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σ-Symmetries and First Integral of Differential Equations

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Abstract. We provide some geometric properties for σ-symmetries of system of ordinary differential equations. According to the corresponding geometric representation of σ-symmetries and solvable structure, we give the first integrals for the system of first-order ordinary differential equations and for the system of n-order ordinary differential equations which has not enough symmetries and λ-symmetries.

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1. Introduction

The study of symmetry properties of ordinary differential equations (ODEs) is a classical and well-established topic (see [1, 6, 13, 19–22]) and provides a powerful tool for both the explicit computation of solutions to the equations and a better understanding of their qualitative behavior. It is well known that if an ODE $\mathcal{E}$ admits a local symmetry, this can be used to reduce the order of $\mathcal{E}$ by one. This procedure, usually referred to as symmetry reduction method, is particular useful when $\mathcal{E}$ is a k-order ODE whose local symmetries form a solvable k-dimensional Lie algebra (one can see [2–4,12,14]). In this case, in fact, $\mathcal{E}$ can be completely integrated by quadratures [9, 13, 19]. Although the Lie symmetry group theory provides a powerful tool for analyzing ordinary (and partial) differential equations [19, 20], not every technique can be based on symmetry analysis [10, 11], and this requires generalizations of classical Lie methods. Muriel and Romero [15] introduced a new class of symmetry based on a new method of prolonging vector fields known as the λ-prolongation, leading to the notion of λ-symmetry, that strictly includes Lie symmetry. For applications of λ-symmetry, one can see [15–18, 24–26].

In recent papers [7, 8], Cicogna et at have introduced a generalized prolongation operation, defined not on single vector fields but on sets of vector fields, and depending on a smooth matrix function $\sigma : J^1M \to Mat(n, R)$. This is called σ-prolongation or joint λ-prolongation to emphasize that it is an extension of the λ-prolongation. Correspondingly, they introduced the notion of σ-symmetry, or joint λ-symmetry, for systems of ODEs: Lie-point vector fields which–after being σ-prolonged up to suitablem order leave a given set of equations $\mathcal{E}$ invariant–are said to be σ-symmetries for $\mathcal{E}$.

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The aim of this paper is to provide some geometric properties for σ-symmetries of system of ordinary differential equations. According to the corresponding geometric representation of σ-symmetries and solvable structure, we give the first integrals for the system of one-order ordinary differential equations and for the system of n-order ordinary differential equations which has not enough symmetries and λ-symmetries. So, in analogy with the symmetries case and λ-symmetries case, σ-symmetries are as useful as symmetries case and λ-symmetries to concern first integrals of differential eqation systems.

This paper is organized as follows. In Section 2, we collect some basic notations and concepts about symmetries, λ-symmetries and σ-symmetries. In Section 3, we give the first integrals of system of a first-order ordinary differential equations by using the σ-Liouville vector field we defined. In Section 4, we give the first integrals of system of n-order ordinary differential equation. In Section 5, we give a summary of the paper.

2. Preliminaries and geometric properties for $\boldsymbol{\lambda}$-symmetries

We will firstly recall some basic notion, also set some general notation to be used in the following.

2.1. Equation, solutions and symmetries

We will only consider ordinary differential equations, the independent variable will be denoted as $x \in R$, the dependent one(s) as $y \in U = R$ or $y^a \in U \subset R^m$ in the multidimensional case. We denote by $M = X \times U$ the phase bundle, and by (J^kM, π^k, M), or J^kM for short, the associated jet bundle of order k.

A differential equation $\mathcal{E}$ of order n is a map $F : J^nM \to R$, and is naturally identified with the solution manifold $S(\mathcal{E}) = F^{-1}(0) \subset J^nM$. In the case of l-dimensional systems $\mathcal{E}$, we have l maps $F^j : J^nM \to R$ (or equivalently a map $F : J^nM \to R^l$) and a solution manifold

$$S(\mathcal{E}) = F^{-1}(\mathbf{0}) = (F^1)^{-1}(0) \cap ... \cap (F^l)^{-1}(0) \subset J^kM.$$

For m-dimensional dependent variables, J^nM has dimension $(m(n+1)+1)$, and a system of l independent equations identifies therefore a solution manifold of dimension $(m(n+1)+1-l)$.

Given a smooth function vector $y = f(x)$, there is an induced function $pr^{(n)}f(x)$, called the n-th prolongation of f, which is defined by the equations

$$\frac{d^\alpha y^a}{dx^\alpha} = \frac{d^\alpha f^a}{dx^\alpha},\ a = 1, ..., m,\ \alpha = 0, ..., n.$$

Thus $pr^{(n)}f(x)$ is a function vector from R^1 to the space J^nM.

A function vector $f : R^1 \to U$ is a solution to the differential equation(s) $\mathcal{E}$ under study if and only if its n-th prolongation lies entirely in the solution manifold $S(\mathcal{E})$.

Let us now consider a vector field Y on J^nM, we say that $\mathcal{E}$ is invariant under Y if and only if its solution manifold has the property $Y : S(\mathcal{E}) \to TS(\mathcal{E})$. This can also be cast as the condition $[Y(\mathcal{E})]_{S(\mathcal{E})} = 0$.

If Y is the prolongation of a (Lie-point) vector field X on $M, Y = X^{(n)}$, we say that X is a symmetry for $\mathcal{E}$ (more precisely, this would be a symmetry generator, we will adopt this standard abuse of notation for ease of language). The condition for X to be a symmetry is therefore $[X^{(n)}(\mathcal{E})]_{S(\mathcal{E})} = 0$.

2.2. Local coordinates

We will consider local coordinates (x, y^a), $a = 1, ..., m$ in $M = X \times U$, and correspondingly local coordinates (x, u^a_k) (with $k = 0, ..., n$), where $y^{a(k)} := (\partial^k y^a / \partial x^k)$, in $J^n M$.

A general vector field on $J^n M$ will be written in local coordinates (here and below we will use the Einstein summation convention, the notation $y^{(k)}$ denotes the k order derivative of y) as

$$Y = \xi(x, y^1, ..., y^m)\frac{\partial}{\partial x} + \psi^a_k(x, y^1, ..., y^m, y^{1(1)}, ..., y^{m(1)}, ..., y^{m(n)})\frac{\partial}{\partial y^{a(k)}}, \quad (1)$$

this is the prolongation of

$$X = \xi(x, y^1, ..., y^m)\frac{\partial}{\partial x} + \phi^a(x, y^1, ..., y^m)\frac{\partial}{\partial y^a} \quad (2)$$

if and only if the ψ^a_k satisfy the (standard) prolongation formula

$$\psi^a_{k+1} = D_x \psi^a_k - y^{a(k+1)} D_x \xi, \quad \psi^a_0 = \phi^a.$$

The notation D_x identifies the total derivative with respect to x,

$$D_x = \frac{\partial}{\partial x} + y^{a(k+1)} \frac{\partial}{\partial y^{a(k)}}.$$

2.3. λ-Prolongations and λ-symmetries

After the work of Muriel and Romero [15, 16] it is common to also consider λ-symmetries of ODEs. A vector field Y written in local coordinates in the form (1) is a λ-prolonged vector field if its coefficients satisfy

$$\psi^a_{k+1} = D_x \psi^a_k - y^{a(k+1)} D_x \xi + \lambda(\psi^a_k - y^{a(k+1)}\xi) \quad (3)$$

with λ being a smooth function $\lambda : J^1 M \to R$. We also say that Y is the nth λ-prolongation of X (see (2)) if $\psi^a_0 = \phi^a$, i.e., if X is the restriction of Y to M.

If the λ-prolongation Y of X leaves an equation (or system) $\mathcal{E}$ invariant, we say that X is a λ-symmetry for $\mathcal{E}$.

2.4. σ-Prolongations and σ-symmetries

Let the set $\chi = \{X_i, i = 1, ..., d\}$ be a vector fields set defined on M, assume that they are in involution and their rank is constant for the sake of simplicity. The involution assumption means that there are smooth functions $\mu^k_{ij} : M \to R, i, j, k = 1, ..., d$ such that

$$[X_i, X_j] = \mu^k_{ij} X_k,$$

we will consider vector fields Y_i on $J^k M$, which is some kind of generalized prolongation of the X_i. In local coordinates, and with $x^{a(k)} := \frac{\partial^k x^a}{\partial t^k}$, these will be written as

$$X_i = \xi_i(t, x^a)\frac{\partial}{\partial t} + \psi^a_{i,0}(t, x^a)\frac{\partial}{\partial x^a},$$
$$Y_i = \xi_i(t, x^a)\frac{\partial}{\partial t} + \psi^a_{i,k}(t, x^a, x^{a(1)}, ..., x^{a(k)})\frac{\partial}{\partial x^{ak}}.$$

Definition 2.1. Let $\{\sigma_i^j, i, j = 1, ..., d\}$ be a set of d^2 smooth real functions on J^1M. The vector fields $\mathcal{Y} = \{Y_i, i = 1, ..., d\}$ on J^kM are said to be *σ-prolongations* of χ if the coefficients $\psi_{i,k}^a$ satisfy, for $k \geq 0$,

$$\psi_{i,k+1}^a = (D_t\psi_{i,k}^a - x^{a(k+1)}D_t\xi_i) + \sigma_i^j(\psi_{j,k}^a - x^{a(k+1)}\xi_j).$$

If the σ-prolongations Y_i of the X_i leave the system of equations $\mathcal{E}$ invariant, then we say that the involution system χ is a σ-symmetry for $\mathcal{E}$.

3. First integrals of system of first-order ordinary differential equations

In this section, we study the system of ordinary differential equations

$$\dot{x} = f(t, x), \tag{4}$$

where $x \in R^n$ and $\dot{x}$ stands for the derivative with respect to the time t. Associated with (4) we consider the dynamical vector field $F = f_j(t, x)\frac{\partial}{\partial x_j}$ and characteristic vector field $\Gamma = \frac{\partial}{\partial t} + f_j(t, x)\frac{\partial}{\partial x_j}$.

