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# Ten-Dimensional Levi Decomposition Lie Algebras with $\mathfrak{sl}(2,\mathbb{R})$ Semi-Simple Factor

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**Abstract.** Turkowski has classified Lie algebras that have a non-trivial Levi decomposition of dimension up to and including nine. In this work the program is continued and completes the classification of the corresponding Lie algebras in dimension ten, for which the semi-simple factor is $\mathfrak{sl}(2,\mathbb{R})$. In the approach adopted here, one begins with a nilpotent Lie algebra *NR*, which will serve as the nilradical of the Levi decomposition algebra $S \rtimes N$ that is ultimately constructed. Here $N$ denotes a solvable extension of *NR*. Two key tools used in obtaining the classification are, the $R$-representation, that is, the action of $\mathfrak{sl}(2,\mathbb{R})$ as endomorphims of *NR* and secondly the algebra of $R$-constants, that is, the subalgebra of $N$ that commutes with the $R$-representation.

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*Key Words:* Semisimple factor, radical, nilradical, R-representation, R-constants Lie algebra.

## 1. Introduction

In this paper we complete our study of ten-dimensional Lie algebras that have a non-trivial Levi decomposition. In two previous articles [1] and [2] we have classified such Lie algebras for which the semi-simple factor is *not* isomorphic to a single copy of $\mathfrak{sl}(2,\mathbb{R})$. The present article is written in such a way as to be almost disjoint from [1] and [2]. Our work is inspired by, and constitutes an extension of, work by Turkowski [16] and [18], who solved the corresponding problem in dimension up to and including nine and we will largely follow his notation.

Another paper of Turkowski that might appear to be closely related to the current article is [15]. Motivated by considerations from string theory, in [15], the author gives a description of the ten-dimensional Levi decomposition Lie algebras that have a compact subalgebra of dimension at least seven. It is not important for his purpose whether the Lie algebras concerned are decomposable. He presents a table of 30 such Lie algebras, all but six of which are decomposable. The indecomposable algebras all have abelian nilradical and the simple factor is isomorphic to either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$. The cases corresponding to $\mathfrak{so}(3)$ appeared as $L_{10.30}$ and $L_{10.31}$ in [1]. There are four cases corresponding to $\mathfrak{sl}(2,\mathbb{R})$ and they are manifested in the current article in subsubsections 12.7.1, 12.7.2, 12.7.3 and 12.7.4.

An outline of this paper is as follows. In Section 2 we consider the problem of constructing Lie algebras that have a Levi decomposition in general. The key point

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is to start from a particular nilpotent Lie algebra, that we denote by $NR$, and look at its Lie algebra of derivations $\operatorname{der}(NR)$. The algebra $NR$ will serve as the nilradical in the Levi decomposition Lie algebra that we ultimately construct. In Section 3 we consider the so called $R$-representation, which in our case will be a representation of $\mathfrak{sl}(2,\mathbb{R})$. When referring to a particular $R$-representation, we will always specify its action on $NR$ rather than the radical $N$ that may contain $NR$. We also examine the subalgebra of $R$-constants, which comprise the elements in $N$ on which the $R$-representation acts trivially. In addition, we consider the conditions that arise in extending $NR$ to the solvable algebra $N$ that will play the role of the radical in the Levi decomposition algebra $L = S \rtimes N$, where $S$ is a semi-simple subalgebra of $\operatorname{der}(NR)$. We discuss the Jacobi identities that must be satisfied. A key point of the current article is that in many cases, a particular class of Lie algebra is determined by its $R$-representation and its algebra of $R$-constants; furthermore, the $R$-constants can be normalized by quoting known facts about Lie algebras of dimension four and five from [11] and [14].

In Section 3 we also list all possible representations of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{gl}(7,\mathbb{R})$ and refer the reader to a specific Levi decomposition Lie algebra for which a particular representation is realized. In Section 4 we carry out a preliminary survey of all nilpotent Lie algebras of dimension up to and including seven and discard some, mostly on the basis that their derivation subalgebras do not contain a semi-simple subalgebra, or for other reasons. We are able also to list the possible $R$-representations that can occur in each case. In Section 5 we begin the classification proper and show that there are no indecomposable Levi decomposition Lie algebras for which dim $NR = 4$. In Sections 6 to 11, we classify Lie algebras for which the dimension of $NR$ varies from five to seven, discussing the cases for which $NR$ is indecomposable and decomposable separately. Finally, in Section 12, we provide all the indecomposable Levi decomposition Lie algebras in a list. In each case, we shall supply the the original Levi decomposition algebra $S \rtimes NR$, once at the start of each subsubsection; in other words we give the first three groups of brackets in equation (1) from Section 3. Then we give the fourth and fifth groups of brackets in equation (1) that involve basis vectors of $N$ obtained in extending $NR$ to $N$, with "bullets", that distinguish the subalgebras within each subsubsection. In each case, at the start of the appropriate list of brackets, we identify for every algebra, its algebra of $R$-constants. These subalgebras may vary between dimension zero and five. Besides the fifth group of brackets in equation (1), the $R$-constants may involve brackets from the third and fourth group.

We do not attempt to assign an overall numbering to the list pending minor adjustments that may be needed later. In this regard, it should be pointed that the algebra $L_{10.14}$ that appeared in [1] has to be removed, because the representation denoted by $R_6$ is in fact conjugate to $2\operatorname{ad}\mathfrak{so}(3)$. Coming back to the present article, the reader will have to pay careful attention to Section 12, since it is closely aligned with the construction of the Lie algebras in Sections 5–11; in the interests of efficiency, we have sought to avoid repeating sets of Lie brackets, since they can be quite lengthy for particular algebras. In our list, we use, to the extent possible, Roman letters at the start of the alphabet for parameters that are needed to describe a family of Lie algebras; these letters usually do *not* coincide with the numbering scheme in [11], even when one of the algebras in our list contains such a subalgebra, particularly its $R$-constants. However, we quote the numbering in [11] so that the reader may be able to identify the $R$-constants in a standard list. Furthermore, in our list the parameters that appear do not usually correspond to parameters that are used in the body of the paper to obtain a particular subalgebra; again, our intention in the list, is to use Roman letters at the start of the alphabet, in a systematic way.

In the course of carrying out our classification of the ten-dimensional Levi decomposition Lie algebras, we shall need to refer to other classes of lower dimension. We follow the numbering as $A_{i,j}$ where $3 \le i \le 6$, and $j$ signifies the $j$th algebra in the list in [11] and [14] for the indecomposable Lie algebras of dimension five and less and also for the indecomposable nilpotent Lie algebras of dimension six. The indecomposable Lie algebras of dimension six that have a five-dimensional nilradical were classified by Mubarakzyanov [10] and are denoted by $g_{6,i}$, where $1 \le i \le 99$; see also [13] for an updated classification. The indecomposable Lie algebras of dimension six that have a four-dimensional nilradical classified by Turkowski [17], are denoted by $N_{6,i}$ where $1 \le i \le 40$. Turkowski denotes by $L_{p,i}$ the classes of indecomposable Levi decomposition Lie algebras of dimension $p$ where $5 \le p \le 9$. We note also that for $p = 8$, one algebra seems to be missing from [16]; it is a semi-direct product of $\mathfrak{sl}(2,\mathbb{R})$ and $H \oplus \mathbb{R}^2$ where here and below, $H$ denotes the three-dimensional Heisenberg Lie algebra. The Lie algebra $H$ is $A_{3,1}$ in [11]. We shall refer to the missing Lie algebra as $L_{8,23}$ and it may be seen in one representation in equation (19) in subsubsection 7.1.1. We shall also need to refer to the Seeley-Gong list [12] and [6], for the indecomposable nilpotent Lie algebras of dimension seven. As well as classifying the Levi decomposition Lie algebras, a natural question is to find faithful matrix representations for them. In [3], we have found such representations for the algebras listed in [1] and intend to continue the program for the algebras that are compiled in the current article in another venue.

For further information about low-dimensional Lie algebras in general, the reader may refer to [14]. There is also a memoir devoted to studying the invariants of the nine-dimensional, Levi decomposition algebras [4]. For abelian Lie algebras of dimension $n$, we usually say "Abelian" and refer to $\mathbb{R}^n$ rather than writing $nA_1$. The trivial representation of dimension $n$ is denoted by $nD_0$. The irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension $n$ is denoted by $D_{\frac{n}{2}+1}$. The notation $K \oplus L$ is used occasionally for the direct sum of two Lie algebras $K$ and $L$, but it will play a very minor role in this paper. However, much more important for us, following Turkowski [18], we shall also write a "diagonal representation" involving, say, $D_m$ and $D_n$ as $D_m \oplus D_n$, which is another representation of $\mathfrak{sl}(2,\mathbb{R})$. Instead of, for example, $D_1 \oplus D_1$ we shall write $2D_1$ and $3D_{\frac{1}{2}}$ in place of $D_{\frac{1}{2}} \oplus D_{\frac{1}{2}} \oplus D_{\frac{1}{2}}$.

The Lie algebra of dimension $2n+1$ that has brackets

$$[e_i, e_{n+j}] = \delta_{ij} e_{2n+1}, \ (1 \le i \le j \le n)$$

is the *Heisenberg* Lie algebra whereas the Lie algebra of dimension $2n+1$ that has brackets

$$[e_i, e_{2n+1}] = \delta_{ij} e_{n+j}, \ (1 \le i \le j \le n)$$

is the *anti-Heisenberg* Lie algebra. Unfortunately the Heisenberg and anti-Heisenberg coincide for $n = 1$ but in any case, that Lie algebra is denoted by $H$. For $n = 3$ we have algebras 7.17 and 7.37$A$ in [6].

The calculations in this article were performed with the symbolic manipulation program MAPLE. Finally, we thank the referee for giving us some useful suggestions for streamlining our presentation.

## 2. A method for constructing Levi-decomposition algebras

We have explained in [1], one way to construct Levi-decomposition algebras. However, we wish now to give a new perspective on the process and propose a quasi-algorithm. We wish to find all Levi-decomposition algebras $L$ of a certain dimension. To that end, suppose first of all, that we do have a non-trivial Levi decomposition algebra $S \rtimes N$. Then we have a representation of $S$ acting by endomorphisms on $N$. We call this representation, the *R-representation* following [18]. We know that any element of $N$ in the complement to $NR$ commutes with $S$. It follows from the fact that any derivation of $N$ maps into $NR$ [8] and that the resulting representation of $S$ is completely reducible, since $S$ is semi-simple. Whenever we give a description of the $R$-representation, we shall always assume that it is acting on $NR$ rather $N$, so as to avoid including unnecessary copies of $D_0$. We know also that $S \rtimes NR$ is itself a Levi decomposition algebra.

