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Left-Symmetric Products on Cosymplectic Lie Algebras

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Abstract. We prove some properties of the cosymplectic Lie algebras and show, in particular, that they support a left-invariant product. We also provide some methods to construct these algebras and classify them in dimensions three and five. These constructions provide a large class of left-symmetric algebras in odd dimensions.

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1. Introduction

Cosymplectic manifolds were introduced by Libermann in 1958. She defined them as follows. An *almost cosymplectic structure* on a manifold M of odd dimension $(2n+1)$ is a pair (α, ω), where α is a 1-form and ω is a 2-form such that $\alpha \wedge \omega^n$ is a volume form on M. The structure is said to be *cosymplectic* if α and ω are closed. Any almost cosymplectic structure (α, ω) uniquely determines a smooth vector field ξ on M, called the *Reeb vector field* of the almost cosymplectic manifold (M, α, ω). It is completely characterized by the following conditions

$$\alpha(\xi) = 1 \quad \text{and} \quad \iota_\xi \omega = 0, \tag{1}$$

where ι denotes the inner product. If we consider the vector bundle morphism $\Phi : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ defined by

$$\Phi(X) = \iota_X \omega + \alpha(X)\alpha, \tag{2}$$

for all $X \in \mathfrak{X}(M)$, then the condition that $\alpha \wedge \omega^n$ is a volume form is equivalent to the condition that Φ is a vector bundle isomorphism. In this case the Reeb vector is given by $\xi = \Phi^{-1}(\alpha)$. For more details on cosymplectic geometry, we refer the reader to the survey article [5], and the references therein.

In this paper, we are interested in Lie groups admitting a cosymplectic structure which is invariant under left translations (left-invariant). Let G be a $(2n+1)$-dimensional real Lie group and let $\mathfrak{g}$ be the corresponding Lie algebra. Denote by e the unit element of G. If G is endowed with a left-invariant differential 1-form α^+ and 2-form ω^+ such that (α^+, ω^+) is a cosymplectic structure, we will say that (G, α^+, ω^+) is a *cosymplectic Lie group* and that $(\mathfrak{g}, \alpha, \omega)$ is a *cosymplectic Lie algebra*, where $\alpha = \alpha^+(e)$ and $\omega = \omega^+(e)$.

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The fact that $(\mathfrak{g}, \alpha, \omega)$ is a cosymplectic Lie algebra this is equivalent to the following conditions:

1. $\alpha([x, y]) = 0,\ \forall x, y \in \mathfrak{g}$.
2. $\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0,\ \forall x, y, z \in \mathfrak{g}$.
3. $\alpha \wedge \omega^n \neq 0$.

The Reeb vector field is the unique left-invariant vector field ξ^+ satisfying $\alpha(\xi) = 1$ and $\iota_\xi \omega = 0$. If ξ^+ is the Reeb vector field of (G, α^+, ω^+), then $\xi = \xi^+(e)$ is called the Reeb vector of $(\mathfrak{g}, \alpha, \omega)$. Note that, because of 1., a semi-simple Lie algebra (or, more generally, a Lie algebra satisfying $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$) cannot support a cosymplectic structure. We refer to [4] for more information on the five-dimensional cosymplectic Lie algebras and their classification.

To begin, we need some basic definitions that we will use throughout the paper. Let $(\mathcal{A}, \cdot)$ be an algebra over $\mathbb{K}$, not necessarily associative or not necessarily finite-dimensional. The associator (x, y, z) of the three elements $x, y, z \in \mathcal{A}$ is defined by

$$(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z).$$

An algebra $(\mathfrak{g}, \cdot)$ over $\mathbb{K}$ with a bilinear product $(x, y) \mapsto x \cdot y$ is called *LSA*, if the product is left-symmetric, i.e., if the identity

$$(x, y, z) = (y, x, z)$$

is satisfied for all $x, y, z \in \mathfrak{g}$. For a left-symmetric algebra $\mathcal{A}$, the commutator

$$[x, y] = x \cdot y - y \cdot x$$

defines a Lie algebra $\mathfrak{g} := \mathfrak{g}_{\mathcal{A}}$, which is called the *sub-adjacent Lie algebra* of $\mathcal{A}$.

A *symplectic Lie algebra* $(\mathfrak{g}, \omega)$ is a real Lie algebra with a skew-symmetric non-degenerate bilinear form ω such that for any $x,\ y,\ z \in \mathfrak{g}$,

$$\oint \omega([x, y], z) = 0, \tag{3}$$

this is to say, ω is a non-degenerate 2-cocycle for the scalar cohomology of $\mathfrak{g}$, where $\oint$ denotes summation over the cyclic permutation. It is known that (see [6] and [11]) given a symplectic Lie algebra $(\mathfrak{g}, \omega)$, the product given by

$$\omega(x * y, z) = -\omega(y, [x, z]), \qquad \forall x, y, z \in \mathfrak{g}, \tag{4}$$

induces a left-symmetric algebra structure "$*$" on $\mathfrak{g}$ that satisfies $x*y - y*x = [x, y]$. In this case, we say that the left-symmetric product " $*$ " is *associated with the symplectic Lie algebra* $(\mathfrak{g}, \omega)$. Geometrically, this is equivalent to the existence of *left-invariant affine structure* (a left-invariant flat torsion-free linear connection) on a symplectic Lie group .

The paper is organized as follows. In Section 2, we show that any cosymplectic Lie algebra supports a left-symmetric product and explore some of their properties. In Section 3, we present some procedures to construct cosymplectic Lie algebras. In particular, we suggest two different constructions of cosymplectic double extensions. In the last section we will prove some results in low dimensions and obtain a classification of three- and five-dimensional cosymplectic Lie algebras.

Notations: Let $\{e_i\}_{1\leq i\leq n}$ be a basis of $\mathfrak{g}$. We denote by $\{e^i\}_{1\leq i\leq n}$ its dual basis on $\mathfrak{g}^*$ and by e^{ij} the 2-form $e^i\wedge e^j\in\wedge^2\mathfrak{g}^*$. We denote by $\langle e\rangle$ the Lie algebra spanned by e.

The software Maple 18® has been used to check all needed calculations.

2. Left-symmetric product associated with cosymplectic Lie algebra

It is known that any symplectic Lie algebra can be equipped with an affine structure. On the other hand, a contact Lie algebra does not necessarily admit a left-symmetric product. Our main result is to show that any cosymplectic Lie algebra supports a left-symmetric product. In the following, we consider a $(2n+1)$-dimensional real cosymplectic Lie algebra $(\mathfrak{g},\alpha,\omega)$ with Reeb vector ξ. Therefore, we have an isomorphism

$$\Phi:\mathfrak{g}\longrightarrow\mathfrak{g}^*,\quad x\longmapsto\iota_x\omega+\alpha(x)\alpha. \tag{5}$$

Throughout the remainder of this paper, we set $\mathfrak{h}=\ker\alpha\in\mathfrak{g}$ and $\omega_{\mathfrak{h}}=\omega_{|\mathfrak{h}\times\mathfrak{h}}$.

Lemma 2.1. *Let $(\mathfrak{g},\alpha,\omega)$ be a cosymplectic Lie algebra with the Reeb vector ξ. Then, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and $(\mathfrak{h},\omega_{\mathfrak{h}})$ is a symplectic Lie algebra.*

Proof. For $x\in\mathfrak{h}$ and $y\in\mathfrak{g}$, we have

$$\alpha([x,y])=-\mathrm{d}\alpha(x,y)=0,$$

where d is the Chevalley-Eilenberg differential. Hence $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. To prove that $(\mathfrak{h},\omega_{\mathfrak{h}})$ is a symplectic Lie algebra, it is clear that $\omega_{\mathfrak{h}}$ is a 2-cocycle. Consider a basis $\{\xi,e_1,...,e_{2n}\}$ of $\mathfrak{g}$, with $\{e_1,...,e_{2n}\}$ is a basis of $\mathfrak{h}$. We have

$$0\neq\alpha\wedge\omega^n(\xi,e_1,...,e_{2n})=\omega_{\mathfrak{h}}^n(e_1,...,e_{2n}).$$

Then, $(\mathfrak{h},\omega_{\mathfrak{h}})$ is a symplectic Lie algebra. ∎

Denote by "$*$" the left-symmetric product associated with the symplectic Lie algebra $(\mathfrak{h},\omega_{\mathfrak{h}})$ and $(\cdot,\cdot,\cdot)^*$ its associator (i.e., $(x,y,z)^*=(x*y)*z-x*(y*z)$, $\forall x,y,z\in\mathfrak{h}$). As in the symplectic framework, the non-degeneration of Φ defines a product "$\cdot$" on $\mathfrak{g}$ by

$$\Phi(x\cdot y)(z)=-\Phi(y)([x,z]),\ \text{for all}\ \ x,y,z\in\mathfrak{g}. \tag{6}$$