Definition 3.1. A differential form ω of degree p (or equivalently p-form) is said to be an invariant form of (4) if it satisfies

$$\left(\frac{\partial}{\partial t} + L_F\right)\omega = 0. \tag{5}$$

Lemma 3.2. *Suppose that a vector field set*

$$\mathcal{Y} = \left\{X_i = \eta_i^a(t, x)\frac{\partial}{\partial x^a},\ i = 1, ..., l,\ a = 1, ..., n\right\}$$

is a σ-symmatry of (4)*. Then it enjoys the property*

$$\left(\frac{\partial}{\partial t} + L_F\right)X_i + \sigma_i^jX_j = 0. \tag{6}$$

Proof. From the definition of σ-symmetry, it requires that

$$Y_i(\dot{x}^a - f_a(t, x)) = 0, \tag{7}$$

for $i = 1, ..., l$, where Y_i is a one-order prolongation of $X_i, i = 1, ..., l$. By the σ-prolongation formula, we have

$$Y_i = \eta_i^a(t, x)\frac{\partial}{\partial x^a} + \psi_{i,1}^a(t, x, \dot{x})\frac{\partial}{\partial x^{a(1)}}, \tag{8}$$

where $\psi_{i,1}^a = D_t\eta_i^a + \sigma_i^j\eta_j^a$. Substituting (8) into (7), we have

$$\begin{aligned} Y_i(\dot{x}^a - f_a(t, x)) &= \left(\eta_i^a(t, x)\frac{\partial}{\partial x^a} + (D_t\eta_i^a + \sigma_i^j\eta_j^a)\frac{\partial}{\partial x^{a(1)}}\right)(\dot{x}^a - f_a(t, x)) \\ &= \frac{\partial\eta_i^a}{\partial t} + f_m\frac{\partial\eta_i^a}{\partial x^m} + \sigma_i^j\eta_j^a - \eta_i^m\frac{\partial f_a}{\partial x^m} = 0. \end{aligned} \tag{9}$$

Multiplying $\frac{\partial}{\partial x^a}$ to (9) and taking sum over $a = 1, ..., n$, we obtain

$$\frac{\partial\eta_i^a}{\partial t}\frac{\partial}{\partial x^a} + f_m\frac{\partial\eta_i^a}{\partial x^m}\frac{\partial}{\partial x^a} + \sigma_i^j\eta_j^a - \eta_i^m\frac{\partial f_a}{\partial x^m}\frac{\partial}{\partial x^a} = 0, \tag{10}$$

which is equality (6).

In fact,

$$\frac{\partial}{\partial t}X_i = \frac{\partial \eta_i^a}{\partial t}\frac{\partial}{\partial x^a},$$

$$[F, X_i] = [f_m\frac{\partial}{\partial x^m}, \eta_i^k\frac{\partial}{\partial x^k}] = f_m\frac{\partial \eta_i^k}{\partial x^m}\frac{\partial}{\partial x^k} - \eta_i^k\frac{\partial f_m}{\partial x^k}\frac{\partial}{\partial x^m},$$

$$\sigma_i^j X_j = \sigma_i^j \eta_i^a \frac{\partial}{\partial x^a}.$$

So, just comparing (10) with $\frac{\partial}{\partial t}X_i + [F, X_i] + \sigma_i^j X_j$, we can get the conclusion. ■

Example 3.3. Consider the linear system

$$\dot{x}^1 = t + x^1 - x^2, \quad \dot{x}^2 = x^2 - x^1. \tag{11}$$

We know the dynamical vector field $F = (t + x^1 - x^2)\frac{\partial}{\partial x^1} + (x^2 - x^1)\frac{\partial}{\partial x^2}$ and the characteristic vector field $\Gamma = \frac{\partial}{\partial t} + (t + x^1 - x^2)\frac{\partial}{\partial x^1} + (x^2 - x^1)\frac{\partial}{\partial x^2}$. It is clear that the vector field set $\{U = \frac{\partial}{\partial x^1}, V = \frac{\partial}{\partial x^2}\}$ is a σ-symmatry of (11) because by Lemma 4.3 we have

$$[U, \Gamma] = U - V, \qquad [V, \Gamma] = -U + V.$$

Thus in this equation we can get that $\sigma_1^1 = 1, \sigma_1^2 = -1, \sigma_2^1 = -1, \sigma_2^2 = 1$, so

$$\Big(\frac{\partial}{\partial t} + L_F\Big)U + \sigma_1^1 U + \sigma_1^2 V = [\Gamma, U] + U - V = 0,$$

$$\Big(\frac{\partial}{\partial t} + L_F\Big)V + \sigma_2^1 U + \sigma_2^2 V = [\Gamma, V] - U + V = 0.$$

Example 3.4. Consider the non-linear system

$$\dot{x}^1 = te^{x^2}, \quad \dot{x}^2 = te^{x^1}. \tag{12}$$

The dynamical vector field $F = te^{x^2}\frac{\partial}{\partial x^1} + te^{x^1}\frac{\partial}{\partial x^2}$ and the characteristic vector field $\Gamma = \frac{\partial}{\partial t} + te^{x^2}\frac{\partial}{\partial x^1} + te^{x^1}\frac{\partial}{\partial x^2}$ are clear. We can verify that the vector field set $\{\bar{U} = e^{x^2}\frac{\partial}{\partial x^1}, \bar{V} = e^{x^1}\frac{\partial}{\partial x^2}\}$ is a σ-symmetry of (12) because by Lemma 4.3 we have

$$[\bar{U}, \Gamma] = -te^{x^1}\bar{U} + te^{x^2}\bar{V}, \quad [\bar{V}, \Gamma] = te^{x^1}\bar{U} - te^{x^2}\bar{V}.$$

Thus in this equation $\sigma_1^1 = -te^{x^1}$, $\sigma_1^2 = te^{x^2}$, $\sigma_2^1 = te^{x^1}$, $\sigma_2^2 = -te^{x^2}$, so

$$\Big(\frac{\partial}{\partial t} + L_F\Big)\bar{U} + \sigma_1^1\bar{U} + \sigma_1^2\bar{V} = [\Gamma, \bar{U}] - te^{x^1}\bar{U} + te^{x^2}\bar{V} = 0,$$

$$\Big(\frac{\partial}{\partial t} + L_F\Big)\bar{V} + \sigma_2^1\bar{U} + \sigma_2^2\bar{V} = [\Gamma, \bar{V}] + te^{x^1}\bar{U} - te^{x^2}\bar{V} = 0.$$

We now define the σ-Liouville vector field set which is a generalization of the λ-Liouville vector field (see [26]) based on C^∞-prolongation.

Definition 3.5. A vector field set $\{X_i = \eta_i^a\frac{\partial}{\partial x^a}, i = 1, ..., l\}$ is called a σ*-Liouville vector field set* of (4), if there exists $\sigma_{ij} \in C^\infty(t, x, \dot{x}), i, j = 1, ..., l$, such that for any $X_i \in \mathcal{Y}$,

$$(\frac{\partial}{\partial t} + L_F)X_i + \sigma_i^j X_j + (\operatorname{div} F)X_i = 0. \tag{13}$$

It is well known that the total mass of an object is unchanged in any case whenever it is experiencing mechanical, physical or chemical motion, which is the celebrated

"Law of conservation of mass". According to the "Law of conservation of mass", if the phase density is p, we can deduce the continuity equation $\frac{\partial p}{\partial t} + \mathrm{div}\,(pF) = 0$, which is equivalent to $\frac{dp}{dt} + (\mathrm{div}\,F)p = 0$. In fact,

$$\begin{aligned}\frac{\partial p}{\partial t} + \mathrm{div}\,(pF) &= \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x^1} f_1 + ... + \frac{\partial p}{\partial x^n} f_n + p\frac{\partial f_1}{\partial x^1} + ... + p\frac{\partial f_n}{\partial x^n} \\ &= \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x^1}\dot{x} + ... + \frac{\partial p}{\partial x^n}\dot{x}^n + p(\mathrm{div}\,F) = \frac{dp}{dt} + (\mathrm{div}\,F)p.\end{aligned}$$

Moreover, because $\frac{dp}{dt} = (\frac{\partial}{\partial t} + L_F)p$, thus the phase density p satisfies the following partial differential equation

$$(\frac{\partial}{\partial t} + L_F)p + (\mathrm{div}\,F)p = 0. \tag{14}$$

The following lemma states a remarkable property of the σ-Liouville vector field set $\{X_i = \eta_i^a \frac{\partial}{\partial x^a}, i = 1, ..., l\}$, which shows the relationship between a σ-Liouville vector field set and a σ-symmetry.

Lemma 3.6. *If we have a σ-symmetry*

$$\mathcal{Y} = \Big\{X_i = \eta_i^a(t,x)\frac{\partial}{\partial x^a}\,,\ i = 1, ..., l,\ a = 1, ..., n\Big\}$$

of (4)*, then the set of vector fields*

$$\Big\{Z_i = pX_i = p\eta_i^a(t,x)\frac{\partial}{\partial x^a}, i = 1, ..., l, a = 1, ..., n\Big\} \tag{15}$$

is a σ-Liouville vector field set of (4).

Proof. We only need to show that the set of vector fields (15) enjoys the property (13), i.e.,

$$\Big(\frac{\partial}{\partial t} + L_F\Big)(pX_i) + \sigma_i^j pX_j + (\mathrm{div}\,F)pX_i = 0. \tag{16}$$

Because a σ-symmetry of (4) has the property (6), from (14) we obtain

$$\begin{aligned}&\Big(\frac{\partial}{\partial t} + L_F\Big)(pX_i) + \sigma_i^j pX_j + (\mathrm{div}\,F)pX_i \\ &\quad = \Big(\frac{\partial}{\partial t} + L_F\Big)(p)X_i + p\Big(\frac{\partial}{\partial t} + L_F\Big)X_i + \sigma_i^j pX_j + (\mathrm{div}\,F)pX_i \\ &\quad = \Big((\frac{\partial}{\partial t} + L_F)(p) + (\mathrm{div}\,F)p\Big)\,X_i + p\,\Big((\frac{\partial}{\partial t} + L_F)X_i + \sigma_i^j X_j\Big) = 0. \qquad \blacksquare\end{aligned}$$

In fact, conversely, we can get another lemma.

Lemma 3.7. *If* $\mathcal{Y} = \{X_i = \eta_i^a(t,x)\frac{\partial}{\partial x^a},\ i = 1, ..., l,\ a = 1, ..., n\}$ *is a σ-Liouville vector field set of* (4)*, then the set of vector fields*

$$\Big\{Z_i = \frac{X_i}{p} = \frac{\eta_i^a(t,x)}{p}\frac{\partial}{\partial x^a},\ i = 1, ..., l,\ a = 1, ..., n\Big\} \tag{17}$$

is a σ-symmetry of (4).