In constructing non-trivial Levi decomposition Lie algebras, we start from some nilpotent Lie algebra $NR$ and look at its space of derivations, $\mathrm{der}(NR)$. If $\mathrm{der}(NR)$ is merely solvable, and there is no semi-simple subalgebra of $\mathrm{der}(NR)$, we stop, because $NR$ cannot be the nilradical of a non-trivial Levi decomposition algebra. In the contrary case, we will obtain a semi-simple Lie algebra, *occurring in some representation*, as endomorphisms of $NR$, [1, 16, 18]. If we are in interested in finding Levi decomposition Lie algebras of a particular dimension, say $n$ ($n = 10$, in our case), we have to find all $R$-representations $S$ in $\mathrm{der}(NR)$, that lead to a Levi decomposition algebra $S \rtimes NR$ of dimension $\leq n$. If $dim(S \rtimes NR) = n$, we have found a Levi decomposition algebra of the required dimension; otherwise, if $dim(S \rtimes NR) < n$, we have to see if it is possible to add some outer derivations $D$ of $NR$, that as we shall see in Section 3, must commute with the $R$-representation. Such derivations cannot come from nilpotent matrices or else we will be extending the nilradical, in view of Engel's Theorem. We will need to find the centralizer of the $R$-representation and it may be useful to apply Schur's Lemma here. Finally we must check if all Jacobi identities are satisfied. We take up this issue again in Section 3.

After we have found a particular Levi decomposition algebra, we can see if it is possible to simplify it by change of basis; for example, by exponentiating derivations $D$ of $NR$, which lead to automorphisms of $N$ that do not change $NR$.

## 3. *R*-representation, *R*-constants, Jacobi identities

**3.1. Structure equations.** Following on from Section 2, we shall develop the main theoretical ideas that will be needed in the classification of the Levi decomposition Lie algebras of dimension ten. If we have such a Lie algebra $L$, not necessarily of dimension ten, we shall write its non-zero Lie brackets in the following way:

$$[e_a, e_b] = C^c_{ab} e_c,\ [e_a, e_i] = C^j_{ai} e_j,\ [e_i, e_j] = C^k_{ij} e_k,\ [e_i, e_\alpha] = C^j_{i\alpha} e_j,\ [e_\alpha, e_\beta] = C^i_{\alpha\beta} e_i. \quad (1)$$

Here the $\{e_a\}$ denote a basis for the semi-simple part $S$ of $L$, whereas $\{e_i\}$ denotes a basis for the nilradical $NR$ and $\{e_\alpha\}$ a basis for a complement in the radical $N$ to $NR$. Furthermore, we use the summation convention on repeated indices, but we do not attempt to give values over which such indices range. The condition $[e_a, e_\alpha] = 0$ results from the fact that any derivation of $N$ maps into $NR$ [8] and that the resulting representation of $S$ is completely reducible, since $S$ is semi-simple.

The brackets $[e_a, e_i] = C^j_{ai} e_j$ comprise the $R$-representation. Our point of view then is that we are starting from the Lie algebra $S \rtimes NR$ and we wish to see if we can extend $NR$ to a solvable algebra so as to obtain a Levi-decomposition algebra of a particular dimension, in our case ten. Thus we will have some "new structure equations" corresponding to the last two brackets in (1). The question now is whether the Jacobi identities hold in this extended algebra.

**3.2. $R$-constants.** Before discussing Jacobi identities, we shall introduce the algebra of $R$-constants. They are the set of elements in $N$ upon which $S$ acts trivially; referring to equation (1), it certainly contains all the $e_\alpha$'s but may also contain other elements in $NR$. We have:

**Proposition 3.1.** *The space of $R$-constants forms a solvable subalgebra of $S \rtimes N$ and the intersection of the $R$-constants and $NR$ is an ideal in this subalgebra. The derived algebra of the $R$-constants is spanned by the intersection of the $R$-constants and $NR$.*

We note also that the Levi-decomposition algebra $L = S \rtimes N$ is a sum of the subalgebra $L = S \rtimes NR$ and the ideal $N$, but the sum is not direct and their intersection is $NR$. Furthermore, the $R$-constants form a complement to the subspace consisting of the semi-simple algebra $S$ and the subspace of $NR$ on which it acts irreducibly, although this subspace need not be a subalgebra.

**3.3. Jacobi identities** Referring to equation (1), we have to consider Jacobi identities coming from the following sets of vectors:

- $e_a, e_b, e_\alpha$,
- $e_a, e_i, e_\alpha$,
- $e_i, e_j, e_\alpha$
- $e_a, e_\alpha, e_\beta$,
- $e_i, e_\alpha, e_\beta$,
- $e_\alpha, e_\beta, e_\gamma$

Taking these conditions in turn, the first and third are already identities. The first of them follows from the condition $[e_a, e_\alpha] = 0$, the third because $\mathrm{ad}(e_\alpha)$ is a derivation of $NR$. The second condition gives

$$C^j_{ai} C^k_{\alpha j} - C^j_{\alpha i} C^k_{aj} = 0 \tag{2}$$

which says that each $\mathrm{ad}(e_\alpha)$ commutes with the $R$-representation on $NR$. The fourth condition gives

$$[e_a, [e_\alpha, e_\beta]] = 0, \tag{3}$$

or equivalently

$$C^i_{\alpha\beta} C^j_{ai} = 0, \tag{4}$$

which says that the bracket of the $R$-constants $[e_\alpha, e_\beta]$ is another $R$-constant. The fifth condition gives

$$C^j_{i\alpha} C^k_{j\beta} - C^j_{i\beta} C^k_{j\alpha} - C^j_{\alpha\beta} C^k_{ij} = 0. \tag{5}$$

The sixth condition gives $$C^j_{i[\alpha}C^i_{\beta\gamma]} = 0, \tag{6}$$
the square parentheses here denoting skew symmetrization.

In practice, it is easy to satisfy equation (2) with the help of Schur's Lemma. Equation (4) is effectively a linear condition since the $R$-representation is assumed to be known and (5) is quadratic in the unknown $e_\alpha$'s.

In the case of Levi decomposition Lie algebras of dimension ten, the nilradical $NR$ could be of dimension four, five six or seven. We shall see that if $NR$ has dimension four, then the Levi decomposition algebra must be decomposable. If $NR$ has dimension seven, then no $e_\alpha$'s are added and if $NR$ has dimension six, then it is apparent from (2,4) and (5), that the Jacobi identities are satisfied. If $NR$ has dimension five then conditions (2) and (4) will have to be satisfied. In this case before satisfying such identities we will sometimes use the term "pre-Lie algebra".

Since we are concerned from now on only with the case where the semi-simple part of the Levi decomposition Lie algebras is $\mathfrak{sl}(2,\mathbb{R})$, we shall need to find all possible representations of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{gl}(7,\mathbb{R})$. Since they are quite large, we shall not give the $7\times7$ matrices that correspond to each such representation; instead we shall refer the reader to particular illustrative Lie algebras in our list in Section 12, for which the given representation occurs as factors in the $R$-representation. The representations are as follows:

- $D_3$: subsubsection 12.7.1
- $D_{\frac{5}{2}} \oplus D_0$: subsubsection 12.5.7
- $D_2 \oplus 2D_0$: subsubsection 12.4.16
- $D_{\frac{3}{2}} \oplus 3D_0$: subsubsection 12.5.5
- $D_1 \oplus 4D_0$: subsubsection 12.4.22
- $D_{\frac{1}{2}} \oplus 5D_0$: subsubsection 12.1.1
- $D_2 \oplus D_{\frac{1}{2}}$: subsubsection 12.7.3
- $D_{\frac{3}{2}} \oplus D_1$: subsubsection 12.7.4
- $D_{\frac{3}{2}} \oplus D_{\frac{1}{2}} \oplus D_0$: subsubsection 12.6.2
- $2D_1 \oplus D_0$: subsubsection 12.3.2
- $D_{\frac{1}{2}} \oplus D_1 \oplus 2D_0$: subsubsection 12.4.13
- $D_1 \oplus 2D_{\frac{1}{2}}$ subsubsection 12.7.2
- $2D_{\frac{1}{2}} \oplus 3D_0$: subsubsection 12.1.1
- $3D_{\frac{1}{2}} \oplus D_0$: subsubsection 12.4.18

We conclude this section with a useful result which greatly simplifies the discussion for $NR = \mathbb{R}^7$ at the end of the body of the paper.

**Theorem 3.2.** *Suppose that a semi-simple Lie algebra $S$ has a faithful representation in $End(N)$ for some vector space $N$. Then there is a Lie algebra $S \rtimes N$ that has a Levi decomposition with $N$ being an abelian radical. Conversely, every Lie algebra that has a Levi decomposition with abelian radical arises in this way. Such a Lie algebra is decomposable if the representation of $S$, being completely reducible, contains a trivial subrepresentation.*

## 4. Initial sorting

In this Section we provide a preliminary list of cases that have to be considered when constructing all possible ten-dimensional indecomposable Levi-decomposition algebras with $\mathfrak{sl}(2,\mathbb{R})$ as semi-simple factor. The nilradical *NR* can have dimension of four to seven inclusive. For each possible such nilpotent Lie algebra, we have computed its list of derivations. Each such nilpotent Lie algebra that does not contain a non-trivial semi-simple subalgebra of derivations, cannot give rise to a Levi-decomposition algebra and is thus to be excluded. For each such nilpotent Lie algebra that does contain a non-trivial semi-simple subalgebra of derivations, we have to find all possible $R$-representations that are subalgebras of the algebra of derivations and are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and they too are listed below. As the title of this Section indicates, here we are only performing a preliminary sorting: it is yet possible that one of the cases does not produce a ten-dimensional indecomposable Levi-decomposition algebra. The only way to ascertain if it does so, is to follow the procedure described in Section 2.

The following simple Proposition will prove to be useful for excluding certain direct sums of nilpotent Lie algebras from giving rise to indecomposable Levi-decomposition Lie algebras.

