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Decomposition of Enveloping Algebras of Simple Lie Algebras and their Related Polynomial Algebras

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Abstract. The decomposition problem of the enveloping algebra of a simple Lie algebra is reconsidered combining both the analytical and the algebraic approach, showing its relation with the internal labelling problem with respect to a nilpotent subalgebra. A lower bound for the number of generators of the commutant as well as the maximal Abelian subalgebra are obtained. The case of rank-two simple Lie algebras is revisited and completed with the analysis of the exceptional Lie algebra G_2.

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1. Introduction

Decomposition theorems for enveloping algebras of (simple) Lie algebras constitute a classical result within the structure theory, and have found extensive application in representation theory [14]. For physical problems, these techniques have passed unnoticed for a long time, until the systematic study of superintegrable and quasi-solvable systems has shown how the detailed analysis of enveloping algebras, as well as its analytical counterpart in terms of differential operators can be of much use in solving these problems [17, 19, 20, 28]. This has motivated various approaches to polynomial algebras, both from the perspective of rings of functions as well as from the purely algebraic formalism (see [6, 11, 15, 16, 23] and references therein).

Polynomial algebras have attracted attention on their own right, in particular, the quadratic case [23], showing to what extent different properties of Lie algebras, such as the Jacobi identity or the Poincaré-Birkhoff-Witt bases are preserved, providing a new scope for building other types of algebraic structures. Polynomial algebras of this type have also found applications within the context of Gröbner bases and adjacent areas [18], the generalization of space-time conformal algebras [29], as well as the description of symmetries and orthogonal polynomials related to quantum models [17, 19]. This shows the diversity of problems in which polynomial algebras emerge naturally. Here, we intend to point out the connection between polynomial algebras and some classical problems of simple Lie algebras and their enveloping algebra.

In this work we reconsider the decomposition problem of enveloping algebras of simple Lie algebras combining the purely algebraic approach with an analytical

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interpretation. More precisely, we show that finding the commutant $C_{\mathcal{U}(\mathfrak{s})}(P)$ of the highest weight vector for the adjoint representation is computationally equivalent to determine the subgroup scalars for a distinguished nilpotent Lie algebra, namely the nilradical $\mathfrak{n}$ of the Borel subalgebra of $\mathfrak{s}$. This allows us to estimate a lower bound for the number of polynomials in the enveloping algebra required to span the commutant, as well as to determine the dimension of its maximal Abelian subalgebra. The case of the three classical rank-two simple Lie algebras, known from the literature (see [3, 10] and references therein) are revisited from this perspective. Further, we solve the decomposition problem for the exceptional Lie algebra G_2, which is considerably involved from the computational point of view. It is shown that the polynomials giving rise to the decomposition of the enveloping algebra determine a 17-dimensional polynomial algebra admitting several non-Abelian subalgebras. In the general context, it is shown that the commutant $C_{\mathcal{U}(\mathfrak{s})}(P)$ always admits a non-Abelian polynomial subalgebra $\mathcal{A}$ spanned by polynomials forming an integrity basis for the system of partial differential equations (PDEs) associated to the reduction chain $\mathfrak{n} \subset \mathfrak{s}$. We finish with some comments on how the procedure can be extended to reduction chains of semisimple Lie algebras not related to the decomposition of enveloping algebras.

2. Presentation of semisimple Lie algebras

One of the most convenient descriptions of complex semisimple Lie algebras is given in terms of the so-called Serre relations, that allow to reconstruct the Lie algebra from its root system and the Dynkin diagram [25].

Let $\mathfrak{s}$ be a complex semisimple Lie algebra of rank ℓ, $\{H_1, \dots, H_\ell\}$ a basis of a Cartan subalgebra $\mathfrak{h}$, $\mathcal{R}$ the corresponding root system and $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ a basis of simple roots. Recall that the Cartan integers are thus given by

$$n(i,j) = \langle \alpha_j, \alpha_i \rangle = \frac{2\,(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)},$$

where $(\alpha_i, \alpha_j) = \kappa\,(H_i, H_j)$ is defined in terms of the nondegenerate Killing form κ. Then we can always find 3ℓ generators H_i, Y_i, Z_i satisfying the relations

$$\begin{gathered}
[H_i, H_j] = 0, \quad [H_i, Y_j] = n(i,j) Y_j, \quad [H_i, Z_j] = -n(i,j) Z_j, \\
[Y_i, Z_j] = \delta_i^j H_i, \quad \mathrm{ad}(Y_i)^{1-n(i,j)} Y_j = 0, \quad \mathrm{ad}(Z_i)^{1-n(i,j)} Z_j = 0 \ (i \neq j).
\end{gathered} \tag{1}$$

Clearly, the generators Y_i are associated to the simple roots α_i, while the Z_i correspond to the opposite roots (see e.g. [25]).

To any linear form $\alpha \in \mathfrak{h}^*$ we associate the weight space $\mathfrak{s}_\alpha$ defined by

$$\mathfrak{s}_\alpha = \{X \in \mathfrak{s} \mid [H, X] = \alpha(H) X, \quad H \in \mathfrak{h}\},$$

Clearly $\mathfrak{h} = \mathfrak{s}^0$ and the α such that $\mathfrak{s}^\alpha \neq 0$ corresponds to the roots in $\mathcal{R}$. In terms of the weight spaces, the Lie algebra $\mathfrak{s}$ can be decomposed as

$$\mathfrak{s} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{s}_\alpha. \tag{2}$$

This actually induces a grading in $\mathfrak{s}$, as for any $\alpha_i, \alpha_j \in R$ the relations (1) imply that $[\mathfrak{s}^{\alpha_i}, \mathfrak{s}^{\alpha_j}] \subset \mathfrak{s}^{\alpha_i + \alpha_j}$.

Let thus $\mathcal{U}(\mathfrak{s})$ be the universal enveloping algebra of $\mathfrak{s}$. For any fixed positive integer p, we denote by $\mathcal{U}_{(p)}(\mathfrak{g})$ the subspace generated by the monomials $X_1^{a_1}\dots X_n^{a_n}$ satisfying the constraint $a_1+a_2+\dots+a_n\leq p$, where $n=\dim\mathfrak{s}$. Using this relation, we say that an element $P\in\mathcal{U}(\mathfrak{s})$ is of degree d if $d=\inf\left\{k\mid P\in\mathcal{U}_{(k)}(\mathfrak{s})\right\}$. As the enveloping algebra is naturally filtered, we have the relation

$$\mathcal{U}_{(0)}(\mathfrak{s})=\mathbb{C},\;\;\mathcal{U}_{(p)}(\mathfrak{s})\mathcal{U}_{(q)}(\mathfrak{s})\subset\mathcal{U}_{(p+q)}(\mathfrak{s}),\quad p,q\geq 0. \tag{3}$$

A direct consequence of the filtration is that each $\mathcal{U}_{(p)}(\mathfrak{s})$ is a finite-dimensional representation of $\mathfrak{s}$, hence the enveloping algebra $\mathcal{U}(\mathfrak{s})$ is a sum of finite-dimensional representations of the simple Lie algebra $\mathfrak{s}$.

From the general structure theory (see e.g. [1, 9]) it is easily seen that the adjoint action of $\mathfrak{s}$ on $\mathcal{U}(\mathfrak{s})$ (respectively, the symmetric algebra $S(\mathfrak{s})$) is given by

$$\begin{aligned} P\in\mathcal{U}(\mathfrak{s})\mapsto\quad & P.X_i:=[X_i,P]=X_iP-PX_i\in\mathcal{U}(\mathfrak{s}),\\ P\left(x_1,\dots,x_n\right)\in S(\mathfrak{g})\mapsto\quad & \widehat{X}_i(P)=C_{ij}^k x_k\frac{\partial P}{\partial x_j}\in S(\mathfrak{g}), \end{aligned}\tag{4}$$

where $\{X_1,\dots,X_n\}$ is a basis of $\mathfrak{s}$. A linear isomorphism $\Lambda:S(\mathfrak{s})\rightarrow\mathcal{U}(\mathfrak{s})$ that commutes with the adjoint action is easily obtained through the symmetrization map

$$\Lambda\left(x_{j_1}\dots x_{j_p}\right)=\frac{1}{p!}\sum_{\sigma\in\mathbf{S}_p}X_{j_{\sigma(1)}}\dots X_{j_{\sigma(p)}}, \tag{5}$$

where $\mathbf{S}(p)$ denotes the symmetric group of order $p!$. For the space $S^{(p)}(\mathfrak{s})$ of homogeneous polynomials having degree p, the relation $\mathcal{U}^{(p)}(\mathfrak{s})=\Lambda\left(S^{(p)}(\mathfrak{s})\right)$ further allows us to decompose the subspace $\mathcal{U}_{(p)}(\mathfrak{g})$ as a direct sum of homogeneous elements: $\mathcal{U}_{(p)}(\mathfrak{s})=\sum_{k=0}^{p}\mathcal{U}^{(k)}(\mathfrak{s})$. Thus for any $P\in\mathcal{U}_{(p)}(\mathfrak{s})$, $Q\in\mathcal{U}_{(q)}(\mathfrak{s})$ the commutator satisfies

$$[P,Q]\in\mathcal{U}_{(p+q-1)}(\mathfrak{s}).$$

Further, as a direct consequence of the Poincaré–Birkhoff–Witt theorem, we have the dimension formula

$$\dim\mathcal{U}^{(p)}(\mathfrak{s})=\dim\frac{\mathcal{U}_{(p)}(\mathfrak{s})}{\mathcal{U}_{(p-1)}(\mathfrak{s})}=\dim S^{(p)}(\mathfrak{s})=\begin{pmatrix}\dim\mathfrak{s}+p-1\\ p\end{pmatrix}. \tag{6}$$

Invariant polynomials of $\mathfrak{s}$ are defined as the centre of $\mathcal{U}(\mathfrak{s})$:

$$Z\left(\mathcal{U}(\mathfrak{s})\right)=\left\{P\in\mathcal{U}(\mathfrak{s})\mid[\mathfrak{s},P]=0\right\}. \tag{7}$$

These elements can be identified with the (polynomial) solutions of the differential operators in(4) (see [1, 9] for details).

The *commutant* $C_{\mathcal{U}(\mathfrak{s})}(P)$ of an element $P\in\mathcal{U}(\mathfrak{s})$ is defined as the centralizer of P in the enveloping algebra, that is, the set of elements in $\mathcal{U}(\mathfrak{g})$ that commute with P

$$C_{\mathcal{U}(\mathfrak{s})}(P)=\left\{Q\in\mathcal{U}(\mathfrak{s})\mid[P,Q]=0\right\}. \tag{8}$$

If $\mathfrak{s}$ is semisimple, the commutant can be shown to be finitely generated (see [9], Chapter 2), a property that allows us to find an integrity basis for it.

2.1. Decomposition of enveloping algebras and commutants

As mentioned before, the adjoint action (4) implies that the universal enveloping algebra $\mathcal{U}(\mathfrak{s})$ can be seen as a representation of $\mathfrak{s}$, which is moreover completely reducible whenever the Lie algebra is semisimple [9]. We therefore have a decomposition

$$\mathcal{U}(\mathfrak{s}) = \bigoplus_{k=0}^{\infty} V_k, \tag{9}$$

where $V_k = \mathcal{U}(\mathfrak{s}).v_{\lambda_k}$ is an irreducible component spanned by a highest vector v_{λ_k} with weight $\lambda_k \in \mathfrak{h}^*$ for the extended adjoint action, i.e.,

$$[H_i, v_{\lambda_k}] = \lambda_k(H_i)v_{\lambda_k},\ H_i \in \mathfrak{h}, \quad [X_i, v_{\lambda_k}] = 0,\ X_i \in \mathfrak{s}_{\alpha_i},\ \alpha_i \in \Delta.$$

Therefore, finding the components in the decomposition (9) amounts to compute the elements in the enveloping algebra $\mathcal{U}(\mathfrak{s})$ that commute with the generators associated to the positive roots of $\mathfrak{s}$. Due to the Serre relations (1), it suffices to consider the elements in $\mathcal{U}(\mathfrak{s})$ that commute with the generators X_i associated to simple roots $\alpha_i \in \Delta$, as any other elements of $\mathfrak{n}$ are obtained by commutators. In this context, we observe that the generators X_i associated to simple roots generate a nilpotent Lie algebra $\mathfrak{n}$ (see equation (1)).[1], so that the decomposition problem corresponds to find the centralizer of $\mathfrak{n}$ in $\mathcal{U}(\mathfrak{s})$:

$$\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n}) = \{P \in \mathcal{U}(\mathfrak{s}) \mid [X, P] = 0, \quad X \in \mathfrak{n}\} \tag{10}$$

By the algebraic properties of enveloping algebras, we can always find a finite number of polynomials $\{P_1, \ldots, P_s\}$ such that, as vector space, $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ is spanned by

$$P_1^{a_1} P_2^{a_2} \ldots P_s^{a_s}, \quad a_i \in \mathbb{N} \cup 0, \tag{11}$$

and where the coefficients a_i are possibly constrained by some relation. It should be observed that these polynomials are not necessarily algebraically independent, but merely linearly independent, and that they generate a non-Abelian polynomial algebra. This implies in practice that suitable dependent polynomials must be added to a functionally independent set of polynomials in order to obtain a basis of the type (11). With respect to a given Cartan subalgebra $\mathfrak{h}$, if the weight of each P_i is given by $(\lambda_1^i, \ldots, \lambda_\ell^i)$, the weight of an element (11) is given by

$$\omega_i = \sum_{i=1}^{s} a_i \left(\lambda_1^i, \ldots, \lambda_\ell^i\right). \tag{12}$$

Any element $P \in \mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ can thus be labeled using its degree d as polynomial and its weight $\mu = (\lambda_1, \ldots, \lambda_\ell)$ with respect to the (fixed) Cartan subalgebra. For given values of a_i, the element in (11) generates an irreducible representation of $\mathfrak{s}$ of dimension $d_{a_1,\ldots a_s}$. In order to fulfill the decomposition (9), the identity

$$\dim \mathcal{U}^{(p)} = \sum_{(a_1,\ldots,a_s)\in\Phi_p} d_{a_1,\ldots a_s},\ \Phi_p = \left\{(a_1, \ldots, a_s) \mid \sum_{k=1}^{s} a_k \mathrm{deg} P_k = p\right\} \bmod \mathcal{S}, \tag{13}$$

must be satisfied for $p \geq 1$, where $\mathcal{S}$ denotes the set of relations that constrain the coefficients a_i.

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[1] This algebra actually corresponds to the nilradical $\mathfrak{n}$ of the Borel subalgebra $\mathfrak{b}(\Delta)$ of $\mathfrak{s}$ (the maximal solvable subalgebra of $\mathfrak{s}$)

As the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ involves generators that do not belong to the subalgebra $\mathfrak{n}$, in addition to the invariants of $\mathfrak{n}$ there will be additional polynomials depending on the variables of $\mathfrak{s}$, in particular the Casimir operators of the simple Lie algebra $\mathfrak{s}$. We can suppose without loss of generalization that for the generic element (11), the (primitive) Casimir operators $I_1, \dots I_\ell$ of $\mathfrak{s}$ satisfy

$$P_{s+1-\ell} = I_1, \dots, P_s = I_\ell. \tag{14}$$

Defining the set
$$\mathcal{B} = \left\{ P_1^{a_1} P_2^{a_2} \dots P_{s-\ell}^{a_{s-\ell}}, \quad a_i \in \mathbb{N} \cup 0 \right\}, \tag{15}$$

it follows easily that $\mathcal{B}$ forms a basis of the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ seen as a free module over $\mathbb{C}[I_1, \dots, I_\ell]$. This is an immediate consequence of the Schur lemma, as the Casimir operators act as scalar matrices on each irreducible $\mathfrak{s}$-representation. As the invariants of $\mathfrak{s}$ have zero highest weight (with respect to a Cartan subalgebra), the highest weight of the vectors in (11) are determined by the polynomials which have a nonvanishing commutator with at least one generator of $\mathfrak{s}$ not belonging to the nilpotent subalgebra $\mathfrak{n}$. In particular, the generator X_β corresponding to the highest root $\beta \in \mathcal{R}^+$ always has a nonzero weight, as it lies in the closure of the Weyl chamber. The Casimir operators of $\mathfrak{n}$, as they do not commute with all generators of $\mathfrak{s}$, also have a nonvanishing weight. In terms of representation theory, this implies that for any finite-dimensional irreducible complex representation $\varphi : \mathfrak{s} \to \mathrm{End}(V)$, the elements $\varphi(P)$ with $P \in \mathcal{B}$ span the space of highest weight elements in $\mathrm{End}(V)$ [10].