Proof. We only need to show that the set of vector fields (17) enjoys the property (6), i.e.,

$$\Big(\frac{\partial}{\partial t} + L_F\Big)\left(\frac{X_i}{p}\right) + \sigma_i^j \cdot \frac{X_j}{p} = 0. \tag{18}$$

From (14) we obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}+L_F\right)\left(\frac{X_i}{p}\right)+\sigma_i^j\cdot\frac{X_j}{p} &= \left(\frac{\partial}{\partial t}+L_F\right)\left(\frac{1}{p}\right)X_i+\frac{1}{p}\left(\frac{\partial}{\partial t}+L_F\right)(X_i)+\sigma_i^j\cdot\frac{X_j}{p}\\ &= \frac{-\left(\left(\frac{\partial}{\partial t}+L_F\right)p\right)\cdot X_i}{p^2}+\frac{1}{p}\left(\frac{\partial}{\partial t}+L_F\right)(X_i)+\sigma_i^j\cdot\frac{X_j}{p}\\ &= \frac{((\operatorname{div}F)p)\cdot X_i}{p^2}+\frac{\left(\frac{\partial}{\partial t}+L_F\right)(X_i)+\sigma_{ij}X_j}{p}\\ &= \frac{(\operatorname{div}F)\cdot X_i}{p}-\frac{(\operatorname{div}F)\cdot X_i}{p}=0.\end{aligned}$$ ∎

Example 3.8. In system (11), we can get that divergence of F is equal to 2 and from the continuity equation, the phase density can be written as $p=e^{-2t}$. In this case, $\frac{dp}{dt}+(\operatorname{div}F)p=-2e^{-2t}+2e^{-2t}=0$.

Now we show that the vector field set $\{pU,pV\}=\{e^{-2t}U,e^{-2t}V\}$ is a σ-Liouville vector field set of (11). By Lemma 3.2 we directly calculate

$$\begin{aligned}&(\frac{\partial}{\partial t}+L_F)(pU)+\sigma_1^1pU+\sigma_1^2pV+(\operatorname{div}F)pU\\ &=(\frac{\partial}{\partial t}+L_F)(p)U+p(\frac{\partial}{\partial t}+L_F)U+pU-pV+2pU\\ &=-2pU+p((\frac{\partial}{\partial t}+L_F)U+U-V)+2pU=-2pU+2pU=0,\end{aligned}$$

$$\begin{aligned}&(\frac{\partial}{\partial t}+L_F)(pV)+\sigma_2^1pV+\sigma_2^2pV+(\operatorname{div}F)pV\\ &=(\frac{\partial}{\partial t}+L_F)(p)V+p(\frac{\partial}{\partial t}+L_F)V-pU+pV+2pV\\ &=-2pV+p((\frac{\partial}{\partial t}+L_F)V-U+V)+2pV=-2pV+2pV=0.\end{aligned}$$

Lemma 3.9. *If* $\mathcal{Y}=\left\{X_i=\eta_i^a(t,x)\frac{\partial}{\partial x^a},\ i=1,...,n,\ a=1,...,n\right\}$ *is a σ-Liouville vector field set of* (4) *and has the property* $\sum_{i=1}^n\sigma_i^j=0$, $j=1,...,n$, *then the $n-1$ form* $\omega=i_{X_1+...+X_n}\Omega$ *is an invariant $n-1$ form, where Ω is the volume form* $dx^1\wedge\cdots\wedge dx^n$.

Proof. It suffices to show that the $n-1$ form ω enjoys the property (5), i.e., $\left(\frac{\partial}{\partial t}+L_F\right)i_{X_1+...+X_n}\Omega=0$. Using the property of Lie derivative, we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+L_F\right)i_{X_1+...+X_n}\Omega=i_{\left(\frac{\partial}{\partial t}+L_F\right)(X_1+...+X_n)}\Omega+i_{X_1+...+X_n}L_F\Omega\\ &=-\sigma_1^ji_{X_j}\Omega-(\operatorname{div}F)i_{X_1}\Omega-\cdots-\sigma_n^ji_{X_j}\Omega-(\operatorname{div}F)i_{X_n}\Omega+(\operatorname{div}F)i_{X_1+...+X_n}\Omega\\ &=-(\sum_{j=1}^n\sigma_j^1)i_{X_1}\Omega-\cdots-(\sum_{j=1}^n\sigma_j^n)i_{X_n}\Omega=0,\end{aligned}$$

where we have used the Cartan's identity and $L_F\Omega=(\operatorname{div}F)\Omega$ (see [23] for more details). ∎

Remark 3.10. For system (11), we know that $\{pU,pV\}=\{e^{-2t}U,e^{-2t}V\}$ is a σ-Liouville vector field set and $\sigma_1^1+\sigma_2^1=1-1=0$, $\sigma_1^2+\sigma_2^2=-1+1=0$.

We calculate

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+L_F\right)\omega=\left(\frac{\partial}{\partial t}+L_F\right)i_{pU+pV}dx^1\wedge dx^2\\
&=i_{\left(\frac{\partial}{\partial t}+L_F\right)(pU+pV)}dx^1\wedge dx^2+i_{pU+pV}\left(\frac{\partial}{\partial t}+L_F\right)dx^1\wedge dx^2\\
&=i_{(-\sigma_1^1pU-\sigma_1^2pV-(\operatorname{div}F)pU-\sigma_2^1pU-\sigma_2^2pV-(\operatorname{div}F)pV)}dx^1\wedge dx^2+(\operatorname{div}F)i_{pU+pV}dx^1\wedge dx^2\\
&=i_{-(\operatorname{div}F)pU-(\operatorname{div}F)pV}dx^1\wedge dx^2+(\operatorname{div}F)i_{pU+pV}dx^1\wedge dx^2=0,
\end{aligned}$$

thus $i_{pU+pV}dx^1\wedge dx^2$ is an invariant 1-form of (11).

Theorem 3.11. *If we have $n-1$ vector fields $Z_1,...,Z_{n-1}$ which have the property*

$$\left(\frac{\partial}{\partial t}+L_F\right)Z_j=span\{X_1+\cdots+X_n\},\quad j=1,...,n,$$

and a σ-Liouville vector field set $\mathcal{Y}=\{X_i=\eta_i^a(t,x)\frac{\partial}{\partial x^a},\ i=1,...,n,\ a=1,...,n\}$ of (4) possessing the property $\sum_{i=1}^n\sigma_i^j=0, j=1,...,n$, then we can get a first integral of (4) which has a form $i_{Z_1}\cdots i_{Z_{n-1}}i_{X_1+\cdots+X_n}\Omega$.

Proof. In fact, by lemma 3.9 and the assumption, we can get

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}+L_F\right)\left(i_{Z_1}\cdots i_{Z_{n-1}}i_{X_1+\cdots+X_n}\Omega\right)\\
&\quad=i_{\left(\frac{\partial}{\partial t}+L_F\right)Z_1}\cdots i_{Z_{n-1}}i_{X_1+\cdots+X_n}\Omega+i_{Z_1}\cdots i_{\left(\frac{\partial}{\partial t}+L_F\right)Z_{n-1}}i_{X_1+\cdots+X_n}\Omega\\
&\quad+i_{Z_1}\cdots i_{Z_{n-1}}\left(\frac{\partial}{\partial t}+L_F\right)\left(i_{X_1+\cdots+X_n}\Omega\right)=0.
\end{aligned}$$

■

Here we give two examples to clarify Theorem 3.11.

Example 3.12. Consider the linear system

$$\dot{x}^1=x^1+x^2,\quad \dot{x}^2=x^1+x^2. \tag{19}$$

The dynamical vector field is $F=(x^1+x^2)\frac{\partial}{\partial x^1}+(x^1+x^2)\frac{\partial}{\partial x^2}$ and the characteristic vector field is $\Gamma=\frac{\partial}{\partial t}+(x^1+x^2)\frac{\partial}{\partial x^1}+(x^1+x^2)\frac{\partial}{\partial x^2}$. We can verify that the vector field set $\{G=x^1\frac{\partial}{\partial x^1},H=x^2\frac{\partial}{\partial x^2}\}$ is a σ-symmetry of (19) because by Lemma 4.3

$$\begin{aligned}
[G,\Gamma]&=G\Gamma-\Gamma G=x^1\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^2}-(x^1+x^2)\frac{\partial}{\partial x^1}\\
&=-x^2\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^2}=\frac{-x^2}{x^1}G+\frac{x^1}{x^2}H\\
[H,\Gamma]&=H\Gamma-\Gamma H=x^2\frac{\partial}{\partial x^1}+x^2\frac{\partial}{\partial x^2}-(x^1+x^2)\frac{\partial}{\partial x^2}\\
&=x^2\frac{\partial}{\partial x^1}-x^1\frac{\partial}{\partial x^2}=\frac{x^2}{x^1}G+\frac{-x^1}{x^2}H
\end{aligned}$$

Thus in this equation we can get that $\sigma_1^1=\frac{-x^2}{x^1},\sigma_1^2=\frac{x^1}{x^2},\sigma_2^1=\frac{x^2}{x^1},\sigma_2^2=\frac{-x^1}{x^2}$, so $\sigma_1^1+\sigma_2^1=0,\sigma_1^2+\sigma_2^2=0$. Due to $\operatorname{div}F=2$, the phase density can be written as $p=e^{-2t}$, hence the vector field set $\{pG,pH\}=\{e^{-2t}G,e^{-2t}H\}$ is a σ-Liouville vector field set and $i_{pG+pH}dx^1\wedge dx^2$ is an invariant 1-form.

We can also verify that the vector field $K = e^{2t}\frac{\partial}{\partial x^1} + e^{2t}\frac{\partial}{\partial x^2}$ satisfies

$$\left(\frac{\partial}{\partial t} + L_F\right) K = 2K - [K, F] = 2K - 2K = 0.$$

At last, we claim that $i_K i_{pG+pH} dx^1 \wedge dx^2$ is a first integral of (19). Calculate

$$\begin{aligned} i_{pG+pH} dx^1 \wedge dx^2 &= e^{-2t}x^1 dx^2 - e^{-2t}x^2 dx^1, \\ i_K i_{pG+pH} dx^1 \wedge dx^2 &= i_{(e^{2t}\frac{\partial}{\partial x^1} + e^{2t}\frac{\partial}{\partial x^2})}(e^{-2t}x^1 dx^2 - e^{-2t}x^2 dx^1) = x^1 - x^2, \\ \frac{d(x^1 - x^2)}{dt} &= \dot{x}^1 - \dot{x}^2 = (x^1 + x^2) - (x^1 + x^2) = 0, \end{aligned}$$

thus, by using theorem 3.11, we find a first integral $x^1 - x^2$ for (19).

