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Characters of the Nullcone Related to Vinberg Groups

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Abstract. Let G be a reductive linear algebraic group defined over an algebraically closed field k of characteristic 0, and let θ be an automorphism of G of order m. We consider the Vinberg pair $(G_0, \mathfrak{g}_1)$, where G_0 is the identity component of the subgroup G^θ of θ-fixed points in G and $\mathfrak{g}_1$ is the ω-eigenspace of $\mathrm{d}\theta$ in $\mathfrak{g} = \mathrm{Lie}(G)$, where ω is a primitive mth root of 1 in k. In particular, we derive a formula for the formal characters of the G_0-modules $k_n[\mathcal{N}]$, where $\mathcal{N}$ is the variety of nilpotent elements in $\mathfrak{g}_1$ and $k_n[\mathcal{N}]$ is the space of polynomials on $\mathcal{N}$ of homogeneous degree n. We use this formula to compute the multiplicities of the simple highest weight modules in $k_n[\mathcal{N}]$. This multiplicity formula is also shown to hold for all n up to a certain maximum when k has positive characteristic.

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1. Introduction

The following setup will be assumed throughout. Let G be a reductive linear algebraic group defined over an algebraically closed field k, and let θ be an automorphism of G of finite order m. Let G^θ be the subgroup of θ-fixed points of G, and let G_0 be the identity component of G^θ. If $\mathrm{char}(k) = p > 0$, we will assume $p \nmid m$. We will also assume that G satisfies the standard hypotheses:

- When $\mathrm{char}(k) = p > 0$, p is a good prime for G. This means p is greater than any coefficient used to write a root of G as a $\mathbb{Z}$-linear combination of simple roots of G.
- The derived subgroup of G is simply connected.
- There is a nondegenerate, G-equivariant, symmetric, bilinear form defined on $\mathfrak{g} = \mathrm{Lie}(G)$.

The second standard hypothesis allows us to assume that G_0 is reductive by [18, Theorem 8.1].

Let $\omega = e^{2\pi i/m}$ in k. The differential, $\mathrm{d}\theta$, of θ is an automorphism of $\mathfrak{g}$ also of order m which induces a grading

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_n,$$

where $\mathfrak{g}_n$ is the ω^n-eigenspace of $\mathrm{d}\theta$ in $\mathfrak{g}$. When it is clear from the context, we will refer to $\mathrm{d}\theta$ simply as θ.

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We have that $\mathrm{Lie}(G_0) = \mathfrak{g}_0$ by [2, Section 9.1], and G_0 acts via the adjoint representation on $\mathfrak{g}_1$. The pair $(G_0, \mathfrak{g}_1)$ is known as a Vinberg pair, as introduced in [19] under the assumption that $k = \mathbb{C}$. Many of Vinberg's results were extended to fields of positive good characteristic in [14]. We will rely on [19] and [14] very heavily below.

The adjoint action of G_0 on $\mathfrak{g}_1$ induces an action of G_0 on the coordinate ring $k[\mathfrak{g}_1]$ given by $(g \cdot f)(x) = f(g^{-1} \cdot x)$ for $g \in G_0$, $f \in k[\mathfrak{g}_1]$, and $x \in \mathfrak{g}_1$. Let $\mathcal{N}$ denote the variety of nilpotent elements in $\mathfrak{g}_1$, which is the zero set of the ideal J^+ of $k[\mathfrak{g}_1]$ generated by the homogeneous G_0-invariant polynomials of positive degree in $k[\mathfrak{g}_1]$. The G_0-action on $k[\mathfrak{g}_1]$ restricts to one on the coordinate ring $k[\mathcal{N}]$.

Each of $k[\mathfrak{g}_1]$, $k[\mathfrak{g}_1]^{G_0}$ (which is the space of G_0-invariant polynomials in $k[\mathfrak{g}_1]$), $k[\mathcal{N}]$, and the ideal J^+ has an $\mathbb{N}$-grading given by homogeneous degree, with homogeneous parts of degree n denoted, respectively, by $k_n[\mathfrak{g}_1]$, $k_n[\mathfrak{g}_1]^{G_0}$, $k_n[\mathcal{N}]$, and J_n^+. The G_0-action on $k[\mathfrak{g}_1]$ restricts to one on these homogeneous parts for each n.

We are interested here in formulas for the formal characters of the G_0-modules $k_n[\mathcal{N}]$ and for the multiplicities of the simple highest weight modules in $k_n[\mathcal{N}]$. When $m = 1$ and $\mathrm{char}(k) = 0$, these formulas were derived by Hesselink in [7] using Kostant's Separation of Variables Theorem ([10, Theorem 11]). Hesselink's results were extended to fields of positive good characteristic in [9, Section 8]. There, Jantzen relies on the Springer resolution of $\mathcal{N}$. The initial investigation into the case $m = 2$ was undertaken by Kostant and Rallis in [11] and included a version of the Separation of Variables Theorem (see [11, Theorem 15]). Levy extended many of the results of [11] to fields of positive good characteristic in [13]. The author adapted Hesselink's formulas to the case $m = 2$ in [5], where the character and multiplicity formulas are derived for all n when $\mathrm{char}(k) = 0$, but only for all n up to a maximum when $\mathrm{char}(k) > 0$.

The main results of this paper are direct extensions of those in [5] and are derived using similar methods. They are contained in Section 3, and they rely on a version of the Separation of Variables Theorem, which is developed in Section 2. Section 4 provides an example of how these results may be applied by using them to decompose symmetric powers of the simple $\mathrm{SL}_9(k)$-module $\bigwedge^3(k^9)$.

Since $\mathrm{char}(k)$ does not divide the order of θ, θ is a semisimple automorphism of G. Thus by [18, Theorem 7.5], there is a θ-stable Borel subgroup B of G and a θ-stable maximal torus T of G contained in B. Let Φ_G be the root system of G relative to T, let Φ_G^+ be the set of positive roots of Φ_G corresponding to B, and let Δ_G be the basis of simple roots in Φ_G^+.

If we let $B_0 = B \cap G_0$ and $T_0 = T \cap G_0$, then the proof of Lemma 5.1 in [16] can be easily adapted to show that B_0 is a Borel subgroup of G_0 and T_0 is a maximal torus of G_0 contained in B_0. We will let Φ denote the root system of G_0 relative to T_0, with positive roots Φ^+ corresponding to B_0 and simple roots Δ.

