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Prime Ideals in Leibniz Algebras

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Abstract. The notions of prime and semi-prime ideals of Leibniz algebras are introduced and the interrelation of these notions with maximal ideals, irreducible ideals and solvable radical are investigated. We prove that a maximal ideal of a Leibniz algebra is prime if and only if its codimension is greater than one. Also, it is shown that if a Leibniz algebra $\mathfrak{g}$ satisfies the maximal condition on ideals, then the intersection of all prime ideals, the intersection of all semi-prime ideals, and the solvable radical of $\mathfrak{g}$ are all equal.

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1. Introduction

A Leibniz algebra is a vector space equipped with a binary operation which has the property of being a derivation of itself. This concept first appeared in papers published in the sixties by Blokh [1], and was popularized three decades later by Loday [6]. In particular, Lie algebras are Leibniz algebras. So, a lot of research on the category of Leibniz algebras is supported by analogous results in Lie algebras.

The notions of prime algebras and prime ideals play an important role on the theories of commutative algebras and associative algebras. These notions have been studied by several authors [5, 7, 9] in the case of Lie algebras. Our aim in this paper is to introduce the concept of prime ideals and investigate analogue results as in Lie algebras and other algebraic structures. The proposed definition generalizes prime ideals in Lie algebras. So, one naturally expects results on Lie algebras to hold in Leibniz algebras. However, many results fail to hold. For instance, it is known that a ring is prime if and only if the zero ideal is a prime ideal. This result also holds in Lie algebras.

However, as shown in this paper, the zero ideal (0) is not a prime ideal. Analogously, a Leibniz algebra $\mathfrak{g}$ is prime if and only if its *Leibniz kernel* $\mathsf{Leib}(\mathfrak{g}) := \langle [x,x] \mid x \in \mathfrak{g} \rangle$ is a prime ideal of $\mathfrak{g}$. As in the case of Lie algebras [5], we provide several characterizations of prime ideals, and study interrelations among prime, semi-prime, maximal and irreducible ideals in Leibniz algebras. We show that, under certain conditions, any semi-prime ideal is an intersection of a finite number of prime ideals (Corollary 3.20), and the solvable radical is equal to the intersection of all semi-prime ideals

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(Corollary 3.8), and also the intersection of all prime ideals (Theorem 3.19). Furthermore, it is known that the *Liezation* $\mathfrak{g}_{\mathsf{Lie}} = \mathfrak{g}/\mathsf{Leib}(\mathfrak{g})$ of a simple Leibniz algebra $\mathfrak{g}$ is a simple Lie algebra, but the converse is not true (see [3]). In this paper, we show that for a Leibniz algebra $\mathfrak{g}$ and an ideal $\mathfrak{I}$ of $\mathfrak{g}$, the quotient algebra $\mathfrak{g}/\mathfrak{I}$ is a prime Lie algebra if and only if $\mathfrak{I}$ is a prime ideal of $\mathfrak{g}$. In particular, a Leibniz algebra $\mathfrak{g}$ is prime if and only if $\mathfrak{g}_{\mathsf{Lie}}$ is a prime Lie algebra.

2. Preliminaries on Leibniz algebras

Throughout the paper, all vector spaces are considered over a fixed field $\mathbb{K}$. A *Leibniz algebra* [6] is a vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, usually called the *Leibniz bracket* of $\mathfrak{g}$, satisfying the *Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad x,y,z \in \mathfrak{g}.$$

A subalgebra $\mathfrak{h}$ of a Leibniz algebra $\mathfrak{g}$ is said to be a *left* (resp. *right*) *ideal* of $\mathfrak{g}$ if $[h,g] \in \mathfrak{h}$ (resp. $[g,h] \in \mathfrak{h}$), for all $h \in \mathfrak{h}$, $g \in \mathfrak{g}$. If $\mathfrak{h}$ is both left and right ideal, then $\mathfrak{h}$ is called an *ideal* of $\mathfrak{g}$. In this case $\mathfrak{g}/\mathfrak{h}$ naturally inherits a Leibniz algebra structure.

For a Leibniz algebra $\mathfrak{g}$, we denote by $\mathsf{Leib}(\mathfrak{g})$ the vector subspace of $\mathfrak{g}$ spanned by all elements of the form $[x,x]$, $x \in \mathfrak{g}$, which is called the Leibniz kernel of $\mathfrak{g}$. One can easily see that $\mathsf{Leib}(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$ which is also the smallest ideal such that the quotient $\mathfrak{g}_{\mathsf{Lie}} = \mathfrak{g}/\mathsf{Leib}(\mathfrak{g})$ is a Lie algebra.

Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be non-empty subsets of $\mathfrak{g}$. Then

$$[\mathfrak{s}_1,\mathfrak{s}_2] = \langle [x,y],[y,x] |\ x \in \mathfrak{s}_1, y \in \mathfrak{s}_2 \rangle$$

is called the commutator of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ ($[x,\mathfrak{s}_2] := [\{x\},\mathfrak{s}_2]$ for $x \in \mathfrak{g}$). In particular, $[\mathfrak{g},\mathfrak{g}]$ is said to be the derived algebra of $\mathfrak{g}$ which is clearly an ideal of $\mathfrak{g}$. Now, consider the sequence $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)},\mathfrak{g}^{(k-1)}]$ for $k \geq 1$. A Leibniz algebra $\mathfrak{g}$ is *solvable*, if there exists $n \geq 0$ such that $\mathfrak{g}^{(n)} = 0$.

As in ring theory, we say that a Leibniz algebra $\mathfrak{g}$ satisfies the *maximal* (resp. *minimal*) *condition* on ideals if any non-empty collection of ideals of $\mathfrak{g}$ has a maximal (resp. minimal) member under the set inclusion (see also [10]).