Example 3.13. Consider the non-linear system

$$\dot{x}^1 = e^{x^2}, \quad \dot{x}^2 = e^{x^1}. \tag{20}$$

The dynamical vector field is $F = e^{x^2}\frac{\partial}{\partial x^1} + e^{x^1}\frac{\partial}{\partial x^2}$ and the characteristic vector field is $\Gamma = \frac{\partial}{\partial t} + e^{x^2}\frac{\partial}{\partial x^1} + e^{x^1}\frac{\partial}{\partial x^2}$. The vector field set $\{\bar{G} = e^{x^2}\frac{\partial}{\partial x^1},\ \bar{H} = e^{x^1}\frac{\partial}{\partial x^2}\}$ is a σ-symmetry of (20) because by Lemma 4.3 we have

$$[\bar{G}, \Gamma] = -e^{x^1}\bar{G} + e^{x^2}\bar{H}, \quad [\bar{H}, \Gamma] = e^{x^1}\bar{G} - e^{x^2}\bar{H}.$$

Thus in this equation we get that $\sigma_1^1 = -e^{x^1}$, $\sigma_1^2 = e^{x^2}$, $\sigma_2^1 = e^{x^1}$, $\sigma_2^2 = -e^{x^2}$, so $\sigma_1^1 + \sigma_2^1 = 0$, $\sigma_1^2 + \sigma_2^2 = 0$. Due to $\operatorname{div} F = 0$, using $\frac{dp}{dt} + (\operatorname{div} F)p = 0$, the phase density p can be written as $p = C$, C is any constant. Hence the vector field set $\{p\bar{G}, p\bar{H}\} = \{C\bar{G}, C\bar{H}\}$ is a σ-Liouville vector field set and $i_{pG+pH} dx^1 \wedge dx^2$ is an invariant 1-form. We can also verify that the vector field $\bar{K} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$ satisfies

$$\left(\frac{\partial}{\partial t} + L_F\right) \bar{K} = -\bar{G} - \bar{H}.$$

At last, we claim that $i_{\bar{K}} i_{C\bar{G}+C\bar{H}} dx^1 \wedge dx^2$ is a first integral of (20). Calculate

$$\begin{aligned} i_{p\bar{G}+p\bar{H}} dx^1 \wedge dx^2 &= Ce^{x^2} dx^2 - Ce^{x^1} dx^1, \\ i_{\bar{K}} i_{p\bar{G}+p\bar{H}} dx^1 \wedge dx^2 &= i_{\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}}(Ce^{x^2} dx^2 - Ce^{x^1} dx^1) = Ce^{x^2} - Ce^{x^1}, \\ \frac{d(Ce^{x^2} - Ce^{x^1})}{dt} &= Ce^{x^2}\dot{x}^2 - Ce^{x^1}\dot{x}^1 = Ce^{x^1+x^2} - Ce^{x^1+x^2} = 0, \end{aligned}$$

thus, by using theorem 3.11, we find a first integral $Ce^{x^2} - Ce^{x^1}$ for (20).

4. First integrals of system of n-order ordinary differential equation

Consider the following nth order ordinary differential equation

$$x^{a(n)} = f^a(t, x^a, x^{a(1)}, ..., x^{a(n-1)}), \tag{21}$$

where $(t, x^a) \in M \subset R^{m+1}$ is some open subset. Clearly, its characteristic vector field is

$$\Gamma = \frac{\partial}{\partial t} + x^{a(1)}\frac{\partial}{\partial x^a} + ... + x^{a(n-1)}\frac{\partial}{\partial x^{a(n-2)}} + f^a\frac{\partial}{\partial x^{a(n-1)}}.$$

Let $\tilde{\Omega} := i_\Gamma(dt \wedge dx^1 \wedge \cdots \wedge dx^m \wedge dx^{1(1)} \wedge \cdots \wedge dx^{m(1)} \wedge \cdots dx^{1(n-1)} \wedge \cdots \wedge dx^{m(n-1)})$ be the characteristic form.

Definition 4.1. Let X and $\mathcal{Y}$ be vector fields and consider a set of vector fields $\{Y_1, ..., Y_m\}$ respectively. Let $\mathcal{D}$ be a distribution, and Θ be a simple differential form defined on generic manifold M.

(1) $\mathcal{Y}$ is called a *σ-symmetry* of X, if there exist functions $\sigma_i^j, i,\ j = 1, ..., m$ defined on M such that for $\forall\, l = 1, ..., m$

$$[Y^l, X] = \sigma_k^l Y^k - (X(\xi^l) + \sigma_k^l \xi^k)X.$$

(2) $\mathcal{Y}$ is called a *σ-symmetry* of $\mathcal{D}$, if for $\forall l = 1, ..., m,\ X \in \mathcal{D}$

$$[Y^l, X] + (X(\xi^l) + \sigma_k^l \xi^k)X \in \mathcal{Y}.$$

(3) $\mathcal{Y}$ is called a *symmetry* of $\mathcal{D}$, if we have $[Y^l, X] \in \mathcal{D}$ for $\forall l = 1, ..., m$.

By the Frobenius theorem, if a distribution $\mathcal{D}$ spans the tangent space to an integral submanifold, which has the same dimension as $\mathcal{D}$, then it is Frobenius integrable provided the dimension is constant. A simple differential form Θ is Frobenius integrable if its kernel is Frobenius integrable and of maximal dimension everywhere.

Definition 4.2. Let $\mathbf{X} = \{X_1, ..., X_r\}$ be a involutive system of independent vector fields and $\mathbf{Y} = \{Y_1, ..., Y_l\}$ be a system of independent vector fields. We say that the system $\{\mathbf{X}, \mathbf{Y}\}$ is *solvable with respect to the involutive system* $\mathbf{X}$ if and only if Y_l $(l = 1, ..., n-r)$ is a symmetry of the system $\{\mathbf{X}, Y_1, ..., Y_{l-1}\}$.

Now, we first give some properties of σ-symmetry and their proofs.

Lemma 4.3. (1) *Let $\mathcal{Y} = \{X_i,\ i = 1, ..., d,\}$ be a σ-symmetry of* (21) *and $\tilde{Y} = \{Y_1, ..., Y_d\}$ be the $n-1$-order σ-prolongation $\mathcal{Y}$. Then for any $Y_i \in \tilde{Y}$,*

$$[Y_i, \Gamma] = \sigma_i^k Y_k - (\Gamma(\xi_i) + \sigma_i^k \xi_k)\Gamma$$

for $\sigma_i^j \in C^\infty(J^1M), i, j = 1, ..., d$, where Y_i is a $n-1$-order σ-prolongation of $\mathcal{Y}$.

(2) *Conversely, if*

$$X_i = \xi_i(t, x^a)\frac{\partial}{\partial t} + \eta_i^a(t, x^a)\frac{\partial}{\partial x^a} + \eta_{i,1}^a(t, x^a, x^{a(1)})\frac{\partial}{\partial x^{a(1)}} + \cdots$$
$$+ \eta_{i,n-1}^a(t, x^a, x^{a(1)}, ..., x^{a(n-1)})\frac{\partial}{\partial x^{a(n-1)}}$$

is a vector field defined on $J^{n-1}M$ such that

$$[X_i, \Gamma] = \sigma_i^k X_k - (\Gamma(\xi_i) + \sigma_i^k \xi_k)\Gamma \tag{22}$$

for $\sigma_i^j \in C^\infty(J^1M), i, j = 1, ..., d$, then the vector field set

$$v_i = \xi_i(t, x^a)\frac{\partial}{\partial t} + \eta_i^a(t, x^a)\frac{\partial}{\partial x^a}, \quad i = 1, ..., d$$

defined on M, denote by $\tilde{\mathcal{Y}}$, is a σ-symmetry of the equation (21)*, and X_i are the $n-1$-order λ-prolongation of $\tilde{\mathcal{Y}}$.*

Proof. (1) Equation (21) considered above has the following form:

$$\begin{cases} \dfrac{d^n x^1}{dt^n} & = f^1(t, x^a, ..., x^{a(n-1)}), \\ & \vdots \\ \dfrac{d^n x^m}{dt^n} & = f^m(t, x^a, ..., x^{a(n-1)}), \end{cases}$$

where $x^a = (x^1, ..., x^m), x^{a(j)} = (x^{1(j)}, ..., x^{m(j)}), j = 1, ..., n-1$ and

$$\begin{aligned} \Gamma = & \frac{\partial}{\partial t} + x^{1(1)}\frac{\partial}{\partial x^1} + \cdots + x^{m(1)}\frac{\partial}{\partial x^m} + x^{1(2)}\frac{\partial}{\partial x^{1(1)}} + \cdots + x^{m(2)}\frac{\partial}{\partial x^{m(1)}} \\ & + \cdots + x^{1(n-1)}\frac{\partial}{\partial x^{1(n-2)}} + \cdots + x^{m(n-1)}\frac{\partial}{\partial x^{m(n-2)}} \\ & + f^1(t, x^a, ..., x^{a(n-1)})\frac{\partial}{\partial x^{1(n-1)}} + \cdots + f^m(t, x^a, ..., x^{a(n-1)})\frac{\partial}{\partial x^{m(n-1)}} \end{aligned}$$

$$\begin{aligned} Y_i = & \xi_i(t, x^a)\frac{\partial}{\partial t} + \psi^1_{i,0}(t, x^a)\frac{\partial}{\partial x^1} + \cdots + \psi^m_{i,0}\frac{\partial}{\partial x^m} + \psi^1_{i,1}(t, x^a, x^{a(1)})\frac{\partial}{\partial x^{1(1)}} + \cdots \\ & + \psi^m_{i,1}\frac{\partial}{\partial x^{m(1)}} + \cdots + \psi^1_{i,n-1}(t, x^a, x^{1(1)}, ..., x^{m(1)}, ..., x^{1(n-1)}, ..., x^{m(n-1)})\frac{\partial}{\partial x^{1(n-1)}} \\ & + \cdots + \psi^m_{i,n-1}(t, x^a, x^{1(1)}, ..., x^{m(1)}, ..., x^{1(n-1)}, ..., x^{m(n-1)})\frac{\partial}{\partial x^{m(n-1)}}, \end{aligned}$$