3. Analytic properties of the commutant

Alternatively to the analysis of the universal enveloping algebra $\mathcal{U}(\mathfrak{s})$, we can use the differential operators in (4) to compute the elements in the commutant analytically. As $\mathfrak{n}$ is a subalgebra, this correspond to the so-called labelling problem with respect to the chain $\mathfrak{n} \subset \mathfrak{s}$ (see [5, 22, 24]).

Starting from the embedding $\mathfrak{n} \subset \mathfrak{s}$, we extend an arbitrary basis $\{X_1, \dots, X_m\}$ of the subalgebra $\mathfrak{n}$ to a basis $\mathfrak{B} = \{X_1, \dots, X_m, X_{m+1} = Y_1, \dots, X_n = Y_{n-m}\}$ of $\mathfrak{s}$, such that the commutators of the subalgebra generators are given by

$$[X_i, X_j] = C_{ij}^k X_k, \quad [X_i, Y_p] = D_{ip}^k X_k + E_{ip}^q Y_q \tag{16}$$

where $i, j, k \in \{1, \dots, m\}$ and $p, q \in \{1, \dots, n-m\}$. The operators that commute with the elements in $\mathfrak{n}$ hence correspond to the solutions of the system of PDEs

$$\widehat{X}_i(F) = -C_{ij}^k x_k \frac{\partial F}{\partial x_j} - (D_{ip}^k x_k + E_{ip}^q y_q) \frac{\partial F}{\partial y_p} = 0, \quad 1 \le i \le m. \tag{17}$$

where $\{x_1, \dots, x_m, y_1, \dots, y_{n-m}\}$ are the coordinates in a dual basis of $\mathfrak{B}$. We observe that the solutions F to the system (17) such that $\frac{\partial F}{\partial y_p} = 0$ for all integers $p \in \{1, \dots, n-m\}$ correspond to the Casimir invariants of the subalgebra, while a genuine subgroup scalar must explicitly depend on the variables $\{y_1, \dots, y_{n-m}\}$ [5].

Now the system (17) has exactly $n - r'$ independent solutions, where r' denotes the rank of the $m \times n$ coefficient matrix of $\mathfrak{n}$ as subalgebra of $\mathfrak{s}$. Although an arbitrary integrity basis for the system (17) does not necessarily generate a polynomial algebra, so that the analytical approach provides at best a lower bound d_0 on the number of required polynomials to obtain the decomposition (9), it may be asked whether for any simple Lie algebra there exists at least an integrity basis for the system (17) such that forms a polynomial algebra. We observe that starting from such a subalgebra, the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ can formally be constructed, adding those elements that are algebraically dependent (but linearly independent) required to satisfy the dimension condition (13).

The following table gives the lower bound d_0 obtained analytically for the complex simple Lie algebras:

Type	A_ℓ	B_ℓ, C_ℓ	D_ℓ	G_2	F_4	E_6	E_7	E_8
d_0	$\frac{\ell(\ell+3)}{2}$	$\ell^2+\ell$	ℓ^2	8	28	42	70	128

(18)

Among these polynomials, the invariants of $\mathfrak{n}$ are included. We denote the number of independent invariants of the latter as $\mathcal{N}(\mathfrak{n})$. As the subalgebra is nilpotent, we can always find a maximal set of functionally independent solutions formed by polynomials, i.e., an integrity basis [8]. In particular, its centre is generated by the generator associated to the highest root in $\mathcal{R}$, and the commutant contains exactly one linear polynomial. From these solutions, $\ell+\mathcal{N}(\mathfrak{n})$ correspond to the Casimir invariants of either $\mathfrak{s}$ or $\mathfrak{n}$,[2] so that the number of available operators is given by $\chi = n - r' - \ell - \mathcal{N}(\mathfrak{n})$. It can be easily shown (see e.g. [22]) that $m = r'$, which implies that $\chi = 2n_0$. It should however be noted that among these $2n_0$ solutions, at most n_0 correspond to operators that commute with each other [22]. We conclude from this that the maximal number of operators that commute with the subalgebra $\mathfrak{n}$ and with each other is given by

$$\xi = n_0 + \ell + \mathcal{N}(\mathfrak{n}) = \frac{\dim\mathfrak{s} - \dim\mathfrak{n} + \ell + \mathcal{N}(\mathfrak{n})}{2}. \tag{19}$$

The remaining independent solutions of system (17) necessarily have some nontrivial commutator. Expressed in other words, we conclude that the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ always contains a maximal Abelian subalgebra of dimension ξ.

As the generators of $\mathfrak{n}$ correspond to the positive roots of $\mathfrak{s}$, we can easily deduce the number of Casimir invariants of $\mathfrak{n}$ making use of the so-called Maurer–Cartan equations of the subalgebra [4]. From (2) we know that $\mathfrak{n} = \sum_{\alpha\in\mathcal{R}^+} s_\alpha$. Let ω_α denote the invariant 1-form corresponding to the generator of the weight space s_α. For any positive root, the Maurer–Cartan equations can be written formally as

$$\mathrm{d}\omega_\alpha = \sum_{\beta,\gamma\in\mathcal{R}^+} \lambda^{\beta,\gamma}\omega_\beta\wedge\omega_\gamma, \tag{20}$$

where $\lambda^{\beta,\gamma} = 0$ whenever $\beta+\gamma\neq\alpha$. In particular, if $\alpha_i\in\Delta$ is a simple root, it is obvious that the form is closed, i.e, $\mathrm{d}\omega_{\alpha_i} = 0$, while for non-simple roots the relations $\mathrm{d}\omega_\alpha \neq 0$ holds. We now define a generic 2-form $\theta = \sum_{\alpha\in\mathcal{R}^+}\mathrm{d}\omega_\alpha$ and the index $j_0(\mathfrak{n})$ as the lowest natural number such that

$$\bigwedge^{j_0(\mathfrak{n})}\theta\neq 0, \qquad \bigwedge^{j_0(\mathfrak{n})+1}\theta\equiv 0. \tag{21}$$

Then the number $\mathcal{N}(\mathfrak{n})$ of invariants is given by $\mathcal{N}(\mathfrak{n}) = \dim\mathfrak{n} - 2j_0(\mathfrak{n})$ [4].

[2] Here we use the fact that the nilradical of the Borel subalgebra and $\mathfrak{s}$ can never have common Casimir invariants.

Proposition 3.1. *Let $\mathfrak{n}$ be the nilradical of the Borel subalgebra of a complex simple Lie algebra $\mathfrak{s}$.*

(1) *If* $\mathfrak{s} \simeq A_\ell$, $\dim \mathfrak{n} = \dfrac{\ell(\ell+1)}{2}$ *and* $\mathcal{N}(\mathfrak{n}) = \left[\dfrac{\ell+1}{2}\right]$ *for* $\ell \geq 1$.

(2) *If* $\mathfrak{s} \simeq B_\ell$, $\dim \mathfrak{n} = \ell^2$ *and* $\mathcal{N}(\mathfrak{n}) = \ell$ *for* $\ell \geq 2$.

(3) *If* $\mathfrak{s} \simeq C_\ell$, $\dim \mathfrak{n} = \ell^2$ *and* $\mathcal{N}(\mathfrak{n}) = \ell$ *for* $\ell \geq 3$.

(4) *If* $\mathfrak{s} \simeq D_\ell$, $\dim \mathfrak{n} = \ell^2 - \ell$ *and* $\mathcal{N}(\mathfrak{n}) = \begin{cases} \ell & \ell = 2p,\ p \geq 2 \\ \ell - 1 & \ell = 2p+1,\ p \geq 2 \end{cases}$

(5) *If* $\mathfrak{s}$ *is an exceptional Lie algebra of rank* ℓ, *then* $\mathcal{N}(\mathfrak{n}) = \ell$.

We prove the assertion for the type A_ℓ, the argument being the same for the remaining types of simple classical algebras. The positive roots $\mathcal{R}^+$ of A_ℓ are given by

$$\beta_{j,k} = \sum_{p=j}^{k} \alpha_p, \quad j,k = 1,\dots \ell,\ j \leq k \tag{22}$$

Let ω_α denote the 1-forms. It is clear from the structure of the root system of A_ℓ that the Maurer–Cartan forms have the generic form

$$\mathrm{d}\omega_{\beta_{j,k}} = \sum_{s=j}^{k} \lambda_{j,k}^{s} \omega_{\beta_{j,s}} \wedge \omega_{\beta_{s+1,k}}, \tag{23}$$

with coefficients $\lambda_{j,k}^{s}$ such that $\mathrm{d}\left(\mathrm{d}\omega_{\beta_{j,k}}\right) = 0$ holds. We separate the analysis according to the parity of ℓ. Let $\ell = 2p$ and consider the 2-form

$$\theta = \mathrm{d}\omega_{\beta_{1,2p}} + \sum_{q=1}^{p-1} \sum_{s=0}^{p-q-\frac{1}{2}} \mathrm{d}\omega_{\beta_{2q,2q+2s+1}} + \sum_{q=1}^{p-1} \sum_{s=0}^{p-q-2} \mathrm{d}\omega_{\beta_{2q+1,2q+2s+2}} \tag{24}$$

A routine computation shows that the form θ satisfies

$$\bigwedge^{p^2} \theta \neq 0, \qquad \bigwedge^{p^2+1} \theta \equiv 0, \tag{25}$$

and that adding additional 2-forms $\mathrm{d}\beta_{j,k}$ does not increase the degree of θ, thus $j_0(\mathfrak{n}) = p^2$. Therefore the number of Casimir invariants of $\mathfrak{n}$ is given by

$$\mathcal{N}(\mathfrak{n}) = 2p^2 + p - 2p^2 = p = \left[\frac{2p+1}{2}\right] \tag{26}$$

For odd $\ell = 2p+1$, we consider the 2-form

$$\theta = \mathrm{d}\omega_{\beta_{1,2p}} + \sum_{q=1}^{p-1} \sum_{s=0}^{p-q-\frac{1}{2}} \mathrm{d}\omega_{\beta_{2q,2q+2s+1}} + \sum_{q=1}^{p-1} \sum_{s=0}^{p-q-1} \mathrm{d}\omega_{\beta_{2q+1,2q+2s+2}} \tag{27}$$

satisfying $\bigwedge^{p^2+p} \theta \neq 0$, $\bigwedge^{p^2+p+1} \theta \equiv 0$, and the addition of other 2-form does not increase the degree $j_0(\mathfrak{n}) = p^2 + p$, from which we deduce that

$$\mathcal{N}(\mathfrak{n}) = 2p^2 + 3p + 1 - 2p^2 - 2p = p + 1 = \left[\frac{2p+2}{2}\right].$$

4. Analysis of rank 2 semisimple Lie algebras

4.1. $D_2 = so(4)$

Let us first consider the semisimple but not simple Lie algebra $D_2 = A_1^2$. In this case, the root system is reducible and indeed the disjoint union of the root systems corresponding to A_1. Choosing a basis adapted to the root system, the commutators are

$$\begin{aligned} &[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = X_1, \\ &[X_4, X_5] = 2X_5, \quad [X_4, X_6] = -2X_6, \quad [X_5, X_6] = X_4. \end{aligned} \tag{28}$$

As expected from the fact that $\mathfrak{n} = \mathbb{R}\langle X_2, X_4\rangle$ is a decomposable algebra, the commutant is also decomposable. There are four algebraically independent polynomials in $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$, given by X_1, X_4 themselves, as well as the two Casimir operators $I_{21} = X_1^2 + 2(X_2X_3 + X_3X_2)$, $I_{22} = X_4^2 + 2(X_5X_6 + X_6X_5)$ of D_2. These elements form an integrity basis for the solutions of the system (17), and generate the (Abelian) commutant. Hence any element in $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ can be written as

$$P = X_1^{a_1} X_4^{a_2} I_{21}^{a_3} I_{22}^{a_4}, \quad a_i \in \mathbb{N} \cup 0. \tag{29}$$

The decomposition (9) is straightforward.

4.2. $A_2 = \mathfrak{sl}(3, \mathbb{C})$

The decomposition of the enveloping algebra of A_2 has already been considered by various authors, using both analytical and algebraic tools ([3, 10] and references therein). Here we proceed somewhat differently, looking for the algebraic structure of commutants.

Starting from the defining representation $[1, 0]$, the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ is best given in terms of the elementary matrices E_{ij} defined by

$$(E_{ij})_{kl} = \delta_i^k \delta_j^l, 1 \leq i, j, k, l \leq 3.$$

and subjected to the constraint $\mathrm{Tr}(X) = 0$ for each $X \in \mathfrak{sl}(3, \mathbb{C})$. Table 1 gives the commutation relations in terms of these generators.

Table 1: Commutators of the elementary matrices

$[\circ, \circ]$	E_{11}	E_{22}	E_{33}	E_{12}	E_{13}	E_{23}	E_{21}	E_{31}	E_{32}
E_{11}	0	0	0	E_{12}	E_{13}	0	$-E_{21}$	$-E_{31}$	0
E_{22}	0	0	0	$-E_{12}$	0	E_{23}	E_{21}	0	$-E_{32}$
E_{33}	0	0	0	0	$-E_{13}$	$-E_{23}$	0	E_{31}	E_{32}
E_{12}	$-E_{12}$	E_{12}	0	0	0	E_{13}	$E_{11} - E_{22}$	$-E_{32}$	0
E_{13}	$-E_{13}$	0	E_{13}	0	0	0	$-E_{23}$	$E_{11} - E_{33}$	E_{12}
E_{23}	0	$-E_{23}$	E_{23}	$-E_{13}$	0	0	0	E_{21}	$E_{22} - E_{33}$
E_{21}	E_{21}	$-E_{21}$	0	$E_{22} - E_{11}$	E_{23}	0	0	0	$-E_{31}$
E_{31}	E_{31}	0	$-E_{31}$	E_{32}	$E_{33} - E_{11}$	$-E_{21}$	0	0	0
E_{32}	0	E_{32}	$-E_{31}$	0	$-E_{12}$	$E_{33} - E_{22}$	E_{31}	0	0

Without loss of generality, we consider the Cartan subalgebra $\mathfrak{h}$ generated by $H_1 = E_{11} - E_{22}$ and $H_2 = E_{22} - E_{33}$. We observe that, due to this choice, E_{12} and E_{23} are the generators associated to the simple roots α_1, α_2 of Δ, while E_{13} is associated to the highest weight $\alpha_1 + \alpha_2$ of the adjoint representation. Similarly, E_{21}, E_{32} and E_{31} correspond to the negative roots $-\alpha_1$, $-\alpha_2$ and $-(\alpha_1 + \alpha_2)$, respectively. Now E_{12} and E_{23} generate a three-dimensional nilpotent algebra $\mathfrak{n}^3$, the centre E_{13} of which, as an invariant, commutes with all generators of $\mathfrak{n}$. There are two other types of elements in the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$, namely the (generalized Casimir) invariants of $\mathfrak{s}$, that commute with all generators of the Lie algebra, and those polynomials that commute with the subalgebra $\mathfrak{n}$, but not with all generators of $\mathfrak{s}$, corresponding to subgroup scalars for the coadjoint representation. Using the analytical method (see [5, 22]) it can be easily shown that an integrity basis of functions that commute with $\mathfrak{n}$ is formed by five independent elements. A direct computation of the polynomials of order at most three in the generators of $\mathfrak{s}$ shows that there are exactly six linearly independent elements, given respectively by

$$\begin{aligned}
B_1 &= E_{13}, \quad B_2 = 3E_{12}E_{23} + (H_1 - H_2)E_{13},\\
C_1 &= E_{32}E_{13}^2 - E_{23}E_{12}^2 + H_2E_{12}E_{13}, \quad C_2 = -E_{21}E_{13}^2 + E_{12}E_{23}^2 + H_1E_{23}E_{13},\\
I_2 &= \frac{1}{3}(H_1^2 + H_1H_2 + H_2^2) + H_1 + H_2 + E_{21}E_{12} + E_{31}E_{13} + E_{32}E_{23},\\
I_3 &= \frac{2}{27}H_1^3 + \frac{1}{9}H_1^2H_2 - \frac{1}{9}H_1H_2^2 - \frac{2}{27}H_2^3 + E_{32}E_{21}E_{13} + E_{31}E_{12}E_{23} - E_{31}E_{13} - E_{32}E_{23}\\
&\quad + \frac{1}{3}E_{31}(H_1 - H_2)E_{13} + \frac{1}{3}E_{21}(H_1 + 2H_2)E_{12} - \frac{1}{3}E_{32}(2H_1 + H_2)E_{23} + E_{21}E_{12}\\
&\quad + \frac{1}{3}(H_1^2 - H_2^2) + \frac{1}{3}(H_1 - H_2). \qquad (30)
\end{aligned}$$