3. Main results

In [9], a Lie algebra L is said to be prime if $[A,B] \neq 0$ for any non-zero ideals A,B of L. Also L is called semiprime if $[A,A] \neq 0$ for any non-zero ideal A of L. Clearly, a semiprime Lie algebra does not contain any non-zero abelian ideal. Moreover in [5], an ideal P of a Lie algebra L is called prime if $[H,K] \subseteq P$ with H,K ideals of L implies either $H \subseteq P$ or $K \subseteq P$. It is easy to see that a Lie algebra is prime (resp. semi-prime) if and only if the zero ideal (0) is prime (resp. semi-prime).

Definition 3.1. (i) A Leibniz algebra $\mathfrak{g}$ is called *prime* if $[\mathfrak{h}_1,\mathfrak{h}_2] \subseteq \mathsf{Leib}(\mathfrak{g})$ for $\mathfrak{h}_1$, $\mathfrak{h}_2$ ideals of $\mathfrak{g}$, then $\mathfrak{h}_1 \subseteq \mathsf{Leib}(\mathfrak{g})$ or $\mathfrak{h}_2 \subseteq \mathsf{Leib}(\mathfrak{g})$.

(ii) A proper ideal $\mathfrak{p}$ of $\mathfrak{g}$ is said to be *prime* if $[\mathfrak{h}_1,\mathfrak{h}_2] \subseteq \mathfrak{p}$ for $\mathfrak{h}_1,\mathfrak{h}_2$ ideals of $\mathfrak{g}$, then $\mathfrak{h}_1 \subseteq \mathfrak{p}$ or $\mathfrak{h}_2 \subseteq \mathfrak{p}$. ■

Clearly, $\mathfrak{g}$ is a prime Leibniz algebra if and only if $\mathsf{Leib}(\mathfrak{g})$ is a prime ideal of $\mathfrak{g}$, and if and only if $\mathfrak{g}_{\mathsf{Lie}}$ is a prime Lie algebra.

For $x \in \mathfrak{g}$, denote by $\langle x^{\mathfrak{g}} \rangle$ the smallest ideal of $\mathfrak{g}$ containing x. Let $\mathfrak{s}_0$ be the vector subspace of $\mathfrak{g}$ spanned by x and $\mathfrak{s}_i = [\mathfrak{s}_{i-1}, \mathfrak{g}]$ for $i \geq 1$. Clearly $\mathfrak{s}_i \subseteq \langle x^{\mathfrak{g}} \rangle$ for all $i \geq 0$ and hence $\sum_{i=0}^{\infty} \mathfrak{s}_i \subseteq \langle x^{\mathfrak{g}} \rangle$. Also $\left[\sum_{i=0}^{\infty} \mathfrak{s}_i, \mathfrak{g}\right] \subseteq \sum_{i=1}^{\infty} \mathfrak{s}_i$ implies that $\sum_{i=0}^{\infty} \mathfrak{s}_i$ is an ideal of $\mathfrak{g}$ containing x. Thus

$$\langle x^{\mathfrak{g}} \rangle = \sum_{i=0}^{\infty} \mathfrak{s}_i. \tag{1}$$

The following result characterizes prime ideals in Leibniz algebras.

Proposition 3.2. *Let $\mathfrak{p}$ be an ideal of $\mathfrak{g}$. Then the following conditions are equivalent:*

(i) *$\mathfrak{p}$ is prime.*

(ii) *If $[x, \mathfrak{h}] \subseteq \mathfrak{p}$ for $x \in \mathfrak{g}$ and an ideal $\mathfrak{h}$ of $\mathfrak{g}$, then $x \in \mathfrak{p}$ or $\mathfrak{h} \subseteq \mathfrak{p}$.*

(iii) *If $[x, \langle y^{\mathfrak{g}} \rangle] \subseteq \mathfrak{p}$ for $x, y \in \mathfrak{g}$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.*

Proof. (i) $\Rightarrow$ (iii) Suppose that $[x, \langle y^{\mathfrak{g}} \rangle] \subseteq \mathfrak{p}$. As $\mathfrak{s}_i = [\mathfrak{s}_{i-1}, \mathfrak{g}]$, using induction on i and Leibniz identity, one may easily prove that $[\mathfrak{s}_i, \langle y^{\mathfrak{g}} \rangle] \subseteq \mathfrak{p}$ for all $i \geq 0$. Now (1) implies that $[\langle x^{\mathfrak{g}} \rangle, \langle y^{\mathfrak{g}} \rangle] \subseteq \mathfrak{p}$, and since $\mathfrak{p}$ is prime, either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

(iii) $\Rightarrow$ (ii) Let $x \notin \mathfrak{p}$ and y be an arbitrary element in $\mathfrak{h}$. Clearly $\langle y^{\mathfrak{g}} \rangle \subseteq \mathfrak{h}$ and so $[x, \langle y^{\mathfrak{g}} \rangle] \subseteq \mathfrak{p}$. Now *(iii)* implies that $y \in \mathfrak{p}$ and hence $\mathfrak{h} \subseteq \mathfrak{p}$.

(ii) $\Rightarrow$ (i) Assume that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are ideals of $\mathfrak{g}$ such that $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{p}$ and $\mathfrak{h}_1 \nsubseteq \mathfrak{p}$. Choose an element $x \in \mathfrak{h}_1 - \mathfrak{p}$. Since $[x, \mathfrak{h}_2] \subseteq \mathfrak{p}$, *(ii)* implies that $\mathfrak{h}_2 \subseteq \mathfrak{p}$ which completes the proof. ■

Note that the definition of prime ideals in Leibniz algebras does not imply that for a prime ideal $\mathfrak{p}$ and $x, y \in \mathfrak{g}$, if $[x, y] \in \mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ (see the prime ideal given in Example 3.15). This is true for semi-prime ideals, as well.