Proposition 2.2. *The product defined by (6), is characterized by the following properties.*

(1) *For all x, $y\in\mathfrak{h}$, we have $x\cdot y=x*y+\omega(x,\mathrm{ad}_\xi y)\xi$.*

(2) *For all $x\in\mathfrak{g}$, we have $\xi\cdot x=\mathrm{ad}_\xi x$ and $x\cdot\xi=0$.*

Proof. (1) For all $x,y\in\mathfrak{h}$, and $z\in\mathfrak{g}$, the relation (6) becomes

$$\omega(x\cdot y,z)+\alpha(x\cdot y)\alpha(z)=-\omega(y,[x,z])-\alpha(y)\alpha([x,z])=-\omega(y,[x,z]).$$

For all $z\in\mathfrak{h}$, we have $\omega_{|\mathfrak{h}}(x\cdot y,z)=-\omega_{|\mathfrak{h}}(y,[x,z])$ and

$$\alpha(x\cdot y)=-\omega(y,[x,\xi])=\omega(x,[\xi,y]),\ \text{for}\ z=\xi.$$

(2) If $x=\xi$ and $y\in\mathfrak{h}$ is arbitrary, the relation (6) becomes

$$\omega(\xi\cdot y,z)+\alpha(\xi\cdot y)\alpha(z)=-\omega(y,[\xi,z]).$$

On the one hand, if $z=\xi$ we obtain $\alpha(\xi\cdot y)=0$, then $\xi\cdot y\in\mathfrak{h}$. On the other hand for all $z\in\mathfrak{h}$, we have $\omega_{\mathfrak{h}}(\xi\cdot y,z)=-\omega_{\mathfrak{h}}(y,[\xi,z])=\omega_{\mathfrak{h}}([\xi,y],z)$. Hence, $\xi\cdot x=\mathrm{ad}_\xi x$, for all $x\in\mathfrak{g}$.

Finally, if $y=\xi$, we also have $\omega(x\cdot\xi,z)+\alpha(x\cdot\xi)\alpha(z)=0$.

Thus, $x\cdot\xi=0$, for all $x\in\mathfrak{g}$. ■

The following lemma shows that ad_ξ is a derivation relatively to the left-symmetric product associated with the symplectic Lie algebra $(\mathfrak{h},\omega_{\mathfrak{h}})$.

Lemma 2.3. *For all $x,\ y\in\mathfrak{h}$, we have* $\mathrm{ad}_\xi(x*y)=\mathrm{ad}_\xi x*y+x*\mathrm{ad}_\xi y$.

Proof. First, it is easy to see that $\mathrm{ad}_\xi x\in\mathfrak{h}$ for all $x\in\mathfrak{h}$. Let $x,y,z\in\mathfrak{h}$. By using relations (3) and (4) and Jacobi identity, we have

$$
\begin{aligned}
&\omega_{\mathfrak{h}}(\mathrm{ad}_\xi(x*y)-\mathrm{ad}_\xi x*y,z)=\omega([\xi,x*y],z)+\omega(y,[[\xi,x],z])\\
&=-\omega([z,\xi],x*y)+\omega(y,[[\xi,x],z])=\omega([x,[z,\xi]],y)-\omega([[\xi,x],z],y)\\
&=\omega([[x,z],\xi],y)=-\omega([\xi,y],[x,z])=\omega_{\mathfrak{h}}(x*\mathrm{ad}_\xi y,z).
\end{aligned}
$$
■

The following theorem shows in particular, that any cosymplectic Lie algebra supports a left-symmetric product.

Theorem 2.4. *The product "$\cdot$" given by $\Phi(x\cdot y)(z)=-\Phi(y)([x,z])$ for all $x,y,z\in\mathfrak{g}$, induces a left-symmetric product on $\mathfrak{g}$.*

Proof. On the one hand, for all $x,y,z\in\mathfrak{h}$, we have

$$
\begin{aligned}
&(x,y,z)=(x\cdot y)\cdot z-x\cdot(y\cdot z)\\
&=(x*y+\omega(x,[\xi,y])\xi)\cdot z-x\cdot(y*z+\omega(y,[\xi,z])\xi)\\
&=(x*y)\cdot z+\omega(x,[\xi,y])\xi\cdot z-x\cdot(y*z)\\
&=(x*y)*z+\omega(x*y,[\xi,z])\xi+\omega(x,[\xi,y])\xi\cdot z-x*(y*z)-\omega(x,[\xi,y*z])\xi\\
&=(x*y)*z-x*(y*z)+\omega(x,[\xi,y])\xi.z+A(x,y)\xi\\
&=(x,y,z)^*+\omega(x,[\xi,y])\xi.z+A(x,y)\xi,
\end{aligned}
$$

where $A(x,y)=\omega(x*y,[\xi,z])-\omega(x,[\xi,y*z])$. It is clear that $(x,y,z)^*=(y,x,z)^*$. Using (3) and the fact that $\omega(\xi,.)=0$, we obtain $\omega(x,[\xi,y])=\omega(y,[\xi,x])$, and

$$
\begin{aligned}
&A(x,y)-A(y,x)=\omega([x,y],[\xi,z])-\omega(x,[\xi,y*z])+\omega(y,[\xi,x*z])\\
&=-\omega(z,[[x,y],\xi])+\omega(y*z,[x,\xi])-\omega(x*z,[y,\xi])\\
&=\omega(z,[\xi,[x,y]])+\omega(z,[y,[\xi,x]])+\omega(z,[x,[y,\xi]])=0.
\end{aligned}
$$

It follows that $(x,y,z)=(y,x,z)$ for all $x,y,z\in\mathfrak{h}$. On the other hand, let $x,y\in\mathfrak{h}$, we have

$$
\begin{aligned}
&(\xi,x,y)-(x,\xi,y)=(\xi\cdot x)\cdot y-\xi\cdot(x\cdot y)-(x\cdot\xi)\cdot y+x\cdot(\xi\cdot y)\\
&=(\mathrm{ad}_\xi x)*y+\omega([\xi,x],[\xi,y])\xi-\mathrm{ad}_\xi(x.y)+x*\mathrm{ad}_\xi y+\omega(x,[\xi,[\xi,y]])\xi.
\end{aligned}
$$

Then, $(\xi,x,y)-(x,\xi,y)=0$ is equivalent to

$$\begin{cases} \mathrm{ad}_\xi(x*y)=\mathrm{ad}_\xi x*y+x*\mathrm{ad}_\xi y\\ \omega([\xi,x],[\xi,y])=-\omega(x,[\xi,[\xi,y]]). \end{cases}$$

The first equation is ensured by Lemma 2.3, while the second follows from the closure of ω and from the fact that $\omega(\xi,.)=0$. ■

It is known that (see, for example, [10] Proposition 1-1) on a Lie group, an affine connection is bi-invariant if and only if its associated left-symmetric product is associative. We obtain the following result.

Corollary 2.5. *Let (G,α^+,ω^+) be a cosymplectic Lie group with the Reeb vector ξ^+. The affine structure associated is bi-invariant if and only if the following conditions are satisfied for all $x,y,z\in\mathfrak{h}$:*

(1) $(x,y,z)^*=\omega(\mathrm{ad}_\xi y,x)\,\mathrm{ad}_\xi z$.

(2) $x*\mathrm{ad}_\xi y=0$.

(3) $(\mathrm{ad}_\xi x)*y=(\mathrm{ad}_\xi y)*x$.

(4) $\mathrm{ad}_\xi^2 x=0$.

Remark 2.6. Note that, if $\mathrm{ad}_\xi x=0$, then the relation 1. becomes $(x,y,z)^*=0$ for all $x,y,z\in\mathfrak{h}$. This means that the left-symmetric product "$*$" associated with the symplectic Lie algebra $(\mathfrak{h},\omega_\mathfrak{h})$ is associative, which is equivalent to saying that $(\mathfrak{h},\omega_\mathfrak{h})$ is a *symplectic Novikov Lie algebra* (see [1] for more details).

3. Cosymplectic double extensions

We suggest two different constructions of cosymplectic double extensions. In the first construction, we combine two characterizations of the cosymplectic Lie algebras. The second way of construction got inspiration from the notion of symplectic double extension.