We observe that, with respect to the Cartan subalgebra $\mathfrak{h}$, the polynomials B_1 and B_2 have weight (1,1) , while C_1 is of weight (3,0) and C_2 of weight (0,3) respectively. The Casimir operators I_2 and I_3 are obviously of weight (0,0). As observed, at most five among these six operators are functionally independent, as follows from the algebraic dependence relation

$$C_1C_2 + B_1^3(I_2 + I_3) - \frac{1}{3}B_1^2B_2(I_2 - 2) - \frac{1}{3}B_1B_2^2 + \frac{1}{27}B_2^3 = 0. \qquad (31)$$

The set (30) contains an integrity basis. As Casimir operators of $\mathfrak{s}$, I_2 and I_3 generate an Abelian ideal, to which B_1, B_2 can be added, thus exhausting the maximal number of commuting elements within the commutant (see equation (19)). Therefore, the pair C_1, C_2 must lead to nonvanishing commutators. We obtain that I_2, B_1, B_2, C_1, C_2 generate a five-dimensional quartic polynomial algebra with nontrivial commutators

$$[B_2, C_1] = 3B_1C_1, \; [B_2, C_2] = -3B_1C_2, \; [C_1, C_2] = -B_1^3I_2 - B_1^2B_2 + \frac{1}{3}B_1B_2^2 \qquad (32)$$

It can be verified (see [10]) that any other polynomial of order $d \geq 4$ is obtained as an algebraic expression in terms of $B_1, B_2, C_1, C_2, I_2, I_3$, showing that any element of the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ can be written as

$$P = B_1^{a_1}B_2^{a_2}C_1^{a_3}C_2^{a_4}I_2^{a_5}I_3^{a_6}, \quad a_i \in \mathbb{N} \cup 0 \qquad (33)$$

subjected to the constraint $a_3a_4 = 0$, as a consequence of equation (31). We also observe that the five elements B_1, B_2, C_1, C_2, I_2 are functionally independent and thus can be chosen as an integrity basis. As follows from the commutators, they also form a polynomial subalgebra $\mathcal{A}$ of the commutant.

[3] It is indeed isomorphic to the Heisenberg algebra $\mathfrak{H}_1$.

4.3. $B_2 = \mathfrak{so}(5,\mathbb{C})$

For the orthogonal Lie algebra $\mathfrak{so}(5,\mathbb{C})$, the root system R has rank two with positive roots

$$R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}\,. \tag{34}$$

Identifying the generators of the positive roots spaces with the elements E_1, E_2, E_3, E_4 and the negative roots with F_1, F_2, F_3, F_4, as well as the generators H_1, H_2 of the Cartan subalgebra $\mathfrak{h}$, the commutators are given in the following table:

Table 2: Commutators of $\mathfrak{so}(5,\mathbb{C})$

$[\circ,\circ]$	H_1	H_2	E_1	E_2	E_3	E_4	F_1	F_2	F_3	F_4
H_1	0	0	0	E_2	E_3	E_4	0	$-F_2$	$-F_3$	$-F_4$
H_2	0	0	E_1	$-E_2$	0	E_4	$-F_1$	F_2	0	$-F_4$
E_1	0	0	0	E_3	E_4	0	H_2	0	$-F_2$	$-F_3$
E_2	$-E_2$	E_2	$-E_3$	0	0	0	0	H_1-H_2	F_1	0
E_3	$-E_3$	0	$-E_4$	0	0	0	$-E_2$	E_1	H_1	F_1
E_4	$-E_4$	$-E_4$	0	0	0	0	$-E_3$	0	E_1	H_1+H_2
F_1	0	F_1	$-H_2$	0	E_2	E_3	0	$-F_3$	$-F_4$	0
F_2	F_2	$-F_2$	0	H_2-H_1	$-E_1$	0	F_3	0	0	0
F_3	F_3	0	F_2	$-F_1$	$-H_1$	$-E_1$	F_4	0	0	0
F_4	F_4	F_4	F_3	0	$-F_1$	$-H_1-H_2$	0	0	0	0

The nilradical $\mathfrak{n}$ of the Borel subalgebra, generated by E_1, E_2, E_3, E_4, is isomorphic to the only indecomposable nilpotent Lie algebra of dimension four $\mathfrak{n}_{4,1}$, and possesses two Casimir operators that are linear and quadratic in the generators, given by $P_1 = E_4$ and $P_2 = E_2E_4 - \frac{1}{2}E_3^2$ respectively (see [27]). The weight of these polynomials are (1,1) and (2,0) respectively. Using formula (17), it can be easily verified that there are six algebraically independent solutions to the system. In addition to the two previous solutions P_1 and P_2, up to order four there are three additional polynomials that commute with the subalgebra $\mathfrak{n}$ but not with the whole orthogonal algebra:

$$\begin{aligned}
P_3 &= E_3 - E_1E_2 + E_4F_1 - H_2E_3,\\
P_4 &= -E_4H_1 + E_4H_2 - 2E_2E_1^2 + 2E_3E_1H_1 - 2E_3E_1H_2 - E_4H_1^2 + 2E_4H_2H_1\\
&\quad - E_4H_2^2 + 2F_2E_3^2 - 4F_2E_4E_2,\\
P_5 &= -2E_4E_3 - \frac{1}{2}E_3^2E_1 + \frac{5}{2}E_4E_2E_1 - E_4E_3H_1 + F_1E_4^2 + \frac{1}{2}E_3E_2E_1^2 - \frac{1}{2}E_3^2E_1H_1\\
&\quad + \frac{1}{2}E_3^2E_1H_2 + \frac{1}{2}E_4E_2E_1H_1 + \frac{1}{2}E_4E_2E_1H_2 - \frac{1}{2}E_4E_3H_2H_1 + \frac{1}{2}E_4E_3H_2^2\\
&\quad - \frac{1}{2}F_1E_4E_3E_1 + \frac{1}{2}F_1E_4^2H_1 - \frac{1}{2}F_1E_4^2H_2\\
&\quad - \frac{1}{2}F_2E_3^3 + F_2E_4E_3E_2 - \frac{1}{2}F_3E_4E_3^2 + F_3E_4^2E_2
\end{aligned} \tag{35}$$

Finally, the two Casimir operators C_2 and C_4 are given by

$$C_2 = H_1 + \frac{1}{3}H_2 - \frac{2}{3}E_1F_1 - \frac{2}{3}E_2F_2 - \frac{2}{3}E_3F_3 - \frac{2}{3}E_4F_4 - \frac{1}{3}H_1^2 - \frac{1}{3}H_2^2,$$

$$\begin{aligned}C_4 = &-H_2 - 2F_1E_1 - 2F_4E_4 - \frac{3}{2}H_2H_1 - H_2^2 - 3F_1E_1H_1 + 2F_2E_2H_2 - 3F_2F_1E_3\\ &- 2F_3E_2E_1 - F_3E_3H_2 - 2F_3F_1E_4 - F_4E_4H_1 - 2F_4E_4H_2 - \frac{1}{2}H_2H_1^2 - \frac{3}{2}H_1H_2^2\\ &- F_1E_1H_1^2 + F_2E_2H_2H_1 - F_2F_1E_2E_1 - F_2F_1E_3H_1 - F_2F_1E_3H_2 + F_2F_1^2E_4\\ &- \frac{1}{2}F_2^2E_2^2 - F_3E_2E_1H_1 - F_3E_2E_1H_2 - F_3E_3H_2^2 - F_3F_1E_4H_1 + F_3F_1E_4H_2\\ &- F_3F_2E_3E_2 - F_3^2E_4E_2 + F_4E_2E_1^2 - F_4E_3E_1H_1 + F_4E_3E_1H_2 - F_4E_4H_2H_1\\ &- F_4F_1E_4E_1 - F_4F_2E_3^2 + F_4F_2E_4E_2 - F_4F_3E_4E_3 - \frac{1}{2}F_4^2E_4^2 - \frac{1}{2}H_1^2H_2^2. \end{aligned} \quad (36)$$

These operators exhaust the number of linearly independent elements in the commutant, and thus generate $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ (see also [2] for a derivation using the Poisson bracket formalism). Among these seven operators, only six are algebraically independent, as follows from the relation (see [2])

$$P_5^2 - P_2P_4 + 2P_1P_3^2 + P_3^3P_4 - 4P_1P_2 + P_1P_2(6 - P_4) - P_2P_4 + P_1^2P_2C_4 = 0. \quad (37)$$

The only nonvanishing commutators among these operators are given by

$$\begin{aligned}&[P_3, P_4] = -\,2P_1P_3 + 4P_5, \quad [P_4, P_5] = 2P_1^2P_3 + P_1P_3P_4 - 4P_1P_5,\\ &[P_3, P_5] = -\,P_1P_2 + \frac{3}{2}P_1P_2C_2 - \frac{1}{2}P_1P_3^2 - \frac{1}{2}P_2P_4.\end{aligned} \quad (38)$$

It follows that any element P in the commutant can be written as

$$P = P_1^{a_1}P_2^{a_2}P_3^{a_3}P_4^{a_4}P_5^{a_5}C_2^{a_6}C_4^{a_7}, \quad a_i \in \mathbb{N} \cup 0 \quad (39)$$

subjected to the constraint $a_5 = 0, 1$ due to relation (37).

We observe that, according to formula (19), at most five of the commutant generators commute with each other. In particular, the polynomials $\{P_1, \dots, P_5, C_2\}$, which can be taken as an integrity basis for the system (17), further generate a polynomial (cubic) subalgebra of $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$.

5. The exceptional Lie algebra G_2

The last of the (reduced) root systems of rank two corresponds to the 14-dimensional exceptional Lie algebra G_2, with the set of positive roots given by

$$R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\} \quad (40)$$

Let X_1, X_2 denote the generators of the Cartan subalgebra and $X_3, X_5, X_7, X_9, X_{11}$, X_{13} be the generators associated to the positive roots, with $X_4, X_6, X_8, X_{10}, X_{12}, X_{14}$ being those corresponding to the negative roots. Over this basis, the commutators are given by

The nilradical $\mathfrak{n}$ of the Borel subalgebra of G_2, generated by X_3, X_5, is of dimension six and isomorphic to the nilpotent Lie algebra $\mathfrak{n}_{6,19}$ listed in [27]. It possesses two Casimir operators of degrees one and two given by the relations $Q_1 = X_{13}$ and $Q_2 = X_9^2 - 3X_3X_{13} + 3X_7X_{11}$, respectively, and their weight with respect to the Cartan subalgebra $\mathfrak{h} = \mathbb{R}\langle X_1, X_2\rangle$ is given by (0,1) and (2,0). The Casimir operators

$$\begin{aligned}C_2 = &\,5X_1 + 15X_2 + X_1^2 + X_{10}X_9 + 3X_{12}X_{11} + 3X_{14}X_{13} + 3X_2X_1 + 3X_2^2\\ &+ X_4X_3 + 3X_6X_5 + X_8X_7\end{aligned}$$

and C_6 of G_2 have degrees 2 and 6 (the explicit expression of the latter operator is skipped because of its length) and both have weight (0,0). These four polynomials commute with each other. As an integrity basis for the system (17) is formed by eight polynomials, four additional independent polynomials must be found. From these eight operators, at most six can commute, as follows from equation (19). We will see that, in this case, the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ requires more than twice this number of operators, making the decomposition of the enveloping algebra of G_2 a computationally demanding problem.

Table 3: Commutators of G_2

[,]	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}
X_1	$2X_3$	$-2X_4$	$-3X_5$	$3X_6$	$-X_7$	X_8	X_9	$-X_{10}$	$3X_{11}$	$-3X_{12}$	0	0
X_2	$-X_3$	X_4	$2X_5$	$-2X_6$	X_7	$-X_8$	0	0	$-X_{11}$	X_{12}	X_{13}	$-X_{14}$
X_3	0	X_1	X_7	0	$2X_9$	$-3X_6$	$-3X_{11}$	$-2X_8$	0	X_{10}	0	0
X_4		0	0	$-X_8$	$3X_5$	$-2X_{10}$	$2X_7$	$3X_{12}$	$-X_9$	0	0	0
X_5			0	X_2	0	X_4	0	0	$-X_{13}$	0	0	X_{12}
X_6				0	$-X_3$	0	0	0	0	X_{14}	$-X_{11}$	0
X_7					0	X_1+3X_2	$-3X_{13}$	$2X_4$	0	0	0	X_{10}
X_8						0	$-2X_3$	$3X_{14}$	0	0	$-X_9$	0
X_9							0	$2X_1+3X_2$	0	$-X_4$	0	$-X_8$
X_{10}								0	X_3	0	X_7	0
X_{11}									0	X_1+X_2	0	$-X_6$
X_{12}										0	X_5	0
X_{13}											0	X_1+2X_2
X_{14}												0

For computational simplicity, and due to the dimension and the length of the polynomials commuting with X_3, X_5, it is convenient to proceed searching for polynomials with a given degree d and weight (λ,μ), that will be denoted by $O_d^{[\lambda,\mu]}$. The strategy is to find an integrity basis formed by polynomials of lowest possible order, so that it contains a set of six independent commuting polynomials, and to complete it to a set of linearly independent polynomials.

Up to degree four, the following seven polynomials in the commutant are linearly independent:

- $d=1$: $O_1^{[0,1]} = Q_1$
- $d=2$: $O_2^{[2,0]} = Q_2$, $O_2^{[0,0]} = C_2$,
- $d=3$:

$$\begin{aligned} O_3^{[3,0]} &= 2X_9^3 + 27X_{11}^2X_5 - 27X_{13}X_{11}X_2 - 27X_{13}^2X_6 + 9X_{11}X_9X_7 - 9X_{13}X_9X_3, \\ O_3^{[1,0]} &= X_{10}X_9^2 + 3(X_{11}X_{10}X_7 + X_{11}X_4X_1 - X_{13}X_{10}X_3) + 9(X_{11}X_4X_2 + X_{11}X_8X_5 \\ &\quad + X_{13}X_6X_4) + 3(X_{13}X_8X_1 + X_7X_3X_2 - X_5X_3^2 + X_7^2X_6 + X_9X_2X_1) + X_9X_4X_3 \\ &\quad + X_9X_8X_7 + X_9X_1^2 - 9X_9X_6X_5 + 2X_9 + 3X_11X_4 - 6X_{13}X_8 - 6X_7X_3 + X_9X_1. \end{aligned}$$

- $d=4$:

$$\begin{aligned} O_4^{[2,0]} &= 2X_9^2X_{13}X_{14} + 2X_9^2X_{11}X_{12} - 3X_8^2X_{13}X_{13} - 2X_7^2X_8X_{11} - X_7X_9X_{10}X_{11} + 6X_7X_{11}X_{13}X_{14} \\ &\quad + 6X_5X_8X_9X_{11} + 9X_6X_{10}X_{13}^2 + X_6X_7^2X_9 + 6X_7X_{11}^2X_{12} - 4X_5X_6X_9^2 - 3X_5X_6X_7X_{11} \\ &\quad - 3X_4^2X_{11}^2 + X_3X_9X_{10}X_{13} + 6X_4X_6X_9X_{13} - 6X_4X_8X_{11}X_{13} - 9X_5X_{10}X_{11}^2 \\ &\quad + 2X_3X_7X_8X_{13} - X_2^2X_9^2 + 2X_3^2X_4X_{13} - X_3^2X_5X_9 - 2X_3X_4X_7X_{11} + 3X_3X_5X_6X_{13} \\ &\quad - 6X_3X_{11}X_{12}X_{13} - 6X_3X_{13}^2X_{14} + 9X_2X_3X_5X_{11} + X_2X_3X_7X_9 - 9X_2X_6X_7X_{13} \end{aligned}$$

$$- 6X_2X_8X_9X_{13} - 3X_2^2X_7X_{11} - 2X_1X_8X_9X_{13} - X_1^2X_3X_{13} + X_1^2X_7X_{11}$$
$$- 6X_1X_2X_3X_{13} + 6X_1X_3X_5X_{11} - 2X_1X_4X_9X_{11} - 6X_1X_6X_7X_{13} + (9X_2X_{10}X_{11}$$
$$- 6X_2^2X_3 + 5X_1X_3)X_{13} - 11X_1X_7X_{11} - 6X_{10}X_{11}X_{13} + 6X_2X_3X_{13} - 4X_1X_9^2$$
$$+ (12X_3X_5 - 15X_2X_7)X_{11} + 2X_3X_7X_9 - 2X_4X_9X_{11} - 3X_6X_7X_{13} - 5X_2X_9^2$$
$$- 5X_7X_{11} + \tfrac{4}{3}X_9^2.$$

$$O_4^{[0,2]} = X_7^2X_9^2 + 4X_7^3X_{11} - 18X_5X_7X_9X_{11} - 18X_1X_5X_{11}X_{13} - 2X_1X_7X_9X_{13} - 12X_3X_4X_{13}^2$$
$$- 4X_5X_9^3 + X_3\left(12X_5X_9 - 4X_7^2\right)X_{13} + 12X_4X_7X_{11}X_{13} + 4X_4X_9^2X_{13} - 3X_1^2X_{13}^2$$
$$- 27X_5^2X_{11}^2 - 6X_1X_{13}^2 - 81X_5X_{11}X_{13} - 3X_7X_9X_{13}.$$