Definition 3.3. (i) A Leibniz algebra $\mathfrak{g}$ is called *semi-prime* if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathsf{Leib}(\mathfrak{g})$ for an ideal $\mathfrak{h}$ of $\mathfrak{g}$, then $\mathfrak{h} \subseteq \mathsf{Leib}(\mathfrak{g})$.

(ii) A proper ideal $\mathfrak{q}$ of $\mathfrak{g}$ is said to be *semi-prime* if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{q}$ for an ideal $\mathfrak{h}$ of $\mathfrak{g}$, then $\mathfrak{h} \subseteq \mathfrak{q}$. ■

Clearly, any prime ideal of a Leibniz algebra is semi-prime. Also, it is easy to see that a semi-prime Leibniz algebra $\mathfrak{g}$ does not contain any non-zero abelian ideal $\mathfrak{h}$ such that $\mathfrak{h} \nsubseteq \mathsf{Leib}(\mathfrak{g})$. Moreover, the Leibniz kernel of $\mathfrak{g}$ is contained in the intersection of all semi-prime ideals of $\mathfrak{g}$.

Lemma 3.4. *If $\mathfrak{g}$ is a Leibniz algebra and $\mathfrak{q}$ is a semi-prime ideal of $\mathfrak{g}$, then $\mathsf{Leib}(\mathfrak{g}) \subseteq \mathfrak{q}$.*

Proof. As $\mathsf{Leib}(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$, we have $[\mathsf{Leib}(\mathfrak{g}), \mathsf{Leib}(\mathfrak{g})] = 0 \subseteq \mathfrak{q}$ and hence $\mathsf{Leib}(\mathfrak{g}) \subseteq \mathfrak{q}$, since $\mathfrak{q}$ is semi-prime. ■

Corollary 3.5. *Let $\mathfrak{g}$ be a non-Lie Leibniz algebra. Then*

(i) *The zero ideal (0) is not semi-prime.*

(ii) *$\mathfrak{p}$ is a prime ideal of $\mathfrak{g}$ if and only if $\mathfrak{g}/\mathfrak{p}$ is a prime Lie algebra.*

(iii) *$\mathfrak{q}$ is a semi-prime ideal of $\mathfrak{g}$ if and only if $\mathfrak{g}/\mathfrak{q}$ is a semi-prime Lie algebra.*

Proof. (i) Since $\mathfrak{g}$ is non-Lie, $\mathsf{Leib}(\mathfrak{g}) \neq 0$. So by the above lemma the zero ideal (0) cannot be semi-prime.

(ii) If $\mathfrak{p}$ is a prime ideal of $\mathfrak{g}$, then Lemma 3.4 implies that $\mathsf{Leib}(\mathfrak{g}) \subseteq \mathfrak{p}$ and hence $\mathfrak{g}/\mathfrak{p}$ is a Lie algebra. Let $\mathfrak{h}_1/\mathfrak{p}$ and $\mathfrak{h}_2/\mathfrak{p}$ be ideals of $\mathfrak{g}/\mathfrak{p}$ such that $[\mathfrak{h}_1/\mathfrak{p}, \mathfrak{h}_2/\mathfrak{p}] = 0$. Then $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{p}$ and so either $\mathfrak{h}_1 \subseteq \mathfrak{p}$ or $\mathfrak{h}_2 \subseteq \mathfrak{p}$, since $\mathfrak{p}$ is prime. Therefore, the Lie algebra $\mathfrak{g}/\mathfrak{p}$ is prime. Conversely, let $\mathfrak{g}/\mathfrak{p}$ be a prime Lie algebra and $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{p}$ for ideals $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}$. Thus

$$[\frac{\mathfrak{h}_1 + \mathfrak{p}}{\mathfrak{p}}, \frac{\mathfrak{h}_2 + \mathfrak{p}}{\mathfrak{p}}] = \frac{[\mathfrak{h}_1, \mathfrak{h}_2] + \mathfrak{p}}{\mathfrak{p}} = 0 \quad \text{and so} \quad \frac{\mathfrak{h}_1 + \mathfrak{p}}{\mathfrak{p}} = 0 \text{ or } \frac{\mathfrak{h}_2 + \mathfrak{p}}{\mathfrak{p}} = 0,$$

since $\mathfrak{g}/\mathfrak{p}$ is prime. This implies that $\mathfrak{h}_1 \subseteq \mathfrak{p}$ or $\mathfrak{h}_2 \subseteq \mathfrak{p}$, which shows that $\mathfrak{p}$ is a prime ideal.

(iii) In the above part set $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathfrak{h}$. ■

The parts (ii) and (iii) of the above result are true for Lie algebras, as well.

Let $\mathfrak{g}$ be a Leibniz algebra and $\mathfrak{s}$ a non-empty subset of $\mathfrak{g}$. Then

$$C_{\mathfrak{g}}(\mathfrak{s}) = \{x \in \mathfrak{g} |\ [x, y] = [y, x] = 0,\ \forall y \in \mathfrak{s}\}$$

is called the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$. Clearly if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then so is $C_{\mathfrak{g}}(\mathfrak{h})$.