Double extensions of Lie algebras: Let $(\overline{\mathfrak{g}},\overline{[\ ,\ ]})$ be a Lie algebra, $\theta\in Z^2(\overline{\mathfrak{g}})$ a 2-cocycle, and $(\overline{\mathfrak{g}}_{(\theta,e)},\overline{[\ ,\ ]}_\theta)$ the central extension of $\overline{\mathfrak{g}}$ by the 2-cocycle θ, i.e.,

$$\overline{\mathfrak{g}}_{(\theta,e)}=(\overline{\mathfrak{g}}\oplus\langle e\rangle,[\ ,\ ]_\theta)\ \text{ with }\ [x,y]_\theta=\overline{[x,y]}+\theta(x,y)e,\ \ x,y\in\overline{\mathfrak{g}}.$$

Let $\mathfrak{g}=\langle d\rangle\oplus\overline{\mathfrak{g}}\oplus\langle e\rangle$ be the direct sum of $\overline{\mathfrak{g}}$ with the one-dimensional vector spaces $\langle e\rangle$ and $\langle d\rangle$. Define an alternating bilinear map, $[\ ,\ ]:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$ as follows

$$\begin{aligned} [x,y]&=\overline{[x,y]}+\theta(x,y)e, && x,y\in\overline{\mathfrak{g}},\\ [d,x]&=\varphi(x)+\lambda(x)e, && x\in\overline{\mathfrak{g}},\\ [d,e]&=v+te, \end{aligned}\tag{7}$$

where $D=(\varphi,\lambda,v,t)\in\mathrm{End}(\overline{\mathfrak{g}})\times\overline{\mathfrak{g}}^*\times\overline{\mathfrak{g}}\times\mathbb{R}$. Set

$$\begin{aligned} \partial\varphi(x,y)&:=\varphi(\overline{[x,y]})-\overline{[\varphi(x),y]}-\overline{[x,\varphi(y)]}\\ \theta_\varphi(x,y)&:=\theta(\varphi(x),y)+\theta(x,\varphi(y)). \end{aligned}$$

We have the following result.

Proposition 3.1. *The alternating map given by* (7) *defines a Lie algebra* $(\mathfrak{g}, [\ ,\ ])$, *if and only if*

(1) $\partial\varphi = \theta v$,

(2) $t\theta - \theta_\varphi = \mathrm{d}\lambda$,

(3) $v \in Z(\overline{\mathfrak{g}}) \cap \ker(\theta)$,

where $\ker(\theta) = \{x \in \overline{\mathfrak{g}} \mid \theta(x,y) = 0,\ \forall y \in \overline{\mathfrak{g}}\}$.

Proof. On the one hand, for all $x, y \in \overline{\mathfrak{g}}$, we have

$$\begin{aligned}\oint [[d,x],y] &= [[d,x],y] + [[x,y],d] + [[y,d],x] \\ &= [\varphi(x) + \lambda(x)e, y] + [\overline{[x,y]} + \theta(x,y)e, d] - [\varphi(y) + \lambda(y)e, x] \\ &= \overline{[\varphi(x), y]} + \theta(\varphi(x), y)e - \varphi(\overline{[x,y]}) - \lambda(\overline{[x,y]})e - \theta(x,y)v \\ &\quad - t\theta(x,y)e - \overline{[\varphi(y), x]} - \theta(\varphi(y), x)e \\ &= \big(\partial\varphi(x,y) - \theta(x,y)v\big) + \big(\theta_\varphi(x,y) - t\theta(x,y) - \lambda(\overline{[x,y]})\big)e.\end{aligned}$$

Hence (1) and (2) hold. On the other hand, we have for all $x \in \overline{\mathfrak{g}}$

$$\begin{aligned}\oint [[x,e],d] &= [[x,e],d] + [[e,d],x] + [[d,x],e] \\ &= 0 + [v + te, x] + 0 = \overline{[v,x]} + \theta(v,x)e.\end{aligned}$$

Hence $v \in Z(\overline{\mathfrak{g}}) \cap \ker(\theta)$. The other possible Jacobi identities are immediately checked. ■

Under the conditions of Proposition 3.1, the Lie algebra $\overline{\mathfrak{g}}(D,\theta) = (\mathfrak{g}, [\ ,\ ])$ is called the *double extension* of $(\overline{\mathfrak{g}}, \overline{[\ ,\ ]})$ by (D,θ), where

$$D = (\varphi, \lambda, v, t) \in \mathrm{End}(\overline{\mathfrak{g}}) \times \overline{\mathfrak{g}}^* \times \overline{\mathfrak{g}} \times \mathbb{R}.$$

Remark 3.2. (1) The Lie algebra $\overline{\mathfrak{g}}(D,\theta)$ is a semi-direct product $\langle d\rangle \ltimes (\overline{\mathfrak{g}} \oplus \langle e\rangle)$ of the abelian Lie algebra $\langle d\rangle$ with the central extension $\overline{\mathfrak{g}} \oplus \langle e\rangle$ relatively to the derivation $D \in \mathrm{Der}(\overline{\mathfrak{g}} \oplus \langle e\rangle)$ which is given by

$$D(x) = \varphi(x) + \lambda(x)e, \quad x \in \overline{\mathfrak{g}}, \quad D(e) = v + te.$$

(2) φ is a derivation of Lie algebra $\overline{\mathfrak{g}}$ if and only if $v = 0$ or $\theta = 0$. In particular, this holds when either $Z(\overline{\mathfrak{g}}) = \{0\}$ or $\ker(\theta) = \{0\}$. ■

3.1. The first construction

We start with two characterizations of symplectic Lie algebras. The first one was given in [9] to characterize in general cosymplectic manifolds. We state here its analogue for cosymplectic Lie algebras.

Proposition 3.3. [9] *Let* $\overline{\mathfrak{g}}$ *be a Lie algebra and* $\overline{\alpha}$, $\overline{\omega}$ *two forms on* $\overline{\mathfrak{g}}$ *with degrees* 1 *and* 2 *respectively. Consider on the direct sum* $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \langle e\rangle$ *the 2-form* $\omega = \overline{\omega} + \overline{\alpha} \wedge e^*$. *Then,* $(\overline{\mathfrak{g}}, \overline{\alpha}, \overline{\omega})$ *is a cosymplectic Lie algebra if and only if* $(\mathfrak{g}, \omega)$ *is a symplectic Lie algebra.*

Remark 3.4. The Lie algebra $\mathfrak{g}=\overline{\mathfrak{g}}\oplus\langle e\rangle$ is a central extension of $\overline{\mathfrak{g}}$ by $\theta=0$. ■

As for the second characterization; see [2], [4] and [8].

Proposition 3.5. *There exists a one-to-one correspondence between $(2n+1)$-dimensional cosymplectic Lie algebras $(\mathfrak{g},\alpha,\omega)$, and $2n$-dimensional symplectic Lie algebras $(\mathfrak{h}=\ker\alpha,\omega_{\mathfrak{h}})$ together with a derivation $D\in\mathrm{Der}(\mathfrak{h})$ satisfying*

$$\omega_{\mathfrak{h}}(Dx,y)=-\omega_{\mathfrak{h}}(x,Dy),\ x,y\in\mathfrak{h}. \tag{8}$$

An endomorphism satisfying (8) is called an infinitesimal symplectic transformation (for short, an i.s.t.). By combining these two characterizations (Propositions 3.3 and 3.5) , we propose the following construction of cosymplectic Lie algebras.

Let $(\overline{\mathfrak{g}},\overline{\alpha},\overline{\omega})$ be a cosymplectic Lie algebra with the Reeb vector $\overline{\xi}$. Then we can always write $\overline{\mathfrak{g}}$ as $\overline{\mathfrak{g}}=\overline{\mathfrak{h}}\oplus\langle\overline{\xi}\rangle$ with $\overline{\mathfrak{h}}=\ker(\overline{\alpha})$. Let $(\overline{\mathfrak{g}}\oplus\langle e\rangle,\omega=\overline{\omega}+\alpha\wedge e^*)$ be a symplectic Lie algebra associated with $(\overline{\mathfrak{g}},\overline{\alpha},\overline{\omega})$ (see Proposition 3.3). A derivation $D\in\mathrm{Der}(\overline{\mathfrak{g}}\oplus\langle e\rangle)$ consists of a 4-tuple $(\varphi,\lambda,v,t)\in\mathrm{End}(\overline{\mathfrak{g}})\times\overline{\mathfrak{g}}^*\times\overline{\mathfrak{g}}\times\mathbb{R}$, such that

$$D(x)=\varphi(x)+\lambda(x)e \text{ for } x\in\overline{\mathfrak{g}},\ \text{and } D(e)=v+te.$$

Lemma 3.6. *The derivation $D\in\mathrm{Der}(\mathfrak{g})$ is an i.s.t if and only if the following conditions are satisfied for all $x\in\mathfrak{h}$.*