These polynomials, jointly with the Casimir operator $O_6^{[0,0]} = C_6$ of G_2, are algebraically independent and can thus be considered as an integrity basis $\mathcal{I}$ of the system (17) associated to the nilpotent Lie algebra $\mathfrak{n}$. Starting from this set $\mathcal{I}$, we analyze the commutators in order to obtain a complete set of generators for the commutant. To this extent, we relabel C_2 as Q_3 and the cubic and quartic polynomials as

$$Q_4 = O_3^{[3,0]}, \quad Q_5 = O_3^{[1,0]}, \quad Q_6 = O_4^{[2,0]}, \quad Q_7 = O_4^{[0,2]}.$$

It is immediate to verify that $Q_1, \dots Q_5$ form an Abelian algebra. Adding the sixth-order Casimir operator of G_2, we obtain the maximal number of operators in $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ that commute with each other (see equation (19)). Considering now the commutator of Q_4 and Q_7, we observe that it decomposes as

$$[Q_4, Q_7] = -54Q_1O_5^{[3,1]}, \tag{41}$$

where

$$O_5^{[3,1]} = 2X_8X_9^2X_{13}^2 + 6X_7X_8X_{11}X_{13}^2 + X_7^2X_9^2X_{11} + 4X_7^3X_{11}^2 + 3X_6X_7X_9X_{13}^2 - 4X_5X_9^3X_{11}$$
$$- 18X_5X_7X_9X_{11}^2 + 27X_5X_6X_{11}X_{13}^2 - 27X_5^2X_{11}^3 + 2X_4X_9^2X_{11}X_{13} + 6X_4X_7X_{11}^2X_{13}$$
$$- 6X_3X_8X_{13}^3 - X_3X_7X_9^2X_{13} - 8X_3X_7^2X_{11}X_{13} + 15X_3X_5X_9X_{11}X_{13} - 6X_3X_4X_{11}X_{13}^2$$
$$+ 4X_3^2X_7X_{13}^2 + X_2(2X_9^3 + 9X_7X_9X_{11})X_{13} + 27X_2X_5X_{11}^2X_{13} - 6X_2X_3X_9X_{13}^2$$
$$- X_1X_7X_9X_{11}X_{13} + (9X_1X_6X_{13}^2 - 9X_1X_5X_{11}^2)X_{13} + \tfrac{2}{3}X_9^3X_{13} + (X_1X_3X_9X_{13}$$
$$+ 9X_1X_2X_{11}X_{13} - 7X_3X_9X_{13} + 24X_{11}X_{13} - 72X_5X_{11}^2)X_{13}.$$

This fifth-order polynomial is linearly independent of $\mathcal{I}$, hence must be added to the set. Two further fifth-order operators are obtained by successive commutators:[4]

$$[Q_5, Q_7] = 27Q_1O_5^{[1,1]}, \quad \left[Q_5, O_5^{[1,1]}\right] = Q_1Q_2Q_3^2 - Q_1Q_5^2 - 6Q_2O_5^{[0,1]}, \tag{42}$$

The operators in $\mathcal{J}_1 = \mathcal{I} \cup \left\{O_5^{[0,1]}, O_5^{[1,1]}, O_5^{[3,1]}\right\}$ are linearly independent, and exhaust the polynomials of order $d \leq 5$ having this property. We label them as $Q_8 = O_5^{[0,1]}$, $Q_9 = O_5^{[1,1]}$, $Q_{10} = O_5^{[3,1]}$. The commutator $[Q_5, Q_6] = 27O_6^{[3,0]}$ provides a linearly independent operator of order six that we label as Q_{11}. The commutator $[Q_6, Q_7] = 12Q_1O_6^{[2,1]}$ gives rise to an operator $Q_{12} = O_6^{[2,1]}$ that is also independent on the previous eleven polynomials. A routine computation shows that Q_{11}, Q_{12} and $Q_{13} = C_6$ are the only sixth-order polynomials linearly independent of the elements

[4] For these as well as the following operators, we omit their explicit expression due to their length.

in $\mathcal{J}_1$. An operator of seventh-order not expressible in terms of the previous elements is obtained from the commutator

$$[Q_5, Q_8] = 6\, O_7^{[1,1]}. \tag{43}$$

We label this polynomial as Q_{14}. Two additional linearly independent operators are obtained from the commutators of the latter as

$$[Q_8, Q_{14}] = -\frac{4}{9} Q_3^2 Q_5 Q_7 - \frac{1}{6} Q_1^2 Q_5 Q_{13} - \frac{4}{3} Q_5 O_8^{[0,2]} - \frac{2}{9} Q_3 O_9^{[1,2]} \tag{44}$$

We denote them as $Q_{15} = O_8^{[0,2]}$ and $Q_{16} = O_9^{[1,2]}$ respectively. Finally, the commutator $[Q_8, Q_{15}] = 4O_{12}^{[0,3]}$ provides a polynomial of order twelve that is linearly independent of $\{Q_1, \dots Q_{16}\}$, and that we denote as Q_{17}. Analyzing higher orders does not lead to new polynomials that are linearly independent of the former. Any other commutator among these operators can hence be expressed in terms of $\{Q_1, \dots Q_{17}\}$. Excluding the commutators above, the remaining non-vanishing commutators are given by [5]

$$
\begin{aligned}
&[Q_4, Q_8] = -27Q_1Q_{11}, \qquad [Q_4, Q_9] = 3Q_1(2Q_2Q_6 - Q_4Q_5 - 6Q_2^2Q_3), \qquad [Q_4, Q_{10}] = Q_1(Q_4^2 - 4Q_2^3),\\
&[Q_4, Q_{12}] = 3Q_1(Q_4Q_6 - 2Q_2^2Q_5 - 3Q_2Q_3Q_4), \qquad [Q_4, Q_{14}] = Q_1(Q_2Q_5^2 - 3Q_2^2Q_3^2 + Q_3Q_4Q_5 + 4Q_2Q_3Q_6),\\
&[Q_4, Q_{15}] = 3Q_1(Q_6Q_9 - 3Q_2Q_3Q_9 + Q_5Q_{17}), \qquad [Q_4, Q_{16}] = 18Q_1(Q_6Q_{12} - Q_2Q_3Q_9) - 243Q_1^2Q_5Q_{11}\\
&\qquad - 27Q_1(Q_3Q_4Q_9 - 3Q_3Q_5Q_{10}),\\
&[Q_4, Q_{17}] = \frac{9}{2}Q_1^3(2Q_3^2Q_5Q_6 - Q_3^4Q_4 + Q_2Q_5(Q_{13} - 9Q_3^3) - Q_3Q_5^3) - \frac{3}{4}Q_1(Q_5^3Q_7 + 3Q_4Q_8^2 - \frac{1}{2}Q_4Q_7Q_{13})\\
&\qquad + \frac{27}{4}Q_1(Q_5Q_9^2 - Q_2Q_3^2Q_5 + Q_3Q_9Q_{12} + \frac{3}{2}Q_4Q_8^2) + \frac{27}{2}Q_1^2(Q_3^2Q_4Q_8 - Q_5Q_6Q_8 + \frac{9}{2}Q_2Q_3Q_5Q_8),\\
&[Q_5, Q_{10}] = \frac{1}{3}Q_1(Q_4Q_5 - 2Q_2Q_6 + 6Q_2^2Q_3), \quad [Q_5, Q_{11}] = \frac{2}{9}(Q_6^2 - Q_3Q_4Q_5 + 2Q_2^2Q_3^2) - \frac{2}{9}Q_2(Q_5^2 - 4Q_3Q_6),\\
&[Q_5, Q_{12}] = 2Q_1(Q_3^2Q_4 + 3Q_2Q_3Q_5 - Q_5Q_6) - 3Q_4Q_8,\\
&[Q_5, Q_{14}] = \frac{1}{3}Q_1Q_2(8Q_3^3 - Q_{13}) + \frac{4}{3}Q_1Q_3(Q_5^2 - Q_3Q_6) + 2Q_6Q_8 - 4Q_2Q_3Q_8, \quad [Q_5, Q_{15}] = (2Q_1Q_3^2 - 3Q_8)Q_9,\\
&[Q_5, Q_{16}] = 12Q_1Q_3(Q_3Q_{12} - Q_5Q_9) + 72Q_1Q_5Q_{14} - 18Q_8Q_{12},\\
&[Q_5, Q_{17}] = -\frac{9}{2}Q_1^3Q_3^4Q_5 + \frac{27}{2}Q_1^2Q_3^2Q_5Q_8 - Q_1(Q_3^2Q_{16} + \frac{9}{2}Q_3Q_9Q_{15} - \frac{1}{8}Q_5Q_7Q_{13} + \frac{81}{8}Q_5Q_8^2 + 3Q_3^3Q_5Q_7)\\
&\qquad + \frac{9}{2}Q_3Q_5Q_7Q_8 + \frac{3}{2}Q_8Q_{16}, \qquad [Q_6, Q_8] = -2Q_5Q_9 - 2Q_3Q_{12},\\
&[Q_6, Q_9] = 2Q_1(Q_3^2Q_4 + 9Q_2Q_3Q_5 - 2Q_5Q_6) - 3Q_4Q_8, \qquad [Q_6, Q_{10}] = \frac{2}{3}Q_1(Q_4Q_6 - 2Q_2^2Q_5 - 3Q_2Q_3Q_4),\\
&[Q_6, Q_{11}] = \frac{4}{9}Q_2(Q_5Q_6 - 4Q_2Q_3Q_5 - \frac{2}{3}Q_3^2Q_4) + \frac{2}{9}Q_4(Q_3Q_6 - Q_5^2), \quad [Q_6, Q_{12}] = 2Q_1Q_6^2 - 6Q_2^2Q_8 + 2Q_1\times\\
&\qquad (2Q_2^2Q_3^2 - 3Q_2Q_5^2 - 3Q_2Q_3Q_6), \qquad [Q_6, Q_{15}] = 2Q_1(Q_3^2Q_{12} + 3Q_3Q_5Q_9 + 9Q_5Q_{14}) - 3Q_8Q_{12},\\
&[Q_6, Q_{14}] = \frac{5}{3}Q_1(Q_2Q_3^2Q_5 + Q_5^3 - Q_3Q_5Q_6 - \frac{2}{5}Q_3^3Q_4 + \frac{1}{10}Q_4Q_{13}) + Q_3Q_4Q_8 + 2Q_2Q_5Q_8,\\
&[Q_6, Q_{16}] = 6Q_1((2Q_2Q_3^2 - Q_5^2 + 2Q_3Q_6)Q_9 - 9Q_3^3Q_{10} - 81Q_8Q_{11}) + 9Q_1(10Q_3Q_5Q_{12} - Q_{10}Q_{13}) + 324Q_1^2\times\\
&\qquad Q_3^2Q_{11} + 81Q_3Q_8Q_{10} - 18Q_2Q_8Q_9, \qquad [Q_6, Q_{17}] = 3Q_1^3(Q_3^4Q_6 - 3Q_2Q_3^5 + 6Q_3^3Q_5^2 - \frac{1}{2}Q_5^2Q_{13})\\
&\qquad + 9Q_1^2(Q_3^2Q_6 + 3Q_2Q_3^3Q_8 - 3Q_3Q_5^2Q_8) - \frac{3}{4}Q_5^2Q_7Q_8 + \frac{9}{4}Q_8Q_9^2 + \frac{27}{4}Q_1(3Q_2Q_3Q_8^2 - Q_6Q_8^2)\\
&\qquad + 5Q_1Q_3^2Q_5^2Q_7 + 3Q_1(Q_3^2Q_9^2 - 3Q_5^2Q_{15} - 9Q_3Q_9Q_{14}) + \frac{1}{4}Q_1(Q_6Q_7Q_{13} - 3Q_2Q_3Q_7Q_{13}),\\
&[Q_7, Q_9] = 2Q_1Q_5Q_7, \quad [Q_7, Q_{10}] = 2Q_1(8Q_1^2Q_2Q_5 - Q_4Q_7), \quad [Q_7, Q_{11}] = -2Q_!Q_4Q_8 + \frac{4}{3}Q_1^2\times\\
&\qquad (Q_3^2Q_4 + 4Q_2Q_3Q_5 - Q_5Q_6), \quad [Q_7, Q_{12}] = 12Q_1^2Q_2(2Q_1Q_3^2 - 3Q_8) + 4Q_1(3Q_2Q_3 - Q_6)Q_7,
\end{aligned}
$$

[5] Due to simplicity, we give the commutators and relations in its symmetric form.