where $i = 1, ..., d,\ \psi^a_{i,k+1} = (D_t\psi^a_{i,k} - x^{a(k+1)}D_t\xi_i) + \sigma^l_i(\psi^a_{l,k} - x^{a(k+1)}\xi_l)$.
Now, we compute

$$\begin{aligned} [Y_i, \Gamma] = & Y_i\Gamma - \Gamma Y_i \\ = & \psi^1_{i,1}(t, x^a)\frac{\partial}{\partial x^1} + \cdots + \psi^m_{i,1}(t, x^a)\frac{\partial}{\partial x^m} + \psi^1_{i,2}\frac{\partial}{\partial x^{1(1)}} + \cdots + \psi^m_{i,1}\frac{\partial}{\partial x^{m(1)}} + \cdots \\ & + \psi^1_{i,n-1}\frac{\partial}{\partial x^{1(n-2)}} + \cdots + \psi^m_{i,n-1}\frac{\partial}{\partial x^{m(n-2)}} + Y_i(f^1)\frac{\partial}{\partial x^{1(n-1)}} + \cdots \\ & + Y_i(f^m)\frac{\partial}{\partial x^{m(n-1)}} - \Gamma(\xi_i)\frac{\partial}{\partial t} - \Gamma(\psi^1_{i,0})\frac{\partial}{\partial x^1} - \cdots - \Gamma(\psi^m_{i,0})\frac{\partial}{\partial x^m} - \Gamma(\psi^1_{i,1})\frac{\partial}{\partial x^{1(1)}} \\ & - \cdots - \Gamma(\psi^m_{i,1})\frac{\partial}{\partial x^{m(1)}} - \cdots - \Gamma(\psi^1_{i,n-1})\frac{\partial}{\partial x^{1(n-1)}} - \cdots - \Gamma(\psi^m_{i,n-1})\frac{\partial}{\partial x^{m(n-1)}}. \end{aligned}$$

So on $S(\Delta) = \{x^{a(n)} - f^a = 0, a = 1, ..., m\}$,

$$\begin{aligned} [Y_i, \Gamma](t) &= -\Gamma(\xi_i) \\ [Y_i, \Gamma](x^1) &= \psi^1_{i,1} - \Gamma(\psi^1_{i,0}) = -x^{1(1)}\Gamma(\xi_i) + \sigma^l_i(\psi^1_{l,0} - x^{1(1)}\xi_l) \\ &\vdots \\ [Y_i, \Gamma](x^m) &= \psi^m_{i,1} - \Gamma(\psi^m_{i,0}) = -x^{m(1)}\Gamma(\xi_i) + \sigma^l_i(\psi^m_{l,0} - x^{m(1)}\xi_l) \\ [Y_i, \Gamma](x^{1(1)}) &= \psi^1_{i,2} - \Gamma(\psi^1_{i,1}) = -x^{1(2)}\Gamma(\xi_i) + \sigma^l_i(\psi^1_{l,1} - x^{1(1)}\xi_l) \\ &\vdots \\ [Y_i, \Gamma](x^{m(1)}) &= \psi^m_{i,2} - \Gamma(\psi^m_{i,1}) = -x^{m(2)}\Gamma(\xi_i) + \sigma^l_i(\psi^m_{l,1} - x^{m(2)}\xi_l) \\ &\vdots \\ [Y_i, \Gamma](x^{1(n-2)}) &= \psi^1_{i,n-1} - \Gamma(\psi^1_{i,n-2}) = -x^{1(n-1)}\Gamma(\xi_i) + \sigma^l_i(\psi^1_{l,n-2} - x^{1(n-2)}\xi_l) \\ &\vdots \end{aligned}$$

$$[Y_i,\Gamma](x^{m(n-2)}) = \psi^m_{i,n-1} - \Gamma(\psi^m_{i,n-2}) = -x^{m(n-1)}\Gamma(\xi_i) + \sigma^l_i(\psi^m_{l,n-2} - x^{m(n-2)}\xi_l)$$
$$[Y_i,\Gamma](x^{1(n-1)}) = Y_i(f^1) - \Gamma(\psi^1_{i,n-1})$$
$$\vdots$$
$$[Y_i,\Gamma](x^{m(n-1)}) = Y_i(f^m) - \Gamma(\psi^m_{i,n-1}).$$

Moreover, because $\mathcal{Y}$ is a σ-symmetry of (21), then on

$$S(\Delta) = \{x^{a(n)} - f^a = 0,\ a = 1,...,m\},$$

for the n-order prolongation $\tilde{\bar{Y}} = \{\bar{Y}_1,...,\bar{Y}_n\}$ of $\mathcal{Y}$, for $i = 1,...,d$, we have

$$\bar{Y}_i(x^{a(n)} - f^a) = 0, \quad i = 1,...,d, a = 1,...,m.$$

Hence $\quad \bar{Y}_i(x^{1(n)}) = \psi^1_{i,n} = Y_i(f^1) = (\Gamma\psi^1_{i,n-1} - f^1\Gamma\xi_i) + \sigma^l_i(\psi^1_{l,n-1} - f^1\xi_l),$

$$\vdots$$
$$\bar{Y}_i(x^{m(n)}) = \psi^m_{i,n} = Y_i(f^m) = (\Gamma\psi^m_{i,n-1} - f^m\Gamma\xi_i) + \sigma^l_i(\psi^m_{l,n-1} - f^m\xi_l),$$

and $\quad [Y_i,\Gamma](x^{1(n-1)}) = -f^1\Gamma\xi_i + \sigma^l_i(\psi^1_{l,n-1} - f^1\xi_l),$

$$\vdots$$
$$[Y_i,\Gamma](x^{m(n-1)}) = -f^m\Gamma\xi_i + \sigma^l_i(\psi^m_{l,n-1} - f^m\xi_l).$$

So we can get that $\quad [Y_i,\Gamma] = \sigma^l_i Y_l - (\Gamma(\xi_i) + \sigma^l_i\xi_l)\Gamma.$

(2) Suppose (22) holds. If we apply both elements of this equation to t, we obtain

$$[X_i,\Gamma](t) = X_i\Gamma(t) - \Gamma X_i(t) = -\Gamma(\xi_i) = \sigma^k_i\xi_k - (\Gamma(\xi_i) + \sigma^k_i\xi_k).$$

Moreover, compute that

$$\begin{aligned}[X_i,\Gamma] &= X_i\Gamma - \Gamma X_i \\ &= \eta^1_{i,1}(t,x^a)\frac{\partial}{\partial x^1} + \cdots + \eta^m_{i,1}(t,x^a)\frac{\partial}{\partial x^m} + \eta^1_{i,2}\frac{\partial}{\partial x^{1(1)}} + \cdots + \eta^m_{i,1}\frac{\partial}{\partial x^{m(1)}} + \cdots \\ &+ \eta^1_{i,n-1}\frac{\partial}{\partial x^{1(n-2)}} + \cdots + \eta^m_{i,n-1}\frac{\partial}{\partial x^{m(n-2)}} + X_i(f^1)\frac{\partial}{\partial x^{1(n-1)}} + \cdots \\ &+ X_i(f^m)\frac{\partial}{\partial x^{m(n-1)}} - \Gamma(\xi_i)\frac{\partial}{\partial t} - \Gamma(\eta^1_{i,0})\frac{\partial}{\partial x^1} - \cdots - \Gamma(\eta^m_{i,0})\frac{\partial}{\partial x^m} - \Gamma(\eta^1_{i,1})\frac{\partial}{\partial x^{1(1)}} \\ &- \cdots - \Gamma(\eta^m_{i,1})\frac{\partial}{\partial x^{m(1)}} - \cdots - \Gamma(\eta^1_{i,n-1})\frac{\partial}{\partial x^{1(n-1)}} - \cdots - \Gamma(\eta^m_{i,n-1})\frac{\partial}{\partial x^{m(n-1)}}.\end{aligned}$$

We apply both elements of the equation (22) to $x^{i(j)}$, $i = 1,...,m$, $j = 0,...,n-2$, and we obtain

$$[X_i,\Gamma](x^1) = \eta^1_{i,1} - \Gamma(\eta^1_{i,0}) = \sigma^k_i\eta^1_k - (\Gamma(\xi_i) + \sigma^k_i\xi_k)x^{1(1)},$$
$$\vdots$$
$$[X_i,\Gamma](x^m) = \eta^m_{i,1} - \Gamma(\eta^m_{i,0}) = \sigma^k_i\eta^m_k - (\Gamma(\xi_i) + \sigma^k_i\xi_k)x^{m(1)},$$
$$[X_i,\Gamma](x^{1(1)}) = \eta^1_{i,2} - \Gamma(\eta^1_{i,1}) = \sigma^k_i\eta^1_{k,1} - (\Gamma(\xi_i) + \sigma^k_i\xi_k)x^{1(2)},$$
$$\vdots$$

$$[X_i,\Gamma](x^{m(1)}) = \eta^m_{i,2} - \Gamma(\eta^m_{i,1}) = \sigma^k_i \eta^m_{k,1} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{m(2)},$$
$$\vdots$$
$$[X_i,\Gamma](x^{1(n-2)}) = \eta^1_{i,n-1} - \Gamma(\eta^1_{i,n-2}) = \sigma^k_i \eta^1_{k,n-2} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{1(n-1)},$$
$$\vdots$$
$$[X_i,\Gamma](x^{m(n-2)}) = \eta^m_{i,n-1} - \Gamma(\eta^m_{i,n-2}) = \sigma^k_i \eta^m_{k,n-2} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{m(n-1)},$$

then we know that

$$\begin{aligned}
\eta^1_{i,1} &= \Gamma(\eta^1_i) + \sigma^k_i \eta^1_k - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{1(1)},\\
&\vdots\\
\eta^m_{i,1} &= \Gamma(\eta^m_{i,0}) + \sigma^k_i \eta^m_k - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{m(1)},\\
\eta^1_{i,2} &= \Gamma(\eta^1_{i,1}) + \sigma^k_i \eta^1_{k,1} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{1(2)},\\
&\vdots\\
\eta^m_{i,2} &= \Gamma(\eta^m_{i,1}) + \sigma^k_i \eta^m_{k,1} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{m(2)},\\
&\vdots\\
\eta^1_{i,n-1} &= \Gamma(\eta^1_{i,n-2}) + \sigma^k_i \eta^1_{k,n-2} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{1(n-1)},\\
&\vdots\\
\eta^m_{i,n-1} &= \Gamma(\eta^m_{i,n-2}) + \sigma^k_i \eta^m_{k,n-2} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)x^{m(n-1)},
\end{aligned}$$

which implies $\{X_i, i = 1, ..., d\}$ is an $n-1$-order σ-prolongation of $\tilde{\mathcal{Y}}$.
Moreover, we apply both elements of the equation (22) to $x^{a(n-1)}$, $a = 1, ..., m$; then on $S(\Delta)$ we can obtain

$$\begin{aligned}
X_i(f^1) &= \Gamma(\eta^1_{i,n-1}) + \sigma^k_i \eta^1_{k,n-1} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)f^1,\\
&\vdots\\
X_i(f^m) &= \Gamma(\eta^m_{i,n-1}) + \sigma^k_i \eta^m_{k,n-1} - (\Gamma(\xi_i) + \sigma^k_i \xi_k)f^m.
\end{aligned}$$