Corollary 3.6. *Let $\mathfrak{g}$ be a non-Lie Leibniz algebra and $\mathfrak{h}$ an ideal of $\mathfrak{g}$. Then*

(i) *If $\mathfrak{g}$ is a prime Leibniz algebra, then either $\mathfrak{h} \subseteq \mathsf{Leib}(\mathfrak{g})$ or $C_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathsf{Leib}(\mathfrak{g})$. In addition, if $\mathfrak{h}$ is prime, then either $\mathfrak{h} = \mathsf{Leib}(\mathfrak{g})$ or $C_{\mathfrak{g}}(\mathfrak{h}) = Z(\mathfrak{h})$.*

(ii) *If $\mathfrak{g}$ is a semi-prime Leibniz algebra, then $\mathfrak{h} \cap C_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathsf{Leib}(\mathfrak{g})$. In particular, $Z(\mathfrak{g}) \subseteq \mathsf{Leib}(\mathfrak{g})$.*

Proof. (i) If $\mathfrak{g}$ is a prime Leibniz, then since $[\mathfrak{h}, C_{\mathfrak{g}}(\mathfrak{h})] = 0 \subseteq \mathsf{Leib}(\mathfrak{g})$, we get $\mathfrak{h} \subseteq \mathsf{Leib}(\mathfrak{g})$ or $C_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathsf{Leib}(\mathfrak{g})$. In addition, if $\mathfrak{h}$ is a prime ideal of $\mathfrak{g}$, it follows from Lemma 3.4 that either $\mathfrak{h} = \mathsf{Leib}(\mathfrak{g})$ or $C_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathsf{Leib}(\mathfrak{g}) \subseteq \mathfrak{h}$, i.e. $C_{\mathfrak{g}}(\mathfrak{h}) = Z(\mathfrak{h})$.

(ii) If $\mathfrak{g}$ is a semi-prime Leibniz algebra, then

$$[\mathfrak{h} \cap C_{\mathfrak{g}}(\mathfrak{h}), \mathfrak{h} \cap C_{\mathfrak{g}}(\mathfrak{h})] \subseteq [\mathfrak{h}, C_{\mathfrak{g}}(\mathfrak{h})] = 0 \subseteq \mathsf{Leib}(\mathfrak{g}),$$

which implies that $\mathfrak{h} \cap C_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathsf{Leib}(\mathfrak{g})$. In particular, if we put $\mathfrak{h} = \mathfrak{g}$, then $Z(\mathfrak{g}) \subseteq \mathsf{Leib}(\mathfrak{g})$. ■

Lemma 3.7. *Let $\mathfrak{q}$ be a semi-prime ideal of $\mathfrak{g}$. Then $\mathfrak{q}$ contains all solvable ideals of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a solvable ideal of $\mathfrak{g}$. Then there is a positive integer k such that $[\mathfrak{h}^{(k-1)}, \mathfrak{h}^{(k-1)}] = \mathfrak{h}^{(k)} = 0 \in \mathfrak{q}$. Since $\mathfrak{q}$ is semi-prime, $[\mathfrak{h}^{(k-2)}, \mathfrak{h}^{(k-2)}] = \mathfrak{h}^{(k-1)} \subseteq \mathfrak{q}$. By iteration, we obtain $\mathfrak{h} \subseteq \mathfrak{q}$. ■

For an ideal $\mathfrak{h}$ of $\mathfrak{g}$, denote by $s_{\mathfrak{g}}(\mathfrak{h})$ the intersection of all semi-prime ideals of $\mathfrak{g}$ containing $\mathfrak{h}$, and let $Rad(\mathfrak{g})$ be the *solvable radical* (the sum of all solvable ideals) of $\mathfrak{g}$ (see [11]). Note that for an arbitrary Leibniz algebra, $Rad(\mathfrak{g})$ need not be solvable. If $\mathfrak{g}$ satisfies the maximal condition on ideals, then $Rad(\mathfrak{g})$ is clearly the unique maximal solvable ideal of $\mathfrak{g}$. For example, if $\mathfrak{g}$ is finite-dimensional, then it obviously satisfies the maximal (and also minimal) condition.

Corollary 3.8. *Let $\mathfrak{g}$ be a Leibniz algebra satisfying the maximal condition on ideals. Then*

(i) *$Rad(\mathfrak{g})$ is semi-prime.*

(ii) $\mathsf{Leib}(\mathfrak{g}) \subseteq s_{\mathfrak{g}}(0) = Rad(\mathfrak{g})$. *Moreover, the equality holds if $\mathfrak{g}$ is a semi-prime Leibniz algebra.*

Proof. (i) Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq Rad(\mathfrak{g})$.

Then $\mathfrak{h}^{(k+1)} \subseteq Rad(\mathfrak{g})^{(k)}$ for all positive integer k. Since $Rad(\mathfrak{g})$ is solvable, we have $\mathfrak{h}^{(k_0+1)} \subseteq Rad(\mathfrak{g})^{(k_0)} = 0$ for some positive integer k_0. So $\mathfrak{h}$ is also solvable and hence $\mathfrak{h} \subseteq Rad(\mathfrak{g})$.

(ii) First notice that by (i), $s_{\mathfrak{g}}(0) \subseteq Rad(\mathfrak{g})$, and by Lemma 3.4, $\mathsf{Leib}(g) \subseteq s_{\mathfrak{g}}(0)$. Moreover, by Lemma 3.7, $Rad(\mathfrak{g}) \subseteq s_{\mathfrak{g}}(0)$ since $Rad(\mathfrak{g})$ is solvable. Now, if $\mathfrak{g}$ is semi-prime, then $\mathsf{Leib}(\mathfrak{g})$ is semi-prime. Thus $s_{\mathfrak{g}}(0) \subseteq \mathsf{Leib}(\mathfrak{g})$. ∎

Lemma 3.9. *Let $\mathfrak{g}$ be a Leibniz algebra satisfying the maximal condition on ideals. Then an ideal $\mathfrak{q}$ of $\mathfrak{g}$ is semi-prime if and only if $Rad(\mathfrak{g}/\mathfrak{q}) = 0$.*