**Proposition 4.1.** *Let $N_1$ and $N_2$ be nilpotent Lie algebras such that the semi-simple part of der $(N_1 \oplus N_2)$ is isomorphic to the the semi-simple part of der $(N_1)$. Then the Levi-decomposition Lie algebra obtained by letting a semi-simple subalgebra of der $(N_1 \oplus N_2)$ act on $N_1 \oplus N_2$ is decomposable.*

### 4.1. NR four-dimensional

#### 4.1.1. NR indecomposable

- $A_{4.1}$: No ten-dimensional indecomposable Levi-decomposition algebra because $\mathrm{der}(A_{4.1})$ contains no non-trivial semi-simple subalgebra.

#### 4.1.2. NR decomposable

- $H \oplus \mathbb{R}$: $R$-Representation $D_{\frac{1}{2}} \oplus 2D_0$
- $\mathbb{R}^4$: $R$-Representations $D_{\frac{1}{2}} \oplus 2D_0, 2D_{\frac{1}{2}}, D_1 \oplus D_0, D_{\frac{3}{2}}$

### 4.2. NR five-dimensional

#### 4.2.1. NR indecomposable

- $A_{5.1}, A_{5.3}, A_{5.4}$: $R$-Representation, $2D_{\frac{1}{2}} \oplus D_0$.
- $A_{5.2}, A_{5.5}, A_{5.6}$: No ten-dimensional indecomposable Levi-decomposition algebra because the derivation algebra contains no non-trivial semi-simple subalgebra.

#### 4.2.2. NR decomposable

- $H \oplus \mathbb{R}^2$: $R$-Representation $D_{\frac{1}{2}} \oplus 5D_0$ in two ways and $2D_{\frac{1}{2}} \oplus 3D_0$.
- $A_{4.1} \oplus \mathbb{R}$: No ten-dimensional indecomposable Levi-decomposition algebra because $\mathrm{der}(A_{4.1})$ contains no non-trivial semi-simple subalgebra.
- $\mathbb{R}^5$: $R$-Representation $D_{\frac{1}{2}} \oplus 3D_0, 2D_{\frac{1}{2}} \oplus D_0, D_1 \oplus 2D_0, D_1 \oplus D_{\frac{1}{2}}, D_{\frac{3}{2}} \oplus D_0, D_2$.

### 4.3. NR six-dimensional

#### 4.3.1. NR indecomposable

- $A_{6.i}$: $1 \leq i \leq 22, i \neq 3,4,5,12$: No ten-dimensional indecomposable Levi-decomposition algebra because $\operatorname{der}(A_{6.i})$ contains no non-trivial semi-simple subalgebra.
- $A_{6.3}$: $R$-Representation $2D_1$ or $2D_{\frac{1}{2}} \oplus 2D_0$.
- $A_{6.4}, A_{6.5}$: $R$-Representation $2D_{\frac{1}{2}} \oplus 2D_0$.
- $A_{6.12}$: $R$-Representation $D_{\frac{1}{2}} \oplus 4D_0$.

#### 4.3.2. NR decomposable

- $H \oplus \mathbb{R}^3$: $R$-Representation, $D_{\frac{1}{2}} \oplus 4D_0, 2D_{\frac{1}{2}} \oplus 2D_0$ (two ways), $D_{\frac{1}{2}} \oplus D_1 \oplus D_0, D_1 \oplus 4D_0$.
- $H \oplus H, A_{5.1} \oplus \mathbb{R}, A_{5.3} \oplus \mathbb{R}$: $R$-Representation $2D_{\frac{1}{2}} \oplus 2D_0$.
- $A_{4.1} \oplus \mathbb{R}^2$: $R$-Representation $D_{\frac{1}{2}} \oplus 4D_0$.
- $A_{5.2} \oplus \mathbb{R}, A_{5.5} \oplus \mathbb{R}, A_{5.6} \oplus \mathbb{R}$: No ten-dimensional indecomposable Levi-decomposition algebra because the derivation algebra contains no non-trivial semi-simple subalgebra.
- $A_{5.4} \oplus \mathbb{R}$: $R$-Representation $D_{\frac{1}{2}} \oplus 4D_0, 2D_{\frac{1}{2}} \oplus 2D_0$ and $D_{\frac{3}{2}} \oplus 2D_0$.
- $\mathbb{R}^6$: $R$-Representation $D_{\frac{1}{2}} \oplus 4D_0$, $2D_{\frac{1}{2}} \oplus 2D_0, 3D_{\frac{1}{2}}$, $2D_1$, $D_{\frac{1}{2}} \oplus D_1 \oplus D_0$, $D_{\frac{1}{2}} \oplus D_{\frac{3}{2}}$, $2D_1$, $D_{\frac{3}{2}} \oplus 2D_0$, $D_2 \oplus 2D_0$, $D_1 \oplus 3D_0$, $D_{\frac{5}{2}}$.

### 4.4. NR seven-dimensional

#### 4.4.1. NR indecomposable

- $7,17$: $R$-Representation $D_{\frac{1}{2}} \oplus 5D_0, 2D_{\frac{1}{2}} \oplus 3D_0, 3D_{\frac{1}{2}} \oplus D_0$, $2D_1 \oplus D_0$, $D_{\frac{3}{2}} \oplus 3D_0$, $D_{\frac{3}{2}} \oplus D_{\frac{1}{2}} \oplus D_0$, $D_{\frac{3}{2}} \oplus D_0$ .
- $7.27A, 7.157, 7.257K, 7.1457A, 7.1457B$: $R$-Representation $D_{\frac{1}{2}} \oplus 5D_0$.
- $7.27B, 7.37A, 7.37C$: $R$-Representation $2D_{\frac{1}{2}} \oplus 3D_0$.
- $7.37D$: $R$-Representation $2D_{\frac{1}{2}} \oplus 3D_0$ and $2D_{\frac{1}{2}} \oplus D_1$.
- $7.37D1$: $R$-Representation $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, not needed for $\mathfrak{sl}(2,\mathbb{R})$.
- $7.247A$: $R$-Representation $3D_{\frac{1}{2}} \oplus D_0$.

#### 4.4.2. NR decomposable

- $H \oplus \mathbb{R}^4$: $R$-Representation, $3D_{\frac{1}{2}} \oplus D_0$ and $D_{\frac{1}{2}} \oplus D_{\frac{3}{2}} \oplus D_0$
- $H \oplus H \oplus \mathbb{R}, A_{4.1} \oplus \mathbb{R}^3, A_{4.1} \oplus H, A_{5.2} \oplus \mathbb{R}^2, A_{5.5} \oplus \mathbb{R}^2, A_{5.6} \oplus \mathbb{R}^2, A_{6.i} \oplus \mathbb{R} (1 \leq i \leq 22)$: excluded by Proposition (4.1)
- $A_{5.1} \oplus \mathbb{R}^2, A_{5.3} \oplus \mathbb{R}^2, A_{5.4} \oplus \mathbb{R}^2$: $R$-Representation $2D_{\frac{1}{2}} \oplus 3D_0$
- $\mathbb{R}^7$: $R$-Representation. In view of Theorem (3.2) it can only be one of $D_3, D_2 \oplus D_{\frac{1}{2}}, D_{\frac{3}{2}} \oplus D_1$ or $D_1 \oplus 2D_{\frac{1}{2}}$

## 5. NR four-dimensional

We continue next the discussion from Section 4 and focus on those nilpotent Lie algebras that could be the nilradical of a ten-dimensional indecomposable Levi-decomposition algebra. The only possibilities for a four-dimensional nilradical, $NR$, are $H \oplus \mathbb{R}$ or $\mathbb{R}^4$. Consider the case $NR = H \oplus \mathbb{R}$ which we shall write as the space spanned by $\{e_4, e_5, e_6, e_7\}$ with bracket $[e_5, e_6] = e_4$. The space of derivations is given by

$$S = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ 0 & s_5 & s_6 & 0 \\ 0 & s_7 & s_1 - s_5 & 0 \\ 0 & s_8 & s_9 & s_{10} \end{bmatrix} \tag{7}$$

whereas the centralizer of the three matrices corresponding to $s_5, s_6, s_7$ is given by

$$T = \begin{bmatrix} z_5 & 0 & 0 & z_4 \\ 0 & z_3 & 0 & 0 \\ 0 & 0 & z_3 & 0 \\ z_2 & 0 & 0 & z_1 \end{bmatrix}. \tag{8}$$

The intersection of the spaces spanned by $S$ and $T$ is three-dimensional but the matrices are nil-dependent. Hence there is no algebra for which $NR = H \oplus \mathbb{R}$. Incidentally, the space spanned by the matrices in $S$ is ten-dimensional with semi-simple part isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, nilradical isomorphic to $A_{5.1}$ with two-dimensional abelian complement. Hence this algebra must appear in our list when we discuss the case that $NR$ is five-dimensional and the semi-simple part is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. We shall encounter it below in subsection 7.1 and subsubsection 12.1.1 in the case where $NR$ is five-dimensional indecomposable.

It remains to discuss the case $NR = \mathbb{R}^4$. If there is to be a Lie algebra that has a non-trivial Levi decomposition with semi-simple part isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, the only possibilities for the $R$-representation, restricted to $NR$, are $D_{\frac{3}{2}}, D_1 \oplus D_0, 2D_{\frac{1}{2}}$ or $D_{\frac{1}{2}} \oplus 2D_0$. We need three nil-independent matrices that commute with the $R$-representation on $NR$. By using Schur's Lemma, we can see that in the case of $D_{\frac{3}{2}}$, there is only one such matrix up to a multiple and only two linearly independent such matrices in the case of $D_1 \oplus D_0$. In the case of $2D_{\frac{1}{2}}$, the Lie algebra of $R$-constants, spanned by $e_8, e_9, e_{10}$, would be abelian. The centralizer of $2D_{\frac{1}{2}}$ is isomorphic to $\mathfrak{gl}(2, \mathbb{R})$ and a maximal abelian subalgebra is of dimension two. Hence none of these three $R$-representations lead to Levi-decomposition algebras.

In case the $R$-representation is $D_{\frac{1}{2}} \oplus 2D_0$ we shall take the algebra in the form:

$[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_3, e_4] = e_5$, $[e_4, e_8] = ae_4$, $[e_4, e_9] = be_4$, $[e_4, e_{10}] = qe_4$, $[e_5, e_8] = ae_5$, $[e_5, e_9] = be_5$, $[e_5, e_{10}] = qe_5$, $[e_6, e_8] = ce_6 + de_7$, $[e_6, e_9] = ke_6 + me_7$, $[e_6, e_{10}] = re_6 + se_7$, $[e_7, e_8] = ee_6 + fe_7$, $[e_7, e_9] = ne_6 + pe_7$, $[e_7, e_{10}] = te_6 + ue_7$, $[e_8, e_9] = ge_6 + he_7$, $[e_8, e_{10}] = -ie_6 - je_7$, $[e_9, e_{10}] = ve_6 + we_7$.