We will need the set of weights $\Phi_{\mathfrak{g}_1}$ of T_0 on $\mathfrak{g}_1$, which can be derived from Φ_G as follows. Let

$$\Phi_0 = \{\alpha|_{T_0} : \alpha \in \Phi_G\} \cup \{0\}.$$

When considering $\mathfrak{g}$ as a G_0-module, we will denote it $\mathfrak{g}^{(G_0)}$.

Then Φ_0 is the set of weights of T_0 on $\mathfrak{g}^{(G_0)}$, and the corresponding weight space decomposition is

$$\mathfrak{g}^{(G_0)} = \bigoplus_{\chi\in\Phi_0} \mathfrak{g}_\chi^{(G_0)},$$

where
$$\mathfrak{g}_\chi^{(G_0)} = \bigoplus_{\{\alpha\in\Phi_G\,:\,\alpha|_{T_0}=\chi\}} \mathfrak{g}_\alpha.$$

Then
$$\Phi_{\mathfrak{g}_1} = \{\chi \in \Phi_0 \,:\, \mathfrak{g}_\chi^{(G_0)} \cap \mathfrak{g}_1 \neq \{0\}\},$$

and the weight space of $\chi \in \Phi_{\mathfrak{g}_1}$ is $(\mathfrak{g}_1)_\chi := \mathfrak{g}_\chi^{(G_0)} \cap \mathfrak{g}_1$.

Let γ be the graph automorphism of Φ_G induced by the automorphism θ, which is defined by $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{\gamma(\alpha)}$. For α and β in Φ_G, $\alpha|_{T_0} = \beta|_{T_0}$ if and only if $\beta = \gamma^i(\alpha)$ for some $i \in \{0, 1, \ldots, m-1\}$. Then by the discussion preceding Lemma 1.2 in [14], $\dim(\mathfrak{g}_1)_\chi = 1$ for all nonzero $\chi \in \Phi_{\mathfrak{g}_1}$.

We will let $X(T_0)$ and $Y(T_0)$ denote the character and cocharacter groups, respectively, of T_0, with $\langle\cdot,\cdot\rangle : X(T_0) \times Y(T_0) \to \mathbb{Z}$ denoting the pairing defined by $(\lambda \circ \psi)(a) = a^{\langle\lambda,\psi\rangle}$ for $a \in k^*$. For each $\alpha \in \Phi$, let $\alpha^\vee \in Y(T_0)$ denote the corresponding coroot, and for each $\alpha_i \in \Delta$, let $\varpi_i \in X(T_0)$ denote the corresponding fundamental weight. The set of dominant weights in $X(T_0)$, i.e., the elements $\lambda \in X(T_0)$ such that $\langle\lambda, \alpha^\vee\rangle \geq 0$ for all $\alpha \in \Delta$, will be denoted $X(T_0)^+$. Equivalently, $X(T_0)^+$ is the set of $\mathbb{Z}$-linear combinations of the fundamental weights with non-negative coefficients.

Let W be the Weyl group of G_0 relative to T_0. We will denote the standard action of W on $X(T_0)$ by $w(\lambda)$ for $w \in W$ and $\lambda \in X(T_0)$. The Weyl group also acts on $X(T_0)$ via the dot action, defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. Every W-orbit in $X(T_0)$ relative to the standard action contains a unique element of $X(T_0)^+$, and every W-orbit relative to the dot action contains a unique element of the set

$$C_0 := \{\lambda \in X(T_0) \,:\, \langle\lambda + \rho, \alpha^\vee\rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

For each $\lambda \in X(T_0)$, let k_λ denote the one-dimensional B_0-module defined by λ, and let $H^0(\lambda)$ be the G_0-module induced from k_λ. Then $H^0(\lambda) \neq 0$ if and only if $\lambda \in X(T_0)^+$ (see [8, Proposition II.2.6]). We also have the dual G_0-module $H^0(-w_0(\lambda))^*$ (where w_0 is the longest element in W) known as the Weyl module corresponding to λ and denoted by $V(\lambda)$. When $\mathrm{char}(k) = 0$ and $\lambda \in X(T_0)^+$, $H^0(\lambda)$ and $V(\lambda)$ are simple G_0-modules, but when $\mathrm{char}(k) = p > 0$, they are not necessarily simple. It is always the case, though, that $H^0(\lambda)$ contains a unique simple submodule ([8, Corollary II.2.3]) and that $V(\lambda)$ has a unique simple quotient. This means that every $\lambda \in X(T_0)^+$ defines a unique simple G_0-module, which we will denote by $L(\lambda)$. Conversely, every simple G_0-module is isomorphic to exactly one $L(\lambda)$ with $\lambda \in X(T_0)^+$. (See [8, Proposition 2.4a].)

There is a partial order $\preccurlyeq$ defined on $X(T_0)$ by

$$\mu \preccurlyeq \lambda \text{ if and only if } \lambda - \mu = \textstyle\sum_{\alpha\in\Delta} c_\alpha \alpha \text{ with } c_\alpha \in \mathbb{N} \cup \{0\} \text{ for each } \alpha.$$

If $\mu \in X(T_0)$ is a weight of T_0 on $L(\lambda)$, then $\mu \preccurlyeq \lambda$ ([8, Proposition II.2.4]), and $L(\lambda)$ is thus called the simple G_0-module of highest weight λ.

Suppose $\mathrm{char}(k) = p > 0$, and let C_p be the set

$$C_p := \{\lambda \in X(T_0) \,:\, 0 \leq \langle\lambda + \rho, \alpha^\vee\rangle \leq p \text{ for all } \alpha \in \Phi^+\}.$$

By [8, Corollary II.5.6], if $\lambda \in C_p \cap X(T_0)^+$, then $L(\lambda) = H^0(\lambda)$. This leads to the following known sufficient condition for the semisimplicity of G_0-modules in positive characteristic, on which we will rely in the next section.

Lemma 1.1. *Suppose M is a G_0-module with the property that for any composition factor $L(\lambda)$ of M, $\lambda \in C_p \cap X(T_0)^+$. Then M is semisimple.*

Proof. Suppose $L(\lambda)$ and $L(\mu)$ are any two composition factors of M. Then $L(\lambda) = H^0(\lambda)$ and $L(\mu) = H^0(\mu)$, so that by [8, Proposition II.2.14],

$$\mathrm{Ext}^1_{G_0}(L(\lambda), L(\mu)) = 0.$$

Therefore M is the direct sum of its composition factors and thus semisimple. ■

2. Separation of variables

The goal of this section is to establish a version of the Separation of Variables Theorem (see [10, Theorem 11] and [11, Theorem 15]), which will be a key tool used to develop the character formula in Section 3.