(i) *φ is an i.s.t on $\overline{\mathfrak{h}}$,*

(ii) *$\lambda(x)=\overline{\omega}(x,\varphi(\overline{\xi}))$,*

(iii) *$\overline{\omega}(v,x)=\overline{\alpha}\circ\varphi(x)$,*

(iv) *$t=-\overline{\alpha}\circ\varphi(\overline{\xi})$.*

Proof. Indeed, for all $x,y\in\overline{\mathfrak{g}}$, we have

$$\omega(Dx,y)+\omega(x,Dy)=\overline{\omega}(\varphi(x),y)+\overline{\omega}(x,\varphi(y))+\lambda(x)\omega(e,y)+\lambda(y)\omega(x,e). \tag{9}$$

Using (9) and the fact thatç D is an i.s.t, that yields to $\overline{\omega}(\varphi(x),y)=-\overline{\omega}(x,\varphi(y))$. Let $y=\xi$ and $x\in\overline{\mathfrak{h}}$ is arbitrary. Once again, by applying (9) and the fact that D is an i.s.t we obtain $\lambda(x)=\overline{\omega}(x,\varphi(\xi))$. Let $y=e$. Then, for all $x\in\overline{\mathfrak{h}}$, we have

$$\begin{aligned}\omega(Dx,e)+\omega(x,De)&=\omega(\varphi(x)+\lambda(x)e,e)+\omega(x,v+te)\\&=\omega(\varphi(x),e)+\overline{\omega}(x,v),\end{aligned}$$

since D is an i.s.t, it follows that $\overline{\omega}(x,v)=\overline{\alpha}\circ\varphi(x)$. Finally, for $x=\overline{\xi}$ and $y=e$, we have

$$\begin{aligned}\omega(D\overline{\xi},e)+\omega(\overline{\xi},De)&=\omega(\varphi(\overline{\xi})+\lambda(\overline{\xi})e,e)+\omega(\overline{\xi},v+te)\\&=\omega(\varphi(\overline{\xi}),e)+t,\end{aligned}$$

therefore, $t=-\overline{\alpha}\circ\varphi(\overline{\xi})$. This completes the proof. ■

Let $D=(\varphi,\lambda,v,t)$ satisfy the conditions of Lemma 3.6. Denote by “ $*$ ” the left-symmetric product associated with the symplectic Lie algebra $(\overline{\mathfrak{h}},\overline{\omega})$ and let $\overline{R}_x$ be the right multiplication by an element x, that is $\overline{R}_x y=y*x$. Then we obtain the following result.

Theorem 3.7. *Let $(\overline{\mathfrak{g}}, \overline{\alpha}, \overline{\omega})$ be a cosymplectic Lie algebra with the Reeb vector $\overline{\xi}$. Let $\mathfrak{g} = \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$ be a double extension of $\overline{\mathfrak{g}}$ by $(D, \theta = 0)$, where $D = (\varphi, \lambda, v, t)$ satisfies the conditions of Lemma* 3.6. *Then*

$$\omega = \overline{\omega} + \overline{\alpha} \wedge e^*, \ \alpha = d^*,$$

defines a cosymplectic structure on $\mathfrak{g}$ if and only if

(1) $\varphi \in \mathrm{Der}(\overline{\mathfrak{g}})$,

(2) $v \in Z(\overline{\mathfrak{g}})$,

(3) $\overline{R}_{\varphi(\overline{\xi})} = 0$ *and* $[\varphi(\overline{\xi}), \overline{\xi}] = 0$.

The Reeb vector is d.

Proof. With $\theta = 0$, Proposition 3.1 implies that $\varphi \in \mathrm{Der}(\overline{\mathfrak{g}})$, $v \in Z(\overline{\mathfrak{g}})$ and $\mathrm{d}\lambda = 0$. Notice that

$$\mathrm{d}\lambda = 0 \Leftrightarrow \begin{cases} \overline{\omega}([x, y], \varphi(\overline{\xi})) = 0, & \forall x, y \in \overline{\mathfrak{h}}, \quad (10) \\ \overline{\omega}([\overline{\xi}, x], \varphi(\overline{\xi})) = 0, & \forall x \in \overline{\mathfrak{h}}. \quad (11) \end{cases}$$

The equation (10) is equivalent to $\overline{\omega}(x, y * \varphi(\overline{\xi})) = 0$, for all $x, y \in \overline{\mathfrak{h}}$. One can then deduce that $\overline{R}_{\varphi(\overline{\xi})} = 0$. Since $\overline{\omega}$ is a 2-cocycle the equation (11) becomes

$$\overline{\omega}([\varphi(\overline{\xi}), \overline{\xi}], x) = 0, \ \forall x \in \overline{\mathfrak{h}}.$$

Since $\overline{\omega}$ is a non-degenerate 2-form on $\overline{\mathfrak{h}}$, it follows that $[\varphi(\overline{\xi}), \overline{\xi}] = 0$. To complete the proof, it remains to verify that $\alpha \wedge \omega^n \neq 0$, which is equivalent to prove that $\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is an isomorphism. Indeed, for all $x \in \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$, we can write $x = x_1 d + \overline{x} + x_2 \overline{\xi} + x_3 e$ with $\overline{x} \in \overline{\mathfrak{h}}$ and $x_1, x_2, x_3 \in \mathbb{R}$. A direct calculation gives

$$\Phi(x, \overline{y}) = \overline{\omega}_{\overline{\mathfrak{h}}}(\overline{x}, \overline{y}), \ \forall \overline{y} \in \overline{\mathfrak{h}},$$

$$\Phi(x, \overline{\xi}) = -x_3, \quad \Phi(x, e) = x_2, \quad \Phi(x, d) = x_1.$$

This means that, if $\Phi(x, .) = 0$, then $x = 0$. ■

3.2. The second construction

Let $(\overline{\mathfrak{g}}, \overline{\alpha}, \overline{\omega})$ be a cosymplectic Lie algebra and $\overline{[\ ,\ ]}$ its Lie bracket and let

$$\mathfrak{g} = \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$$

be a double extension of $(\overline{\mathfrak{g}}, \overline{[\ ,\ ]})$ by (D, θ). We define a 2-form ω and a 1-form α on the vector space $\mathfrak{g}$ by requiring that

$$\omega = \overline{\omega} + d^* \wedge e^*, \quad \text{and} \quad \alpha(x) = \overline{\alpha}(x), \ \forall x \in \overline{\mathfrak{g}}, \ \alpha(d) \in \mathfrak{g}, \ \alpha(e) \in \mathfrak{g}.$$

On the one hand, if we assume that α is a 1-cocycle, we obtain

$$\begin{cases} \alpha([x, y]) = \theta(x, y)\alpha(e) = 0, \\ \alpha([d, x]) = \overline{\alpha}(\varphi(x)) + \lambda(x)\alpha(e) = 0, \\ \alpha([d, e]) = \alpha(v) + t\alpha(e) = 0. \end{cases} \quad (12)$$

On the other hand, since ω is a 2-cocycle, for any $x, y \in \overline{\mathfrak{g}}$,

$$\begin{aligned}\oint \omega([x,y],d) &= \omega(\overline{[x,y]},d) + \theta(x,y)\omega(e,d) + \overline{\omega}(\varphi(x),y) + \lambda(x)\omega(e,y)\\ &\quad - \overline{\omega}(\varphi(y),x) - \lambda(y)\omega(d,y) \qquad (13)\\ &= -\theta(x,y) + \overline{\omega}(\varphi(x),y) + \overline{\omega}(x,\varphi(y)) = 0,\end{aligned}$$

and

$$\oint \omega([x,e],d) = \overline{\omega}(v,x) = 0. \qquad (14)$$

We distinguish two cases.

The first case: $\alpha(e) = 0$. We use relations (12), (13) and (14) to obtain

$$\overline{\alpha} \circ \varphi = 0 \quad \text{and} \quad \alpha(v) = 0, \quad \theta = \overline{\omega}_\varphi \quad \text{and} \quad v \in \ker(\overline{\omega}).$$

Note that, $\alpha(v) = 0$ and $v \in \ker(\overline{\omega})$ implies that $v = 0$. By combining the relations (12), (13) and (14) with Proposition 3.1, we obtain the following result.

Theorem 3.8. *Let $(\overline{\mathfrak{g}}, \overline{\alpha}, \overline{\omega})$ be a cosymplectic Lie algebra with the Reeb vector $\overline{\xi}$. Let $\mathfrak{g} = \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$ be a double extension of $\overline{\mathfrak{g}}$ by $(D, \overline{\omega}_\varphi)$ with*

$$D = (\varphi, \lambda, 0, t) \in \mathrm{Der}(\overline{\mathfrak{g}}) \times \overline{\mathfrak{g}}^* \times \overline{\mathfrak{g}} \times \mathbb{R}.$$

Then

$$\omega = \overline{\omega} + d^* \wedge e^*, \quad \alpha(x) = \overline{\alpha}(x),\ \forall x \in \overline{\mathfrak{g}},\ \alpha(d) \in \mathbb{R},$$

defines a cosymplectic structure on $\mathfrak{g}$ if and only if

(1) $\alpha(e) = 0$,

(2) $\overline{\alpha} \circ \varphi = 0$,

(3) $t\omega_\varphi - \omega_{\varphi,\varphi} = \mathrm{d}\lambda$,

where $\omega_{\varphi,\varphi} = \theta_\varphi$, and the Reeb vector is $\overline{\xi}$.