Proof. By Corollary 3.5, if $\mathfrak{q}$ is semi-prime, then $\mathfrak{g}/\mathfrak{q}$ is a semi-prime Lie algebra. Using Corollary 3.8 and the fact that $\mathfrak{g}/\mathfrak{q}$ is a Lie algebra, we get immediately $Rad(\mathfrak{g}/\mathfrak{q}) = \mathsf{Leib}(\mathfrak{g}/\mathfrak{q}) = 0$. Conversely, assume that $Rad(\mathfrak{g}/\mathfrak{q}) = 0$. Then the zero ideal of $\mathfrak{g}/\mathfrak{q}$ is semi-prime and $\mathsf{Leib}(\mathfrak{g}/\mathfrak{q}) \subseteq s_{\mathfrak{g}/\mathfrak{q}}(0) = 0$ by Corollary 3.8. This implies that $\mathfrak{g}/\mathfrak{q}$ is a semi-prime Lie algebra. It follows from Corollary 3.5 that $\mathfrak{q}$ is semi-prime. ∎

Definition 3.10. An ideal $\mathfrak{I}$ of $\mathfrak{g}$ is said to be *irreducible* if $\mathfrak{I} = \mathfrak{h}_1 \cap \mathfrak{h}_2$ for $\mathfrak{h}_1$, $\mathfrak{h}_2$ ideals of $\mathfrak{g}$, then $\mathfrak{I} = \mathfrak{h}_1$ or $\mathfrak{I} = \mathfrak{h}_2$.

Lemma 3.11. *Let $\mathfrak{g}$ be a Leibniz algebra and $\mathfrak{I}$ an ideal of $\mathfrak{g}$.*

(i) *If $\mathfrak{I}$ is prime, then $\mathfrak{I}$ is irreducible.*

(ii) *If $\mathfrak{I}$ is maximal, then $\mathfrak{I}$ is irreducible.*

Proof. (i) Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be ideals of $\mathfrak{I}$ such that $\mathfrak{I} = \mathfrak{h}_1 \cap \mathfrak{h}_2$. Note that $\mathfrak{I} \subseteq \mathfrak{h}_1$ and $\mathfrak{I} \subseteq \mathfrak{h}_2$. If $\mathfrak{I}$ is prime, then $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1 \cap \mathfrak{h}_2 = \mathfrak{I}$ implies that $\mathfrak{I} = \mathfrak{h}_1$ or $\mathfrak{I} = \mathfrak{h}_2$.

(ii) If $\mathfrak{I}$ is maximal and $\mathfrak{I} \neq \mathfrak{h}_1$, then $\mathfrak{h}_1 = \mathfrak{g}$ and hence $\mathfrak{I} = \mathfrak{h}_1 \cap \mathfrak{h}_2 = \mathfrak{h}_2$. ∎

The following example shows that the converses of the statements of Lemma 3.11 are not true.

Example 3.12. Let $\mathfrak{g}$ be the 4-dimensional non-Lie Leibniz algebra spanned by $\{e_1, e_2, e_3, x\}$ with non-zero products given by $[e_1, e_1] = e_3$, $[e_2, x] = e_2, [x, e_2] = -e_2$ and $[x, x] = -2e_3$ (see [2, Proposition 3.10 ($\mathcal{L}_{16}$)]). Clearly, the ideal $\mathfrak{I} = \langle e_2 \rangle$ is irreducible. However $\mathfrak{I}$ is not prime since $[\langle e_3 \rangle, \langle e_1, e_3 \rangle] = 0 \subseteq \mathfrak{I}$, but $e_1, e_3 \notin \mathfrak{I}$. Moreover, $\mathfrak{I}$ is not maximal since $\mathfrak{I} \subsetneq \langle e_2, e_3, x \rangle$. ∎

Theorem 3.13. *Let $\mathfrak{p}$ be an ideal of $\mathfrak{g}$. Then*

(i) *$\mathfrak{p}$ is prime if and only if $\mathfrak{p}$ is irreducible and semi-prime.*

(ii) *If $\mathfrak{g}$ satisfies the minimal condition on ideals, then $\mathfrak{p}$ is prime if and only if there is a smallest ideal $\mathfrak{m}$ containing $\mathfrak{p}$ properly, such that $\mathfrak{m}/\mathfrak{p}$ is not abelian.*

Proof. (i) Let $\mathfrak{p}$ be irreducible and semi-prime, and let $\mathfrak{h}_1, \mathfrak{h}_2$ be ideals of $\mathfrak{g}$ such that $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{p}$. Set $\mathfrak{n} := (\mathfrak{h}_1 + \mathfrak{p}) \cap (\mathfrak{h}_2 + \mathfrak{p})$. Clearly, $\mathfrak{p} \subseteq \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{p}$. This implies that $\mathfrak{n} \subseteq \mathfrak{p}$ since $\mathfrak{p}$ is semi-prime, and thus $\mathfrak{p} = \mathfrak{n} = (\mathfrak{h}_1 + \mathfrak{p}) \cap (\mathfrak{h}_2 + \mathfrak{p})$. Hence $\mathfrak{p} = \mathfrak{h}_1 + \mathfrak{p}$ or $\mathfrak{p} = \mathfrak{h}_2 + \mathfrak{p}$ since $\mathfrak{p}$ is irreducible. Therefore $\mathfrak{h}_1 \subseteq \mathfrak{p}$ or $\mathfrak{h}_2 \subseteq \mathfrak{p}$. The converse holds by Lemma 3.11 and since prime ideals are obviously semi-prime.