Let $\mathfrak{c}$ be a Cartan subspace (i.e., a maximal Abelian subspace consisting of semisimple elements) of $\mathfrak{g}_1$, and let $W_\mathfrak{c} = N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c})$. When $\mathrm{char}(k) = 0$, by [19, Theorem 8], $W_\mathfrak{c}$ is generated by pseudoreflections, which are transformations of finite order which fix a hyperplane. The Chevalley-Shephard-Todd Theorem thus implies that $k[\mathfrak{c}]^{W_\mathfrak{c}}$, the $W_\mathfrak{c}$-invariant polynomials in the coordinate ring $k[\mathfrak{c}]$, is generated by a set of r algebraically independent homogeneous polynomials of positive degree, where $r = \dim(\mathfrak{c})$. Vinberg also proves a version of the Chevalley Restriction Theorem ([19, Theorem 7]), namely that the restriction function $k[\mathfrak{g}_1]^{G_0} \to k[\mathfrak{c}]^{W_\mathfrak{c}}$ is an isomorphism. Thus, $k[\mathfrak{g}_1]^{G_0}$ is also generated by r algebraically independent homogeneous polynomials of positive degree. The degrees of these generating polynomials are called the characteristic degrees of $k[\mathfrak{g}_1]^{G_0}$ and can be found in [17, Table VII].

In [14], Levy extended the results in the previous paragraph to fields of positive good characteristic. In particular, by [14, Theorem 2.18], we have that the inclusion $\mathfrak{c} \hookrightarrow \mathfrak{g}_1$ induces an isomorphism of varieties $\mathfrak{c}/W_\mathfrak{c} \to \mathfrak{g}_1 /\!/ G_0 := \mathrm{Spec}(k[\mathfrak{g}_1]^{G_0})$, and thus the restriction $k[\mathfrak{g}_1] \to k[\mathfrak{c}]$ maps $k[\mathfrak{g}_1]^{G_0}$ isomorphically onto $k[\mathfrak{c}]^{W_\mathfrak{c}}$.

Still assuming positive good characteristic, in [14, Proposition 4.21], Levy also proves that when G is almost simple, $W_\mathfrak{c}$ is generated by pseudoreflections. He necessarily avoids the Chevalley-Shephard-Todd Theorem in favor of case-checking by root system type to conclude in [14, Theorem 4.22] that when G satisfies the standard hypotheses, $k[\mathfrak{c}]^{W_\mathfrak{c}}$, and hence $k[\mathfrak{g}_1]^{G_0}$, is generated by $r = \dim(\mathfrak{c})$ algebraically independent homogeneous polynomials of positive degree. By [3, V.5.1, Corollary], the list of degrees $d_1, \ldots, d_r$ of these generating polynomials in $k[\mathfrak{g}_1]^{G_0}$ is the same as the list of characteristic degrees obtained above when $\mathrm{char}(k) = 0$.

When $m = 1$, $W_\mathfrak{c}$ is just the Weyl group of Φ_G. When $m = 2$, Richardson constructed a root system associated to θ which he denoted Φ_A for which $W_\mathfrak{c}$ is the Weyl group (see [16, Section 4]). In either case, then, we have the identity

$$\sum_{w \in W_\mathfrak{c}} z^{l(w)} = \prod_{i=1}^{r} \frac{1 - z^{d_i}}{1 - z}, \tag{1}$$

where $l(w)$ is the length of w relative to the simple roots in Φ_A. This identity is true when $\mathrm{char}(k) = 0$ and when $\mathrm{char}(k)$ is positive and good. (See [4, Theorem 3] when $m = 1$ and [13, Lemma 4.11] when $m = 2$.)

Recall that J^+ is the ideal of $k[\mathfrak{g}_1]$ whose zero set is $\mathcal{N}$.

Lemma 2.1. *If* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k) > 0$ *and G satisfies the standard hypotheses, then J^+ is a radical ideal.*

Proof. The proof of Theorem 14 in [11] can be adapted as long as we can verify that (1) each irreducible component of $\mathcal{N}$ has codimension $r := \dim \mathfrak{c}$ in $\mathfrak{g}_1$ and that (2) for a set of algebraically independent homogeneous polynomials $u_1, \ldots, u_r$ which generates $k[\mathfrak{g}_1]^{G_0}$, the differentials $du_1, \ldots, du_r$ are linearly independent at each regular element $x \in \mathfrak{g}_1$. We can appeal to a few results from [14] which are stated for groups defined over fields of positive characteristic but which still hold when $\mathrm{char}(k) = 0$. Namely, (1) follows from [14, Corollary 2.20] (where we identify $\mathcal{N}$ with $\pi^{-1}(\pi(0))$), and (2) is implicit in the proof of [14, Proposition 5.2]. ∎

Remark 2.2. In the case that $\mathrm{char}(k) = 0$ and G is semisimple, we also get that J^+ is radical by [12, Corollary 4.8].

Assume $\mathrm{char}(k) = 0$. We first summarize Wallach's generalization of the Separation of Variables Theorem from [21] to our current context and then show how it can be adapted to the positive characteristic setting in a limited way.

Since $\mathrm{char}(k) = 0$, $k_n[\mathfrak{g}_1]$ is a semisimple G_0-module for each $n \geq 0$, which means for each $n \geq 0$, there exists a G_0-module H_n such that $k_n[\mathfrak{g}_1] = J_n^+ \oplus H_n$. We can then define the graded G_0-module $H = \bigoplus_{n \geq 0} H_n$. By [21, Corollary 1], the map

$$k[\mathfrak{g}_1]^{G_0} \otimes H \to k[\mathfrak{g}_1], \tag{2}$$

defined by $f \otimes g \mapsto fg$, is an isomorphism of G_0-modules, and as G_0-modules,

$$H \cong k[\mathfrak{g}_1]/J^+.$$

By Lemma 2.1, we can conclude that, as graded G_0-modules, $H \cong k[\mathcal{N}]$.

Assume now that $\mathrm{char}(k) = p > 0$, that p does not divide the order m of θ, and that p does not divide the order of $W_{\mathfrak{c}}$. It need not be the case that $k_n[\mathfrak{g}_1]$ is a semisimple G_0-module for all $n \geq 0$, so we cannot repeat the construction of the graded G_0-module H from above. We will show, though, that under these conditions on $\mathrm{char}(k)$, Wallach's result can be partially recovered.