Proof. It suffices to verify that $\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is an isomorphism. Let $x \in \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$ and write $x = x_1 d + \overline{x} + x_2\overline{\xi} + x_3 e$, with $\overline{x} \in \overline{\mathfrak{h}}$ and x_1, x_2, $x_3 \in \mathbb{R}$. After a short calculation we obtain

$$\begin{aligned}&\Phi(x,\overline{y}) = \overline{\omega}_{\overline{\mathfrak{h}}}(\overline{x},\overline{y}),\ \forall \overline{y} \in \overline{\mathfrak{h}}, && \Phi(x,e) = x_1,\\ &\Phi(x,\overline{\xi}) = x_1\alpha(d) + x_2, && \Phi(x,d) = -x_3 + x_1\alpha^2(d) + x_2\alpha(d).\end{aligned}$$

In other words, if $\Phi(x,.) = 0$, then $x = 0$. ∎

The second case: $\alpha(e) \neq 0$. To simplify notation, we can assume whithout loss of generality that $\alpha(e) = -1$. We use relations (12), (13) and (14) to obtain

$$\begin{gathered}\overline{\omega}_\varphi = \theta = 0 \quad \text{and} \quad \varphi \in \mathrm{Der}(\overline{\mathfrak{g}}),\\ \lambda = \overline{\alpha} \circ \varphi \quad \text{and} \quad t = \overline{\alpha}(v),\\ v \in \ker(\overline{\omega}).\end{gathered}$$

Together with Proposition 3.1 and relations (12), (13) and (14), we therefore have:

Theorem 3.9. *Let* $(\overline{\mathfrak{g}}, \overline{\alpha}, \overline{\omega})$ *be a cosymplectic Lie algebra with the Reeb vector* $\overline{\xi}$*. Let* $\mathfrak{g} = \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$ *be a double extension of* $\overline{\mathfrak{g}}$ *by* $(D, \theta = 0)$*, where* $D = (\varphi, \overline{\alpha}\circ\varphi, v, \overline{\alpha}(v)) \in \mathrm{Der}(\overline{\mathfrak{g}}) \times \overline{\mathfrak{g}}^* \times \overline{\mathfrak{g}} \times \mathbb{R}$*. Then*

$$\omega = \overline{\omega} + d^* \wedge e^*, \quad \alpha(x) = \overline{\alpha}(x),\ \forall x \in \overline{\mathfrak{g}},\ \alpha(d) \in \mathbb{R},\ \alpha(e) = -1,$$

defines a cosymplectic structure on $\mathfrak{g}$ *if and only if*

(1) $\overline{\omega}_\varphi = 0$,

(2) $\overline{\alpha}\circ\varphi([x,y]) = 0,\ \forall x, y \in \overline{\mathfrak{g}}$,

(3) $v \in Z(\overline{\mathfrak{g}}) \cap \ker(\overline{\omega})$.

The Reeb vector is $\overline{\xi}$*.*

Proof. Similarly to the proof of Theorem 3.8, it suffices to verify that $\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is an isomorphism. Let $x \in \langle d\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e\rangle$ and write $x = x_1 d + \overline{x} + x_2\overline{\xi} + x_3 e$, with $\overline{x} \in \overline{\mathfrak{h}}$ and x_1, x_2, $x_3 \in \mathbb{R}$. We have

$$\Phi(x, \overline{y}) = \overline{\omega}_{\overline{\mathfrak{h}}}, (\overline{x}, \overline{y}),\ \forall \overline{y} \in \overline{\mathfrak{h}}, \qquad \Phi(x, \overline{\xi}) = x_1\alpha(d) + x_2 - x_3,$$
$$\Phi(x, e) = x_1 - x_1\alpha(d) - x_2 + x_3, \quad \Phi(x, d) = -x_3 + x_1\alpha^2(d) + x_2\alpha(d) - x_3\alpha(d).$$

As a result, if $\Phi(x, .) = 0$, then $x = 0$. ■

Examples 3.10. To illustrate and compare the two constructions presented above, we end this chapter by obtaining deferent five-dimensional cosymplectic Lie algebras from the same three-dimensional cosymplectic Lie algebra. We consider the following three-dimensional cosymplectic Lie algebra:

$$\overline{\mathfrak{g}} : [e_1, e_2] = e_1,\ \overline{\alpha} = e^3,\ \overline{\omega} = e^{12}.$$

Let $\mathfrak{g} = \langle e_4\rangle \oplus \overline{\mathfrak{g}} \oplus \langle e_5\rangle$ be a double extension of $(\overline{\mathfrak{g}}, \overline{[\ ,\ ]})$ by (D, θ), where we have $D = (\varphi, \lambda, v, t)$.

It is straightforward to show that any derivation on Lie algebra $\overline{\mathfrak{g}}$ has the following form

$$\varphi = \begin{pmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & c & f \end{pmatrix},\ a, b, c, f \in \mathbb{R}.$$

Set $v = ze_3 \in Z(\overline{\mathfrak{g}})$ and $\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \overline{\mathfrak{g}}^*$, where $z, \lambda_i \in \mathbb{R}$. Simple calculations yield the following results.

The first construction gives the cosymplectic Lie algebra

$$\begin{aligned} \mathfrak{g}_1 : [e_1, e_2] &= e_1, \quad [e_4, e_3] = fe_3 + \lambda_3 e_5, \\ [e_4, e_2] &= be_1, \quad [e_4, e_5] = ze_3 - fe_5. \\ \alpha &= e^4,\ \omega = e^{12} + e^{35},\ \xi = e_4. \end{aligned}$$

The second construction:

(i) In the first case, $\alpha(e)=0$,

$$\mathfrak{g}_2: \begin{array}{ll} [e_1,e_2]=e_1+ae_5, & [e_4,e_2]=be_1+\lambda_2e_5, \\ [e_4,e_1]=ae_1+(a^2-ta)e_5, & [e_4,e_3]=\lambda_3e_5, \\ [e_4,e_5]=te_5. & \end{array}$$

$$\alpha=e^3+xe^4,\ \omega=e^{12}+e^{45},\ \xi=e_3,\ x\in\mathbb{R}.$$

(ii) In the second case, $\alpha(e)\neq 0$,

$$\mathfrak{g}_3: \begin{array}{ll} [e_1,e_2]=e_1, & [e_4,e_3]=f(e_3+e_5), \\ [e_4,e_2]=be_1+c(e_3+e_5), & [e_4,e_5]=t(e_3+e_5). \end{array}$$

$$\alpha=e^3+xe^4-e^5,\ \omega=e^{12}+e^{45},\ \xi=e_3,\ x\in\mathbb{R}.$$

Let us denote by $\mathcal{D}(\mathfrak{g}_i)$ the derived Lie algebra of $\mathfrak{g}_i$, $i=1,\dots,3$. It is clear that for a suitable choice of the constants (b,f,λ_3,z) we have $\dim(\mathcal{D}(\mathfrak{g}_1))=3$, while $\dim(\mathcal{D}(\mathfrak{g}_2))$ and $\dim(\mathcal{D}(\mathfrak{g}_3))$ are less than or equal to two. Therefore, the cosymplectic Lie algebra $\mathfrak{g}_1$ never comes from the second construction. ■

4. Classification in low dimension

4.1. Three-dimensional cosymplectic Lie algebras.

Let $\mathfrak{g}$ be a three-dimensional Lie algebra. Denote by $\alpha=a_1e^1+a_2e^2+a_3e^3$ a 1-form and by $\omega=a_{12}e^{12}+a_{13}e^{13}+a_{23}e^{23}$ a 2-form on $\mathfrak{g}$. On each three-dimensional Lie algebra we compute the 1- and 2-cocycle conditions for α and ω respectively. Next, we calculate the rank of Φ. If Φ has a maximum rank, then $\mathfrak{g}$ will be endowed with a cosymplectic structure.