We are at liberty to change basis by taking linear combinations of $e_8, e_9, e_{10}$. Moreover, not all of $a, b, q$ can be zero or else the algebra will be decomposable. Accordingly, we may assume that $a = 0$, $b = 0$, $q = 1$. Now equation (5) amounts to the mutual commuting of the three matrices $\left[\begin{smallmatrix} a & g \\ b & h \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} c & i \\ d & j \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} e & k \\ f & m \end{smallmatrix}\right]$ and the first two of them and any linear combinations of them, must be nil-independent. As such the first two may be assumed to be in canonical form and we may subtract multiples of them from the third matrix, so as to reduce the latter matrix to zero. There are only two canonical forms since neither of the non-zero matrices can be nilpotent. In the first form we may suppose that $c = 1$, $d = 0$, $e = 0$, $f = 0$, $p = 1$, $n = 0$, $k = 0$, $m = 0$. At this point equation (5) reduces to $j = v = 0$. Finally we make a change of basis according to which $e_8, e_9, e_{10}$ become $e_8 - he_7$, $e_9 + ge_6$, $e_{10} - ie_6 + we_7$ and the Lie algebra changes to

$[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_3, e_4] = e_5$, $[e_4, e_{10}] = e_4$, $[e_5, e_{10}] = e_5$, $[e_6, e_8] = e_6$, $[e_7, e_9] = e_7$,

which is a decomposable Lie algebra. The case of the second canonical form is similar. In conclusion, it is impossible to have an *indecomposable* ten-dimensional Levi-decomposition Lie algebra for which the nilradical is four-dimensional.

## 6. NR five-dimensional indecomposable

Now we suppose that the dimension of $NR$ is five. There are six indecomposable nilpotent Lie algebras of dimension five [14]. According to Section 4 we need only consider $A_{5.1}, A_{5.3}$ and $A_{5.4}$.

**6.1.** $NR = A_{5.1}$**.** The Lie algebra is given by: $[e_6, e_8] = e_4, [e_7, e_8] = e_5$.

The space of derivations is given by

$$\begin{bmatrix} s_1 & 0 & s_2 & s_3 & s_4 \\ 0 & s_1 & s_5 & s_6 & s_7 \\ 0 & 0 & s_8 & 0 & s_9 \\ 0 & 0 & 0 & s_8 & s_{10} \\ 0 & 0 & 0 & 0 & s_1 - s_8 \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & 0 & 0 & 0 \\ t_3 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & t_1 & t_2 & 0 \\ 0 & 0 & t_3 & -t_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

The semi-simple subalgebra produces an $R$-representation of $2D_{\frac{1}{2}}$ and together with $NR = A_{5.1}$ forms an eight-dimensional Levi-decomposition Lie algebra, 8.14 in [16]. We seek to extend it by using the solvable ideal of derivations. The set of such derivations that commute with the $R$-representation is

$$\begin{bmatrix} z_1 & 0 & z_2 & 0 & 0 \\ 0 & z_1 & 0 & z_2 & 0 \\ 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & 0 & z_1 - z_3 \end{bmatrix}, \tag{10}$$

and we obtain a ten-dimensional Levi-decomposition pre-Lie algebra as:

$[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_1, e_6] = e_6$, $[e_1, e_7] = -e_7$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_2, e_7] = e_6$, $[e_3, e_4] = e_5$, $[e_3, e_6] = e_7$, $[e_4, e_9] = ae_4$, $[e_5, e_9] = ae_5$, $[e_6, e_8] = e_4$, $[e_6, e_9] = be_4$, $[e_6, e_{10}] = e_6$, $[e_7, e_8] = e_5$, $[e_7, e_9] = be_7$, $[e_7, e_{10}] = e_7$, $[e_8, e_9] = (a - b)e_8$, $[e_8, e_{10}] = (c - d)e_8$, $[e_9, e_{10}] = fe_8$.

All that is required to satisfy the Jacobi identity is to put $f = c(d-e)$. Thereafter, the change of basis that replaces $e_9$ by $e_9 + ce_8$ results in $c$ being set to zero and we obtain a ten-dimensional algebra that is free of parameters where $a = d = 1$ and $b = e = 0$.

**6.2.** $NR = A_{5.3}$**.** We take the algebra as: $[e_6, e_7] = e_5$, $[e_6, e_8] = e_4$, $[e_7, e_8] = e_6$. The space of derivations is given by

$$\begin{bmatrix} 3s_1 & 0 & s_2 & s_3 & s_4 \\ 0 & 3s_1 & s_5 & s_6 & s_7 \\ 0 & 0 & 2s_1 & s_2 & -s_5 \\ 0 & 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & 0 & s_1 \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & 0 & 0 & 0 \\ t_3 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t_1 & t_3 \\ 0 & 0 & 0 & t_2 & t_1 \end{bmatrix}. \tag{11}$$

We can combine the semi-simple factor with $A_{5.3}$ so as to obtain an eight-dimensional Levi-decomposition Lie algebra. However, we see from the solvable factor that there is only one nil-independent derivation and so it is not possible to extend to a ten-dimensional Levi-decomposition algebra.

We note, in passing, that the derivation algebra of $A_{5.3}$ gives a ten-dimensional Levi-decomposition Lie algebra with semi-simple factor $\mathfrak{sl}(2,\mathbb{R})$ appearing in the representation $2D_{\frac{1}{2}}$ and whose nilradical is six-dimensional and isomorphic to $H \oplus \mathbb{R}^3$. We also obtain a representation of this algebra that we shall encounter below in subsubsections 9.6.4 and 12.4.12 in the case where $NR$ is six-dimensional decomposable.

**6.3.** $NR = A_{5.4}$**.** In fact $A_{5.4}$ is the five-dimensional Heisenberg Lie algebra. We shall write it in the form

$$[e_5, e_7] = e_4,\ [e_6, e_8] = e_4. \tag{12}$$

The space of derivations is given by

$$\begin{bmatrix} 2s_5 & s_1 & s_2 & s_3 & s_4 \\ 0 & s_5 & 0 & 0 & 0 \\ 0 & 0 & s_5 & 0 & 0 \\ 0 & 0 & 0 & s_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & t_5 & t_6 \\ 0 & t_3 & t_4 & t_6 & t_7 \\ 0 & t_8 & t_9 & -t_1 & -t_3 \\ 0 & t_9 & t_{10} & -t_2 & -t_4 \end{bmatrix}. \tag{13}$$

It is a 15-dimensional Levi-decomposition algebra with semi-simple factor $\mathfrak{sp}(4)$. We can find, up to conjugacy, each of the representations $D_{\frac{1}{2}} \oplus 3D_0$, $2D_{\frac{1}{2}} \oplus D_0$ and $D_{\frac{3}{2}} \oplus D_0$ inside $\mathfrak{sp}(4)$.

**6.3.1. $R$-representation $D_{\frac{1}{2}} \oplus 3D_0$.**

We take the $D_{\frac{1}{2}} \oplus 3D_0$ from equation (13) as $t_1 = t_5 = t_8 = 1$ and all the remaining $t_i$'s as zero giving as $R$-representation:

$[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_2, e_3] = e_1$, $[e_1, e_5] = e_5$, $[e_1, e_7] = -e_7$, $[e_2, e_7] = e_5$, $[e_3, e_5] = e_7$, $[e_5, e_7] = e_4$, $[e_6, e_8] = e_4$.

As such, the most general matrix of the form (13) that commutes with the $R$-representation is given by

$$\begin{bmatrix} 2(a+b) & 0 & e & 0 & f \\ 0 & a+b & 0 & 0 & 0 \\ 0 & 0 & 2a & 0 & c \\ 0 & 0 & 0 & a+b & 0 \\ 0 & 0 & d & 0 & 2b \end{bmatrix}. \tag{14}$$

Accordingly, we add to the $R$-representation the following brackets:
$[e_4, e_9] = 2(a+b)e_4$, $[e_4, e_{10}] = 2(g+h)e_4$, $[e_5, e_9] = (a+b)e_5$, $[e_5, e_{10}] = (g+h)e_5$,
$[e_6, e_9] = ee_4 + 2ae_6 + ce_8$, $[e_6, e_{10}] = ke_4 + 2ge_6 + ie_8$, $[e_7, e_9] = (a+b)e_7$,
$[e_7, e_{10}] = (g+h)e_7$, $[e_8, e_9] = fe_4 + de_6 + 2be_8$, $[e_8, e_{10}] = me_4 + je_6 + 2he_8$,
$[e_9, e_{10}] = pe_4 + qe_6 + re_8$.

The Jacobi identities that are yet to be satisfied are given only by condition (5). There are actually five independent conditions, two of which are easily fulfilled by setting:

$$q = 2am - dk + ej - 2fg, \; r = cm - 2bk + 2eh - fi.$$

Passing over the remaining Jacobi identities, which amount to the commuting of the matrices $\begin{bmatrix} 2a & d \\ c & d \end{bmatrix}$ and $\begin{bmatrix} 2g & j \\ i & 2h \end{bmatrix}$, we shall now consider the algebra of $R$-constants, which is spanned by $\{e_4, e_6, e_8, e_9, e_{10}\}$. We note that that $\mathrm{ad}(e_9)$ and $\mathrm{ad}(e_{10})$ are nilpotent if and only if their restrictions to the $R$-constant subalgebra are nilpotent, hence the $R$-constants is five-dimensional with three-dimensional nilradical isomorphic to $H$. Such an algebra is necessarily indecomposable. As such, we invoke the classification of the *indecomposable*, five-dimensional solvable Lie algebras with nilradical isomorphic to $H$; see [11]. There are just two such Lie algebras, $A_{5.36}$ and $A_{5.37}$. Furthermore, the change of basis necessary to put the algebra of $R$-constants into standard form, leaves the rest of the ten-dimensional algebra intact, which is now in its final form. For $A_{5.36}$ we have:
$a = h = -g = \frac{1}{2}$, $b = c = d = e = f = i = j = k = m = p = q = r = 0$
and for $A_{5.37}$ we have:
$a = d = \frac{1}{2}$, $i = -1, j = 1$, $b = c = e = f = g = h = k = m = p = q = r = 0$.