Lemma 2.3. *The coordinate ring $k[\mathfrak{c}]$ is a free $k[\mathfrak{c}]^{W_{\mathfrak{c}}}$-module.*

Proof. This is a special case of Theorem 1 in Section V.5.2 of [3] with $V = \mathfrak{c}$ and $G = W_{\mathfrak{c}}$. ∎

Corollary 2.4. *The coordinate ring $k[\mathfrak{g}_1]$ is a free $k[\mathfrak{g}_1]^{G_0}$-module.*

Proof. For a vector space V over k, a subspace U of V, and a graded subalgebra I of $k[V]$, Proposition 1.1 in [1] states that if

(1) the restriction morphism $\mathrm{res} : I \to k[U]$ is injective, and

(2) $k[U]$ is a free module over $\mathrm{res}(I)$,

then $k[V]$ is a free module over $k[V/U] \otimes I$.

Letting $V = \mathfrak{g}_1$, $U = \mathfrak{c}$, and $I = k[\mathfrak{g}_1]^{G_0}$, we have that (1) holds by [14, Theorem 2.18] and that (2) holds by Lemma 2.3. Thus $k[\mathfrak{g}_1]$ is free over $k[\mathfrak{g}_1/\mathfrak{c}] \otimes k[\mathfrak{g}_1]^{G_0}$, and the result follows. ■

Let $\mathcal{S}$ be the set of maximal weights in $\Phi_{\mathfrak{g}_1}$ relative to the partial order $\preccurlyeq$ on $X(T_0)$. Let N_p be the largest non-negative integer such that $N_p\delta \in C_p$ for all $\delta \in \mathcal{S}$. For an integer n and $\delta \in \mathcal{S}$, $n\delta \in C_p$ if and only if $\langle n\delta + \rho, \mu \rangle \leq p$, where μ is the coroot of the highest short root in Φ. Since $\langle \rho, \mu \rangle = h - 1$, where h is the Coxeter number of Φ, we have that

$$N_p = \min_{\delta \in \mathcal{S}} \left\lfloor \frac{p - h + 1}{\langle \delta, \mu \rangle} \right\rfloor .$$

The idea for the following lemma comes from [6, Proposition 4.4].

Lemma 2.5. *For $n \leq N_p$, $k_n[\mathfrak{g}_1]$ is a semisimple G_0-module.*

Proof. The possible composition factors of $k_n[\mathfrak{g}_1] = S^n(\mathfrak{g}_1^*)$ are the duals of those of $S^n(\mathfrak{g}_1)$. The dual G_0-module $L(\lambda)^*$ is isomorphic to $L(-w_0(\lambda))$, where w_0 is the longest element in W (see [8, Corollary II.2.5]). Thus, the possible composition factors of $k_n[\mathfrak{g}_1]$ are the simple G_0-modules $L(-w_0(\lambda))$, where λ is a dominant weight such that $\lambda \preccurlyeq n\delta$ for some $\delta \in \mathcal{S}$. Since $n\delta$ is in $C_p \cap X(T_0)^+$, so is any such λ. Because $-w_0(\rho) = \rho$, $-w_0(C_p) = C_p$, which means that if $\lambda \preccurlyeq n\delta$ for $\delta \in \mathcal{S}$, then $-w_0(\lambda) \in C_p \cap X(T_0)^+$, and the result now follows from Lemma 1.1. ■

For each $n \geq 0$, we define H_n to be a subspace of $k_n[\mathfrak{g}_1]$ such that $k_n[\mathfrak{g}_1] \cong J_n^+ \oplus H_n$. By Lemma 2.5, when $n \leq N_p$, we may assume H_n is G_0-invariant and that $k_n[\mathfrak{g}_1]$ and $J_n^+ \oplus H_n$ are isomorphic as G_0-modules. Let $H = \bigoplus_{n \geq 0} H_n$. Then by Lemma 2.1, H is isomorphic to $k[\mathcal{N}]$ as a graded vector space, and H_n is isomorphic to $k_n[\mathcal{N}]$ as a G_0-module when $n \leq N_p$.

We let $(k[\mathfrak{g}_1]^{G_0} \otimes H)_n$ denote $\bigoplus_{i \leq n}(k_i[\mathfrak{g}_1]^{G_0} \otimes H_{n-i})$. Clearly, when $n \leq N_p$, $(k[\mathfrak{g}_1]^{G_0} \otimes H)_n$ is also a G_0-module. We now have the following adaptation of (2):

Theorem 2.6. *For $n \leq N_p$, the map $(k[\mathfrak{g}_1]^{G_0} \otimes H)_n \to k_n[\mathfrak{g}_1]$, given by $f \otimes g \mapsto fg$, is a G_0-module isomorphism.*

Proof. We will follow the argument in [11, Lemma 19] to first prove that for each $n \geq 0$, there is a vector space isomorphism

$$\psi_n : k_n[\mathfrak{g}_1] \to (k[\mathfrak{g}_1]^{G_0} H)_n = \bigoplus_{i \leq n} (k_i[\mathfrak{g}_1]^{G_0} H_{n-i})$$

which is a G_0-module isomorphism when $n \leq N_p$. We proceed by induction on n, the result being trivial when $n = 0$. Suppose now that the maps ψ_j are isomorphisms for all $j < n$. Let f be in $k_n[\mathfrak{g}_1]$. By the definition of H_n, we have a vector space isomorphism $\phi : k_n[\mathfrak{g}_1] \to J_n^+ \oplus H_n$ which is G_0-invariant when $n \leq N_p$. Suppose $\phi(f) = \sum_i v_i q_i + h$, where $h \in H_n$ and for each i, v_i is a degree-n_i polynomial in $k[\mathfrak{g}_1]^{G_0}$ with $n_i > 0$, and $q_i \in k_{n-n_i}[\mathfrak{g}_1]$. Assuming by the induction hypothesis that ψ_{n-n_i} is an isomorphism for each i and $\psi_{n-n_i}(q_i) \in (k[\mathfrak{g}_1]^{G_0} H)_{n-n_i}$, we can now define $\psi_n : k_n[\mathfrak{g}_1] \to (k[\mathfrak{g}_1]^{G_0} H)_n$ by

$$\psi_n(f) = \sum_i v_i \psi_{n-n_i}(q_i) + h.$$

Then ψ_n is a vector space isomorphism for all n and is G_0-invariant when $n \leq N_p$. The theorem now follows immediately since $k[\mathfrak{g}_1]$ is a free $k[\mathfrak{g}_1]^{G_0}$-module by Corollary 2.4. ■

3. Character and multiplicity formulas

The multiplicity of the simple G_0-module $L(\lambda)$ in a G_0-module M is the number of times $L(\lambda)$ occurs as a composition factor of M. This section contains our main result, which is a method for computing the multiplicity of $L(\lambda)$ (for each $\lambda \in X(T_0)^+$) in $k_n[\mathcal{N}]$ that works for all n when $\mathrm{char}(k) = 0$ and for $n \leq N_p$ when $\mathrm{char}(k) = p > 0$.