Proposition 4.1. *Let $(\mathfrak{g},\alpha,\omega)$ be a three-dimensional cosymplectic real Lie algebra. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras equipped with a cosymplectic structure:*

$\mathfrak{g}_{2.1}\oplus\mathfrak{g}_1:$ $[e_1,e_2]=e_1$ *(decomposable solvable, Bianchi III)*
$\alpha=a_2e^2+a_3e^3,\quad \omega=a_{12}e^{12}+a_{23}e^{23},\quad a_3a_{12}\neq 0.$

$\mathfrak{g}_{3.1}:$ $[e_2,e_3]=e_1$ *(Weyl algebra, nilpotent, Bianchi II)*
$\alpha a_2e^2+a_3e^3,\quad \omega=a_{12}e^{12}+a_{13}e^{13}+a_{23}e^{23},\quad a_3a_{12}-a_2a_{13}\neq 0.$

$\mathfrak{g}_{3.4}^{-1}:$ $[e_1,e_3]=e_1$ *and* $[e_2,e_3]=-e_2$ *(solvable, Bianchi VI, $a=-1$)*
$\alpha=a_3e^3,\quad \omega=a_{12}e^{12}+a_{13}e^{13}+a_{23}e^{23},\quad a_3a_{12}\neq 0.$

$\mathfrak{g}_{3.5}^{0}:$ $[e_1,e_3]=-e_2$ *and* $[e_2,e_3]=e_1$ *(solvable, Bianchi VII, $\beta=0$)*
$\alpha=a_3e^3,\quad \omega=a_{12}e^{12}+a_{13}e^{13}+a_{23}e^{23},\quad a_3a_{12}\neq 0.$

Two cosymplectic Lie algebra $(\mathfrak{g}_1,\alpha_1,\omega_1)$ and $(\mathfrak{g}_2,\alpha_2,\omega_2)$ are said to be isomorphic if there exists a Lie algebra isomorphism $L:\mathfrak{g}_1\longrightarrow\mathfrak{g}_2$, such that,

$$L^*(\alpha_2)=\alpha_1 \quad\text{and}\quad L^*(\omega_2)=\omega_1.$$

The following proposition gives a complete classification up to isomorphism of three-dimensional cosymplectic real Lie algebras.

Proposition 4.2. *Let $(\mathfrak{g}, \alpha, \omega)$ be a three-dimensional cosymplectic Lie algebra. Then, $(\mathfrak{g}, \alpha, \omega)$ is isomorphic to one of the following cosymplectic Lie algebras:*

- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 : \ (\alpha, \omega) = (e^3, e^{12})$.
  $(\alpha, \omega) = (\lambda e^3, e^{12} + e^{23}), \ \ \lambda \in \mathbb{R} - \{0\}$.
- $\mathfrak{g}_{3.1} : \ (\alpha, \omega) = (\lambda e^2, e^{13}), \quad \lambda \in \mathbb{R} - \{0\}$.
- $\mathfrak{g}_{3.4}^{-1} : \ (\alpha, \omega) = (\lambda e^3, e^{12}), \quad \lambda > 0$.
- $\mathfrak{g}_{3.5}^{0} : \ (\alpha, \omega) = (\lambda e^3, e^{12}), \quad \lambda > 0$.

Proof. We proceed as follows. We act by the automorphisms of $\mathfrak{g}$ on ω to find the simplest possible form ω_0, then we seek all the a utomorphisms which transform ω into ω_0, and finally we act by these automorphisms on α in order to simplify it. We will give the proof for Lie algebra $\mathfrak{g}_{4.3}^{-1}$: $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_3$, since all cases must be treated in the same way. Cosymplectic structures on $\mathfrak{g}_{4.3}^{-1}$ are given by the following family

$$\alpha = a_3 e^3, \quad \omega = a_{12} e^{12} + a_{13} e^{13} + a_{23} e^{23}, \quad a_3 a_{12} \neq 0.$$

In this case, the automorphisms are given by

$$T_1 = \begin{pmatrix} t_{11} & 0 & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & t_{12} & t_{13} \\ t_{21} & 0 & t_{23} \\ 0 & 0 & -1 \end{pmatrix}.$$

The automorphism $L = \begin{pmatrix} 1 & 0 & \frac{a_{23}}{a_{12}} \\ 0 & \frac{1}{a_{12}} & -\frac{a_{13}}{a_{12}} \\ 0 & 0 & 1 \end{pmatrix}$ satisfies $L^*(\omega) = e^{12}$;
all automorphisms that satisfy $L^*(\omega) = e^{12}$ are

$$L_1 = \begin{pmatrix} t_{11} & 0 & \frac{a_{23}}{a_{12}} \\ 0 & \frac{1}{t_{11} a_{12}} & -\frac{a_{13}}{a_{12}} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} 0 & t_{12} & -\frac{a_{23}}{a_{12}} \\ -\frac{1}{t_{12} a_{12}} & 0 & \frac{a_{13}}{a_{12}} \\ 0 & 0 & -1 \end{pmatrix}.$$

A direct calculation gives $L_1^*(\alpha) = \alpha$ and $L_2^*(\alpha) = -\alpha$. Therefore, we can take $\alpha = \lambda e^3$, with $\lambda > 0$. ■

By Proposition 2.2, we have the following consequence.

Corollary 4.3. *The left-symmetric product associated to the three-dimensional cosymplectic Lie algebras is given by*

- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 : \ e_1 \cdot e_2 = e_1, \ \ e_2 \cdot e_2 = e_2$.
- $\mathfrak{g}_{3.1} : \ e_2 \cdot e_2 = \frac{1}{\lambda^2} e_3, \ \ e_3 \cdot e_2 = -e_1$.
- $\mathfrak{g}_{3.4}^{-1} : \ e_1 \cdot e_2 = e_2 \cdot e_1 = \frac{1}{\lambda^2} e_3, \ \ e_3 \cdot e_1 = -e_1, \ \ e_3 \cdot e_2 = e_2$.
- $\mathfrak{g}_{3.5}^{0} : \ e_1 \cdot e_1 = \frac{1}{\lambda^2} e_3, \ \ e_2 \cdot e_2 = \frac{1}{\lambda^2} e_3, \ \ e_3 \cdot e_1 = e_2, \ \ e_3 \cdot e_2 = -e_1$.

4.2. Five-dimensional cosymplectic Lie algebras

In [4] the authors give a classification of five-dimensional cosymplectic Lie algebras, starting from four-dimensional symplectic Lie algebras endowed with a derivation. They did not complete the correspondence with the well-known five-dimensional Lie algebras in the literature [12]. We propose here a direct method (case by case), by searching among the 40 five-dimensional Lie algebras [12], those which are cosymplectic and we give their structures.

Proposition 4.4. *Let $(\mathfrak{g},\alpha,\omega)$ be a five-dimensional cosymplectic real Lie algebra. Then, $(\mathfrak{g},\alpha,\omega)$ is isomorphic to one of the following cosymplectic Lie algebras*

$A_{5,1}$: $[e_3,e_5]=e_1,\ [e_4,e_5]=e_2$ *(nilpotent)*

$\alpha=a_3e^3+a_4e^4+a_5e^5$

$\omega=a_{13}e^{13}+a_{23}e^{14}+a_{15}e^{15}+a_{23}e^{23}+a_{24}e^{24}+a_{25}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45}$,

$a_3a_{15}a_{24}-a_3a_{23}a_{25}+a_4a_{13}a_{25}-a_4a_{15}a_{23}-a_5a_{13}a_{24}+a_5a_{23}^2\neq 0$.

$A_{5,2}$: $[e_2,e_5]=e_1,\ [e_3,e_5]=e_2,\ [e_4,e_5]=e_3$ *(nilpotent)*

$\alpha=a_4e^4+a_5e^5$

$\omega=-a_{23}e^{14}+a_{15}e^{15}+a_{23}e^{23}+a_{25}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45}$,

$a_{23}(a_4a_{15}+a_5a_{23})\neq 0$.

$A_{5,5}$: $[e_3,e_4]=e_1,\ [e_2,e_5]=e_1,\ [e_3,e_5]=e_2$ *(nilpotent)*

$\alpha=a_3e^3+a_4e^4+a_5e^5$

$\omega=a_{24}e^{15}+a_{23}e^{23}+a_{24}e^{24}+a_{25}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45}$,

$a_{24}(a_3a_{24}-a_4a_{23})\neq 0$.

$A_{5,6}$: $[e_3,e_4]=e_1,\ [e_2,e_5]=e_1,\ [e_3,e_5]=e_2,\ [e_4,e_5]=e_3$ *(nilpotent)*

$\alpha=a_4e^4+a_5e^5$

$\omega=-a_{23}e^{14}+a_{24}e^{15}+a_{23}e^{23}+a_{24}e^{24}+a_{25}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45}$,

$a_{23}(a_4a_{24}+a_5a_{23})\neq 0$.

$A_{5,7}^{a,-a,-1}$: $[e_1,e_5]=e_1,\ [e_2,e_5]=ae_2,\ [e_3,e_5]=-ae_3,\ [e_4,e_5]=-e_4,\ a\in]-1,0[\cup]0,1[$.

$\alpha=a_5e^5$

$\omega=a_{14}e^{14}+a_{15}e^{15}+a_{23}e^{23}+a_{24}e^{24}+a_{25}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45}$,

$a_5a_{14}a_{23}\neq 0$.