$$[Q_7,Q_{14}]=-\frac{2}{3}Q_1(Q_{16}+4Q_3Q_5Q_7)-6Q_1^3Q_3^2Q_5,\quad [Q_7,Q_{16}]=3Q_1(8Q_3Q_7+18Q_1^2Q_3^2)Q_9+18Q_1Q_7Q_{14}$$
$$-81Q_1^2Q_8,\quad [Q_8,Q_9]=9Q_1Q_5Q_8-\frac{8}{3}Q_3Q_5Q_7-6Q_1^2Q_3^2Q_5-\frac{2}{3}Q_{16},$$
$$[Q_8,Q_9]=\frac{2}{3}Q_1^2(Q_3^2Q_4-Q_5Q_6+4Q_2Q_3Q_5)-Q_1Q_4Q_8,\quad [Q_8,Q_{11}]=\frac{8}{9}Q_1(Q_2Q_3^2+\frac{1}{3}Q_5^2)Q_5+\frac{1}{27}Q_1\times$$
$$Q_4Q_{13}-\frac{4}{9}Q_1Q_3Q_5Q_6,\quad [Q_8,Q_{12}]=4Q_1^2Q_2Q_3^3-6Q_1Q_2Q_3Q_8-Q_9^2+(2Q_2Q_3^2+\frac{1}{3}Q_5^2-\frac{2}{3}Q_2Q_6)Q_7,$$
$$[Q_8,Q_{16}]=18Q_1^2Q_3^2(Q_3Q_9+3Q_{14})+4Q_3^2Q_7Q_9-27Q_1(Q_3Q_8Q_9-3Q_8Q_{14})+30Q_3Q_7Q_{14}-\frac{3}{2}Q_1^2Q_9Q_{13},$$
$$[Q_8,Q_{17}]=2Q_1^4Q_3^6-9Q_1^3Q_3^4Q_8+Q_1^2(Q_3^5Q_7+\frac{27}{2}Q_3^2Q_8^2+\frac{1}{6}Q_3^2Q_7-12Q_3^3Q_{15}+Q_{13}Q_{15})-\frac{27}{4}Q_1Q_8^3$$
$$-3Q_1Q_3(Q_3^2Q_7-6Q_8Q_{15})+\frac{1}{12}Q_3Q_7(Q_7Q_{13}+27Q_8^2-36Q_3Q_{15})-\frac{1}{4}Q_1Q_4Q_7Q_{13}+6Q_{15}^2,$$
$$[Q_9,Q_{10}]=-Q_1(Q_5Q_{10}+Q_4Q_9),\quad [Q_9,Q_{11}]=3Q_1Q_5Q_{11}+\frac{1}{3}Q_1(Q_4Q_9+5Q_5Q_{10})+\frac{2}{9}(Q_2Q_5Q_9-Q_6Q_{12}),$$
$$[Q_9,Q_{12}]=3Q_1((3Q_2Q_3-Q_6)Q_9-2Q_3^2Q_{10})+9Q_8Q_{10},\quad [Q_9,Q_{14}]=\frac{4}{3}Q_1Q_3(Q_5Q_9+Q_3Q_{12})+Q_1Q_5Q_{14}$$
$$-2Q_8Q_{12},\quad [Q_9,Q_{15}]=Q_5Q_7Q_8-\frac{2}{3}Q_1Q_3^2Q_5Q_7,$$
$$[Q_9,Q_{16}]=9Q_1^3(4Q_2Q_3^4-Q_3^2Q_5^2)+Q_1(108Q_2Q_3^2+\frac{27}{2}Q_5^2)Q_8+81Q_1(Q_2Q_8^2+Q_9Q_{14})-6Q_6Q_7Q_8$$
$$+Q_1(4Q_3^2Q_6Q_7+3Q_2Q-7Q_{13})$$
$$[Q_9,Q_{17}]=\frac{9}{2}Q_1^2Q_3^2(Q_1Q_3^2-3Q_8)Q_9+Q_1Q_3(Q_3^2Q_7Q_9+3Q_3Q_7Q_{14}-\frac{9}{2}Q_9Q_{15})-\frac{3}{2}Q_3Q_7Q_8Q_9-\frac{9}{2}Q_7Q_8Q_{14}$$
$$+\frac{1}{8}Q_1(Q_7Q_9Q_{13}+81Q_8^2Q_9),\quad [Q_{10},Q_{11}]=-Q_1Q_4Q_{11},\quad [Q_{10},Q_{12}]=Q_1(Q_6-3Q_2Q_3)Q_{10}$$
$$+2Q_1Q_2^2Q_9,\quad [Q_{10},Q_{14}]=\frac{3}{2}Q_1^2Q_5Q_{11}-\frac{2}{9}Q_1(Q_6Q_{12}+2Q_2Q_5Q_9-3Q_3Q_5Q_{10}),$$
$$[Q_{10},Q_{15}]=-\frac{4}{3}Q_1^3Q_2Q_3^2Q_5+2Q_1^2Q_2Q_5Q_8+Q_1(Q_9Q_{12}-Q_2Q_3Q_5Q_7+\frac{1}{3}Q_5Q_6Q_7),$$
$$[Q_{10},Q_{16}]=Q_1^3(7Q_5^2Q_6-4Q_3^2Q_4Q_5+3Q_2^2Q_{13}+3Q_2Q_3(4Q_3Q_6+6Q_5^2-12Q_2Q_3^2))+6Q_1^2(Q_4Q_5-3Q_2Q_6)Q_8$$
$$+54Q_1^2Q_2^2Q_3Q_8+Q_1Q_2Q_3(8Q_6-9Q_2Q_3)Q_7-\frac{1}{3}Q_1(Q_2Q_5^2+5Q_6^2)Q_7+Q_1(6Q_2Q_9^2-5Q_3Q_4Q_5Q_7),$$
$$[Q_{10},Q_{17}]=\frac{3}{2}Q_1^3Q_3(Q_5^2+5Q_2Q_3^2-\frac{2}{3}Q_3Q_6)Q_9+2Q_1^2Q_3^2(Q_5Q_{12}-\frac{9}{4}Q_3^2Q_{10})-\frac{1}{2}Q_1^3Q_2Q_9Q_{13}-\frac{3}{4}Q_1Q_9^3$$
$$+\frac{3}{2}Q_1^2(Q_6-\frac{15}{2}Q_2Q_3)Q_8Q_9-3Q_1^2(Q_5Q_8Q_{12}-\frac{9}{2}Q_3^2Q_8Q_{10})+\frac{3}{4}Q_1Q_3(Q_3Q_5Q_7Q_12-Q_6Q_7Q_9)$$
$$+\frac{3}{4}Q_1(Q_5^2+3Q_2Q_3^2)Q_7Q_9-\frac{3}{8}Q_1(Q_7Q-13+27Q_8Q_{10}^2)$$
$$[Q_{11},Q_{12}]=3Q_1(Q_6-3Q_2Q_3)Q_{11}+\frac{2}{9}Q_2(Q_6-3Q_2Q_3)Q_9-\frac{1}{9}Q_4Q_5Q_9+\frac{1}{3}(Q_5^2+6Q_2Q_3^2-2Q_3Q_6)Q_{10},$$
$$[Q_{11},Q_{14}]=\frac{1}{9}Q_3^2(Q_5Q_{10}-Q_4Q_9)+\frac{2}{27}(Q_3Q_6-Q_5^2)Q_{12}+\frac{2}{27}(Q_5Q_6-4Q_2Q_3Q-5)Q_9,$$
$$[Q_{11},Q_{15}]=\frac{2}{9}Q_1^2(Q_3^4Q_4+13Q_2Q_3^3Q_5+6Q_3Q_5^3-4Q_3^2Q_5Q_6)-\frac{1}{3}Q_1^2Q_2Q_5Q_{13}-\frac{13}{3}Q_1Q_2Q_3Q_5Q_8-\frac{1}{3}Q_5Q_9^2$$
$$+\frac{2}{3}Q_1(2Q_5Q_6-Q_3^2Q_4)Q_8-\frac{1}{9}Q_3(Q_5Q_6Q_7+3Q_9Q_{12})+\frac{1}{54}(Q_4Q_{13}+2Q_5^3)Q_7+\frac{1}{2}Q_4Q_8^2$$
$$+\frac{1}{3}Q_2Q_3^2Q_5Q_7,$$
$$[Q_{11},Q_{16}]=\frac{1}{3}Q_1^2(68Q_2^2Q_3^4+8Q_3^3Q_4+26Q_2Q_3^2Q_5^2+2(Q_2Q_6-Q_4Q_5)Q_{13}+10Q_5^4-2Q_3(Q_5^2+24Q_2Q_3^2)Q_6)$$
$$+\frac{8}{3}Q_1^2Q_3^2Q_6^2-2Q_1^2Q_2^2Q_3Q_{13}-4Q_1(Q_6^2+Q_2Q_5^2+Q_3Q_4Q_5-6Q_2Q_3Q_6-8Q_2^2Q_3^2)Q_8+Q_6Q_9^2$$
$$+3Q_2^2(2Q_3^3Q_7-Q_8^2-\frac{1}{27}Q_7Q_{13})-\frac{1}{9}(Q_5^2Q_6-10Q_3Q_6^2+4Q_2Q_3Q_5^2-6Q_3^2Q_4Q_5+48Q_2Q_3^2Q_6)Q_7$$
$$-2Q-2Q_3Q_9^2,$$
$$[Q_{11},Q_{17}]=\frac{1}{2}Q_1^2(Q_3^3Q_6-\frac{4}{3}Q_3^2Q_5^2-\frac{22}{3}Q_2Q_3^4+\frac{1}{2}Q_2Q_3Q_{13}-\frac{1}{6}Q_6Q_{13})Q_9+\frac{27}{2}Q_1^2Q_3^2(Q_8Q_{11}-\frac{7}{81}Q_3Q_5Q_{12})$$
$$+Q_1Q_3^3(Q_1Q_3^2-3Q_8)Q_{10}+\frac{1}{6}(Q_3^2Q_6-\frac{4}{3}Q_3Q_5^2-3Q_2Q_3^3)Q_7Q_9-\frac{1}{6}Q_3^2Q_5Q_7Q_{12}-\frac{9}{2}Q_1^3Q_3^4Q_{11}$$
$$+\frac{1}{4}Q_1(7Q_2Q_3^2-3Q-3Q_6+8Q_5^2))Q_8Q_9+\frac{3}{2}Q_2Q_8^2Q_9+\frac{9}{4}Q_3Q_8^2Q_{10}+\frac{7}{4}Q_1Q_3Q_5Q_8Q_{12}-\frac{81}{8}Q_1\times$$
$$Q_8^2Q_{11}+\frac{1}{12}(Q_3Q_7+3Q_1^2Q_3^2)Q_{10}Q_{13}+\frac{1}{18}Q_2Q_7Q_9Q_{13}-\frac{3}{8}Q_1Q_8Q_{10}Q_{13}-Q_1^2Q_3Q_5^2Q_{14}-\frac{1}{6}\times$$
$$(Q_5^2Q_7-9Q_9^2Q_{14}),$$

$$
\begin{aligned}
[Q_{12},Q_{14}] =&\frac{9}{2}Q_1^2Q_3^2Q_{11}-\frac{1}{6}Q_1(Q_2Q_3^2+8Q_5^2-3Q_3Q_6)Q_9-2Q_2Q_8Q_9+(3Q_3Q_8-2Q_1Q_3^3)Q_{10}\\
&-\frac{27}{4}Q_1Q_8Q_{11}-\frac{1}{2}Q_1Q_3Q_5Q_{12}+\frac{1}{4}Q_1Q_{10}Q_{13},\\
[Q_{12},Q_{15}] =&-2Q_1^3Q_2Q_3^4+6Q_1^2Q_2Q_3^2Q_8+2Q_1(Q_2Q_3^3+\frac{3}{4}Q_3Q_5^2-\frac{1}{3}Q_3^2Q_6)Q_7+Q_6(Q_6-3Q_2Q_3)Q_7Q_8\\
&-\frac{1}{2}Q_1Q_2(Q_7Q_{13}+9Q_8^2)-\frac{3}{2}Q_1Q-3Q-9^2-18Q_1Q_9Q_{14},\\
[Q_{12},Q_{16}] =&3Q_1^3(7Q_3^4Q_4+25Q_2Q_3^3Q_5-4Q_3^2Q_5Q_6-3Q_2Q_5Q_{13})+9Q_1^2(Q_5Q_6-7Q_3^2Q_4-\frac{45}{2}Q_2Q_3Q_5)Q_8\\
&+Q_1(6Q_3^3Q_4+\frac{89}{2}Q_2Q_3^2Q_5-\frac{9}{2}Q_5^3-\frac{27}{2}Q_3Q_5Q_6)Q_7-(9Q_3Q_4+6Q_2Q_5)Q_7Q_8-\frac{9}{2}Q_1Q_5Q_9^2\\
&+\frac{189}{4}Q_1Q_4Q_8^2-\frac{27}{2}Q_1Q_3Q_9Q_{12}+\frac{3}{4}Q_1Q_4Q_7Q_{13},\\
[Q_{12},Q_{17}] =&\frac{3}{2}Q_1^3Q_5Q_9Q_{13}-Q_1Q_3^2(Q_5Q_7+9Q_1^2Q_3Q_5)Q_9+9Q_1(Q_5Q_9+Q_3Q_{12})Q_{15}+\frac{27}{2}Q_1^2Q_3Q_5Q_8Q_9\\
&-\frac{1}{4}Q_1(Q_7Q_{12}Q_{13}+27Q_8^2Q_{12})-3Q_1^2Q_3^2(Q_1Q_3^2-3Q_8)Q_{12}+\frac{3}{2}Q_5Q_7Q_8Q_9,\\
[Q_{14},Q_{15}] =&\frac{2}{3}(Q_8-\frac{2}{3}Q_1Q_3^2)Q_{16}-Q_1^3Q_3^4Q_5+3Q_1^2Q_3^2Q_5Q_8-\frac{10}{9}Q_1Q_3^2Q_5Q_7+\frac{5}{3}Q_3Q_5Q_7Q_8-\frac{9}{4}Q_1\times\\
&Q_5Q_8^2+\frac{1}{12}Q_1Q_5(Q_7Q_{13}-36Q_3Q_{15}),\\
[Q_{14},Q_{16}] =&\ 2Q_1^3(Q_5^2-2Q_2Q_3^2)Q_{13}-12Q_1^3Q_3^3(Q_5^2-3Q_2Q_3^2)-13Q_1^3Q_3^4Q_6+6Q_1^2Q_2(Q_{13}-18Q_3^3)Q_8\\
&+18(Q_1^2Q_3Q_5^2-Q_2Q_3^2Q_7)Q_8-\frac{1}{12}Q_1(Q_6Q_7Q_{13}+7Q_3^2Q_5^2Q_7)-\frac{16}{3}Q_1Q_3^2Q_6Q_7+6Q_8Q_9^2\\
&+39Q_1^2Q_3^2Q_6Q_8-\frac{117}{4}Q_1Q_6Q_8^2+17Q_1Q_5^2Q_{15}+2(Q_5^2+4Q_3Q_6)Q_7Q_8+12Q_1Q_2Q_3^4Q_7\\
&-Q_1Q_2Q_3(Q_7Q_{13}-81Q_8^2)-\frac{7}{4}Q_1Q_3^2Q_9^2,\\
[Q_{14},Q_{17}] =&\frac{1}{8}Q_1^2(2Q_1Q_3^2Q_9-3Q_8Q_9)Q_{13}+(\frac{9}{2}Q_1^2Q_3^2Q_8-\frac{3}{2}Q_1^3Q_3^4)(Q_3Q_9+3Q_{14})+\frac{9}{2}Q_1Q_3Q_{14}Q_{15}\\
&+\frac{1}{2}(Q_1Q_3^2-6Q_8)Q_9Q_{15}+\frac{3}{2}(Q_3Q_7-\frac{27}{4}Q_1Q_8)Q_8Q_{14}-\frac{1}{8}Q_1(Q_7Q_{13}+8Q_3^3Q_7)Q_{14}\\
&+\frac{1}{24}Q_1Q_3Q_7Q_9Q_{13}-\frac{1}{3}Q_1Q_3^4Q_7Q_9+\frac{1}{2}Q_3^2Q_7Q_8Q_9-\frac{27}{8}Q_1Q_3Q_8^2Q_9,\\
[Q_{15},Q_{16}] =&18Q_1^2Q_3^2(Q_1Q_3^2-3Q_8)Q_9+(2Q_1Q_3^3Q_7-3Q_3Q_7Q_8+\frac{81}{2}Q_1Q_8^2)Q_9-12Q_1Q_3^2Q_7Q_{14}\\
&+\frac{3}{2}Q_1Q_7Q_9Q_{13}+18Q_7Q_8Q_{14}+162Q_1Q_{14}Q_{15},\\
[Q_{15},Q_{17}] =&\ Q_1^3Q_3^4(9Q_{15}-Q_3^2Q_7)+\frac{27}{8}(Q_8-2Q_1Q_3^2)Q_7Q_8^2+\frac{1}{4}Q_1(Q_7Q_{13}+81Q_8^2-36Q_3Q-15)Q_{15}\\
&+\frac{9}{2}Q_1^2Q_3^4Q_7Q_8+\frac{1}{8}(Q_8-\frac{2}{3}Q_1Q_3^2)Q_7^2Q_{13}+3Q_1Q_3^2(Q_3Q_7-9Q_1Q_8)Q_{15}-\frac{3}{2}Q_3Q_7Q_8Q_{15},\\
[Q_{16},Q_{17}] =&27Q_1^5Q_3^6Q_5-\frac{243}{2}Q_1^4Q_3^4Q_5Q_8+\frac{729}{4}Q_1^2(Q_1Q_3^2Q_5-\frac{1}{2}Q_5Q_8)Q_8^2+\frac{243}{2}Q_1^2Q_3Q_5Q_8Q_{15}\\
&+9Q_1Q_3^4(Q_5Q_7+4Q_1^2Q_3Q_5)Q_7+81Q_1Q_5(Q_{15}^2+Q_3Q_7Q_8^2)+Q_1^3Q_3^3(\frac{9}{2}Q_3Q_{16}-81Q_5Q_{15})\\
&-108Q_1^2Q_3^3Q_5Q_7Q_8+\frac{9}{8}Q_1^2Q_5(Q_7Q_8-\frac{2}{3}Q_1Q_3^2Q_7)Q_{13}+\frac{27}{2}Q_5(Q_1^3Q_{13}Q_{15}-Q_3^2Q_7^2Q_8)\\
&+9Q_5Q_7(Q_8-\frac{2}{3}Q_1Q_3^2)Q_{15}+\frac{9}{2}Q_3(Q_1Q_{15}-Q_7Q_8)Q_{16}-\frac{1}{8}Q_1(Q_7Q_{13}-81Q_8^2)Q_{16}\\
&+3Q_1Q_3^2(Q_3Q_7-\frac{9}{2}Q_1Q_8)Q_{16}.
\end{aligned}
$$