(ii) Assume that $\mathfrak{g}$ satisfies the minimal condition on ideals. Then

$$M_{\mathfrak{p}} := \{\mathfrak{m} |\ \mathfrak{m} \text{ is a minimal ideal of } \mathfrak{g} \text{ such that } \mathfrak{p} \subsetneqq \mathfrak{m}\} \neq \emptyset.$$

Now assume that $\mathfrak{p}$ is prime and $|M_{\mathfrak{p}}| > 1$, and let $\mathfrak{m}, \mathfrak{m}' \in M_{\mathfrak{p}}$ with $\mathfrak{m} \neq \mathfrak{m}'$. Then $\mathfrak{p} \subseteq \mathfrak{m} \cap \mathfrak{m}' \subsetneqq \mathfrak{m}$, so $\mathfrak{m} \cap \mathfrak{m}' \notin M_{\mathfrak{p}}$, and thus $[\mathfrak{m}, \mathfrak{m}'] \subseteq \mathfrak{m} \cap \mathfrak{m}' = \mathfrak{p}$, which implies that $\mathfrak{m} = \mathfrak{p}$ or $\mathfrak{m}' = \mathfrak{p}$, which is a contradiction. Hence, $M_{\mathfrak{p}}$ contains a unique element $\mathfrak{m}_0$ which is therefore the smallest ideal containing $\mathfrak{p}$ properly. Moreover, $\mathfrak{m}_0/\mathfrak{p}$ is not abelian, since $\mathfrak{m}_0 \nsubseteq \mathfrak{p}$ and so $[\mathfrak{m}_0, \mathfrak{m}_0] \nsubseteq \mathfrak{p}$ and hence $[\mathfrak{m}_0/\mathfrak{p}, \mathfrak{m}_0/\mathfrak{p}] \neq 0$.

For the converse, assume that $\mathfrak{p}$ is not prime and let $\mathfrak{h}_1, \mathfrak{h}_2$ be ideals of $\mathfrak{g}$ such that $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{p}$, $\mathfrak{h}_1 \nsubseteq \mathfrak{p}$ and $\mathfrak{h}_2 \nsubseteq \mathfrak{p}$. If $\mathfrak{m}$ is the smallest ideal containing $\mathfrak{p}$ properly and such that $[\mathfrak{m}/\mathfrak{p}, \mathfrak{m}/\mathfrak{p}] \neq 0$, then $\mathfrak{m} \subseteq \mathfrak{h}_1 + \mathfrak{p}$ and $\mathfrak{m} \subseteq \mathfrak{h}_2 + \mathfrak{p}$, since $\mathfrak{p} \subsetneqq \mathfrak{h}_1 + \mathfrak{p}$ and $\mathfrak{p} \subsetneqq \mathfrak{h}_2 + \mathfrak{p}$. So $[\mathfrak{m}, \mathfrak{m}] \subseteq [\mathfrak{h}_1 + \mathfrak{p}, \mathfrak{h}_2 + \mathfrak{p}] \subseteq \mathfrak{p}$, and thus $[\mathfrak{m}/\mathfrak{p}, \mathfrak{m}/\mathfrak{p}] = 0$. This contradiction completes the proof. ∎

The next result shows that a maximal ideal of a Leibniz algebra is prime if and only if its codimension is greater than one.

Theorem 3.14. *Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{g}$. Then the following conditions are equivalent:*

(i) $\mathfrak{m}$ *is prime.*

(ii) $\mathfrak{m}$ *is semi-prime.*

(iii) $\dim(\mathfrak{g}/\mathfrak{m}) > 1$.

Proof. (i)⇒(ii) Trivial.

(ii) ⇒ (iii) Let $\mathfrak{m}$ be a semi-prime ideal of $\mathfrak{g}$. If $\dim(\mathfrak{g}/\mathfrak{m}) = 1$, then $\mathfrak{g}/\mathfrak{m}$ is abelian and so $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{m}$. Hence $\mathfrak{g} \subseteq \mathfrak{m}$ which is impossible.

(iii) ⇒ (i) If $\mathfrak{m}$ is not prime, then there exist ideals $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}$ such that $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{m}$ but neither $\mathfrak{h}_1$ nor $\mathfrak{h}_2$ is contained in $\mathfrak{m}$. Then maximality of $\mathfrak{m}$ implies that $\mathfrak{m} + \mathfrak{h}_1 = \mathfrak{m} + \mathfrak{h}_2 = \mathfrak{g}$ and hence $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{m} + \mathfrak{h}_1, \mathfrak{m} + \mathfrak{h}_2] \subseteq \mathfrak{m}$. Now assume that the elements $x + \mathfrak{m}, y + \mathfrak{m}$ belong to a basis of $\mathfrak{g}/\mathfrak{m}$. Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{m}$, so $\langle x, \mathfrak{m} \rangle$ is an ideal of $\mathfrak{g}$, which is neither equal to $\mathfrak{m}$ nor $\langle x, y, \mathfrak{m} \rangle$. This contradiction completes the proof. ∎

Example 3.15. Consider the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C}) = \langle x, y, z \rangle$ with Lie products $[x, y] = z$, $[x, z] = 2x$ and $[y, z] = -2y$. Let $\mathcal{V} = \langle v_0, v_1, v_2, v_3 \rangle$ be a 4-dimensional vector space and define the action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathcal{V}$ by

$$v_i^x = (i^2 - 4i)v_{i-1}, \quad v_i^y = v_{i+1} \text{ and } v_i^z = (3 - 2i)v_i,$$

where $0 \leq i \leq 3$ and $v_4 = 0$. One can check that $\mathcal{V}$ is an irreducible module of $\mathfrak{sl}_2(\mathbb{C})$ (see also [4, 8]).