$A_{5,7}^{1,-1,-1}$: $[e_1,e_5]=e_1,\ [e_2,e_5]=e_2,\ [e_3,e_5]=-e_3,\ [e_4,e_5]=-e_4$

$\alpha=a_5e^5$

$\omega=a_{13}e^{13}+a_{14}e^{14}+a_{15}e^{15}+a_{23}e^{23}+a_{24}e^{24}+a_{25}e^{25}+a_{35}e^{35}+a_{45}e^{45}$,

$a_5(a_{24}a_{13}-a_{14}a_{23})\neq 0$.

$A_{5,8}^{-1}$: $[e_2,e_5]=e_1,\ [e_3,e_5]=e_3,\ [e_4,e_5]=-e_4$

$\alpha=a_2e^2+a_5e^5$

$\omega=a_{12}e^{12}+a_{15}e^{15}+a_{25}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45}$,

$a_{34}(a_5a_{12}-a_2a_{15})\neq 0$.

$A_{5,9}^{0,-1}$: $[e_1, e_5] = e_1\,,\ [e_2, e_5] = e_1 + e_2\,,\ [e_4, e_5] = -e_4$

$\alpha = a_3 e^3 + a_5 e^5$

$\omega = a_{15} e^{15} + a_{24} e^{24} + a_{25} e^{25} + a_{35} e^{35} + a_{45} e^{45},$

$a_3 a_{15} a_{24} \neq 0.$

$A_{5,13}^{-1,0,q}$: $[e_1, e_5] = e_1\,,\ [e_2, e_5] = -e_2\,,\ [e_3, e_5] = -q e_4\,,\ [e_4, e_5] = q e_3$

$\alpha = a_5 e^5$

$\omega = a_{12} e^{12} + a_{15} e^{15} + a_{25} e^{25} + a_{34} e^{34} + a_{35} e^{35} + a_{45} e^{45},$

$a_5 a_{12} a_{34} \neq 0.$

$A_{5,14}^{0}$: $[e_2, e_5] = e_1\,,\ [e_3, e_5] = -e_4\,,\ [e_4, e_5] = e_3$

$\alpha = a_2 e^2 + a_5 e^5$

$\omega = a_{12} e^{12} + a_{15} e^{15} + a_{25} e^{25} + a_{34} e^{34} + a_{35} e^{35} + a_{45} e^{45},$

$a_{34}(a_5 a_{12} - a_2 a_{15}) \neq 0.$

$A_{5,15}^{-1}$: $[e_1, e_5] = e_1\,,\ [e_2, e_5] = e_1 + e_2\,,\ [e_3, e_5] = -e_3\,,\ [e_4, e_5] = e_3 - e_4$

$\alpha = a_5 e^5$

$\omega = -a_{23} e^{14} + a_{15} e^{15} + a_{23} e^{23} + a_{24} e^{24} + a_{25} e^{25} + a_{35} e^{35} + a_{45} e^{45},$

$a_5 a_{23} \neq 0.$

$A_{5,17}^{1,p,-p}$: $[e_1, e_5] = p e_1 - e_2,\ [e_2, e_5] = e_1 + p e_2,\ [e_3, e_5] = -p e_3 - e_4,\ [e_4, e_5] = e_3 - p e_4,\ p \neq 0.$

$\alpha = a_5 e^5$

$\omega = a_{24} e^{13} - a_{23} e^{14} + a_{15} e^{15} + a_{23} e^{23} + a_{24} e^{24} + a_{25} e^{25} + a_{35} e^{35} + a_{45} e^{45},$

$a_5(a_{23}^2 + a_{24}^2) \neq 0.$

$A_{5,17}^{1,0,0}$: $[e_1, e_5] = -e_2\,,\ [e_2, e_5] = e_1\,,\ [e_3, e_5] = -e_4\,,\ [e_4, e_5] = e_3$

$\alpha = a_5 e^5$

$\omega = a_{12} e^{12} + a_{24} e^{13} - a_{23} e^{14} + a_{15} e^{15} + a_{23} e^{23} + a_{24} e^{24} + a_{25} e^{25} + a_{34} e^{34}$
$+ a_{35} e^{35} + a_{45} e^{45},\ a_5(a_{12} a_{34} - a_{23}^2 - a_{24}^2) \neq 0.$

$A_{5,17}^{-1,p,-p}$: $[e_1, e_5] = p e_1 - e_2\,,\ [e_2, e_5] = e_1 + p e_2\,,\ [e_3, e_5] = -p e_3 + e_4\,,$
$[e_4, e_5] = -e_3 - p e_4,\ p \neq 0.$

$\alpha = a_5 e^5$

$\omega = -a_{24} e^{13} + a_{23} e^{14} + a_{15} e^{15} + a_{23} e^{23} + a_{24} e^{24} + a_{25} e^{25} + a_{35} e^{35} + a_{45} e^{45},$

$a_5(a_{23}^2 + a_{24}^2) \neq 0.$

$A_{5,17}^{-1,0,0}$: $[e_1, e_5] = -e_2\,,\ [e_2, e_5] = e_1\,,\ [e_3, e_5] = e_4\,,\ [e_4, e_5] = -e_3$

$\alpha = a_5 e^5$

$\omega = a_{12} e^{12} - a_{24} e^{13} + a_{23} e^{14} + a_{15} e^{15} + a_{23} e^{23} + a_{24} e^{24} + a_{25} e^{25} + a_{34} e^{34}$
$+ a_{35} e^{35} + a_{45} e^{45},\ a_5(a_{12} a_{34} + a_{23}^2 + a_{24}^2) \neq 0.$

$A_{5,17}^{s,0,0}$: $[e_1, e_5] = -e_2\,,\ [e_2, e_5] = e_1\,,\ [e_3, e_5] = -se_4\,,\ [e_4, e_5] = se_3\,,\ s \notin \{-1, 0, 1\}.$
$\alpha = a_5 e^5$
$\omega = a_{12}e^{12} + a_{15}e^{15} + a_{25}e^{25} + a_{34}e^{34} + a_{35}e^{35} + a_{45}e^{45},$
$a_5 a_{12} a_{34} \neq 0.$

$A_{5,18}^{0}$: $[e_1, e_5] = -e_2\,,\ [e_2, e_5] = e_1\,,\ [e_3, e_5] = e_1 - e_4\,,\ [e_4, e_5] = e_2 + e_3$
$\alpha = a_5 e^5$
$\omega = a_{24}e^{13} + a_{15}e^{15} + a_{24}e^{24} + a_{25}e^{25} + a_{34}e^{34} + a_{35}e^{35} + a_{45}e^{45},$
$a_5 a_{24} \neq 0.$

$A_{5,19}^{1,-1}$: $[e_1, e_5] = e_1\,,\ [e_2, e_3] = e_1\,,\ [e_2, e_5] = e_2\,,\ [e_4, e_5] = -e_4$
$\alpha = a_3 e^3 + a_5 e^5$
$\omega = a_{23}e^{15} + a_{23}e^{23} + a_{24}e^{24} + a_{25}e^{25} + a_{35}e^{35} + a_{45}e^{45},$
$a_3 a_{23} a_{24} \neq 0.$

$A_{5,19}^{\frac{1}{2},-1}$: $[e_2, e_3] = e_1\,,\ [e_1, e_5] = \frac{1}{2}e_1\,,\ [e_2, e_5] = e_2\,,\ [e_3, e_5] = -\frac{1}{2}e_3\,,\ [e_4, e_5] = -e_4$
$\alpha = a_5 e^5$
$\omega = a_{13}e^{13} + a_{15}e^{15} + 2a_{15}e^{23} + a_{24}e^{24} + a_{25}e^{25} + a_{35}e^{35} + a_{45}e^{45},$
$a_5 a_{13} a_{24} \neq 0.$

$A_{5,19}^{-1,2}$: $[e_2, e_3] = e_1\,,\ [e_1, e_5] = -e_1\,,\ [e_2, e_5] = e_2\,,\ [e_3, e_5] = -2e_3\,,\ [e_4, e_5] = 2e_4$
$\alpha = a_5 e^5$
$\omega = a_{12}e^{12} - a_{23}e^{15} + a_{23}e^{23} + a_{25}e^{25} + a_{34}e^{34} + a_{35}e^{35} + a_{45}e^{45},$
$a_5 a_{12} a_{34} \neq 0.$

$A_{5,30}^{1}$: $[e_2, e_4] = e_1\,,\ [e_3, e_4] = e_2\,,\ [e_1, e_5] = 2e_1\,,\ [e_2, e_5] = e_2\,,\ [e_4, e_5] = e_4$
$\alpha = a_3 e^3 + a_5 e^5$
$\omega = 2a_{24}e^{15} + a_{24}e^{24} + a_{34}e^{25} + a_{34}e^{34} + a_{35}e^{35} + a_{45}e^{45},$
$a_3 a_{24} \neq 0.$