**6.3.2. $R$-representation $2D_{\frac{1}{2}} \oplus D_0$.**

We take the $R$-representation as being given by the following space of matrices:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & u_2 & 0 \\ 0 & 0 & u_1 & 0 & u_2 \\ 0 & u_3 & 0 & -u_1 & 0 \\ 0 & 0 & u_3 & 0 & -u_1 \end{bmatrix}. \tag{15}$$

This time we find that the most general matrix of the form (13) that commutes with the $R$-representation is given by

$$\begin{bmatrix} 2a & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 \\ 0 & -b & a & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & -b & a \end{bmatrix}. \tag{16}$$

As such we find immediately the Lie algebra.

**6.3.3.** $R$**-representation** $D_{\frac{3}{2}} \oplus D_0$**.**

We take the $R$-representation as being determined by the following matrices:

$$\begin{bmatrix} u_1 & 0 & 2u_2 & u_3 & 0 \\ 0 & -3\,u_1 & u_3 & 0 & 0 \\ 2\,u_3 & 3\,u_2 & -u_1 & 0 & 0 \\ 3\,u_2 & 0 & 0 & 3\,u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

This time we find that there is only one linearly independent matrix of the form (13) that commutes with the $R$-representation. Accordingly, there is no indecomposable ten-dimensional Levi decomposition in this case.

## 7. NR five-dimensional decomposable

According to Section 4, there are two possibilities for a five-dimensional decomposable $NR$, that is, $H \oplus \mathbb{R}^2$ and $\mathbb{R}^5$.

**7.1.** $NR = H \oplus \mathbb{R}^2$ The semi-simple part of the derivation algebra of $H \oplus \mathbb{R}^2$ consists of the direct sum of the the derivation algebras of $H$ and $\mathbb{R}^2$, that is, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. More specifically, if we have a basis $\{e_4, e_5, e_6, e_7, e_8\}$ and only non-zero bracket $[e_5, e_6] = e_4$ then the decomposition of the derivation algebra into its semi-simple and solvable parts is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & s_1 & s_2 & 0 & 0 \\ 0 & s_3 & -s_1 & 0 & 0 \\ 0 & 0 & 0 & s_4 & s_5 \\ 0 & 0 & 0 & s_6 & -s_4 \end{bmatrix} + \begin{bmatrix} 2\,t_1 & t_2 & t_3 & t_4 & t_5 \\ 0 & t_1 & 0 & 0 & 0 \\ 0 & 0 & t_1 & 0 & 0 \\ 0 & t_6 & t_7 & t_8 & 0 \\ 0 & t_9 & t_{10} & 0 & t_8 \end{bmatrix}. \tag{18}$$

There are thus three $R$-representations, that we have to consider: $2D_{\frac{1}{2}}$ and two kinds of $D_{\frac{1}{2}}$, depending whether the action occurs on $H$ or $\mathbb{R}^2$.

**7.1.1.** $2D_{\frac{1}{2}} \oplus D_0$**.**

We start from an eight-dimensional Lie algebra, whose brackets are

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6, \\ &[e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5, \\ &[e_2, e_8] = e_7,\ [e_3, e_5] = e_6,\ [e_3, e_7] = e_8,\ [e_5, e_6] = e_4. \end{aligned} \tag{19}$$

The Lie algebra comprised in (19) is missing from [16] and we shall refer it as $L_{8,23}$.

The space of derivations that commute with the $R$-representation and are derivations of $H \oplus \mathbb{R}^2$, are of the form

$$\begin{bmatrix} 2a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & c & 0 & b & 0 \\ 0 & 0 & c & 0 & b \end{bmatrix}. \tag{20}$$

To obtain a ten-dimensional Levi decomposition Lie algebra from algebra (19), we may assume that it has the same brackets as in (19) as well as the following:

$$\begin{aligned} &[e_4, e_9] = 2ae_4,\ [e_4, e_{10}] = 2ce_4,\ [e_5, e_9] = ae_5,\ [e_5, e_{10}] = ce_5 + ee_7, \\ &[e_6, e_9] = ae_6,\ [e_6, e_{10}] = ce_6 + ee_8,\ [e_7, e_9] = be_7,\ [e_7, e_{10}] = de_7, \\ &[e_8, e_9] = be_8,\ [e_8, e_{10}] = de_8,\ [e_9, e_{10}] = fe_4. \end{aligned}$$

This form is dictated by the fact that only two of the matrices obtained from (20) can be nil-independent. In order to satisfy the Jacobi identity, it is only necessary to have $(a-b)e = 0$ so either $b = a$ or $e = 0$. If $e = 0$, we can reduce to $a = 1, b = 0, c = 0, d = 1$. Finally, replacing $e_{10}$ by $e_{10} + \frac{f}{2}e_4$ reduces $f$ to zero.

If $b = a$ then we may assume that $b = a = 1, c = 0, d = 1$. Then replacing $e_{10}$ by $e_{10} + \frac{f}{2}e_4$ reduces $f$ to zero. If $e \neq 0$, we can reduce $e$ to 1, by replacing $e_4, e_5, e_6$ by $e^2e_4, ee_5, ee_6$.

We note in passing that the derivation algebra of (19) is a ten-dimensional Levi decomposition Lie algebra with semi-simple factor $\mathfrak{sl}(2,\mathbb{R})$ in the representation $2D_{\frac{1}{2}}$, seven-dimensional radical and nilradical $A_{5.1}$ and so it must appear in our list. Its Lie brackets are:

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = -e_4,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6, \\ &[e_1, e_7] = e_7,\ [e_2, e_3] = e_1,\ [e_2, e_4] = -e_5,\ [e_2, e_6] = -e_7,\ [e_3, e_5] = -e_4, \\ &[e_3, e_7] = -e_6,\ [e_4, e_{10}] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_8] = e_4,\ [e_6, e_9] = e_6, \\ &[e_7, e_8] = e_5,\ [e_7, e_9] = e_7,\ [e_8, e_9] = -e_8,\ [e_8, e_{10}] = e_8. \end{aligned} \tag{21}$$

After replacing $(e_4, e_5, e_6, e_7, e_9, e_{10})$ by $(-e_5, e_4, -e_7, e_6, e_{10}, e_9)$, we do indeed obtain the algebra appearing in subsubsection 12.1.1.

#### 7.1.2. $D_{\frac{1}{2}} \oplus 3D_0, D_{\frac{1}{2}}$ acting on $H$.

This time we begin with an eight-dimensional Lie algebra spanned by $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, whose brackets are

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6, \\ &[e_2, e_3] = e_1,\ [e_2, e_6] = e_5, [e_3, e_5] = e_6, [e_5, e_6] = e_4. \end{aligned} \tag{22}$$

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_{10} & 0 & 0 & z_9 & z_8 \\ 0 & z_7 & 0 & 0 & 0 \\ 0 & 0 & z_7 & 0 & 0 \\ z_6 & 0 & 0 & z_5 & z_4 \\ z_3 & 0 & 0 & z_2 & z_1 \end{bmatrix} \tag{23}$$

and its intersection with (20) consists of matrices of the form

$$\begin{bmatrix} 2a & 0 & 0 & b & e \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & c & f \\ 0 & 0 & 0 & d & g \end{bmatrix}. \tag{24}$$

As such the ten-dimensional pre-algebra that we need to consider is obtained by adding to (22) the following Lie brackets:

$$\begin{aligned} &[e_4, e_9] = 2ae_4,\ [e_5, e_9] = ae_5,\ [e_6, e_9] = ae_6,\ [e_7, e_9] = be_4 + ce_7 + de_8, \\ &[e_8, e_9] = ee_4 + fe_7 + ge_8,\ [e_4, e_{10}] = 2he_4,\ [e_5, e_{10}] = he_5,\ [e_6, e_{10}] = he_6, \\ &[e_7, e_{10}] = ie_4 + je_7 + ke_8,\ [e_8, e_{10}] = me_4 + ne_7 + pe_8,\ [e_9, e_{10}] = qe_4 + re_7 + se_8. \end{aligned} \tag{25}$$

Algebra (25) contains 17 parameters that we shall endeavor to normalize.

The algebra of $R$-constants is spanned by $\{e_4, e_7, e_8, e_9, e_{10}\}$ and is five-dimensional solvable, with three-dimensional abelian nilradical. If the Jacobi identity for the $R$-constants is satisfied, then it is satisfied for the ten-dimensional algebra. In fact, the Jacobi identity amounts to the commuting of $\mathrm{ad}(e_9)$ and $\mathrm{ad}(e_{10})$ on the nilradical. Since they commute and the nilradical is three-dimensional, they must have a common eigenvector. Hence the form of the five-dimensional solvable is completely general and we may quote the classification of the *indecomposable*, five-dimensional solvable Lie algebras with three-dimensional nilradical Lie algebras [11]. There are five such classes of Lie algebra, labelled as $A_{5.33}, A_{5.34}, A_{5.35}, A_{5.38}, A_{5.39}$; in the first three, the nilradical has a two-dimensional abelian complement, whereas in the latter two, the complement is not abelian. We also have to include brackets involving $e_5$ and $e_6$. To obtain the ten-dimensional Lie algebras, all that is required is to append the Lie brackets that appear in (22).

We continue by supposing that the algebra of $R$-constants is decomposable, as it must be, if we are not just to recover cases already considered. Now, as abstract Lie algebras, the only decomposable five-dimensional solvable codimension two nilradical Lie algebras are $A_{4.12} \oplus \mathbb{R}, A_{3.m} \oplus A_{2.1}$ where $2 \leq m \leq 7$ or $A_{2.1} \oplus A_{2.1} \oplus \mathbb{R}$. Looking at the adjoint matrices of the $R$-constants, in order to obtain $A_{4.12} \oplus \mathbb{R}$ we may assume that $c = g = n = 1, d = f = j = p = 0, k = -1$ and $r = s = 0$. Here we are putting $A_{4.12}$ into standard form changing basis in the subspace spanned by $e_7, e_8, e_9, e_{10}$. In order to have a direct sum with $\mathbb{R}$ spanned by $e_4$, we shall need also that $a = e = h = i = m = q = 0$. However, in that case the entire ten-dimensional Lie algebra is decomposable.