As only eight among these seventeen polynomials are functionally independent, a certain number of algebraic dependence relations are expected. In contrast to the previous examples, the computation of a complete set of relations for G_2 is rather long and cumbersome, ultimately leading to 57 constraints.

$$
\begin{aligned}
&Q_4Q_9+3Q_5Q_{10}-2Q_2Q_{12}=0, \qquad 6Q_2Q_{14}-Q_2Q_3Q_9+Q_6Q_9+Q_5Q_{12}=0,\\
&Q_5^2Q_7+3Q_9^2-12Q_2Q_15=0, \qquad 3Q_1^2Q_5^3+Q_5Q_6Q_7-3Q_4Q_{15}+Q_2Q_{16}=0,\\
&6Q_1^2(Q_5^3-Q_2Q_3^2Q_5)-3Q_2Q_3Q_5Q_7+Q_5Q_6Q_7+9Q_1Q_2Q_5Q_8+3Q_9Q_{12}-6Q_4Q_{15}=0,\\
&6Q_1^2Q_3^2Q_5Q_9+2Q_3Q_5Q_7Q_9-9Q_1Q_5Q_8Q_9+6Q_5Q_7Q_{14}+6Q_{12}Q_{15}+12Q_2Q_{17}=0,\\
&6Q_1^2Q_3^2Q_5Q_9+4Q_3Q_5Q_7Q_9-9Q_1Q_5Q_8Q_9-6Q_5Q_7Q_{14}-12Q_{12}Q_{15}+2Q_9Q_{16}=0,
\end{aligned}
$$

$2Q_2^2Q_9 - 9Q_2Q_3Q_{10} + 3Q_6Q_{10} + 27Q_1Q_2Q_{11} - Q_4Q_{12} = 0,$

$2Q_2Q_5Q_9 - Q_3Q_4Q_9 - 9Q_3Q_5Q_{10} + 27Q_1Q_5Q_{11} + 2Q_6Q_{12} + 6Q_4Q_{14} = 0,$

$3Q_{12}^2 - 6Q_1^2Q_3^2Q_4Q_5 - 3Q_1^2Q_5^2Q_6 - 3Q_3Q_4Q_5Q_7 - Q_2Q_5^2Q_7 + 9Q_1Q_4Q_5Q_8 - Q_4Q_{16} = 0,$

$36Q_1^2Q_2^2Q_3^3 - 12Q_1^2Q_2Q_3^2Q_6 + (9Q_2^2Q_3^2 - Q_2Q_5^2 - 6Q_2Q_3Q_6 + Q_6^2)Q_7 + 18Q_1Q_2(Q_6 - 3Q_2Q_3)Q_8$
$- 3Q_2Q_9^2 + 3Q_{12}^2 - 3Q_1^2Q_2^2Q_{13} = 0,$

$6Q_1^2Q_3^2Q_4Q_5 - 12Q_1^2Q_2^2Q_3^3 + 12Q_1^2Q_2Q_3Q_5^2 + (12Q_1^2Q_2Q_3^2 - 6Q_1^2Q_5^2)Q_6 - 6Q_2^2Q_3^2Q_7 - 2Q_6^2Q_7 + 2Q_2Q_5^2Q_7$
$+ 2(Q_3Q_4Q_5 + 4Q_2Q_3Q_6)Q_7 + 18Q_1(Q_2^2Q_3 - \frac{1}{2}Q_4Q_5 - Q_2Q_6)Q_8 + 54Q_{10}Q_{14} = 0,$

$12Q_1^2Q_2^2Q_3^2 - 12Q_1^2Q_2Q_5^2 + 6Q_2^2Q_3Q_7 + Q_4Q_5Q_7 - 2Q_2Q_6Q_7 - 18Q_1Q_2^2Q_8 - 9Q_9Q_{10} = 0,$

$12Q_1Q_2^2Q_3^3 + 2Q_1Q_3^2Q_4Q_5 + 12Q_1Q_2Q_3Q_5^2 - 4Q_1Q_2Q_3^2Q_6 - 2Q_1Q_5^2Q_6 - 18Q_2^2Q_3Q_8 - 3Q_4Q_5Q_8$
$+ 6Q_2Q_6Q_8 - 2Q_1Q_2^2Q_{13} + 27Q_9Q_{11} = 0,$

$3(Q_3Q_5Q_7 - 3Q_1Q_5Q_8)Q_{12} + 3Q_1^2(2Q_3^2Q_5Q_{12} - Q_3Q_5^2Q_9 + 6Q_5^2Q_{14}) - Q_5^2Q_7Q_9 + Q_{12}Q_{16} + 6Q_4Q_{17} = 0,$

$36Q_1^2Q_2Q_3^3Q_9 - 12Q_1^2Q_3^2Q_6Q_9 + 9Q_2Q_3^2Q_7Q_9 - Q_5^2Q_7Q_9 - 5Q_3Q_6Q_7Q_9 - 54Q_1Q_2Q_3Q_8Q_9 - 6Q_6Q_7Q_{14}$
$+ 18Q_1Q_6Q_8Q_9 - 3Q_9^3 + 6Q_1^2Q_3^2Q_5Q_{12} + 3Q_3Q_5Q_7Q_{12} - 9Q_1Q_5Q_8Q_{12} - 3Q_1^2Q_2Q_9Q_{13} + 2Q_{12}Q_{16} = 0,$

$6Q_1Q_3^2Q_{10} - 2Q_1Q_6Q_9 - 9Q_8Q_{10} + 9Q_7Q_{11} - 4Q_1Q_5Q_{12} = 0,$

$6Q_1^2Q_2Q_3^2Q_9 - 6Q_1^2Q_5^2Q_9 + 3Q_2Q_3Q_7Q_9 - Q_6Q_7Q_9 - 9Q_1Q_2Q_8Q_9 + Q_5Q_7Q_{12} - 18Q_{10}Q_{15} = 0,$

$4Q_5^2Q_9 + 4Q_3Q_6Q_9 + 54Q_1Q_3^2Q_{11} - 81Q_8Q_{11} + 12Q_3Q_5Q_{12} - 3Q_{10}Q_{13} + 12Q_6Q_{14} = 0,$

$6Q_1Q_2Q_3^3Q_9 + 4Q_1Q_3Q_5^2Q_9 - 2Q_1Q_3^2Q_6Q_9 - 9Q_2Q_3Q_8Q_9 + 3Q_6Q_8Q_9 + 2Q_1Q_3^2Q_5Q_{12} - 3Q_5Q_8Q_{12}$
$- Q_1Q_2Q_9Q_{13} + 12Q_1Q_5^2Q_{14} + 54Q_{11}Q_{15} = 0,$

$12Q_1^2Q_2Q_3^4 - 18Q_1^2Q_3^2Q_5^2 - 9Q_3Q_5^2Q_7 - 36Q_1Q_2Q_3^2Q_8 + 27Q_1Q_5^2Q_8 + 27Q_2Q_8^2 - 3Q_3Q_9^2 + Q_2Q_7Q_{13}$
$+ 18Q_9Q_{14} - 2Q_5Q_{16} = 0,$

$12Q_1^2Q_2Q_3^4 - 12Q_1^2Q_3^2Q_5^2 - 6Q_3Q_5^2Q_7 - 36Q_1Q_2Q_3^2Q_8 + 18Q_1Q_5^2Q_8 + 27Q_2Q_8^2 - 6Q_3Q_9^2 + Q_2Q_7Q_{13}$
$+ 36Q_9Q_{14} + 12Q_6Q_{15} = 0,$

$12Q_1^2Q_3^4Q_9 - 36Q_1Q_3^2Q_8Q_9 + 27Q_8^2Q_9 + Q_7Q_9Q_{13} - 12Q_3Q_9Q_{15} + 72Q_{14}Q_{15} + 24Q_5Q_{17} = 0,$

$36Q_1Q_3^2Q_4Q_8 - 12Q_1^2Q_3^4Q_4 - 144Q_1^2Q_2Q_3^3Q_5 + 48Q_1^2Q_3^2Q_5Q_6 - 36Q_2Q_3^2Q_5Q_7 + 24Q_3Q_5Q_6Q_7 + 12Q_5Q_9^2$
$+ 216Q_1Q_2Q_3Q_5Q_8 - 72Q_1Q_5Q_6Q_8 - 27Q_4Q_8^2 + 12Q_1^2Q_2Q_5Q_{13} - Q_4Q_7Q_{13} + 4Q_6Q_{16} = 0,$

$12Q_1^2Q_3^4Q_4 + 72Q_1^2Q_2Q_3^3Q_5 - 12Q_1^2Q_3^2Q_5Q_6 + 2(Q_5^3 + 9Q_2Q_3^2Q_5 - 3Q_3Q_5Q_6)Q_7 - 36Q_1Q_3^2Q_4Q_8 - 6Q_5Q_9^2$
$- 108Q_1Q_2Q_3Q_5Q_8 + 18Q_1Q_5Q_6Q_8 + 27Q_4Q_8^2 - 6Q_3Q_9Q_{12} - 6Q_1^2Q_2Q_5Q_{13} + Q_4Q_7Q_{13} + 36Q_{12}Q_{14} = 0,$

$12Q_3Q_5Q_7Q_{14} - (36Q_1^2Q_3^3Q_5 - 8Q_3^2Q_5Q_7 + 54Q_1Q_3Q_5Q_8)Q_9 + 3Q_1^2Q_5Q_9Q_{13} + 36Q_1^2Q_3^2Q_5Q_{14} + 12Q_{14}Q_{16}$
$- 54Q_1Q_5Q_8Q_{14} + 12Q_5Q_9Q_{15} - 12Q_3Q_{12}Q_{15} - 24Q_6Q_{17} = 0,$

$24Q_1^2Q_3^3Q_5Q_9 + 4Q_3^2Q_5Q_7Q_9 - 36Q_1Q_3Q_5Q_8Q_9 + 12Q_1^2Q_3^4Q_{12} - 36Q_1Q_3^2Q_8Q_{12} + (Q_7Q_{12} - 3Q_1^2Q_5Q_9)Q_{13}$
$+ 27Q_8^2Q_{12} + (108Q_1^2Q_3^2Q_5 + 48Q_3Q_5Q_7 - 162Q_1Q_5Q_8)Q_{14} - 12Q_5Q_9Q_{15} - 12Q_3Q_{12}Q_{15} + 12Q_{14}Q_{16} = 0,$

$108Q_3Q_5Q_7Q_{15} - 12Q_1^2Q_3^4Q_5Q_7 + 36Q_1Q_3^2Q_5Q_7Q_8 - 27Q_5Q_7Q_8^2 - Q_5Q_7^2Q_{13} + 216Q_1^2Q_3^2Q_5Q_{15} + 24Q_{15}Q_{16}$
$- 324Q_1Q_5Q_8Q_{15} + 72Q_9Q_{17} = 0,$

$Q_4^2Q_7 - 12Q_1^2Q_2Q_4Q_5 - 12Q_1^2Q_2^2Q_6 - 4Q_2^3Q_7 + 27Q_{10}^2 = 0,$

$8Q_1Q_2^3Q_3^2 - 2Q_1Q_3^2Q_4^2 - 12Q_1Q_2Q_3Q_4Q_5 - 8Q_1Q_2^2Q_5^2 - 12Q_1Q_2^2Q_3Q_6 + 2Q_1Q_4Q_5Q_6 + 4Q_1Q_2Q_6^2 - 12Q_2^3Q_8$
$+ 3Q_4^2Q_8 + 81Q_{10}Q_{11} = 0,$

$Q_4Q_6Q_7 - 6Q_1^2Q_2Q_3^2Q_4 - 6Q_1^2Q_2Q_5Q_6 - 3Q_2Q_3Q_4Q_7 - 2Q_2^2Q_5Q_7 + 9Q_1Q_2Q_4Q_8 + 9Q_{10}Q_{12} = 0,$

$9Q_3Q_5Q_7Q_{10} - 6Q_1^2Q_3^2Q_4Q_9 - 6Q_1^2Q_5Q_6Q_9 - 3Q_3Q_4Q_7Q_9 - 2Q_2Q_5Q_7Q_9 + 9Q_1Q_4Q_8Q_9 + 18Q_1^2Q_3^2Q_5Q_{10}$
$- 27Q_1Q_5Q_8Q_{10} + 2Q_6Q_7Q_{12} + 6Q_{10}Q_{16} = 0,$

$12Q_1^2(Q_3^2Q_5Q_6 - Q_3^4Q_4 - 6Q_2Q_3^3Q_5)Q_7 + -18Q_2Q_3^2Q_5Q_7^2 - 2Q_5^3Q_7^2 + 6Q_3Q_5Q_6Q_7^2 + 36Q_1Q_3^2Q_4Q_7Q_8$
$+ 108Q_1Q_2Q_3Q_5Q_7Q_8 - 18Q_1Q_5Q_6Q_7Q_8 - 27Q_4Q_7Q_8^2 + 36Q_1^2Q_3Q_5Q_9^2 + 18Q_5Q_7Q_9^2 + 36Q_1^2Q_3^2Q_9Q_{12}$
$+ 18Q_3Q_7Q_9Q_{12} - 54Q_1Q_8Q_9Q_{12} + 6Q_1^2Q_2Q_5Q_7Q_{13} - Q_4Q_7^2Q_{13} - 216Q_1^2Q_5Q_9Q_{14} + 216Q_{10}Q_{17} = 0,$

$12Q_3Q_4Q_5Q_6 - 36Q_2Q_3^2Q_4Q_5 - 48Q_2^2Q_3Q_5^2 - 4Q_4Q_5^3 - (36Q_2^2Q_3^2 + 12Q_2Q_5^2)Q_6 + 24Q_2Q_3Q_6^2 - 4Q_6^3 + 4Q_2^3Q_{13}$
$- Q_4^2Q_{13} + 729Q_{11}^2 = 0,$

$6Q_1Q_2Q_3^3Q_4 + 4Q_1Q_2^2Q_3^2Q_5 - 4Q_1Q_2Q_5^3 - 2Q_1(Q_3^2Q_4 + 3Q_2Q_3Q_5)Q_6 + 2Q_1Q_5Q_6^2 - 9Q_2Q_3Q_4Q_8 - 6Q_2^2Q_5Q_8$
$+ 3Q_4Q_6Q_8 - Q_1Q_2Q_4Q_{13} + 27Q_{11}Q_{12} = 0,$

$12Q_1Q_2^2Q_3^4 - 4Q_1Q_3^3Q_4Q_5 + 8Q_1Q_2Q_3^2Q_5^2 + 4Q_1Q_5^4 - 16Q_1Q_2Q_3^3Q_6 - 8Q_1Q_3Q_5^2Q_6 + 4Q_1Q_3^2Q_6^2 - 18Q_2^2Q_3^2Q_8$
$+ 6(Q_3Q_4Q_5 + Q_2Q_5^2 + 4Q_2Q_3Q_6 - 6Q_6^2)Q_8 - 2Q_1Q_2^2Q_3Q_{13} + Q_1(Q_4Q_5 + 2Q_2Q_6)Q_{13} + 162Q_{11}Q_{14} = 0,$

$6Q_1Q_3^3Q_4Q_9 + 4Q_1Q_2Q_3^2Q_5Q_9 + 4Q_1Q_5^3Q_9 + 12Q_1Q_3Q_5Q_6Q_9 - 9Q_3Q_4Q_8Q_9 - 6Q_2Q_5Q_8Q_9 + 6Q_6Q_8Q_{12}$
$- 18Q_1Q_3^3Q_5Q_{10} + 27Q_3Q_5Q_8Q_{10} + 162Q_1^2Q_3^2Q_5Q_{11} - 243Q_1Q_5Q_8Q_{11} + 36Q_1Q_3Q_5^2Q_{12} - 4Q_1Q_3^2Q_6Q_{12}$
$- Q_1Q_4Q_9Q_{13} - 6Q_1Q_5Q_{10}Q_{13} + 18Q_{11}Q_{16} = 0,$

$Q_6Q_7Q_{13} - 36Q_1^2Q_3^3Q_5^2 + 12Q_1^2Q_3^4Q_6 - 9Q_3^2Q_5^2Q_7 + Q_1(54Q_3Q_5^2 - 36Q_3^2Q_6)Q_8 + 27Q_6Q_8^2 - 3Q_3^2Q_9^2 + 3Q_1^2Q_5^2Q_{13}$
$+ 36Q_3Q_9Q_{14} + 12Q_5^2Q_{15} - 108Q_{14}^2 = 0,$