$A_{5,33}^{0,-1}$: $[e_1, e_4] = e_1\,,\ [e_3, e_4] = -e_3\,,\ [e_2, e_5] = e_2$
$\alpha = a_4 e^4 + a_5 e^5$
$\omega = a_{13}e^{13} + a_{14}e^{14} + a_{25}e^{25} + a_{34}e^{34} + a_{45}e^{45},$
$a_4 a_{13} a_{25} \neq 0.$

$A_{5,33}^{-1,0}$: $[e_1, e_4] = e_1\,,\ [e_2, e_5] = e_2\,,\ [e_3, e_5] = -e_3$
$\alpha = a_4 e^4 + a_5 e^5$
$\omega = a_{14}e^{14} + a_{23}e^{23} + a_{25}e^{25} + a_{35}e^{35} + a_{45}e^{45},$
$a_5 a_{14} a_{23} \neq 0.$

$A_{5,36}$: $[e_2, e_3] = e_1\,,\ [e_1, e_4] = e_1\,,\ [e_2, e_4] = e_2\,,\ [e_2, e_5] = -e_2\,,\ [e_3, e_5] = e_3$
$\alpha = a_4 e^4 + a_5 e^5$
$\omega = a_{23}e^{14} + a_{23}e^{23} - a_{25}e^{24} + a_{25}e^{25} + a_{35}e^{35} + a_{45}e^{45},$
$a_5 a_{23} \neq 0.$

$A_{5,37}:$ $[e_2,e_3]=e_1\,,\ [e_1,e_4]=2e_1\,,\ [e_2,e_4]=e_2\,,\ [e_3,e_4]=e_3\,,\ [e_2,e_5]=-e_3\,,$
$[e_3,e_5]=e_2$
$\alpha=a_4e^4+a_5e^5$
$\omega=2a_{23}e^{14}+a_{23}e^{23}+a_{35}e^{24}-a_{34}e^{25}+a_{34}e^{34}+a_{35}e^{35}+a_{45}e^{45},$
$a_5a_{23}\neq 0.$

Remark 4.5. Every five-dimensional cosymplectic Lie algebra is necessarily solvable. ■

Unlike the three-dimensional Heisenberg algebra, the five-dimensional Heisenberg algebra does not support any cosymplectic structure. As a general result, we have the following:

Proposition 4.6. *Let* $\mathfrak{h}_{2n+1}$ *be the* $(2n+1)$*-dimensional Heisenberg Lie algebra generated by elements* $\{e_i,f_i,z\}_{\{1\leq i\leq n\}}$*, with the relations* $[e_i,f_i]=z$*. Then* $\mathfrak{h}_{2n+1}$ *supports a cosymplectic structure if and only if* $n=1$*.*

Proof. Proposition 4.2 shows that $\mathfrak{h}_3=\mathfrak{g}_{3.1}$ admits cosymplectic structures. Let $n\geq 2$, and $\alpha\in\mathfrak{h}^*_{2n+1}$, $\omega\in\wedge^2\mathfrak{h}^*_{2n+1}$ be a 1-cocycle and 2-cocycle respectively. On the one hand by using the Maurer-Cartan equations of $\mathfrak{h}_{2n+1}$ we get $\alpha(z)=0$, on the other hand. For all $e_i,e_j,f_j,\ j\neq i$, we have

$$\omega(e_i,z)=\omega(e_i,[e_j,f_j])=\oint\omega(e_i,[e_j,f_j])=0.$$

In the same way we find $\omega(f_i,z)=0$ for all $1\leq i\leq n$. It follows that $\Phi(e_i,z)=\Phi(f_i,z)=0$ for all $1\leq i\leq n$. ■

Denote by $\mathfrak{aff}(2,\mathbb{R})$ the affine Lie algebra, generated by elements $\{e_1,...,e_6\}$, with the relations:

$$\begin{array}{lllll}[e_1,e_3]=-e_1, & [e_2,e_4]=-e_1, & [e_3,e_4]=e_4, & [e_4,e_5]=e_3-e_6, & [e_5,e_6]=-e_5,\\ [e_1,e_5]=-e_2, & [e_2,e_6]=-e_2, & [e_3,e_5]=-e_5, & [e_4,e_6]=e_4. & \end{array}$$

Proposition 4.7. *The only non-solvable cosymplectic Lie algebra of dimension less than or equal to seven is isomorphic to*

$$(\mathfrak{aff}(2,\mathbb{R})\ltimes\langle e_7\rangle,\alpha,\omega).$$

Its Lie brackets are given by those of $\mathfrak{aff}(2,\mathbb{R})$*, to which we add these new entries:* $[e_7,e_6]=\lambda e_2$*,* $[e_7,e_4]=\lambda e_1$*,* $\lambda\in\mathbb{R}$*, and the cosymplectic structure is* $\alpha=e^7$*,* $\omega=e^{15}+e^{26}+e^{34}+e^{46}$*.*

Proof. Let $(\mathfrak{g},\alpha,\omega)$ be a five-dimensional cosymplectic non-solvable Lie algebra. It is well known (see [11]) that any four-dimensional symplectic Lie algebra is solvable. By Proposition 3.5, we deduce that $\mathfrak{g}$ contains an ideal of codimension 1, and this contradicts the fact that $\mathfrak{g}$ is non-solvable. Now let $(\mathfrak{g},\alpha,\omega)$ be a seven-dimensional cosymplectic non-solvable Lie algebra. By Proposition 3.5, $(\mathfrak{h},\omega_{\mathfrak{h}})$

is a six-dimensional non-solvable symplectic Lie algebra. On the one hand, it is known that (see [3]) the only non-solvable six-dimensional Lie algebra is (up to isomorphism) $(\mathfrak{aff}(2,\mathbb{R}),\omega_0)$ with

$$\omega_0 = e^{15} + e^{26} + e^{34} + e^{46}.$$

On the other hand, on $\mathfrak{aff}(2,\mathbb{R})$ any derivation D is inner (see [7]). Hence, there exists $x \in \mathfrak{h}$ such that $D = \mathrm{ad}_x$. The 2-cocycle condition of ω_0 gives that

$$\omega_0(\mathrm{ad}_x y, z) + \omega_0(y, \mathrm{ad}_x z) = \omega_0(x, [y, z]).$$

Since D is an i.s.t, it follows that $\omega_0(x,[y,z]) = 0$, $\forall y, z \in \mathfrak{h}$. By direct computation we find that $x = \lambda e_2$, $\lambda \in \mathbb{R}$ and, using Proposition 3.5, we complete the proof. ■

References

[1] T. Aït Aissa, M. W. Mansouri: *Symplectic Novikov Lie algebras*, Comm. Algebra 50/7 (2022) 2921–2933.

[2] G. Bazzoni, M. Fernandez, V. Muñoz: *Non-formal co-symplectic manifolds*, Trans. Amer. Math. Soc. 367 (2015) 4459–4481.

[3] O. Bouzour, M. W. Mansouri: *Bi-Lagangian structure on the symplectic affine Lie algebra* $\mathbb{R}^2$, J. Geom. Symm. Physics 56 (2020) 45–57.

[4] G. Calvaruso, A. Perrone: *Cosymplectic and* α*-cosymplectic Lie algebras*, Complex Manifolds 3/1 (2016) 252–270.

[5] B. Cappelletti-Montano, A. De Nicola, I. Yudin: *A survey on cosymplectic geometry*, Rev. Math. Phys. 25/10 (2013), art. no. 1343002, 55 pp.

[6] B. Chu: *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. 197 (1974) 145–159.

[7] A. Diatta, B. Manga: *On properties of principal elements of Frobenius Liealgebras*, J. Lie Theory 24/3 (2014) 849–864.

[8] A. Fino, L. Vezzoni: *Some results on cosymplectic manifolds*, Geom. Dedicata 151 (2011) 41–58.

[9] M. de León, M. Saralegi: *Cosymplectic reduction for singular momentum maps*, J. Phys. A 26/19 (1993) 5033–5043.

[10] A. Medina: *Flat left-invariant connections adapted to the automorphism structure of a Lie group*, J. Diff. Geom. 16 (1981) 445–474.

[11] A. Medina, P. Revoy: *Groupes de Lie à structure symplectique invariante*, in: *Symplectic Geometry, Groupoids, and Integrable Systems*, Sémin. Sud-Rhodan. Geom. VI, Berkeley 1989, Math. Sci. Res. Inst. Publ. 20 (1991) 247–266.

[12] J. Patera, R. T. Sharp, P. Winternitz: *Invariants of real low dimension Lie algebras*, J. Math. Phys. 17 (1976) 986.

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