We continue by supposing that the matrices $\left[\begin{smallmatrix} c & f \\ d & g \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} j & n \\ k & p \end{smallmatrix}\right]$ are linearly dependent. Then by taking linear combinations of $e_9$ and $e_{10}$, we may assume that $j = k = n = p = 0$. In order to have a three-dimensional nilradical, we must have $h \neq 0$ and so we may suppose that $h = \frac{1}{2}$ and $a = 0$. By subtracting multiples of $e_1$ from $e_2, e_3$ and $e_4$, we may assume that $i = m = q = 0$. So far, we have not satisfied all the Jacobi identities, but they are just $b = e = 0$. If $cg - df \neq 0$, then by subtracting multiplies of $e_2$ and $e_3$ from $e_5$, we can reduce $r$ and $s$ to zero. As such the $R$-constants subalgebra is decomposable as sum of an $A_{2.1}$ spanned by $e_4$ and $e_{10}$

and a three-dimensional solvable Lie algebra spanned by $e_7, e_8$ and $e_9$. Moreover, the ten-dimensional solvable Lie algebra is decomposable. If $cg - df = 0$ then by subtracting a multiple of $e_2$ from $e_3$, it may be assumed that $f = g = 0$. Now if $c \neq 0$ the Lie algebra is isomorphic to $A_{5.38}$ and if $c = 0$ the nilradical is no longer of dimension three, but rather four. To summarize, if the algebra of $R$-constants is decomposable, then the ten-dimensional Lie algebra is either decomposable or its nilradical will be of dimension four, so there will no addition to our list.

### 7.1.3. $D_{\frac{1}{2}} \oplus 3D_0, D_{\frac{1}{2}}$ acting on $\mathbb{R}^2$.

This time we begin with an eight-dimensional Lie algebra spanned by $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, whose brackets are:

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,$$
$$[e_2, e_3] = e_1,\ [e_2, e_8] = e_7,\ [e_3, e_7] = e_8,\ [e_5, e_6] = e_4. \tag{26}$$

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_1 & z_2 & z_3 & 0 & 0 \\ z_4 & z_5 & z_6 & 0 & 0 \\ z_7 & z_8 & z_9 & 0 & 0 \\ 0 & 0 & 0 & z_{10} & 0 \\ 0 & 0 & 0 & 0 & z_{10} \end{bmatrix} \tag{27}$$

and its intersection with (20) consists of matrices of the form

$$\begin{bmatrix} a & d & f & 0 & 0 \\ 0 & b & g & 0 & 0 \\ 0 & e & a-b & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & c \end{bmatrix}. \tag{28}$$

As such the ten-dimensional pre-Lie algebra that we need to consider is obtained by adding to (26) the following Lie brackets:

$$[e_4, e_9] = (a+d)e_4,\ [e_4, e_{10}] = (h+i)e_4,\ [e_5, e_9] = ee_4 + ae_5 + ce_6,$$
$$[e_5, e_{10}] = je_4 + he_5 + ke_6,\ [e_6, e_9] = fe_4 + be_5 + de_6,$$
$$[e_6, e_{10}] = me_4 + pe_5 + ie_6,\ [e_7, e_9] = ge_7,\ [e_7, e_{10}] = qe_7,\ [e_8, e_9] = ge_8,$$
$$[e_8, e_{10}] = qe_8,\ [e_9, e_{10}] = te_4 + re_5 + se_6. \tag{29}$$

Algebra (29) contains 17 parameters that we shall endeavor to normalize. The Jacobi identity comprises five conditions, two of which are easily satisfied:

$$r = am - bj + ep - fh, s = cm - dj + ei - fk. \tag{30}$$

The remaining Jacobi identities turn out to be equivalent to the commuting of the following $2 \times 2$ blocks:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} h & p \\ k & i \end{bmatrix}. \tag{31}$$

Now we consider the algebra of $R$-constants, which is spanned by $\{e_4, e_5, e_6, e_9, e_{10}\}$. We shall assume in the first instance that this algebra has a three-dimensional nilradical. Such an algebra is necessarily indecomposable. Again, we invoke the classification of the *indecomposable*, five-dimensional solvable Lie algebras with nilradical isomorphic to $H$, see [11]. There are just two such Lie algebras, $A_{5.36}$ and $A_{5.37}$. We also have to include brackets involving $e_7$ and $e_8$ giving:

- $A_{5.36}$: $[e_5,e_6]=e_4,\ [e_4,e_9]=e_4,\ [e_5,e_9]=e_5,\ [e_5,e_{10}]=-e_5,\ [e_6,e_{10}]=e_6,$ $[e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=ae_8$
- $A_{5.37}$: $[e_5,e_6]=e_4,\ [e_4,e_9]=2e_4,\ [e_5,e_9]=e_5,\ [e_6,e_9]=e_6,\ [e_5,e_{10}]=-e_6,$ $[e_6,e_{10}]=e_5,\ [e_7,e_{10}]=ae_7,\ [e_8,e_{10}]=ae_8.$

Since we are assuming that $g=0$, if $q=0$, these algebras comprise the only possibility for the $R$-constants. However, if $q\neq 0$, we could have that $\left[\begin{smallmatrix} h & p\\ k & i\end{smallmatrix}\right]$ is nilpotent, whereas $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ cannot be, without the ten-dimensional algebra failing to have a five-dimensional nilradical. In this case $\{e_4,e_5,e_6,e_9\}$ must span a four-dimensional subalgebra and $e_{10}$ must commute with it. However, the ten-dimensional algebra will be decomposable.

### 7.2. $NR=\mathbb{R}^5$

According to Section 4 there are six possible $R$-representations that we consider in turn.

#### 7.2.1. $R$-representation $=D_2$

In the case of $D_2$, the $R$-representation on $\mathbb{R}^5$ is irreducible. By Schur's Lemma, the only $5\times 5$ matrix commuting with the representation is a multiple of the identity, so we could obtain at most one linearly independent nil-independent matrix and so there can be no ten-dimensional Levi decomposition Lie algebra.

#### 7.2.2. $R$-representation $=D_{\frac{1}{2}}\oplus D_1$

In the case of $D_1\oplus D_{\frac{1}{2}}$ the space of $5\times5$ matrices commuting with the representation is of dimension two and the Lie algebra is easily obtained.

#### 7.2.3. $R$-representation $=D_{\frac{3}{2}}\oplus D_0$

In the case of $D_{\frac{3}{2}}\oplus D_0$, the space of $5\times5$ matrices commuting with the representation is of dimension two and we obtain:

$$[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=3e_4,\ [e_1,e_5]=e_5,\ [e_1,e_6]=-e_6,$$
$$[e_1,e_7]=-3e_7,\ [e_2,e_3]=e_1,\ [e_2,e_5]=3e_4,\ [e_2,e_6]=2e_5,\ [e_2,e_7]=e_6,$$
$$[e_3,e_4]=e_5,\ [e_3,e_5]=2e_6,\ [e_3,e_6]=3e_7,\ [e_4,e_9]=e_4,\ [e_5,e_9]=e_5,$$
$$[e_6,e_9]=e_6,[e_7,e_9]=e_7,[e_8,e_{10}]=e_8,[e_9,e_{10}]=ae_8.$$

However, this algebra is decomposable via a transformation that replaces $e_9$ by $e_9-ae_8$ and so must be excluded.

#### 7.2.4. $2D_{\frac{1}{2}}\oplus D_0$

We begin with an eight-dimensional decomposable Levi decomposition Lie algebra

$$[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=e_4,\ [e_1,e_5]=-e_5,\ [e_1,e_6]=e_6,$$
$$[e_1,e_7]=-e_7,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_4,\ [e_2,e_7]=e_6,\ [e_3,e_4]=e_5,[e_3,e_6]=e_7.$$

The space of $5\times5$ matrices that commute with the $R$-representation is of the form:

$$\begin{bmatrix} z_1 & 0 & z_2 & 0 & 0\\ 0 & z_1 & 0 & z_2 & 0\\ z_3 & 0 & z_4 & 0 & 0\\ 0 & z_3 & 0 & z_4 & 0\\ 0 & 0 & 0 & 0 & z_5\end{bmatrix}. \tag{32}$$

For the pre-Lie algebra we supplement the eight-dimensional algebra by the following Lie brackets:

$[e_4, e_9] = ae_4 + be_6,\ [e_4, e_{10}] = fe_4 + ge_6,\ [e_5, e_9] = ae_5 + be_7,\ [e_5, e_{10}] = fe_5 + ge_7,$
$[e_6, e_9] = ce_4 + de_6,\ [e_6, e_{10}] = he_4 + ie_6,\ [e_7, e_9] = ce_5 + de_7,\ [e_7, e_{10}] = he_5 + ie_7,$
$[e_8, e_9] = ee_8, [e_8, e_{10}] = je_8, [e_9, e_{10}] = ke_8.$

Since we are assuming that $NR = \mathbb{R}^5$, the ad-matrices $\mathrm{ad}(e_9)$ and $\mathrm{ad}(e_{10})$ of the extra basis vectors $e_9$ and $e_{10}$, that we are endeavoring to add, must commute on $NR$, which is the sum total of the Jacobi identities that remain to be satisfied. Now there are two possibilities: either the upper $4 \times 4$ blocks in the two matrices of the form (32) are linearly dependent or are linearly independent. If they are linearly dependent, we may assume that the second such block is zero, that is, $f = g = h = i = 0, j = 1$. However, if we replace $e_9$ by $je_9 - ke_8 - ee_{10}$ we obtain a decomposable Lie algebra, so the first possibility leads to a null result.

In the second case, where the $4 \times 4$ blocks are linearly independent, we may put the blocks into simultaneous canonical form. We may reduce to three subcases:

- $b = c = d = f = g = h = 0, a = i = 1$
- $a = d = g = 1, b = c = f = 0, h = -1$
- $a = c = d = h = j = 1, b = f = g = i = 0.$

In each of these cases, we can reduce $k$ to 0 or 1 by scaling $e_8$, and in the third case we must have $j \neq 0$ and we may assume that $j = 1$ by scaling $e_8$.

