widehat{\otimes} \pi) \circ \left(\alpha \widehat{\otimes} \alpha\right) = 1_{H \widehat{\otimes} H}$. Put $K = \ker\left(\pi \widehat{\otimes} \pi\right)$, and note that K is closed since it is a preimage of a point under a continuous function. Moreover, $\pi \widehat{\otimes} \pi$ is a continuous function of compact Hausdorff spaces, so it is a closed map. Therefore, $H \widehat{\otimes} H$ is closed, and hence $G \widehat{\otimes} G \simeq K \rtimes (H \widehat{\otimes} H)$.

(ii) By (i) above we have $G \widehat{\otimes} G \simeq K \rtimes (C \widehat{\otimes} C)$. Furthermore,

$$G \widehat{\otimes} G \simeq K \rtimes (C \widehat{\otimes} C) \simeq K \rtimes C$$

since $C \widehat{\otimes} C \simeq C$ by [36, Theorem 1.1] and $\mathbb{Z}\left(p^n\right) \otimes \mathbb{Z}\left(p^n\right) \simeq \mathbb{Z}\left(p^n\right)$ for all $n \in \mathbb{N}$. Since $C = \overline{\langle x \rangle}$ for some $x \in C$, it follows that $C \widehat{\otimes} C$ acts trivially on K, and so

$$G \widehat{\otimes} G \simeq K \times (C \widehat{\otimes} C).$$

Moreover, by [34, Corollary 2.2], $C \wedge C \simeq 1$, so that $\widehat{\nabla}(C) \simeq C \widehat{\otimes} C$. By [13, Proposition 2.3(i)] we have $\nabla(G) = \nabla(C)$, therefore $\widehat{\nabla}(G) = \widehat{\nabla}(C)$. Now

$$K \simeq (G \widehat{\otimes} G)/(C \widehat{\otimes} C) \simeq (G \widehat{\otimes} G)/\widehat{\nabla}(C) \simeq (G \widehat{\otimes} G)/\widehat{\nabla}(G) \simeq G \widehat{\wedge} G,$$

and hence $G \widehat{\otimes} G \simeq \left(G \widehat{\wedge} G\right) \times C$. ■

Observe that the proof of Theorem 1.5 (i) goes through even when G is an arbitrary compact group that is not a pro-p-group. However, that of Theorem 1.5 (ii) may not hold because we do not know whether [36, Theorem 1.1] holds for general compact groups.

Acknowledgements. We thank the referee, who gave relevant suggestions and indicated significant references on the free topological groups. We also thank the National Research Foundation of South Africa for grants with Reference Numbers 150555, 113144, 118517 and Shuttleworth Postgraduate Scholarship at University of Cape Town (Cape Town, South Africa).

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Received January 27, 2023
and in final form June 6, 2023