Let us check that $\tilde{\mathcal{Y}}$ is a σ-symmetry of the equation (21), i.e., on $S(\Delta)$,

$$V_i(x^{a(n)} - f^a) = 0, \quad a = 1, ..., m,$$

where the vector fields set $\{V_i, i = 1, ..., d\}$ is the n-order σ-prolongation of $\tilde{\mathcal{Y}}$. In fact, on $S(\Delta)$,

$$\begin{aligned}
V_i(x^{a(n)} - f^a) &= \psi^a_{i,n} - X_i(f^a) = (\Gamma\psi^a_{i,n-1} - f^a\Gamma\xi_i) + \sigma^k_i(\psi^a_{k,n-1} - f^a\xi_k) - X_i(f^a)\\
&= (\Gamma\eta^a_{i,n-1} - f^a\Gamma\xi_i) + \sigma^k_i(\eta^a_{k,n-1} - f^a\xi_k) - X_i(f^a) = 0, \quad a = 1, ..., m.
\end{aligned}$$

■

Thus, we have the following

Theorem 4.4. *Let $\mathcal{Y} = \{X_i, i = 1, ..., d\}$ be a σ-symmetry of (21) such that the $n-1$-order σ-prolongation $\tilde{Y} = \{Y_1, ..., Y_d\}$ of $\mathcal{Y}$ is in involution and $\{Y_1, ..., Y_d, \Gamma\}$ are linearly independent everywhere. Then $i_{Y_1} \cdots i_{Y_l}\tilde{\Omega}$ is Frobenius integrable.*

Proof. Since $\mathcal{Y}$ is a σ-symmetry of (21), $\tilde{Y}$ is a σ-symmetry of $\ker\tilde{\Omega}$. So the Lie bracket is closed on $\mathrm{span}(\tilde{Y} \cup ker\tilde{\Omega}) = keri_{Y_1}\cdots i_{Y_l}\tilde{\Omega}$, and then $i_{Y_1}\cdots i_{Y_l}\tilde{\Omega}$ is Frobenius integrable by definition. ■

We know that when $\mathcal{Y} = \{X_i,\ i = 1, ..., m\}$ are in involution, its σ-prolongation $\{W_i, i = 1, ..., m\}$ doesn't have to be in involution; so we give the following theorem which can be fond in [7].

Theorem 4.5. *Let $\mathcal{Y} = \{X_i = \eta_i^a(t, x^a)\frac{\partial}{\partial x^a},\ i = 1, ..., m,\ a = 1, ..., n\}$ be a σ-symmetry of* (21) *satisfying $[X_i, X_j] = \mu_{ij}^k X_k$ for μ_{ij}^k smooth functions on M. If σ_i^j satisfy, for all $k = 1, ..., m$, the equations*

$$\left(D_x(\mu_{ij}^l) + \sigma_i^k\mu_{kj}^l - \sigma_j^k\mu_{ki}^l + W_i(\sigma_j^l) - W_j(\sigma_i^l) - \mu_{ij}^k\sigma_k^l\right)\psi_{l,q-1}^a = 0, \tag{23}$$

then, the qth σ-prolongations W_i satisfy the same involution relations, $q = 1, 2, \ldots$, i.e. $[W_i, W_j] = \mu_{ij}^k W_k$. Moreover, if the $n-1$-order σ-prolongation $\tilde{Y} = \{Y_1, ..., Y_m\}$ of $\mathcal{Y}$ and Γ are linearly independent everywhere, then $i_{Y_1}\cdots i_{Y_l}\tilde{\Omega}$ is Frobenius integrable.

Proof. This follows from an explicit computation. We proceed by induction on the order of the prolongation. Denoting by Z_i the $(q-1)$th σ-prolongation of X_i, we have

$$\begin{aligned}
[W_i, W_j] &= [W_i, Z_j + \psi_{j,q}^a\frac{\partial}{\partial x^{a,(q)}}] \\
&= [W_i, Z_j] + [W_i, \psi_{j,q}^a\frac{\partial}{\partial x^{a,(q)}}] \\
&= [Z_i + \psi_{i,q}^a\frac{\partial}{\partial x^{a,(q)}}, Z_j] + [W_i, \psi_{j,q}^a\frac{\partial}{\partial x^{a,(q)}}] \\
&= [Z_i, Z_j] - Z_j(\psi_{i,q}^a)\frac{\partial}{\partial x^{a,(q)}} + W_i(\psi_{j,q}^a)\frac{\partial}{\partial x^{a,(q)}} - \psi_{j,q}^a\frac{\partial\psi_{i,q}^a}{\partial x^{a,(q)}}\frac{\partial}{\partial x^{a,(q)}} \\
&= [Z_i, Z_j] + (W_i(\psi_{j,q}^a) - W_j(\psi_{i,q}^a))\frac{\partial}{\partial x^{a,(q)}}.
\end{aligned}$$

Thus, assuming $[Z_i, Z_j] = \mu_{ij}^k Z_k$ (i.e. the involution relations are satisfied for $q-1$th prolongations), the requirement that $[W_i, W_j] = \mu_{ij}^k W_k$ is equivalent to the requirement that

$$W_i(\psi_{j,q}^a) - W_j(\psi_{i,q}^a) = \mu_{ij}^k\psi_{k,q}^a = \mu_{ij}^k D_x\psi_{k,q-1}^a + \mu_{ij}^k\sigma_k^l\psi_{l,q-1}^a. \tag{24}$$

Using $W_i\psi_{j,q-1}^a = Z_i\psi_{j,q-1}^a$, with standard computation we obtain

$$\begin{aligned}
&W_i(\psi_{j,q}^a) - W_j(\psi_{i,q}^a) \\
&\quad= W_i(D_x\psi_{j,q-1}^a + \sigma_j^k\psi_{k,q-1}^a) - W_j(D_x\psi_{i,q-1}^a + \sigma_i^k\psi_{k,q-1}^a) \\
&\quad= W_iD_x\psi_{j,q-1}^a - W_jD_x\psi_{i,q-1}^a + \psi_{k,q-1}^aW_i(\sigma_j^k) - \psi_{k,q-1}^aW_j(\sigma_i^k) \\
&\qquad+ \sigma_j^kW_i(\psi_{k,q-1}^a) - \sigma_i^kW_j(\psi_{k,q-1}^a) \\
&\quad= D_xW_i\psi_{j,q-1}^a + \sigma_i^kW_k\psi_{j,q-1}^a - D_xW_j\psi_{i,q-1}^a - \sigma_j^kW_k\psi_{i,q-1}^a \\
&\qquad+ \psi_{k,q-1}^aW_i(\sigma_j^k) - \psi_{k,q-1}^aW_j(\sigma_i^k) + \sigma_j^kW_i(\psi_{k,q-1}^a) - \sigma_i^kW_j(\psi_{k,q-1}^a) \\
&\quad= D_xZ_i\psi_{j,q-1}^a + \sigma_i^kZ_k\psi_{j,q-1}^a - D_xZ_j\psi_{i,q-1}^a - \sigma_j^kZ_k\psi_{i,q-1}^a \\
&\qquad+ \psi_{k,q-1}^aW_i(\sigma_j^k) - \psi_{k,q-1}^aW_j(\sigma_i^k) + \sigma_j^kZ_i(\psi_{k,q-1}^a) - \sigma_i^kZ_j(\psi_{k,q-1}^a)
\end{aligned}$$

$$
\begin{aligned}
&= D_x(Z_i\psi^a_{j,q-1} - Z_j\psi^a_{i,q-1}) + \sigma^k_i(Z_k\psi^a_{j,q-1} - Z_j\psi^a_{k,q-1})\\
&\quad + \psi^a_{k,q-1}(W_i(\sigma^k_j) - W_j(\sigma^k_i)) + \sigma^k_j(Z_i(\psi^a_{k,q-1}) - Z_k(\psi^a_{i,q-1}))\\
&= D_x(\mu^k_{ij}\psi^a_{k,q-1}) + \sigma^k_i(\mu^l_{kj}\psi^a_{l,q-1}) - \sigma^k_j(\mu^l_{ki}\psi^a_{l,q-1}) + \psi^a_{k,q-1}(W_i(\sigma^k_j) - W_j(\sigma^k_i))\\
&= D_x(\mu^k_{ij})\psi^a_{k,q-1} + \mu^k_{ij}D_x(\psi^a_{k,q-1}) + \sigma^k_i(\mu^l_{kj}\psi^a_{l,q-1}) - \sigma^k_j(\mu^l_{ki}\psi^a_{l,q-1})\\
&\quad + \psi^a_{k,q-1}(W_i(\sigma^k_j) - W_j(\sigma^k_i)).
\end{aligned}
$$

Comparing with (24), we must require

$$
\begin{aligned}
D_x(\mu^k_{ij})\psi^a_{k,q-1} + \mu^k_{ij}D_x(\psi^a_{k,q-1}) + \sigma^k_i(\mu^l_{kj}\psi^a_{l,q-1}) - \sigma^k_j(\mu^l_{ki}\psi^a_{l,q-1})&\\
+ \psi^a_{k,q-1}(W_i(\sigma^k_j) - W_j(\sigma^k_i)) = \mu^k_{ij}D_x\psi^a_{k,q-1} + \mu^k_{ij}\sigma^l_k\psi^a_{l,q-1}&.
\end{aligned}
$$

That is, eliminating equal terms on both sides and renaming the summation indices,

$$D_x(\mu^l_{ij})\psi^a_{l,q-1} + \sigma^k_i(\mu^l_{kj}\psi^a_{l,q-1}) - \sigma^k_j(\mu^l_{ki}\psi^a_{l,q-1}) + \psi^a_{l,q-1}(W_i(\sigma^l_j) - W_j(\sigma^l_i)) - \mu^k_{ij}\sigma^l_k\psi^a_{l,q-1} = 0.$$

We can now collect the $\psi^a_{l,q-1}$ terms, and finally obtain

$$\left(D_x(\mu^l_{ij}) + \sigma^k_i\mu^l_{kj} - \sigma^k_j\mu^l_{ki} + W_i(\sigma^l_j) - W_j(\sigma^l_i) - \mu^k_{ij}\sigma^l_k\right)\psi^a_{l,q-1} = 0,$$

which is condition (23). So if condition (23) is satisfied, we know that $\tilde{Y}$ is in involution.