**7.2.5.** $D_1 \oplus 2D_0$**.** We take the pre-Lie algebra in the form:

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6,\ [e_2, e_3] = e_1,$
$[e_2, e_5] = 2e_4,\ [e_2, e_6] = e_5,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_4, e_9] = ae_4,$
$[e_4, e_{10}] = be_4,\ [e_5, e_9] = ae_5,\ [e_5, e_{10}] = be_5,\ [e_6, e_9] = ae_6,\ [e_6, e_{10}] = be_6,$
$[e_7, e_9] = ce_7 + de_8,\ [e_7, e_{10}] = ge_7 + he_8,\ [e_8, e_9] = ee_7 + fe_8,\ [e_8, e_{10}] = ie_7 + je_8,$
$[e_9, e_{10}] = ke_7 + me_8.$

The only Jacobi condition left to satisfy is equation (5), which requires that the matrices

$$\begin{bmatrix} c & e \\ d & f \end{bmatrix}, \begin{bmatrix} g & i \\ h & j \end{bmatrix} \tag{33}$$

should commute. If these matrices are linearly dependent, then we could assume that the second is zero, but then the ten-dimensional algebra is decomposable. Accordingly, we may put these two matrices into simultaneous canonical form for which:

- $c = j = 1,\ d = e = f = g = h = i = 0$
- $c = f = i = 1,\ h = -1, d = e = g = h = j = 0$
- $b = c = f = i = 1,\ d = e = g = h = j = 0$

In the first case, we may replace $e_9$ and $e_{10}$ by $e_9 + me_8$ and $e_{10} - ke_7$, respectively, which reduces $k$ and $m$ to zero. We must further have $ab \neq 0$ in order to avoid having a decomposable algebra.

In the second case, we may replace $e_{10}$ by $e_{10} - ke_7 - me_8$, which reduces $k$ and $m$ to zero. We must further have $a^2 + b^2 \neq 0$ in order to avoid having a decomposable algebra.

In the third case, we may also assume that $b=1$ and the same change in $e_{10}$ as in the second case reduces $k$ and $m$ to zero.

**7.2.6.** $D_{\frac{1}{2}} \oplus 3D_0$**.** We take the pre-Lie algebra in the form:

$[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=e_4,\ [e_1,e_5]=-e_5,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_4,$
$[e_3,e_4]=e_5,\ [e_4,e_9]=\alpha e_4,\ [e_4,e_{10}]=\beta e_4,\ [e_5,e_9]=\alpha e_5,\ [e_5,e_{10}]=\beta e_5,$
$[e_6,e_9]=ae_6+be_7+ce_8,\ [e_6,e_{10}]=je_6+ke_7+me_8,\ [e_7,e_9]=de_6+ee_7+fe_8,$
$[e_7,e_{10}]=ne_6+pe_7+qe_8,\ [e_8,e_9]=ge_6+he_7+ie_8,\ [e_8,e_{10}]=re_6+se_7+te_8,$
$[e_9,e_{10}]=ue_6+ve_7+we_8.$

The only Jacobi condition left to satisfy is equation (5), which requires that the matrices

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}, \begin{bmatrix} j & n & r \\ k & p & s \\ m & q & t \end{bmatrix} \tag{34}$$

should commute. We may also assume that $\beta=0$.

Now we shall consider the algebra of $R$-constants, which is spanned by

$$\{e_6,e_7,e_8,e_9,e_{10}\}.$$

It is solvable and has at least a three-dimensional abelian nilradical. Let us suppose in the first instance that it is five-dimensional indecomposable with three-dimensional abelian nilradical. We may quote the classification of such algebras [11] and we obtain $A_{5.33}$, $A_{5.34}$, $A_{5.35}$, $A_{5.38}$, $A_{5.39}$; they may be seen in a suitable basis in subsubsection 12.2.7.

- $A_{5.33}$: $a=p=1,\ b=c=d=e=f=g=h=j=k=m=n=q=r=s=u=v=w=0$
- $A_{5.34}$: $e=i=j=s=1,\ b=c=d=f=g=h=k=m=n=q=r=u=v=w=0$
- $A_{5.35}$: $e=i=s=1,\ q=-1,\ b=c=d=f=g=h=k=m=n=r=u=v=w=0$
- $A_{5.38}$: $a=p=w=1,\ b=c=d=e=f=g=h=i=j=k=m=n=q=r=s=u=v=0$
- $A_{5.39}$: $a=e=n=w=1,\ k=-1, b=c=d=f=g=h=i=j=m=p=q=r=s=t=u=v=0.$

A different possibility occurs if one of the the matrices in (34) is nilpotent. It could happen if the reduced form of the matrices is:

$$\begin{bmatrix} 0 & d & g \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} j & 0 & r \\ 0 & j & 0 \\ 0 & 0 & j \end{bmatrix}. \tag{35}$$

Now all depends on the value of $d$; if $d \neq 0$, the five-dimensional algebra may be shown to be isomorphic to $A_{5.32}$ in [11]. On the other hand, if $d=0$, then the five-dimensional algebra may be shown to be isomorphic to $A_{5.19_{a=1,b=1}}$, $A_{5.27}$ or $A_{5.28a=1}$ in [11].

- $A_{5.19a=1,b=1}$: $g=j=p=t=1,\ \beta=0, a=b=c=d=e=f=h=i=k=m=n=q=r=s=u=v=w=0$, ($R$-constants $e_6,e_8,e_9,e_7,e_{10}$)

- $A_{5.27}$: $g=j=n=p=s=t=1,\ a=b=c=d=e=f=h=i=k=m=q=r=u=v=w=0$, ($R$-constants $e_6, -e_9, e_8, e_7, e_{10}$)
- $A_{5.28}$: $g=j=p=s=t=1,\ a=b=c=d=e=f=h=i=k=m=n=q=r=u=v=w=0$, ($R$-constants $e_6, -e_9, e_8, e_7, e_{10}$)
- $A_{5.32}$: $d=h=j=p=t=1,\ a=b=c=d=e=f=g=i=k=m=n=q=s=u=v=w=0$, ($R$-constants $e_6, e_7, e_8, e_9, e_{10}$)

The full algebras may be seen in subsubsection 12.2.7.

## 8. $NR$ six-dimensional indecomposable

If we assume that the nilradical of a ten-dimensional Levi decomposition algebra $NR$ is spanned by $\{e_4, e_5, e_6, e_7, e_8, e_9\}$, then $\mathrm{ad}\,(e_{10})$ is a $7\times 7$ matrix, subject to the conditions that its bottom row and last column should be zero. Of the 24 six-dimensional indecomposable nilpotent Lie algebras, only four have a non-zero semi-simple subalgebra in their space of derivations. Here are those four nilpotent algebras, together with a semi-simple algebra that is isomorphic to its subalgebra of semi-simple derivations:

$$\bullet\ A_{6,3},\ \mathfrak{sl}(3,\mathbb{R}), \quad \bullet\ A_{6,4},\ \mathfrak{sl}(2,\mathbb{R}), \quad \bullet\ A_{6,5},\ \mathfrak{so}(3,1), \quad \bullet\ A_{6,12},\ \mathfrak{sl}(2,\mathbb{R}).$$

We consider each of these four cases in turn. It should be noted that when $NR$ is six-dimensional, the Lie algebra $S\rtimes NR$ is nine-dimensional. If we extend $S\rtimes NR$ by one dimension, using a derivation of $NR$ that commutes with the $R$-representation, we shall necessarily obtain a ten-dimensional Lie algebra. This fact has to contrasted with the case where the dimension of $NR$ is five, because when extending by two dimensions certain extra Jacobi identity conditions have to be satisfied. Accordingly, there is no need when $NR$ is six-dimensional, to refer to a "pre-Lie algebra".

**8.1.** $NR = A_{6.3}$**.** We take $NR = A_{6.3}$ in the form

$$[e_4,e_5]=e_9,\ [e_4,e_6]=-e_8,\ [e_5,e_6]=e_7.$$

The space of derivations of $A_{6.3}$ is given by

$$\begin{bmatrix} s_{10} & 0 & 0 & 0 & 0 & 0\\ 0 & s_{10} & 0 & 0 & 0 & 0\\ 0 & 0 & s_{10} & 0 & 0 & 0\\ s_1 & s_2 & s_3 & 2\,s_{10} & 0 & 0\\ s_4 & s_5 & s_6 & 0 & 2\,s_{10} & 0\\ s_7 & s_8 & s_9 & 0 & 0 & 2\,s_{10} \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & t_3 & 0 & 0 & 0\\ t_4 & t_5 & t_6 & 0 & 0 & 0\\ t_7 & t_8 & -t_1-t_5 & 0 & 0 & 0\\ 0 & 0 & 0 & -t_1 & -t_4 & -t_7\\ 0 & 0 & 0 & -t_2 & -t_5 & -t_8\\ 0 & 0 & 0 & -t_3 & -t_6 & t_1+t_5 \end{bmatrix}, \tag{36}$$

the first matrix displaying the solvable ideal and the second its semi-simple subalgebra, so that the $R$-representation is $2\ \mathrm{ad}\mathfrak{sl}(3,\mathbb{R})$.

Up to conjugacy, there are two possible $R$-representations that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, that is, $2D_{\frac{1}{2}}\oplus 2D_0$ and $2D_1$. We consider them in turn.

**8.1.1. $R$-representation $2D_{\frac{1}{2}}$.** The $R$-representation appears as

$$\begin{bmatrix} u_1 & u_2 & 0 & 0 & 0 & 0 \\ u_3 & -u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u_1 & -u_3 & 0 \\ 0 & 0 & 0 & -u_2 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_8 & 0 & 0 & 0 & -z_7 & 0 \\ 0 & z_8 & 0 & z_7 & 0 & 0 \\ 0 & 0 & z_6 & 0 & 0 & z_5 \\ 0 & -z_4 & 0 & z_3 & 0 & 0 \\ z_4 & 0 & 0 & 0 & z_3 & 0 \\ 0 & 0 & z_2 & 0 & 0 & z_1 \end{bmatrix}. \tag{38}$$

Taking the intersection of (36) and (38) gives

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & c & 0 & a+b & 0 & 0 \\ -c & 0 & 0 & 0 & a+b & 0 \\ 0 & 0 & d & 0 & 0 & 2\,a \end{bmatrix} \tag{39}$$

and leads to the following Lie algebra:

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_7] = -e_7, \\ &[e_1, e_8] = e_8,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = -e_8,\ [e_3, e_4] = e_5, \\ &[e_3, e_8] = -e_7,\ [e_4, e_5] = e_9,\ [e_4, e_6] = -e_8,\ [e_4, e_{10}] = ae_4 - ce_8,\ [e_5, e_6] = e_7, \\ &[e_5, e_{10}] = ae_5 + ce_7,\ [e_6, e_{10}] = de_9 + be_6,\ [e_7, e_{10}] = (a+b)e_7, \\ &[e_8, e_{10}] = (a+b)e_8,\ [e_9, e_{10}] = 2ae_9. \end{aligned}$$

If we make a change of basis according to which $e_{10}$ is replaced by $e_{10} - ce_6$, we can set $c$ to zero. If we make another change, according to which $e_6$ is replaced by $e_6 + \alpha e_9$, the only effect on the algebra is to transform $d$ into $d + (2a - b)\alpha$. Accordingly, if $b \neq 2a$, we can reduce $d$ to zero. If $b = 2a$, we may assume by scaling that $a = 1, b = 2$; then scaling $e_6$ by $\alpha$ changes $d$ into $d\alpha$. Hence we reduce $d$ to 0 or 1.