Now, consider the 7-dimensional Leibniz algebra $\mathfrak{g} = \langle x, y, z, v_0, v_1, v_2, v_3 \rangle$ with non-zero multiplications:

$$[x, y] = z, \quad [x, z] = 2x, \quad [y, z] = -2y,$$

$$[v_i, x] = (i^2 - 4i)v_{i-1}, \quad [v_i, y] = v_{i+1}, \quad [v_i, z] = (3 - 2i)v_i,$$

where $0 \leq i \leq 3$ and $v_4 = 0$. Clearly, $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathcal{V}$ (as vector spaces) and $\mathsf{Leib}(\mathfrak{g}) \cong \mathcal{V}$ is a maximal ideal of $\mathfrak{g}$ with $\dim(\mathfrak{g}/\mathsf{Leib}(\mathfrak{g})) = 3$. Therefore, Theorem 3.14 implies that $\mathsf{Leib}(\mathfrak{g})$ is a prime ideal of $\mathfrak{g}$, and hence $\mathfrak{g}$ is a prime Leibniz algebra. ∎

A Leibniz algebra $\mathfrak{g}$ is called *perfect*, if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Corollary 3.16. *Every maximal ideal of a perfect Leibniz algebra is prime.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathfrak{g}$. By Theorem 3.14, it is enough to show that $\mathfrak{m}$ is semi-prime. If it is not semi-prime, then there exists an ideal $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{m}$ and $\mathfrak{h} \not\subseteq \mathfrak{m}$. Hence $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and so $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{h} + \mathfrak{m}, \mathfrak{h} + \mathfrak{m}] \subseteq \mathfrak{m}$, which is a contradiction. ∎

A non-Lie Leibniz algebra $\mathfrak{g}$ cannot be *simple* in the classical sense, since $\mathfrak{g}$ contains the non-zero proper ideal $\mathsf{Leib}(\mathfrak{g})$. In fact, a Leibniz algebra $\mathfrak{g}$ is called simple if $[\mathfrak{g}, \mathfrak{g}] \neq \mathsf{Leib}(\mathfrak{g})$ and 0, $\mathsf{Leib}(\mathfrak{g})$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$ (see [3, 8]). Clearly if $\mathfrak{g}$ is a Lie algebra, then this definition gives the notion of simple Lie algebra, since in this case we have $\mathsf{Leib}(\mathfrak{g}) = 0$.

Corollary 3.17. *Let $\mathfrak{g}$ be a Leibniz algebra such that $\mathfrak{g} = \bigoplus_{i \geq 1} \mathfrak{g}_i$ (with at least two direct summands), where $\mathfrak{g}_i$ is simple for every $i \geq 1$. If $\mathfrak{I}$ is a proper irreducible ideal of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{I} \oplus \mathfrak{g}_j$ for some $j \geq 1$. Moreover, $\mathfrak{I} \oplus \mathsf{Leib}(\mathfrak{g}_j)$ is a maximal and prime ideal of $\mathfrak{g}$.*

Proof. Let I be the smallest subset of $\mathbb{N}$ such that $|I| \geq 2$ and $\mathfrak{g} = \mathfrak{I} \oplus (\bigoplus_{i \in I} \mathfrak{g}_i)$. Then $\mathfrak{I} \neq \mathfrak{I} \oplus \mathfrak{g}_i$ for all $i \in I$, and $\mathfrak{I} = \bigcap_{i \in I} (\mathfrak{I} \oplus \mathfrak{g}_i)$. Now, the irreducibility of $\mathfrak{I}$ yields a contradiction and so I has only one element j say. Thus $\mathfrak{g} = \mathfrak{I} \oplus \mathfrak{g}_j$. Moreover since $\mathfrak{g}_j$ is simple, $\mathfrak{I} \oplus \mathsf{Leib}(\mathfrak{g}_j)$ is a maximal ideal of $\mathfrak{g}$. Also since $\mathfrak{g}$ is perfect, $\mathfrak{I} \oplus \mathsf{Leib}(\mathfrak{g}_j)$ is a prime ideal of $\mathfrak{g}$, by the above corollary. ∎

For an ideal $\mathfrak{h}$ of $\mathfrak{g}$, let $r_{\mathfrak{g}}(\mathfrak{h})$ denote the intersection of all prime ideals of $\mathfrak{g}$ containing $\mathfrak{h}$. We close this section by some results on this notion.

Lemma 3.18. *Let $\mathfrak{g}$ be a Leibniz algebra satisfying the maximal condition on ideals, and $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then there exists a finite set $\{\mathfrak{p}_i\}_{i=1}^n$ of prime ideals such that $r_{\mathfrak{g}}(\mathfrak{h}) = \bigcap_{i=1}^n \mathfrak{p}_i$. Moreover, this set is unique if no prime ideal of $\mathfrak{g}$ is contained in another.*

Proof. If $\mathfrak{h}$ is prime, then the result is clear. Otherwise there are ideals $\mathfrak{h}_1, \mathfrak{h}_2$ such that $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}$ and $\mathfrak{h}_1, \mathfrak{h}_2 \not\subseteq \mathfrak{h}$. Suppose on the contrary that $r_{\mathfrak{g}}(\mathfrak{h}) = \bigcap_{i=1}^{\infty} \mathfrak{p}_i$. Since $[\mathfrak{h} + \mathfrak{h}_1, \mathfrak{h} + \mathfrak{h}_2] \subseteq \mathfrak{h} \subseteq \mathfrak{p}_i$ for every $i \geq 1$, we get $\mathfrak{h} + \mathfrak{h}_1 \subseteq \mathfrak{p}_i$ or $\mathfrak{h} + \mathfrak{h}_2 \subseteq \mathfrak{p}_i$. Therefore, at least one of these ideals ($\mathfrak{h} + \mathfrak{h}_1$ say) is contained in infinitely many prime ideals $\mathfrak{p}_i$. The same conclusion can be obtained for $\mathfrak{h}+\mathfrak{h}_1$, and hence we will get the ascending chain $\mathfrak{h} \subsetneqq \mathfrak{h}+\mathfrak{h}_1 \subsetneqq \cdots$ of ideals of $\mathfrak{g}$, which contradicts the maximal condition.