$24Q_1^3(Q_3^6Q_4 + 6Q_2Q_3^5Q_5 + 6Q_3^3Q_5^3 - 3Q_3^4Q_5Q_6) + Q_1(36Q_2Q_3^4Q_5 + 40Q_3^2Q_5^3 - 12Q_3^3Q_5Q_6)Q_7 - 48Q_1Q_5^3Q_{15}$
$- 108Q_1^2(Q_3^4Q_4 + 4Q_2Q_3^3Q_5 2Q_3Q_5^3 - 2Q_3^2Q_5Q_6)Q_8 - 6(Q_5^3 - 3Q_3Q_5Q_6 + 9Q_2Q_3^2Q_5)Q_7Q_8 + 162Q_1Q_3^2Q_4Q_8^2$
$+ (324Q_1Q_2Q_3Q_5 - 162Q_1Q_5Q_6)Q_8^2 - 81Q_4Q_8^3 + 54Q_5Q_8Q_9^2 - (36Q_1Q_3^3Q_9 + 54Q_3Q_8Q_9)Q_{12} - 12Q_1^3Q_2Q_3^2Q_5Q_{13}$
$- 12Q_1^3Q_5^3Q_{13} + 2Q_1Q_3^2Q_4Q_7Q_{13} - 4Q_1Q_5Q_6Q_7Q_{13} + 18Q_1^2Q_2Q_5Q_8Q_{13} - 3Q_4Q_7Q_8Q_{13} + 6Q_1Q_9Q_{12}Q_{13}$
$- 216Q_1Q_3Q_5Q_9Q_{14} + 648Q_{11}Q_{17} = 0,$

$144Q_1^2Q_2(Q_3^5Q_7 + 2Q_1^2Q_3^6) - 324Q_1^2Q_3^3Q_5^2(Q_7 + Q_1^2Q_3) + 324(Q_3Q_7Q_9 - 3Q_1Q_8Q_9)Q_{14} - 36Q_9^2Q_{15} - 9Q_1^2Q_9^2Q_{13}$
$+ 12Q_1^2Q_3^4Q_6Q_7 - 81Q_3^2Q_5^2Q_7^2 + 324Q_1^3(3Q_3^2Q_5^2 - 4Q_2Q_3^4)Q_8 + (486Q_1Q_3Q_5^2 - 432Q_1Q_2Q_3^3)Q_7Q_8 - 729Q_1^2Q_5^2Q_8^2$
$- 36Q_1Q_3^2Q_6Q_7Q_8 + 1944Q_1^2Q_2Q_3^2Q_8^2 + 27(Q_6 + 12Q_2Q_3)Q_7Q_8^2 - 972Q_1Q_2Q_8^3 - 27Q_3^2Q_7Q_9^2 + 24Q_1^2Q_2Q_3^2Q_7Q_{13}$
$+ 12Q_2Q_3Q_7^2Q_{13} + Q_6Q_7^2Q_{13} - 36Q_1Q_2Q_7Q_8Q_{13} + 648Q_(1^2)Q_(3^2)Q_9Q_{14} + 4Q_{16}^2 = 0,$

$12Q_1^2Q_3^3(3Q_5^2 - 12Q_3Q_6)Q_7 + 9Q_3^2Q_5^2Q_7^2 - 54Q_1Q_3Q_5^2Q_7Q_8 + 36Q_1Q_3^2Q_6Q_7Q_8 - 27Q_6Q_7Q_8^2 - 9(Q_3^2Q_7 + 8Q_1^2Q_3^3)Q_9^2$
$+ 108Q_1Q_3Q_8Q_9^2 - 3Q_1^2Q_5^2Q_7Q_{13} - Q_6Q_7^2Q_{13} + 9Q_1^2Q_9^2Q_{13} - 216Q_1^2Q_3^2Q_9Q_{14} - 108Q_3Q_7Q_9Q_{14} + 324Q_1Q_8Q_9Q_{14}$
$- 12Q_5^2Q_7Q_{15} + 36Q_9^2Q_{15} + 72Q_{12}Q_{17} = 0,$

$144Q_1Q_3^3Q_5Q_7Q_8 - 486Q_1^2Q_3^2Q_5Q_8^2 - 108Q_3Q_5Q_7Q_8^2 + 243Q_1Q_5Q_8^3 - 72Q_1^4Q_3^6Q_5 - 48Q_1^2Q_3^5Q_5Q_7 + 324Q_1^3Q_3^4Q_5Q_8$
$- 6Q_1^2Q_3^2Q_5Q_7Q_{13} - 4Q_3Q_5Q_7^2Q_{13} + 9Q_1Q_5Q_7Q_8Q_{13} - 216Q_1^2Q_3^3Q_5Q_{15} + 324Q_1Q_3Q_5Q_8Q_{15} + 36Q_1^2Q_5Q_{13}Q_{15}$
$+ 144Q_5Q_{15}^2 - 24Q_1^2Q_3^4Q_{16} + 72Q_1Q_3^2Q_8Q_{16} - 54Q_8^2Q_{16} - 2Q_7Q_{13}Q_{16} + 24Q_3Q_{15}Q_{16} + 432Q_{14}Q_{17} = 0,$

$9Q_1Q_7Q_8Q_9Q_{13} + 243Q_1Q_8^3Q_9 - 6Q_1^2Q_3^2Q_7Q_9Q_{13} - 4Q_3Q_7^2Q_9Q_{13} + 72Q_1^2Q_3^4Q_7Q_{14} - 216Q_1Q_3^2Q_7Q_8Q_{14}$
$- 72Q_1^4Q_3^6Q_9 - 48Q_1^2Q_3^5Q_7Q_9 + (324Q_1^3Q_3^4 + 144Q_1Q_3^3Q_7)Q_8Q_9 - 486Q_1^2Q_3^2Q_8^2Q_9 - 108Q_3Q_7Q_8^2Q_9 + 162Q_7Q_8^2Q_{14}$
$+ 6Q_7^2Q_{13}Q_{14} - 216Q_1^2Q_3^3Q_9Q_{15} + 324Q_1Q_3Q_8Q_9Q_{15} + 36Q_1^2Q_9Q_{13}Q_{15} - 1296Q_1^2Q_3^2Q_{14}Q_{15} - 648Q_3Q_7Q_{14}Q_{15}$
$+ 1944Q_1Q_8Q_{14}Q_{15} + 144Q_9Q_{15}^2 + 48Q_{16}Q_{17} = 0,$

$144Q_1^4Q_3^8Q_7 - 864Q_1^3Q_3^6Q_7Q_8 + 1944Q_1^2Q_3^4Q_7Q_8^2 - 1944Q_1Q_3^2Q_7Q_8^3 + 729Q_7Q_8^4 + 24Q_1^2Q_3^4Q_7^2Q_{13} - 72Q_1Q_3^2Q_7^2Q_8Q_{13}$
$+ 54Q_7^2Q_8^2Q_{13} + Q_7^3Q_{13}^2 - 1728Q_1^4Q_3^6Q_{15} - 864Q_1^2Q_3^5Q_7Q_{15} + 7776Q_1^3Q_3^4Q_8Q_{15} + 2592Q_1Q_3^3Q_7Q_8Q_{15} - 1728Q_{15}^3$
$- 11664Q_1^2Q_3^2Q_8^2Q_{15} - 1944Q_3Q_7Q_8^2Q_{15} + 5832Q_1Q_8^3Q_{15} - 72(Q_3Q_7^2 + 2Q_1^2Q_3^2Q_7 - 3216Q_1Q_7Q_8)Q_{13}Q_{15}$
$+ 5184Q_1^2Q_3^3Q_{15}^2 + 1296Q_3^2Q_7Q_{15}^2 - 7776Q_1Q_3Q_8Q_{15}^2 - 432Q_1^2Q_{13}Q_{15}^2 + 1728Q_{17}^2 = 0,$

$- 4Q_2^3Q_9 + Q_4^2Q_9 + 18Q_2^2Q_3Q_{10} + 3Q_4Q_5Q_{10} - 6Q_2Q_6Q_{10} - 54Q_1Q_2^2Q_{11} = 0,$

$- 6Q_3Q_4Q_5Q_9 - 4Q_2Q_5^2Q_9 - 6Q_2Q_3Q_6Q_9 + 2Q_6^2Q_9 - 18Q_3Q_5^2Q_{10} - 54Q_1Q_2Q_3^2Q_{11} + 81Q_2Q_8Q_{11}$
$+ 2Q_5Q_6Q_{12} + 3Q_2Q_{10}Q_{13} = 0,$

$4Q_2Q_5Q_6Q_9 + 108Q_2Q_3^2Q_5Q_{10} - 54Q_3Q_5Q_6Q_{10} - 54Q_1Q_3^2Q_4Q_{11} - 24Q_2^2Q_3Q_5Q_9 - 4Q_4Q_5^2Q_9 - 6Q_3Q_4Q_6Q_9$
$- 324Q_1Q_2Q_3Q_5Q_{11} + 54Q_1Q_5Q_6Q_{11} + 81Q_4Q_8Q_{11} + 4Q_6^2Q_{12} + 3Q_4Q_{10}Q_{13} = 0,$

$36Q_1Q_2^2Q_3^2Q_8 - 18Q_1Q_2Q_5^2Q_8 - 27Q_2^2Q_8^2 - 12Q_1^2Q_2^2Q_3^4 + 12Q_1^2Q_2Q_3^2Q_5^2 + 6Q_2Q_3Q_5^2Q_7 - Q_5^2Q_6Q_7 + 3Q_6Q_9^2$
$- Q_2^2Q_7Q_{13} + 6Q_5Q_9Q_{12} = 0,$

$18Q_2Q_3Q_5Q_6Q_7 - 2Q_5Q_6^2Q_7 + 36Q_1Q_2Q_3^2Q_4Q_8 - 12Q_1^2Q_2Q_3^4Q_4 - 144Q_1^2Q_2^2Q_3^3Q_5 + 36Q_1^2Q_2Q_3^2Q_5Q_6$
$- 36Q_2^2Q_3^2Q_5Q_7 + 216Q_1Q_2^2Q_3Q_5Q_8 - 54Q_1Q_2Q_5Q_6Q_8 - 27Q_2Q_4Q_8^2 + 12Q_2Q_5Q_9^2 + 12Q_1^2Q_2^2Q_5Q_{13}$
$- Q_2Q_4Q_7Q_{13} + 6Q_6Q_9Q_{12} = 0,$

$12Q_1^2(Q_3^4Q_4^2 - 4Q_2^3Q_3^4 + 12Q_2Q_3^3Q_4Q_5 + 8Q_2^2Q_3^2Q_5^2 - 4Q_2Q_5^4 + 12Q_2^2Q_3^3Q_6 - 2Q_3^2Q_4Q_5Q_6 - 4Q_2Q_3^2Q_6^2 + Q_5^2Q_6^2)$
$+ 36Q_2Q_3^2Q_4Q_5Q_7 + 48Q_2^2Q_3Q_5^2Q_7 + 4Q_4Q_5^3Q_7 + 36Q_2^2Q_3^2Q_6Q_7 - 12Q_2Q_5^2Q_6Q_7 - 24Q_2Q_3Q_6^2Q_7 + 4Q_6^3Q_7$
$+ 144Q_1Q_2^3Q_3^2Q_8 - 36Q_1Q_3^2Q_4^2Q_8 - 216Q_1Q_2Q_3Q_4Q_5Q_8 - 144Q_1Q_2^2Q_5^2Q_8 - 216Q_1Q_2^2Q_3Q_6Q_8 + 27Q_4^2Q_8^2$
$+ 36Q_1Q_4Q_5Q_6Q_8 + 72Q_1Q_2Q_6^2Q_8 - 108Q_2^3Q_8^2 - 12Q_1^2Q_2Q_4Q_5Q_{13} - 12Q_1^2Q_2^2Q_6Q_{13} - 4Q_2^3Q_7Q_{13} + Q_4^2Q_7Q_{13}$
$- 12Q_3Q_4Q_5Q_6Q_7 = 0,$

$12Q_1^2(Q_2^2Q_5^2Q_6 + 12Q_2^4Q_3^3 + 2Q_2^2Q_3^2Q_4Q_5 - Q_2Q_4Q_5^3 - 4Q_2^3Q_3^2Q_6) + 36Q_2^4Q_3^2Q_7 + 12Q_2^2Q_3Q_4Q_5Q_7 + Q_4^2Q_5^2Q_7$
$- 24Q_2^3Q_3Q_6Q_7 + 4Q_2^2Q_6^2Q_7 - 216Q_1Q_2^4Q_3Q_8 - 36Q_1Q_2^2Q_4Q_5Q_8 + 72Q_1Q_2^3Q_6Q_8 - 12Q_2^3Q_9^2 - 12Q_1^2Q_2^4Q_{13}$
$+ 3Q_4^2Q_9^2 - 4Q_2Q_4Q_5Q_6Q_7 = 0,$

$$36Q_2Q_3^2Q_4Q_5Q_9 + 48Q_2^2Q_3Q_5^2Q_9 + 4Q_4Q_5^3Q_9 + 36Q_2^2Q_3^2Q_6Q_9 - 12Q_2Q_5^2Q_6Q_9 - 24Q_2Q_3Q_6^2Q_9 + 324Q_1Q_2^2Q_3^3Q_{11}$$
$$+ 4Q_6^3Q_9 + 54Q_1Q_3^2Q_4Q_5Q_{11} + 324Q_1Q_2Q_3Q_5^2Q_{11} - 108Q_1Q_2Q_3^2Q_6Q_{11} - 54Q_1Q_5^2Q_6Q_{11} - 486Q_2^2Q_3Q_8Q_{11}$$
$$- 81Q_4Q_5Q_8Q_{11} + 162Q_2Q_6Q_8Q_{11} - 4Q_2^3Q_9Q_{13} + Q_4^2Q_9Q_{13} - 54Q_1Q_2^2Q_{11}Q_{13} - 12Q_3Q_4Q_5Q_6Q_9 = 0,$$
$$12Q_1^2Q_2Q_3^4Q_9 - 12Q_1^2Q_3^2Q_5^2Q_9 - 5Q_3Q_5^2Q_7Q_9 - 36Q_1Q_2Q_3^2Q_8Q_9 + 18Q_1Q_5^2Q_8Q_9 + 27Q_2Q_8^2Q_9 + Q_2Q_7Q_9Q_{13}$$
$$- 3Q_3Q_9^3 - 6Q_5^2Q_7Q_{14} + 18Q_9^2Q_{14} - 12Q_5Q_{12}Q_{15} = 0,$$
$$12Q_1^2(Q_2^3Q_3^4 - 2Q_2^2Q_3^2Q_5^2 + Q_2Q_5^4) - 12Q_2^2Q_3Q_5^2Q_7 - Q_4Q_5^3Q_7 + 3Q_2Q_5^2Q_6Q_7 - 36Q_1Q_2^3Q_3^2Q_8 + 36Q_1Q_2^2Q_5^2Q_8$$
$$+ 27Q_2^3Q_8^2 - 3Q_4Q_5Q_9^2 - 3Q_2Q_6Q_9^2 + Q_2^3Q_7Q_{13} = 0,$$
$$12Q_1^2(Q_2^2Q_3^4Q_4 + 12Q_2^3Q_3^3Q_5 - 2Q_2^2Q12Q_1^2Q_2Q_5^3Q_6) + (36Q_2^3Q_3^2Q_5 - 24Q_2^2Q_3Q_5Q_6 - Q_4Q_5^2Q_6 + 4Q_2Q_5Q_6^2)Q_7$$
$$+ 72Q_1(Q_2^2Q_5Q_6 - \frac{1}{2}Q_2^2Q_3^2Q_4 - 3Q_2^3Q_3Q_5)Q_8 + 27Q_2^2Q_4Q_8^2 - 12Q_2^2Q_5Q_9^2 - 12Q_1^2Q_2^3Q_5Q_{13} - 3Q_4Q_6Q_9^2$$
$$+ Q_2^2Q_4Q_7Q_{13} = 0,$$
$$12Q_1^2Q_2Q_3^4Q_4Q_5 + 144Q_1^2Q_2^2Q_3^3Q_5^2 - 12Q_1^2(Q_2^2Q_3^42Q_2Q_3^2Q_5^2)Q_6 + 36Q_2^2Q_3^2Q_5^2Q_7 - 12Q_2Q_3Q_5^2Q_6Q_7 - 27Q_2^2Q_6Q_8^2$$
$$+ Q_5^2Q_6^2Q_7 + 36Q_1(Q_2^2Q_3^2Q_6 - Q_2Q_3^2Q_4Q_5 + Q_2Q_5^2Q_6)Q_8 - 216Q_1Q_2^2Q_3Q_5^2Q_8 + 27Q_2Q_4Q_5Q_8^2 - 12Q_2Q_5^2Q_9^2$$
$$+ 3Q_6^2Q_9^2 - 12Q_1^2Q_2^2Q_5^2Q_{13} + Q_2Q_4Q_5Q_7Q_{13} - Q_2^2Q_6Q_7Q_{13} = 0,$$
$$48Q_1^2Q_2^2(Q_3^2Q_6^2 + 10Q_2^2Q_3^4) - 12Q_1^2Q_2Q_3^4Q_4^2 - 72Q_1^2Q_2^2Q_3^3Q_4Q_5 - 96Q_1^2Q_2^3Q_3^2Q_5^2 - 36Q_1^2Q_2Q_3Q_4Q_5^3 + 48Q_1^2Q_2^2Q_5^4$$
$$- 288Q_1^2Q_2^3Q_3^3Q_6 + 24Q_1^2Q_2Q_3^2Q_4Q_5Q_6 + 36Q_1^2Q_2^2Q_3Q_5^2Q_6 - 12Q_1^2Q_2Q_5^2Q_6^2 + 108Q_2^4Q_3^3Q_7 - 48Q_2^3Q_3Q_5^2Q_7$$
$$+ 3Q_3Q_4^2Q_5^2Q_7 - 4Q_2Q_4Q_5^3Q_7 - 108Q_2^3Q_3^2Q_6Q_7 + 12Q_2^2Q_5^2Q_6Q_7 + 36Q_2^2Q_3Q_6^2Q_7 - 4Q_2Q_6^3Q_7 - 792Q_1Q_2^4Q_3^2Q_8$$
$$+ 36Q_1(Q_2Q_3^2Q_4^2 + 3Q_2^2Q_3Q_4Q_5 + 4Q_2^3Q_5^2 + 12Q_2^3Q_3Q_6Q - Q_2Q_4Q_5Q_6 - 2Q_2^2Q_6^2)Q_8 + (108Q_2^4 - 27Q_2Q_4^2)Q_8^2$$
$$- 36Q_2^3Q_3Q_9^2 + 9Q_3Q_4^2Q_9^2 - (36Q_1^2Q_2^4Q_3 + 12Q_1^2Q_2^2Q_4Q_5 + 12Q_1^2Q_2^3Q_6 + 4Q_2^4Q_7 - Q_2Q_4^2Q_7)Q_{13} = 0.$$