Given a T_0-module M and $\lambda \in X(T_0)$, we denote the λ-weight space of T_0 on M by M_λ. The formal character of M is an element in the group ring $\mathbb{Z}[X(T_0)]$ defined by

$$\mathrm{ch}(M) = \sum_{\lambda \in X(T_0)} \dim(M_\lambda) e^\lambda,$$

where the elements e^λ, with $\lambda \in X(T_0)$, are the basis elements of $\mathbb{Z}[X(T_0)]$, and $e^\lambda e^\mu = e^{\lambda+\mu}$. If $V = \bigoplus_{n \geq 0} V_n$ is an $\mathbb{N}$-graded T_0-module, then we let $\mathrm{ch}_z(V) = \sum_{n \geq 0} \mathrm{ch}(V_n) z^n$.

For T_0-modules M and M',

$$\mathrm{ch}(M \otimes M') = \mathrm{ch}(M)\mathrm{ch}(M') \text{ and } \mathrm{ch}(M \oplus M') = \mathrm{ch}(M) + \mathrm{ch}(M').$$

It follows that for graded T_0-modules V and V', $\mathrm{ch}_z(V \otimes V') = \mathrm{ch}_z(V)\mathrm{ch}_z(V')$.

The action of the Weyl group W extends linearly to $\mathbb{Z}[X(T_0)]$. Let D be the additive group endomorphism of $\mathbb{Z}[X(T_0)]$ defined by $D(e^\lambda) = \sum_{w \in W} \det(w) e^{w(\lambda)}$, where $\det(w) = (-1)^{l(w)}$.

In the rest of this section, we will let $V = \bigoplus_{n \geq 0} V_n$ be an $\mathbb{N}$-graded T_0-module. We will also let $p_n(\lambda) = \dim((V_n)_\lambda)$ if λ is a weight of T_0 on V_n and $p_n(\lambda) = 0$ otherwise.

Lemma 3.1. *For each $n \geq 0$,*

$$\mathrm{ch}(V_n) = \sum_{\lambda \in X(T_0)} p_n(\lambda) D(e^{\lambda+\rho})/D(e^\rho).$$

Proof. We have that each element $w \in W$ permutes the weights of T_0 on V_n, and $p_n(\lambda) = p_n(w(\lambda))$ for all $w \in W$. Thus, since $\mathrm{ch}(V_n) = \sum_{\lambda \in X(T_0)} p_n(\lambda) e^\lambda$,

$$\begin{aligned} D(e^\rho)\mathrm{ch}(V_n) &= \sum_{w \in W} \left(\sum_{\lambda \in X(T_0)} p_n(\lambda) \det(w) e^{w(\rho)+\lambda} \right) \\ &= \sum_{w \in W} \left(\sum_{\lambda \in X(T_0)} p_n(w(\lambda)) \det(w) e^{w(\rho)+w(\lambda)} \right) \\ &= \sum_{\lambda \in X(T_0)} \left(\sum_{w \in W} p_n(w(\lambda)) \det(w) e^{w(\rho)+w(\lambda)} \right) \\ &= \sum_{\lambda \in X(T_0)} p_n(\lambda) D(e^{\lambda+\rho}). \end{aligned}$$

■

Theorem 3.2. *Suppose* $\mathrm{char}(k) = 0$. *Let* $m_n(\lambda)$ *denote the multiplicity of the simple* G_0*-module* $L(\lambda)$ *in* V_n. *For any* $n \geq 0$ *and* $\lambda \in X(T_0)^+$,

$$m_n(\lambda) = \sum_{w\in W} \det(w) p_n(w(\lambda+\rho)-\rho).$$

Proof. Recall that the set C_0 defined in Section 1 is a fundamental domain for the dot action of W on $X(T_0)$. It follows from Corollary II.5.5a and Proposition II.5.10 in [8] that if $\lambda \in C_0 \setminus X(T_0)^+$, then $D(e^{\lambda+\rho})/D(e^\rho) = 0$. Also, since Weyl's Character Formula holds when $\mathrm{char}(k) = 0$, we have $D(e^{\lambda+\rho})/D(e^\rho) = \mathrm{ch}(L(\lambda))$ for $\lambda \in X(T_0)^+$. Lastly, notice that for any $w \in W$ and $\lambda \in X(T_0)$,

$$D(e^{w(\lambda)}) = \sum_{\overline{w}\in W} \det(\overline{w}w^{-1}) e^{\overline{w}w^{-1}(w(\lambda))} = \det(w) D(e^\lambda).$$

Now by Lemma 3.1, for $n \geq 0$ we have

$$\begin{aligned}
\mathrm{ch}(V_n) &= \sum_{\lambda\in X(T_0)} p_n(\lambda) D(e^{\lambda+\rho})/D(e^\rho) \\
&= \sum_{\lambda\in C_0} \left(\sum_{w\in W} p_n(w(\lambda+\rho)-\rho) D(e^{w(\lambda+\rho)})/D(e^\rho) \right) \\
&= \sum_{\lambda\in C_0} \left(\sum_{w\in W} \det(w) p_n(w(\lambda+\rho)-\rho) \right) D(e^{\lambda+\rho})/D(e^\rho) \\
&= \sum_{\lambda\in X(T_0)^+} \left(\sum_{w\in W} \det(w) p_n(w(\lambda+\rho)-\rho) \right) \mathrm{ch}(L(\lambda)),
\end{aligned}$$

and the formula for $m_n(\lambda)$ follows. ∎

To obtain our desired formula for the multiplicity of $L(\lambda)$ in $k_n[\mathcal{N}]$, we will need the character formula for $k[\mathcal{N}]$, which is provided by the following theorem.