Moreover if the $n-1$-order σ-prolongation $\tilde{Y} = \{Y_1, ..., Y_m\}$ of $\mathcal{Y}$ and Γ are linearly independent everywhere, according to theorem 4.4, $i_{Y_1}\cdots i_{Y_l}\tilde{\Omega}$ is Frobenius integrable by definition. ■

Inspired by the idea in Ferraioli and Morando [5] and Sherring and Prince [21], the problem of finding first integrals can be reduced to finding Frobenius integrable 1-forms and the corresponding symmetries that can reduce the problems to first-order cases. We will show that σ-symmetries together with symmetries can be used to be derive first integrals, when differential equations do not possess enough symmetries and λ-symmetries.

Theorem 4.6. *Suppose*

(1) $\mathcal{X} = \{X_i,\ i = 1, ..., d\}$ *is a* σ*-symmetry of* (21) *and* $\mathcal{Y} = \{Y_i, i = 1, ..., d\}$ *is the* $n-1$*-order* σ*-prolongation of* $\mathcal{X}$*;*

(2) $Z_j,\ j = d+1, ..., mn$ *is symmetries of* (21) *and* Y_j *is the* $n-1$*-order standard-prolongation of* $Z_j,\ j = d+1, ..., mn$*;*

(3) $Y_j,\ j = d+1, ..., mn$ *satisfy the condition*

$$[Y_j, Y_l] \in span\{Y_1, ..., Y_d, Y_{d+1}, ..., Y_{j-1}, Y_{j+1}, \cdots, Y_{mn}, \Gamma\}$$

for any $j = d+1, ..., mn$ *and* $l = 1, ..., mn$*;*

(4) $\mathbf{Y}_j = \{Y_1, ..., Y_{j-1}, Y_{j+1}, ..., Y_{mn}, \Gamma\},\ j = d+1, ..., mn$ *are involutive systems.*

(5) $Y_1, ..., Y_{mn}$ *and* Γ *are linearly independent everywhere. Put*

$$\sigma^j = i_{Y_{d+1}}\cdots i_{Y_{j-1}}i_{Y_{j+1}}\cdots i_{Y_{mn}}i_{Y_1}\cdots i_{Y_d}\tilde{\Omega},\ j = d+1, ..., mn,$$

then $\omega^j = \dfrac{\sigma^j}{i_{Y_j}\sigma^j}$, $j = d+1, ..., mn$ *are closed, and locally provide* $mn-d$ *functionally independent first integrals of* (21)*.*

Proof. Similar to Theorem 4.4, we know that Lie bracket is closed on

$$\text{span}\{Y_1, ..., \quad Y_d, Y_{d+1}, ..., Y_{j-1}, Y_{j+1}, ..., Y_{mn}, \Gamma\} = \ker \sigma^j, \ j = d+1, ..., mn,$$

thus, σ^j is a Frobenius integrable 1-form. Since $Y_1, ..., Y_d, Y_{d+1}, ..., Y_{mn}$ and Γ are linearly independent, we have $i_{Y_j}\tilde{\Omega} \neq 0, i_{Y_k}\tilde{\Omega} \neq 0, j = 1, ..., d, k = d+1, ..., mn$, which implies $i_{Y_j}\sigma^j \neq 0$, and $\sigma^j, j = d+1, ..., mn$ is locally simple. Therefore, ω^i is closed according to the result in Sherring and Prince [21]. Then locally, $\omega^i = dI_i$, $i = d+1, ..., mn$, where I_i are functionally independent first integrals of (21).

The proof for ω^j is closed can be given by direct calculation. Without loss of generality, we show that ω^{d+1} is closed. Since Z_{d+1} is a symmetry of (21), there exists μ_{d+1} defined on $J^{n-1}M$ such that $L_{Y_{d+1}}\tilde{\Omega} = \mu_{d+1}\tilde{\Omega}$. Because of (3), there exist μ_l such that

$$\begin{aligned}
L_{Y_{d+1}}\sigma^{d+1} &= L_{Y_{d+1}}(i_{Y_{d+2}} \cdots i_{Y_{mn}} i_{Y_1} \cdots i_{Y_d}\tilde{\Omega}) \\
&= i_{L_{Y_{d+1}}Y_{d+2}} \cdots i_{Y_{mn}} i_{Y_1} \cdots i_{Y_d}\tilde{\Omega} + \cdots + i_{Y_{d+2}} \cdots i_{L_{Y_{d+1}}Y_{mn}} i_{Y_1} \cdots i_{Y_d}\tilde{\Omega} \\
&\quad + i_{Y_{d+2}} \cdots i_{Y_{mn}} i_{L_{Y_{d+1}}Y_1} \cdots i_{Y_d}\tilde{\Omega} + \cdots + i_{Y_{d+2}} \cdots i_{Y_{mn}} i_{Y_1} \cdots i_{L_{Y_{d+1}}Y_d}\tilde{\Omega} \\
&\quad + i_{Y_{d+2}} \cdots i_{Y_{mn}} i_{Y_1} \cdots i_{Y_d} L_{Y_{d+1}}\tilde{\Omega} \\
&= \sum_{l=1}^{mn} \mu_l \sigma^{d+1}.
\end{aligned}$$

Then we can calculate

$$\begin{aligned}
(i_{Y_{d+1}}\sigma^{d+1})^2 d\omega^{d+1} &= (i_{Y_{d+1}}\sigma^{d+1})d\sigma^{d+1} - d(i_{Y_{d+1}}\sigma^{d+1}) \wedge \sigma^{d+1} \\
&= (i_{Y_{d+1}}\sigma^{d+1})d\sigma^{d+1} - (L_{Y_{d+1}}\sigma^{d+1} - i_{Y_{d+1}}d\sigma^{d+1}) \wedge \sigma^{d+1} \\
&= i_{Y_{d+1}}(\sigma^{d+1} \wedge d\sigma^{d+1}) - \sum_{l=1}^{mn} \mu_l \sigma^{d+1} \wedge \sigma^{d+1}.
\end{aligned}$$

Because $\sigma^{d+1} \wedge d\sigma^{d+1} = 0$ by Frobenius integrability and $\sigma^{d+1} \wedge \sigma^{d+1} = 0$ by degree. ∎

Remark 4.7. If the symmetry $Z_j, j = d+1, ..., mn$ of (21) in theorem 4.6 has the form $\mathcal{Y}$ in theorem 4.5 satisfying conditions $[Z_i, Z_j] = 0, i, j = d+1, ..., mn$ and (23), then the condition (3) in theorem 4.6 can be replaced by (3'): $Y_j, j = d+1, ..., mn$ satisfy the condition $[Y_j, Y_l] \in span\{Y_1, ..., Y_d, Y_{d+1}, ..., Y_{j-1}, Y_{j+1}, \cdots, Y_{mn}, \Gamma\}$ for any $j = d+1, ..., mn$ and $l = 1, ..., d$;

In Theorem 4.6, if we don't have σ-symmetry of (21) but have a σ-symmetry of $\{Y_i, \ i = d+1, ..., mn\}$, we can get another theorem.

Theorem 4.8. *Suppose*

(1) $Z_j, \ j = d+1, ..., mn$ *are symmetries of* (21) *and* Y_j *are the* $n-1$*-order standard-prolongation of* $Z_j, \ j = d+1, ..., mn$*;*

(2) $\mathcal{Y} = \{Y_i, \ i = 1, ..., d\}$ *is a* σ*-symmetry of* $\{Y_j, \ j = d+1, ..., mn\}$*;*

(3) $Y_j, \ j = d+1, ..., mn$ *satisfy the condition*

$$[Y_j, Y_l] \in span\{Y_1, ..., Y_d, Y_{d+1}, ..., Y_{j-1}, Y_{j+1}, \cdots, Y_{mn}, \Gamma\}$$

for any $j = d+1, ..., mn$ *and* $l = 1, ..., mn$*;*

(4) $\mathbf{Y}_j = \{Y_1, ..., Y_{j-1}, Y_{j+1}, ..., Y_{mn}, \Gamma\}, j = d+1, ..., mn$ *are involutive systems.*

(5) $Y_1, ..., Y_{mn}$ *and* Γ *are linearly independent everywhere. Put*

$$\sigma^j = i_{Y_{d+1}} \cdots i_{Y_{j-1}} i_{Y_{j+1}} \cdots i_{Y_{mn}} i_{Y_1} \cdots i_{Y_d} \tilde{\Omega}, \quad j = d+1, ..., mn,$$

then $\omega^j = \dfrac{\sigma^j}{i_{Y_j}\sigma^j}$, $j = d+1, ..., mn$ *are closed, and locally provide* $mn - d$ *functionally independent first integrals of* (21)*.*

Proof. Similar to the proof of theorem 4.6. ■

Theorem 4.9. *Suppose*

(1) $\mathcal{X} = \{X_i, i = 1, ..., d\}$ *is a* σ*-symmetry of* (21) *and* $\mathcal{Y} = \{Y_i, i = 1, ..., d\}$ *is the* $n-1$*-order* σ*-prolongation of* $\mathcal{X}$ *which is in involution;*

(2) $\{\mathbf{Y} := \{Y_1, ..., Y_d, \Gamma\},\ \mathbf{Z} := \{Z_{d+1}, ..., Z_{mn}\}\}$ *is a solvable structure with respect to the involutive system* $\mathbf{Y}$*;*