Any element in the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ can thus be expressed as the product

$$P = \prod_{i=1}^{17} Q_i^{a_i}, \quad (a_1, \ldots, a_{17}) \in \mathcal{S}, \tag{45}$$

where the set $\mathcal{S}$ of constraints consists of the 53 conditions

$$a_{10} = 0,1, \quad a_{11} = 0,1, \quad a_{12} = 0,1, \quad a_{14} = 0,1, \quad a_{16} = 0,1, \quad a_{17} = 0,1,$$
$$a_2a_{12} = 0, \quad a_5a_{12} = 0, \quad a_2a_{15} = 0, \quad a_2a_{16} = 0, \quad a_2a_{17} = 0, \quad a_4a_{12} = 0, \quad a_4a_{14} = 0,$$
$$a_4a_{15} = 0, \quad a_4a_{16} = 0, \quad a_4a_{17} = 0, \quad a_5a_{16} = 0, \quad a_5a_{17} = 0, \quad a_6a_{14} = 0, \quad a_6a_{15} = 0,$$
$$a_6a_{16} = 0, \quad a_6a_{17} = 0, \quad a_7a_{11} = 0, \quad a_9a_{10} = 0, \quad a_9a_{11} = 0, \quad a_9a_{16} = 0, \quad a_9a_{17} = 0,$$
$$a_{10}a_{11} = 0, \quad a_{10}a_{12} = 0, \quad a_{10}a_{14} = 0, \quad a_{10}a_{15} = 0, \quad a_{10}a_{16} = 0, \quad a_{10}a_{17} = 0, \quad a_{11}a_{12} = 0,$$
$$a_{11}a_{14} = 0, \quad a_{11}a_{15} = 0, \quad a_{11}a_{16} = 0, \quad a_{11}a_{17} = 0, \quad a_{12}a_{14} = 0, \quad a_{12}a_{16} = 0, \quad a_{12}a_{17} = 0,$$
$$a_{14}a_{16} = 0, \quad a_{14}a_{17} = 0, \quad a_{16}a_{17} = 0, \quad a_2a_6a_{10} = 0, \quad a_2a_{10}a_{13} = 0, \quad a_4a_{10}a_{13} = 0,$$
$$a_5a_9a_{12} = 0, \; a_6a_9a_{12} = 0, \; a_5a_{12}a_{15} = 0, \; a_2a_4a_5a_6a_7 = 0, \; a_3a_4a_5a_6a_7 = 0, \; a_3a_4a_5a_6a_9 = 0,$$

as well as the requirement $(a_1, \ldots, a_{17}) \notin \mathbb{Z}[v_1, v_2, v_3, v_4]$, where

$$v_1 = \left(2,3,4,0^{14}\right), \; v_2 = \left(2,4,4,0^{14}\right), \; v_3 = \left(2,2,4,1,0^{13}\right), \; v_4 = \left(2,2,3,0,2,0^{12}\right). \tag{46}$$

For any admissible $(a_1, \ldots, a_{17}) \in \mathcal{S}$, the weight $\Lambda = [\lambda, \mu]$ of the corresponding irreducible representation of G_2 is given by

$$\begin{aligned}\Lambda = [&2a_2 + 3a_4 + a_5 + 2a_6 + a_9 + 3a_{10} + 3a_{11} + 2a_{12} + a_{14} + a_{16},\\ &a_1 + 2a_7 + a_8 + a_9 + a_{10} + a_{12} + a_{14} + 2a_{15} + 2a_{16} + 3a_{17}]\end{aligned} \tag{47}$$

with dimension

$$d_{a_1,\ldots a_{17}} = \frac{1}{120}(\lambda+1)(\mu+1)(\lambda+\mu+2)(\lambda+2\mu+3)(\lambda+3\mu+4)(2\lambda+3\mu+5) \tag{48}$$

A long but straightforward computation shows that the identity (13) is satisfied for any $p \geq 2$. For low values of p, the terms in the decomposition (9) are explicitly given by: [6]

$$
\begin{aligned}
p=2:\ \mathcal{U}^{(2)} =& [0,0]\oplus[1,0]\oplus[0,1],\\
p=3:\ \mathcal{U}^{(3)} =& [1,0]\oplus[0,1]\oplus[3,0]\oplus[0,3]\oplus[2,1],\\
p=4:\ \mathcal{U}^{(4)} =& [0,0]\oplus[2,0]^2\oplus[0,2]^2\oplus[1,1]\oplus[4,0]\oplus[0,4]\oplus[3,1]\oplus[2,2],\\
p=5:\ \mathcal{U}^{(5)} =& [1,0]\oplus[0,1]^2\oplus[1,1]\oplus[1,2]\oplus[2,1]^2\oplus[3,0]^2\oplus[0,3]^2\oplus[3,1]\oplus[3,2]\oplus[2,3]\\
& \oplus[4,1]\oplus[5,0]\oplus[0,5],\\
p=6:\ \mathcal{U}^{(6)} =& [0,0]^2\oplus[2,0]^3\oplus[0,2]^3\oplus[1,1]\oplus[3,0]^2\oplus[2,1]\oplus[1,2]\oplus[4,0]^3\oplus[0,4]^2\oplus[2,2]^2\\
& \oplus[3,1]^2\oplus[1,3]\oplus[3,2]\oplus[6,0]\oplus[5,1]\oplus[4,2]\oplus[3,3]\oplus[2,4]\oplus[0,6].
\end{aligned}
$$

The total number of terms in the decomposition for the values $1 \leq p \leq 16$ is

$$1, 3, 5, 10, 17, 31, 48, 80, 122, 187, 274, 404, 569, 805, 1106, 1512,$$

while the number of nonequivalent irreducible representations intervening is

$$1, 3, 5, 8, 13, 19, 25, 35, 44, 55, 66, 80, 93, 109, 124, 142.$$

An interesting property of the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ of G_2 that distinguishes it from previous cases is the existence of several non-Abelian polynomial subalgebras. As commented before, the polynomials $\{Q_1, \dots Q_5, Q_{13}\}$ generate a six-dimensional Abelian subalgebra $\mathcal{A}_0$. To comment only on some possibilities, we observe that if we extend $\mathcal{A}_0$ adjoining Q_6, the commutators show that $\mathcal{A}_1 = \{Q_1, \dots Q_6, Q_{11}, Q_{13}\}$ determines an eight-dimensional polynomial algebra. As these eight polynomials are functionally independent, they form an integrity basis for the system (17). The subalgebra $\mathcal{A}_1$ suffices to describe the decomposition (9) up to $p = 3$, and fails for $p = 4$ due to be absence of Q_7. If we adjoin the latter element, the resulting polynomial algebra is easily seen to be isomorphic to the whole commutant. However, adjoining Q_{10} to $\mathcal{A}_1$ leads to a nine-dimensional subalgebra $\mathcal{A}_2$ that is maximal, as the adjunction of any other element again generates the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$. On the other hand, the elements $\{Q_1, Q_2, Q_4, Q_{10}\}$ themselves generate a minimal non-Abelian polynomial algebra.

6. Conclusions

The problem of determining the decomposition of the universal enveloping algebra of a simple complex Lie algebra $\mathfrak{s}$ has been revisited combining both the purely algebraic and the analytical approach. It has been observed that determining the polynomials that commute with the elements associated to the simple roots of $\mathfrak{s}$ is formally equivalent to computing the subgroup scalars for the reduction chain $\mathfrak{n} \subset \mathfrak{s}$, where $\mathfrak{n}$ is the maximal nilpotent ideal of the Borel subalgebra of $\mathfrak{s}$. This fact allows us to deduce that the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ always possesses a maximal Abelian subalgebra, the dimension of which is completely determined by $\mathfrak{n}$. With the analyticalapproach, the commutant can be constructed successively

[6] Conventions on representations are adapted to those used in [21].

starting from an integrity basis for the system of PDEs associated to the reduction chain. In this context, the question whether any integrity basis forms a non-Abelian polynomial algebra has been shown to be false in general, although for any of the cases considered, a polynomial subalgebra $\mathcal{A}$ of the commutant $\mathcal{C}_{\mathcal{U}(\mathfrak{s})}(\mathfrak{n})$ and whose elements form an integrity basis has been shown to exist.

The case of rank two simple Lie algebra has been analyzed in detail. For the classical algebras A_2 and B_2, that have already been considered in the literature (see [2, 3, 10]), additional properties of the commutant have been described. The case of the exceptional Lie algebra G_2, which has been solved for the first time, is much more complicated. The first notable difference with the classical algebras is that number of linearly polynomials required for the decomposition (9) is more than twice the cardinal of an integrity basis.

Besides the computational difficulties due to the degree and length of the polynomials involved, for G_2 the number of algebraic dependence relations is unexpectedly high, namely 57, in contrast with the classical algebras, where the number of relations is low. It has been further shown that, among the three classical Lie algebras of rank two, for A_2 and B_2 the polynomial subalgebra $\mathcal{A}$ is actually maximal of codimension one, for D_2 it does not exist as the commutant is itself Abelian, while for G_2 the subalgebra is of codimension nine and not maximal, as it can be extended nontrivially.

In view of the analogies with the so-called internal label problem [26], we may ask whether the procedure can be adapted to other reduction chains $\mathfrak{s}' \subset \mathfrak{s}$ of semisimple Lie algebras in order to construct polynomial algebras. It shall however taken into account that polynomials P in the enveloping algebra of $\mathfrak{s}$ commuting with the generators of $\mathfrak{s}'$ are not suitable to describe the decomposition (9), as the eigenvalue of P with respect to the Cartan subalgebra of $\mathfrak{s}'$ is always zero. Nonetheless, the ansatz can be of use for the analysis of integrable systems [7,16] endowed with some internal symmetry, as well as for (quasi-)exactly solvable (QES) systems [6,12].

In [6], the case of $\mathfrak{gl}(3)$ was exploited to provide new algebraic (QES) systems, while in [7], $\mathfrak{su}(3)$ was used to provide ab algebraic setting for a superintegrable Hamiltonian on the sphere. This points out how the results of this paper may be as well of interest for different areas of mathematical physics, as well as it may also reveal to be of interest in the description of wider types of representations of simple Lie algebras [13].

As an illustrating example, let us consider the simple Lie algebra A_2 in the basis (1) and the regular subalgebra $\mathfrak{sl}(2,\mathbb{C})$ generated by H_1, E_{12}, E_{21}. In this case, the generators are no more related to highest weight vectors of A_2, but formally, the commutant of the subalgebra $\mathfrak{sl}(2,\mathbb{C})$ in $\mathcal{U}(\mathfrak{sl}(3,\mathbb{C}))$ has also a polynomial structure. An integrity basis for the system (17) associated to the subalgebra generators contains five elements, as can be easily verified. A short computation shows that there exist six linearly independent elements that span the commutant, given by the polynomials

$$
\begin{aligned}
P_1 &= H_1 + 2H_2,\\
P_2 &= E_{13}E_{31} + E_{23}E_{32} - H_2 - \frac{1}{2}H_1,\\
P_3 &= H_1^2 + 4E_{12}E_{21} - 2H_1,
\end{aligned}
$$

$$
\begin{aligned}
P_4 &= E_{12}E_{31}^2 - E_{21}E_{32}^2 - H_1E_{31}E_{32},\\
P_5 &= E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32} + H_1E_{13}E_{31} - H_2E_{23}E_{32} + E_{23}E_{32} - E_{12}E_{21}\\
&\quad - H_2^2 - H_1H_2 - \frac{1}{2}H_1^2 - H_2,\\
P_6 &= E_{12}E_{23}^2 - E_{13}E_{21}^2 + H_1E_{13}E_{23} - 2E_{13}E_{23}.
\end{aligned}
\tag{49}
$$

The six operators generate a polynomial algebra with nonvanishing commutators

$$
\begin{aligned}
&[P_1,P_4] = -6P_4,\ [P_1,P_6] = 6P_6,\ [P_2,P_3] = 3P_4 + P_1P_4,\ [P_2,P_6] = 3P_6 - P_1P_6,\\
&[P_4,P_5] = \left(-\tfrac{1}{2}P_1^2 - 5P_1 + 4P_2 - \tfrac{1}{2}P_3 - 12\right)P_4,\\
&[P_4,P_6] = P_1P_2P_3 - P_1(P_2^2 - P_2) + (2P_2 - P_3)P_5 - \frac{1}{4}P_1P_3,\\
&[P_5,P_6] = 4P_2P_6 - \frac{1}{2}(P_1^2 + P_3)P_6 + 5P_1P_6 - 12P_6.
\end{aligned}
\tag{50}
$$

The polynomials are further algebraically dependent through the relation

$$
\begin{aligned}
&P_1^2P_2^2 - P_2^2P_3 - 4P_1P_2P_5 + 2P_1P_2P_3 - 2P_1P_2^2 - P_1^2P_2 - \frac{1}{4}P_1^2P_3 - 4P_4P_6\\
&+4P_5^2 + 2\left(P_1 + 2P_2 - P_3\right)P_5 + 2P_1P_2 - P_2P_3 + \frac{1}{4}\left(P_3 - 2P_1\right)P_3 = 0,
\end{aligned}
\tag{51}
$$

showing that any element in the commutant can be written as

$$
P = P_1^{a_1}P_2^{a_2}P_3^{a_3}P_4^{a_4}P_5^{a_5}P_6^{a_6},\quad a_i \in \mathbb{N}\cup 0,\quad a_5 = 0,1. \tag{52}
$$

We observe that the weight of the polynomials with respect to the Cartan subalgebra of $\mathfrak{sl}(3,\mathbb{C})$ is (0,0) for P_1, P_2, P_3 and P_5, while P_4 has weight $(0,3)$ and P_6 has weight $(0,-3)$.

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