Theorem 3.3. *Suppose* $\mathrm{char}(k) = 0$. *Let* $r = \dim(\mathfrak{c})$ *and* $R = \dim(\mathfrak{g}_1)_0$. *Let* $d_1, \ldots, d_r$ *be the characteristic degrees of* $k[\mathfrak{g}_1]^{G_0}$. *Then*

$$\mathrm{ch}_z(k[\mathcal{N}]) = (1-z)^{-R} \prod_{i=1}^{r} (1-z^{d_i}) \prod_{\chi\in\Phi_{\mathfrak{g}_1}\setminus\{0\}} (1-e^\chi z)^{-1}.$$

Proof. By (2) and the fact that $H \cong k[\mathcal{N}]$ as graded G_0-modules,

$$\mathrm{ch}_z(k[\mathcal{N}]) = \frac{\mathrm{ch}_z(k[\mathfrak{g}_1])}{\mathrm{ch}_z(k[\mathfrak{g}_1]^{G_0})}. \tag{3}$$

Using the weight space decomposition of $\mathfrak{g}_1$ and the fact stated in Section 1 that $\dim(\mathfrak{g}_1)_\chi = 1$ for each $\chi \in \Phi_{\mathfrak{g}_1} \setminus \{0\}$, we have

$$\mathrm{ch}(\mathfrak{g}_1) = Re^0 + \sum_{\chi\in\Phi_{\mathfrak{g}_1}\setminus\{0\}} e^\chi,$$

and thus

$$\mathrm{ch}_z(k[\mathfrak{g}_1]) = (1-z)^{-R} \prod_{\chi\in\Phi_{\mathfrak{g}_1}\setminus\{0\}} (1-e^\chi z)^{-1}. \tag{4}$$

Since $k[\mathfrak{g}_1]^{G_0}$ is generated by r algebraically independent homogeneous polynomials with degrees $d_1, \ldots, d_r$,

$$\mathrm{ch}_z(k[\mathfrak{g}_1]^{G_0}) = \prod_{i=1}^{r}(1-z^{d_i})^{-1}.$$

The result now follows immediately from (3). ∎

Remark 3.4. The identity (1) in Section 2 allows us to simplify the above character formula to the one in [7, Section 3 Lemma] when $m=1$ and the one in [5, Proposition 4.2.1] when $m=2$.

Remark 3.5. When $n \leq d_i$ for $1 \leq i \leq r$, $\mathrm{ch}(k_n[\mathcal{N}]) = \mathrm{ch}(k_n[\mathfrak{g}_1])$.

When $V = k[\mathcal{N}]$, Theorem 3.3 provides a way to find the numbers $p_n(\lambda)$ needed to compute the multiplicities from Theorem 3.2. Let $t_n(\lambda)$ represent the number of maps $f : \Phi_{\mathfrak{g}_1} \setminus \{0\} \to \mathbb{N} \cup \{0\}$ such that

$$\sum_{\chi \in \Phi_{\mathfrak{g}_1} \setminus \{0\}} f(\chi)\chi = \lambda \quad \text{and} \quad \sum_{\chi \in \Phi_{\mathfrak{g}_1} \setminus \{0\}} f(\chi) = n.$$

Then
$$\prod_{\chi \in \Phi_{\mathfrak{g}_1} \setminus \{0\}} (1-e^{\chi}z)^{-1} = \sum_{n \geq 0} \left(\sum_{\lambda \in X(T_0)} t_n(\lambda) e^{\lambda} \right) z^n.$$

Now for each $n \geq 0$, if we let s_n be the integer such that

$$(1-z)^{-R} \prod_{i=1}^{r}(1-z^{d_i}) = \sum_{n \geq 0} s_n z^n,$$

then
$$p_n(\lambda) = \sum_{i=0}^{n} s_i t_{n-i}(\lambda). \tag{5}$$

For the rest of this section, we will now assume that $\mathrm{char}(k) = p > 0$ and, as always, that G satisfies the standard hypotheses. In particular, we assume p is good for G. The next theorem shows that Theorem 3.2 extends partially to this case. The proof very closely follows that of [6, Proposition 4.4(ii)]. The numbers $p_n(\lambda)$ are defined by the character formula in Theorem 3.3; i.e., $p_n(\lambda)$ is the dimension of the λ-weight space in $k_n[\mathcal{N}]$ when λ is a weight of T_0, and $p_n(\lambda) = 0$ otherwise.

Theorem 3.6. *Suppose* $\mathrm{char}(k) = p > 0$ *and that* p *divides neither* m *nor the order of* $W_{\mathfrak{c}}$. *When* $n \leq N_p$, *the formula for* $m_n(\lambda)$ *agrees with the formula in the case when* $\mathrm{char}(k) = 0$. *In other words, for* $\lambda \in X(T_0)^+$,

$$m_n(\lambda) = \sum_{w \in W} \det(w) p_n(w(\lambda+\rho)-\rho).$$

Proof. Let F be any field, and let $\mathscr{K}_F$ be the category of finite-dimensional rational G_0-modules over F. Let $K(\mathscr{K}_F)$ be the corresponding Grothendieck group. The class in $K(\mathscr{K}_F)$ containing a G_0-module V is denoted $[V]$.

Let k_0 be an algebraically closed field of characteristic 0, and let k_p be an algebraically closed field satisfying the hypotheses of the theorem. The simple G_0-modules when $F = k_0$, respectively when $F = k_p$, are precisely the highest weight modules $L(\lambda)_{k_0}$, respectively $L(\lambda)_{k_p}$, with $\lambda \in X(T_0)^+$. Thus the free Abelian groups $K(\mathscr{K}_{k_0})$ and $K(\mathscr{K}_{k_p})$ have bases

$$\{[L(\lambda)_{k_0}] : \lambda \in X(T_0)^+\} \quad \text{and} \quad \{[L(\lambda)_{k_p}] : \lambda \in X(T_0)^+\},$$

respectively. The map $[L(\lambda)_{k_0}] \mapsto [L(\lambda)_{k_p}]$ for $\lambda \in X(T_0)^+$ thus defines an isomorphism $K(\mathscr{K}_{k_0}) \to K(\mathscr{K}_{k_p})$.