Moreover, if there is another set $\{\mathfrak{q}_i\}_{i=1}^m$ of prime ideals such that $r_\mathfrak{g}(\mathfrak{h}) = \bigcap_{i=1}^m \mathfrak{q}_i$, then

$$[\cdots[[\mathfrak{q}_1,\mathfrak{q}_2],\mathfrak{q}_3],\ldots,\mathfrak{q}_m] \subseteq \bigcap_{i=1}^m \mathfrak{q}_i \subseteq \mathfrak{p}_i$$

implies that $\mathfrak{q}_j \subseteq \mathfrak{p}_i$ for some $1 \le j \le m$. This completes the proof. ■

Theorem 3.19. *Let $\mathfrak{g}$ be a Leibniz algebra. Then $Rad(\mathfrak{g}) \subseteq r_\mathfrak{g}(0)$. Moreover, $Rad(\mathfrak{g}) = s_\mathfrak{g}(0) = r_\mathfrak{g}(0)$ if $\mathfrak{g}$ satisfies the maximal condition on ideals.*

Proof. Lemma 3.7 implies that every prime ideal of $\mathfrak{g}$ contains all solvable ideals. Thus the sum of all solvable ideals of $\mathfrak{g}$ is contained in $r_\mathfrak{g}(0)$. For the second part, suppose on the contrary that $r_\mathfrak{g}(0)$ is not solvable. Let $\mathcal{C}$ be a family of ideals $\mathfrak{I}$ such that $r_\mathfrak{g}(0)^{(k)} \nsubseteq \mathfrak{I}$ for all integer $k \ge 1$. Clearly $(0) \in \mathcal{C}$, and so $\mathcal{C}$ has a maximal element $\mathfrak{p}$, since $\mathfrak{g}$ satisfies the maximal condition.

We show that $\mathfrak{p}$ is a prime ideal of $\mathfrak{g}$.

Assume on the contrary that there are ideals $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}$ such that $[\mathfrak{h}_1,\mathfrak{h}_2] \subseteq \mathfrak{p}$, $\mathfrak{h}_1 \nsubseteq \mathfrak{p}$ and $\mathfrak{h}_2 \nsubseteq \mathfrak{p}$. Therefore $\mathfrak{h}_1+\mathfrak{p}, \mathfrak{h}_2+\mathfrak{p} \notin \mathcal{C}$ and hence there is an integer $m \ge 1$ such that $r_\mathfrak{g}(0)^{(m)} \subseteq (\mathfrak{h}_1+\mathfrak{p}) \cap (\mathfrak{h}_2+\mathfrak{p})$. Thus $r_\mathfrak{g}(0)^{(m+1)} \subseteq [\mathfrak{h}_1+\mathfrak{p}, \mathfrak{h}_2+\mathfrak{p}] \subseteq \mathfrak{p}$, which is a contradiction since $\mathfrak{p} \in \mathcal{C}$. Hence $\mathfrak{p}$ is prime and so $r_\mathfrak{g}(0) \subseteq \mathfrak{p}$ which is a contradiction again since $\mathfrak{p} \in \mathcal{C}$. Therefore $r_\mathfrak{g}(0)$ is solvable and so $r_\mathfrak{g}(0) \subseteq Rad(\mathfrak{g})$. Now, Corollary 3.8 completes the proof. ■

Corollary 3.20. *Let $\mathfrak{g}$ be a Leibniz algebra satisfying the maximal condition on ideals and $\mathfrak{q}$ be an ideal of $\mathfrak{g}$. Then the following statements are equivalents:*

(i) *$\mathfrak{q}$ is semi-prime.*

(ii) $\mathfrak{q} = r_\mathfrak{g}(\mathfrak{q})$.

(iii) *$\mathfrak{q}$ is a finite intersection of prime ideals of $\mathfrak{g}$.*

Proof. (i) $\Rightarrow$ (ii) Lemma 3.9 implies that $Rad(\mathfrak{g}/\mathfrak{q}) = 0$. Hence by Theorem 3.19, we have $r_{\mathfrak{g}/\mathfrak{q}}(0) = 0$, which shows that $\mathfrak{q} = r_\mathfrak{g}(\mathfrak{q})$.

(ii) $\Rightarrow$ (iii) It follows from Lemma 3.18.

(iii) $\Rightarrow$ (i) It is easy to see that an intersection of (prime and so) semi-prime ideals is semi-prime. ■

Problem 3.21. Let $\mathfrak{p}$ be a prime ideal of a Leibniz algebra $\mathfrak{g}$. As mentioned above, $[x,y] \in \mathfrak{p}$ (for elements $x, y \in \mathfrak{g}$) does not necessarily imply either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Such an implication holds for prime ideals in commutative rings

$$(xy \in \mathfrak{p} \Rightarrow x \in \mathfrak{p} \text{ or } y \in \mathfrak{p}),$$

and is a fundamental tool in proving the *prime avoidance lemma* in this structure. So, a question arises to whether the following Leibniz version of this lemma is correct?

Let $\{\mathfrak{p}_i\}_{i=1}^n$ be an arbitrary finite set of prime ideals and $\mathfrak{I}$ be an ideal of $\mathfrak{g}$ such that $\mathfrak{I} \subseteq \bigcup_{i=1}^n \mathfrak{p}_i$. Then $\mathfrak{I} \subseteq \mathfrak{p}_i$ for some $1 \le i \le n$.

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