(3) $Y_1, ..., Y_d,\ Z_{d+1}, ..., Z_{mn}$ *and* Γ *are linearly independent everywhere. Put* σ^j *and* ω^j *as before. Then* $d\omega^{d+1} = 0, d\omega^j = 0\ mod\ \omega^{d+1}, ..., \omega^{j-1},\ j = 2, ..., mn$*.*

So locally

$$\begin{aligned} \omega^{d+1} &= dI_1, \\ \omega^{d+2} &= dI_2 - Z_{d+1}(I_2)dI_1, \\ &\vdots \\ \omega^{mn} &= dI_{mn-d}\ mod\ \omega^{d+1}, ..., \omega^{mn-1}, \end{aligned}$$

where $I_1, ..., I_{mn-d}$ *are linearly independent first integrals of* (21)*, and they are also first integrals of vector fields* $Y_1, ..., Y_d$*.*

In particular, if $Y_j, j = 1, ..., d$ *are symmetries of span*$\{Y_{j+1}, ..., Y_d, \Gamma\}$ *and in smooth matrix function* $\sigma : J^1M \to Mat(n, R)$*,* $\sigma_d^i = 0, i = 1, ..., d$*, put*

$$\sigma^j = i_{Y_{d+1}} \cdots i_{Y_{mn}} i_{Y_1} \cdots i_{Y_{j-1}} i_{Y_{j+1}} \cdots i_{Y_d} \tilde{\Omega}, \quad \omega^j = \frac{\sigma^j}{i_{Y_j}\sigma^j}, \quad j = 1, ..., d,$$

then locally

$$\begin{aligned} \omega^1 &= dI_{mn-d+1}\ mod\ \omega^{d+1}, ..., \omega^{mn}, \\ &\vdots \\ \omega^d &= dI_{mn}\ mod\ \omega^{d+1}, ..., \omega^{mn}, \omega^1, \cdots, \omega^{d-1}, \end{aligned}$$

hence, we find a complete system of first integrals $I_1, ..., I_{mn}$ *for* (21)*.*

Proof. By assumption, span$\{Y_1, ..., Y_d, \Gamma\}$ is a Frobenius integral with symmetry Z_{mn}, therefore span$\{Z_{mn}, Y_1, ..., Y_d, \Gamma\}$ is a Frobenius integral and, recursively, span$\{Z_i, ..., Z_{mn}, Y_1, ..., Y_d, \Gamma\}$ is a Frobenius integral for $i = d+1, ..., mn$. Then σ^{d+1} is Frobenius integrable with symmetry Z_{d+1}, so ω^{d+1} is closed and locally $\omega^{d+1} = dI_1$. Consider the foliation $\mathcal{F}$ of kerω^{d+1} which is spanned by $Y_1, ..., Y_d, Z_{d+2}, Z_{d+3}, ..., Z_{mn}$ and Γ. Restrict σ^{d+2} to some leaf L_1 of $\mathcal{F}^1$. Then σ^{d+2} is Frobenius integrable on L_1 with symmetry Z_{d+2}, so ω^{d+2} is closed on L_1. Because of the decomposition of the exterior derivative, we have $d_1\omega^{d+2} = 0$, $d\omega^{d+2} = 0\ mod\ \omega^{d+1}$ and locally $\omega^{d+2} = d_1 I_2 = dI_2 - Z_1(I_2)dI_1$, where d_1 is the restriction of d to L_1. Then the result follows by induction and we have $\omega^i = dI_{i-d} - \sum_{l=1}^{i-d-1} Z_{d+l}(I_{i-d})\omega^{d+l}, i = d+2, ..., mn-d$.

In particular, if $Y_j, j = 1, ..., d$ are symmetries of span$\{Y_{j+1}, ..., Y_d, \Gamma\}$ and $\sigma_d^i = 0$, $i = 1, ..., d$, then Y_d is a symmetry of Γ, thus we can continue with the steps above until ω^d; therefore we can get

$$\omega^1 = dI_{mn-d+1} - \sum_{l=1}^{mn-d} Z_{d+l}(I_{mn-d+1})\omega^{d+l},$$
$$\vdots$$
$$\omega^d = dI_{mn} - \sum_{l=1}^{mn-d} Z_{d+l}(I_{mn})\omega^{d+l} - \sum_{k=1}^{d-1} Y_k(I_{mn})\omega^k. \qquad \blacksquare$$

Remark 4.10. If the σ-symmetry $\mathcal{X}$ of (21) in theorem 4.9 has the form $\mathcal{Y}$ in theorem 4.5 satisfying conditions $[X_i, X_j] = 0, i, j = 1, ..., d$ and (23), in smooth matrix function $\sigma : J^1M \to Mat(n, R)$, $\sigma_d^i = 0, i = 1, ..., d$, then we can also find a complete system of first integrals $I_1, ..., I_{mn}$ for (21). Moreover, in theorem 4.9 if $\mathcal{Y}$ and $\mathbf{Z}$ is a generalized solvable structure(see [14]) of kth-order ODE (21) ($k \geq 3$), we can find a complete system of first integrals $I_1, ..., I_{mn}$ for (21) too and the examples of this case can be found in [14].

Here, we give an example of how we can use our theory to find the first integral.

Example 4.11. Consider the system

$$\begin{cases} \dot{x}^1 = tx^2 + t^2x^1 - t^2x^3 \\ \dot{x}^2 = 2tx^2 + 2t^2x^1 - 2t^2x^3 \\ \dot{x}^3 = tx^2 + t^2x^1 - t^2x^3 \end{cases} \tag{25}$$

so the characteristic vector field is

$$\Gamma = \frac{\partial}{\partial t} + (tx^2 + t^2x^1 - t^2x^3)\frac{\partial}{\partial x^1} + (2tx^2 + 2t^2x^1 - 2t^2x^3)\frac{\partial}{\partial x^2} + (tx^2 + t^2x^1 - t^2x^3)\frac{\partial}{\partial x^3},$$

and the characteristic form is $\tilde{\Omega} = i_\Gamma dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. We can verify that the vector field set $\{X_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2},\ X_2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}\}$ is a σ-symmetry of (25) because by Lemma 4.3 we have

$$[X_1, \Gamma] = X_1\Gamma - \Gamma X_1 = t^2\frac{\partial}{\partial x^1} + 2t^2\frac{\partial}{\partial x^2} + t^2\frac{\partial}{\partial x^3} = t^2X_1 + t^2X_2,$$
$$[X_2, \Gamma] = X_2\Gamma - \Gamma X_2 = t\frac{\partial}{\partial x^1} + 2t\frac{\partial}{\partial x^2} + t\frac{\partial}{\partial x^3} = tX_1 + tX_2.$$

In this equation $\sigma_1^1 = t^2$, $\sigma_1^2 = t^2$, $\sigma_2^1 = t$, $\sigma_2^2 = t$ and clearly $[X_1, X_2] = 0$. We can also verify that $X_3 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3}$ is a symmetry of $\{X_1, X_2, \Gamma\}$, in fact we have $[X_3, X_2] = [X_3, X_1] = 0$, and

$$[X_3, \Gamma] = X_3\Gamma - \Gamma X_3 = -t^2\frac{\partial}{\partial x^1} - 2t^2\frac{\partial}{\partial x^2} - t^2\frac{\partial}{\partial x^3} = -t^2X_1 - t^2X_2.$$

Then by using Theorem 4.6, (also can use theorem 4.8 or theorem 4.9), we can get a first integral of (25). Calculate

$$\begin{aligned} & i_{X_2}i_{X_1}i_\Gamma(dt \wedge dx^1 \wedge dx^2 \wedge dx^3) \\ & = i_{X_2}i_{X_1}(dx^1 \wedge dx^2 \wedge dx^3 - (tx^2 + t^2x^1 - t^2x^3)dt \wedge dx^2 \wedge dx^3 \\ & \quad + (2tx^2 + 2t^2x^1 - 2t^2x^3)dt \wedge dx^1 \wedge dx^3 - (tx^2 + t^2x^1 - t^2x^3)dt \wedge x^1 \wedge x^2) \end{aligned}$$

$$= i_{X_2}(dx^2 \wedge dx^3 - (2tx^2 + 2t^2x^1 - 2t^2x^3)dt \wedge dx^3 + (tx^2 + t^2x^1 - t^2x^3)dt \wedge dx^2 - dx^1 \wedge dx^3 + (tx^2 + t^2x^1 - t^2x^3)dt \wedge dx^3 - (tx^2 + t^2x^1 - t^2x^3)dt \wedge dx^1) = dx^3 - dx^2 + dx^1.$$

Moreover, $i_{X_3} i_{X_2} i_{X_1} i_\Gamma(dt \wedge dx^1 \wedge dx^2 \wedge dx^3) = 2$, so

$$\frac{i_{X_2} i_{X_1} i_\Gamma(dt \wedge dx^1 \wedge dx^2 \wedge dx^3)}{i_{X_3} i_{X_2} i_{X_1} i_\Gamma(dt \wedge dx^1 \wedge dx^2 \wedge dx^3)} = d\,\frac{x^1 - x^2 + x^3}{2},$$

and it is obvious that

$$\frac{d\frac{x^1-x^2+x^3}{2}}{dt} = \frac{1}{2}(tx^2 + t^2x^1 - t^2x^3 - (2tx^2 + 2t^2x^1 - 2t^2x^3) + tx^2 + t^2x^1 - t^2x^3) = 0.$$

Thus, $\frac{x^1 - x^2 + x^3}{2}$ is a first integral of (25). ■

5. Conclusions

In this paper, σ-Liouville vector field have been introduced, and providing a general method to obtain the first integrals of system of a first-order ODE

For giving the first integrals of system of n-order ODE, we consider a solvable structure structure not only for the vector field associated to an ODE system, but for the involutive distribution formed by such vector field and σ-symmetry of the system. This idea was also behind the paper [14], but instead of a σ-symmetry, which is much more general, the authors considered the symmetry algebra $\mathfrak{sl}(2)$ and the structure underlying the symmetry algebra $\mathfrak{sl}(2)$ can be exploited to complete the integration.

According to Lemma 4.3, Theorems 4.4, and 4.5, if the ODE considered doesn't have a symmetry algebra $\mathfrak{sl}(2)$, we can also use σ-symmetry to find some first integrals.

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