Identifying $K(\mathscr{K}_{k_0})$ with $K(\mathscr{K}_{k_p})$, we can obtain the desired equality of multiplicities for k_0 and k_p if we can show that $[(k_0)_n[\mathcal{N}]] = [(k_p)_n[\mathcal{N}]]$ for $n \leq N_p$. (Recall that in this case, $L(\lambda)_{k_p} = H^0(\lambda)$, so that $\mathrm{ch}(L(\lambda)_{k_p}) = \mathrm{ch}(L(\lambda)_{k_0})$.) This can be done by induction on n, with the case $n = 0$ being trivial since $[k_0] = [k_p] = 0$ in $K(\mathscr{K}_{k_0}) = K(\mathscr{K}_{k_p})$.

Suppose for a given $n \leq N_p$ that $[(k_0)_j[\mathcal{N}]] = [(k_p)_j[\mathcal{N}]]$ for all $j < n$. As shown in the proof of Lemma 2.5, the weights of the composition factors of $(k_p)_n[\mathfrak{g}_1]$ lie in $C_p \cap X(T_0)^+$. Formula (4) in the proof of Theorem 3.3 depends only on the weight space decomposition of $\mathfrak{g}_1$ and is thus true for both k_0 and k_p. Therefore the formal character of $(k_0)_n[\mathfrak{g}_1]$ coincides with that of $(k_p)_n[\mathfrak{g}_1]$. Thus $[(k_0)_n[\mathfrak{g}_1]] = [(k_p)_n[\mathfrak{g}_1]]$.

Now by Theorem 2.6, which is true for both k_0 and k_p,

$$[(k_0)_n[\mathfrak{g}_1]] = \bigoplus_{i=0}^{n} \dim_{k_0}(J_i^+)[(k_0)_{n-i}[\mathcal{N}]] \tag{6}$$

and

$$[(k_p)_n[\mathfrak{g}_1]] = \bigoplus_{i=0}^{n} \dim_{k_p}(J_i^+)[(k_p)_{n-i}[\mathcal{N}]]. \tag{7}$$

As mentioned in Section 2, the characteristic degrees of $k_0[\mathfrak{g}_1]^{G_0}$ are the same as those of $k_p[\mathfrak{g}_1]^{G_0}$, and thus $\dim_{k_0}(J_i^+) = \dim_{k_p}(J_i^+)$ for all i. Therefore by the induction hypothesis, the summands of (6) and (7) agree for each $i > 0$. Since the left sides of (6) and (7) also agree, we can conclude that $[(k_0)_n[\mathcal{N}]] = [(k_p)_n[\mathcal{N}]]$. ■

4. An example

Suppose $\mathrm{char}(k) = 0$, and let V be a 9-dimensional vector space over k. The elements of the third exterior power $\bigwedge^3 V$ of V are called the trivectors of V. In [20], Vinberg and Èlashvili classified the orbits of trivectors of V under the natural action of $\mathrm{SL}(V)$. In particular, they found that the nullcone $\mathcal{N}$ of $\bigwedge^3 V$ consists of 102 orbits of nilpotent trivectors. Their methods involved realizing $(\mathrm{SL}(V), \bigwedge^3 V)$ as a Vinberg pair. We will use their construction along with the character and multiplicity formulas from Section 3 to describe $k_2[\mathcal{N}]$ and $k_3[\mathcal{N}]$ as $\mathrm{SL}(V)$-modules. By Remark 3.5, this will amount to descriptions of the $\mathrm{SL}(V)$-modules $S^2[\bigwedge^3 V]$ and $S^3[\bigwedge^3 V]$.

Following Vinberg et al. [20, Section 2.2], let $\mathfrak{g}$ be the $\mathbb{Z}/3\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = L_0(V) \oplus \bigwedge^3 V \oplus \bigwedge^3 V^*$, where $L_0(V)$ is the space of endomorphisms of V of trace 0 and V^* is the dual space of V. Let $\mathfrak{h}$ be the subalgebra of $L_0(V)$ consisting of the diagonal transformations relative to a fixed basis of V. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. If $\varepsilon_1, \dots, \varepsilon_9$ are the diagonal coordinate functions on $\mathfrak{h}$, then the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ is $\{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k) : i, j, k \text{ distinct}\}$, which is of Type E_8. Further calculation shows that $\mathfrak{g}$ is semisimple and is thus also of Type E_8. This graded Lie algebra is summarized in row 9 of the table in [19, Section 9].

Now following [20, Section 2.3], let $\omega = e^{2\pi i/3}$. Define an automorphism of order 3 of $\mathfrak{g}$ by $\theta(x_0, x_1, x_2) = (x_0, \omega x_1, \omega^2 x_2)$. Then $\mathfrak{g}_0 = L_0(V)$, $\mathfrak{g}_1 = \bigwedge^3 V$, and $\mathfrak{g}_2 = \bigwedge^3 V^*$. Let G be a reductive algebraic group defined over k of Type E_8 with $\mathrm{Lie}(G) = \mathfrak{g}$. Then θ is the differential of a uniquely determined (necessarily inner) automorphism of G with $\mathrm{Lie}(G_0) = \mathfrak{g}_0$. There is homomorphism $\pi : \mathrm{SL}(V) \to G_0$ which is onto and has a kernel of order 3 contained in the center. Thus, we can identify the action of G_0 on $\mathfrak{g}_1$ with that of $\mathrm{SL}(V)$ on $\mathfrak{g}_1$. Also, this action coincides with the natural action of $\mathrm{SL}(V)$ on $\bigwedge^3 V$.

We can now apply the formulas of Section 3 to the Vinberg pair $(G_0, \mathfrak{g}_1)$ to describe $k_2[\mathcal{N}]$ as G_0-module. By [19, Section 9], we have that $W_{\mathfrak{c}}$ is group No. 32 in Shephard and Todd's classification in [17], $r = 4$, and the characteristic degrees of $k[\mathfrak{g}_1]^{G_0}$ are 12, 18, 24, and 30.

Further, since $\mathrm{rank}(G_0) = \mathrm{rank}(G) = 8$, the maximal tori T_0 and T of G_0 and G, respectively, coincide. This means $\Phi_0 = \Phi_G$. Using the root spaces determined in [20, Section 2.2], we thus have that $\Phi_{\mathfrak{g}_1} = \{\varepsilon_i + \varepsilon_j + \varepsilon_k : i < j < k\}$. Also, since $\mathfrak{h} \subseteq \mathfrak{g}_0$, $R = \dim(\mathfrak{g}_1)_0 = 0$. By Theorem 3.3, we thus have

$$\mathrm{ch}_z(k[\mathcal{N}]) = (1 - z^{12})(1 - z^{18})(1 - z^{24})(1 - z^{30}) \prod_{i<j<k} (1 - e^{\varepsilon_i + \varepsilon_j + \varepsilon_k} z)^{-1}.$$

This allows us to determine the values of the integers $p_n(\lambda)$ for $\lambda \in X(T_0)$. In particular, by (5), $p_2(\lambda)$ is the number of ways λ can be written as the sum of two weights in $\Phi_{\mathfrak{g}_1}$. Suppose $\lambda = \sum c_i \varepsilon_i$. Then $p_2(\lambda) \neq 0$ if and only if $\sum c_i = 6$ and $0 \le c_i \le 2$ for all i. There are 4 W-conjugacy classes of such weights (where W is the symmetric group on 9 elements) which correspond to the 4 partitions of 6 into parts that are at most 2. The table below lists the dominant representatives λ from each class along with the values of $p_2(\lambda)$.

λ	$p_2(\lambda)$
$2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$	1
$2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4$	1
$2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5$	3
$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6$	10

Using GAP to carry out the lengthy calculation, we get with Theorem 3.2 (taking $\rho = 4\varepsilon_1 + 3\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4 - \varepsilon_6 - 2\varepsilon_7 - 3\varepsilon_8 - 4\varepsilon_9$) the result

$$m_2(\lambda) = \begin{cases} 1 & \text{if } \lambda = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 \text{ or } \lambda = 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 \\ 0 & \text{otherwise.} \end{cases}$$

In terms of the fundamental weights $\varpi_i = \sum_{j=1}^{i} \varepsilon_j$ for $1 \le i \le 8$, $\mathfrak{g}_1 = L(\varpi_3)$, and we have thus shown that

$$S^2(L(\varpi_3)) = L(\varpi_1 + \varpi_5) \oplus L(2\varpi_3). \tag{8}$$

We note for the moment that by Weyl's Dimension Formula, $\dim(L(\varpi_3)) = 84$, $\dim(L(\varpi_1+\varpi_5)) = 1050$, and $\dim(L(2\varpi_3)) = 2520$.

Now suppose $\mathrm{char}(k) = p \geq 7$. This guarantees that p is good for the Type E_8 group G and that p divides neither the order of θ nor $|W_{\mathfrak{c}}| = 2^7 \cdot 3^5 \cdot 5$. The unique maximal weight in $\Phi_{\mathfrak{g}_1}$ relative to the partial order on $X(T_0)$ is $\varepsilon_1+\varepsilon_2+\varepsilon_3$, the Coxeter number h in Type A_8 is 9, and the highest root is $\varepsilon_1 - \varepsilon_9$. Thus

$$N_p = \begin{cases} p-8 & \text{if } p \geq 11 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore by Theorem 3.6, the decomposition in (8) stills holds when $p \geq 11$.

In fact, the work of Lübeck in [15] shows that we can lower this bound on $\mathrm{char}(k) = p$ a little further. When we reduce a Weyl module $V(\lambda)$ modulo p, there is a chance that the resulting module will not be simple, meaning that its unique simple quotient $L(\lambda)$ will have dimension less than that of $V(\lambda)$. However, this is usually not the case for the Weyl modules involved in (8). In particular, [15, Appendix A.12] shows that $\dim(L(\varpi_3)) = 84$ for any p, $\dim(L(\varpi_1+\varpi_5)) = 1050$ when $p \geq 5$, and $\dim(L(2\varpi_3)) = 2520$ when $p \neq 3$. Therefore, when $p \geq 5$, reducing the Weyl modules $L(\varpi_3)$, $L(\varpi_1+\varpi_5)$, and $L(2\varpi_3)$ modulo p results in simple $\mathrm{SL}(V)$-modules, and we see that the decomposition (8) also holds when $p = 5$ and $p = 7$.

Thus, the results of [15] indicate that for a given n, the lower bound on p given by Theorem 3.6 is a sufficient but not necessary condition. However, the algorithm presented here would allow us to extend the results of [15] by decomposing $S^n(L(\varpi_3))$ for n-values that would result in simple modules of higher dimension than those considered in [15].

For example, assuming for the moment that $\mathrm{char}(k) = 0$, we can consider the module $S^3(L(\varpi_3))$, which has dimension 102,340. Applying our algorithm requires that we find the values of $p_3(\lambda)$ for $\lambda \in X(T_0)$. By (5) again, we have that $p_3(\lambda)$ is the number of ways λ can be written as the sum of three weights in $\Phi_{\mathfrak{g}_1}$. There are 12 W-conjugacy classes of weights with nonzero values for p_3. For a given representative λ from each class, we can use a computer to find the value of $p_3(\lambda)$. The results are listed below.

λ	$p_3(\lambda)$
$3\varepsilon_1+3\varepsilon_2+3\varepsilon_3$	1
$3\varepsilon_1+3\varepsilon_2+2\varepsilon_3+\varepsilon_4$	1
$3\varepsilon_1+3\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5$	1
$3\varepsilon_1+2\varepsilon_2+2\varepsilon_3+2\varepsilon_4$	1
$3\varepsilon_1+2\varepsilon_2+2\varepsilon_3+\varepsilon_4+\varepsilon_5$	3
$3\varepsilon_1+2\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6$	6
$2\varepsilon_1+2\varepsilon_2+2\varepsilon_3+2\varepsilon_4+\varepsilon_5$	6
$2\varepsilon_1+2\varepsilon_2+2\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6$	16
$2\varepsilon_1+2\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7$	40
$3\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7$	15
$2\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8$	105
$\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon_7+\varepsilon_8+\varepsilon_9 = 0$	280

Theorems 3.2 and 3.3 now give

$$S^3(L(\varpi_3)) = L(3\varpi_3) \oplus L(\varpi_1 + \varpi_3 + \varpi_5) \oplus L(2\varpi_1 + \varpi_7) \oplus L(\varpi_3 + \varpi_6). \qquad (9)$$

We have that

$$\dim(L(3\varpi_3)) = 41{,}580, \qquad \dim(L(\varpi_1 + \varpi_3 + \varpi_5)) = 53{,}460,$$
$$\dim(L(2\varpi_1 + \varpi_7)) = 1540, \qquad \dim(L(\varpi_3 + \varpi_6)) = 5760.$$

We therefore cannot use [15] to check when (9) holds in positive characteristic since the simple $\mathrm{SL}(V)$-modules of dimension greater than 4000 are not included in [15, Appendix A.12]. However, Theorem 3.6 guarantees that (9) at least holds when $\mathrm{char}(k) \geq 11$.

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