**8.1.2. $R$-representation $2D_1$.** The $R$-representation appears as

$$\begin{bmatrix} 2\,u_1 & 2\,u_2 & 0 & 0 & 0 & 0 \\ u_3 & 0 & u_2 & 0 & 0 & 0 \\ 0 & 2\,u_3 & -2\,u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\,u_1 & -u_3 & 0 \\ 0 & 0 & 0 & -2\,u_2 & 0 & -2\,u_3 \\ 0 & 0 & 0 & 0 & -u_2 & 2\,u_1 \end{bmatrix}. \tag{40}$$

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_1 & 0 & 0 & 0 & 0 & z_2 \\ 0 & z_1 & 0 & 0 & -\frac{1}{2}z_2 & 0 \\ 0 & 0 & z_1 & z_2 & 0 & 0 \\ 0 & 0 & z_3 & z_4 & 0 & 0 \\ 0 & -2\,z_3 & 0 & 0 & z_4 & 0 \\ z_3 & 0 & 0 & 0 & 0 & z_4 \end{bmatrix}. \tag{41}$$

In order to obtain a derivation of the Lie algebra $NR = A_{6.3}$, we require that $z_2 = 0$, $z_4 = 2z_1$. By scaling we may assume that $z_1 = 1$. We obtain the Levi decomposition Lie algebra

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6, \\ &[e_1, e_7] = -2e_7,\ [e_1, e_9] = 2e_9,\ [e_2, e_5] = 2e_4,\ [e_2, e_6] = e_5,\ [e_2, e_7] = -2e_8, \\ &[e_2, e_8] = -e_9,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_3, e_8] = -e_7,\ [e_3, e_9] = -2e_8, \\ &[e_4, e_{10}] = e_4 + a_9,\ [e_5, e_{10}] = e_5 - 2ae_8,\ [e_6, e_{10}] = e_6 + ae_7, \\ &[e_7, e_{10}] = 2e_7,\ [e_8, e_{10}] = 2e_8,\ [e_9, e_{10}] = 2e_9. \end{aligned}$$

The transformation which replaces $e_4, e_5, e_6$ by $e_4 - ae_9, e_5 + 2ae_8, e_6 - ae_7$ reduces $a$ to zero.

**8.2.** $NR = A_{6.4}$**.** We take $NR = A_{6.4}$ in the form

$$[e_4, e_5] = e_8,\ [e_4, e_6] = e_9,\ [e_5, e_7] = e_9.$$

The Levi decomposition of the space of derivations of $A_{6.4}$ is given by

$$\begin{bmatrix} s_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_4 & 0 & 0 & 0 & 0 \\ s_5 & s_6 & s_7 & 0 & 0 & 0 \\ s_8 & s_9 & 0 & s_7 & 0 & 0 \\ s_{10} & s_{11} & s_{12} & s_{13} & 2\,s_4 & 0 \\ s_{14} & s_{15} & s_{16} & s_{17} & s_6 - s_8 & s_4 + s_7 \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & 0 & 0 & 0 & 0 \\ t_3 & -t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t_1 & -t_3 & 0 & 0 \\ 0 & 0 & -t_2 & t_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{42}$$

the first matrix displaying the solvable ideal and the second its semi-simple subalgebra, so that the $R$-representation is $2D_{\frac{1}{2}}$.

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_1 & 0 & 0 & -\,z_2 & 0 & 0 \\ 0 & z_1 & z_2 & 0 & 0 & 0 \\ 0 & z_3 & z_4 & 0 & 0 & 0 \\ -z_3 & 0 & 0 & z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & z_6 \\ 0 & 0 & 0 & 0 & z_7 & z_8 \end{bmatrix}. \tag{43}$$

Taking the intersection of (42) and (43) gives

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & -c & b & 0 & 0 & 0 \\ c & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\,a & 0 \\ 0 & 0 & 0 & 0 & -2\,c & a+b \end{bmatrix} \tag{44}$$

and leads to the following Lie algebra:

$$\begin{aligned} &[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=e_4,\ [e_1,e_5]=-e_5,\ [e_1,e_6]=-e_6,\\ &[e_1,e_7]=e_7,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_4,\ [e_2,e_6]=-e_7, [e_3,e_4]=e_5,\\ &[e_3,e_7]=-e_6,\ [e_4,e_5]=e_8,\ [e_4,e_6]=e_9,\ [e_4,e_{10}]=ae_4+ce_7,\\ &[e_5,e_7]=e_9,\ [e_5,e_{10}]=ae_5-ce_6,\ [e_6,e_{10}]=be_6,\ [e_7,e_{10}]=be_7,\\ &[e_8,e_{10}]=2ae_8-2ce_9,\ [e_9,e_{10}]=(a+b)e_9. \end{aligned}$$

The remaining issue now concerns the simplification of the parameters $a,b,c$. We make a change of basis according to which $e_4,e_5,e_8$ become

$$e_4-\alpha e_7, e_5+\alpha e_6, e_8+2\alpha e_9$$

and the only brackets that change are:

$$\begin{aligned} &[e_4,e_{10}]=((b-a)\alpha+c)e_7+ae_4,\ [e_5,e_{10}]=((a-b)\alpha-c)e_6+ae_5,\\ &[e_8,e_{10}]=(2(a-b)\alpha-c)e_9+2ae_8. \end{aligned}$$

We see that provided $a\neq b$, we can set $c$ to zero.

If $a=b$, by scaling we may assume that $a=b=1$. In that case we may change $e_6,e_7,e_9$ to $\beta e_6,\beta e_7,\beta e_9$, where $\beta\neq 0$. The only effect on the Lie algebra is to replace $c$ by $\frac{c}{\beta}$. Accordingly, we may assume that $c=0$ or $c=1$.

**8.3.** $NR=A_{6.5}$**.** We take $A_{6.5}$ in the form

$$[e_4,e_6]=e_8, [e_4,e_7]=e_9, [e_5,e_8]=-e_9, [e_5,e_7]=e_8.$$

The Levi decomposition of the space of derivations of $A_{6.5}$ is given by

$$\begin{bmatrix} s_1 & s_2 & 0 & 0 & 0 & 0 \\ -s_2 & s_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & -s_2 & 0 & 0 \\ 0 & 0 & s_2 & s_1 & 0 & 0 \\ s_3 & s_4 & s_5 & s_6 & 2\,s_1 & -2\,s_2 \\ s_7 & s_8 & s_9 & s_{10} & 2\,s_2 & 2\,s_1 \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & 0 & 0 \\ -t_2 & t_1 & t_4 & -t_3 & 0 & 0 \\ t_5 & t_6 & -t_1 & t_2 & 0 & 0 \\ t_6 & -t_5 & -t_2 & -t_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{45}$$

the first matrix displaying the solvable ideal and the second its semi-simple subalgebra. In this case, the $R$-representation is isomorphic to $\mathfrak{so}(3,1)$, since it is real-indecomposable and semi-simple. It was shown in [5], that any two subalgebras of $\mathfrak{so}(3,1)$ that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ are conjugate.

Accordingly, we shall take as $\mathfrak{sl}(2,\mathbb{R})$ semi-simple factor the $R$-representation

$$\begin{bmatrix} u_1 & 0 & u_2 & 0 & 0 & 0 \\ 0 & u_1 & 0 & -u_2 & 0 & 0 \\ u_3 & 0 & -u_1 & 0 & 0 & 0 \\ 0 & -u_3 & 0 & -u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{46}$$

The $R$-representation (46) appears as $2D_{\frac{1}{2}}$.
The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_1 & z_2 & 0 & 0 & 0 & 0 \\ z_3 & z_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & -z_2 & 0 & 0 \\ 0 & 0 & -z_3 & z_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & z_6 \\ 0 & 0 & 0 & 0 & z_7 & z_8 \end{bmatrix}. \tag{47}$$

Taking the intersection of (45) and (47) gives

$$\begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ -b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & -b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 2a & -2b \\ 0 & 0 & 0 & 0 & 2b & 2a \end{bmatrix} \tag{48}$$

and leads to the Lie algebra. It only remains to remark that by scaling, we may assume that $a = 1$ or $b = 1$.

**8.4.** $NR = A_{6.12}$**.** We take $A_{6.12}$ in the form

$$[e_4, e_6] = e_7,\ [e_4, e_7] = e_9,\ [e_5, e_8] = e_9.$$

The Levi decomposition of the space of derivations of $A_{6.12}$ is given by

$$\begin{bmatrix} s_1 & 0 & 0 & 0 & 0 & 0 \\ s_2 & s_3 & 0 & 0 & 0 & 0 \\ s_4 & 0 & -2s_1+2s_3 & 0 & 0 & 0 \\ s_5 & s_6 & s_7 & -s_1+2s_3 & -s_2 & 0 \\ s_6 & 0 & 0 & 0 & s_3 & 0 \\ s_8 & s_9 & s_{10} & s_7 & s_{11} & 2s_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_3 & 0 & 0 & -t_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

the first matrix displaying the solvable ideal and the second its semi-simple subalgebra. In this case, the $R$-representation is isomorphic to $D_{\frac{1}{2}}$.

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_6 & 0 & z_5 & z_4 & 0 & z_3 \\ 0 & z_{11} & 0 & 0 & 0 & 0 \\ z_2 & 0 & z_1 & z_{17} & 0 & z_{16} \\ z_{15} & 0 & z_{14} & z_{13} & 0 & z_{12} \\ 0 & 0 & 0 & 0 & z_{11} & 0 \\ z_{10} & 0 & z_9 & z_8 & 0 & z_7 \end{bmatrix}. \tag{50}$$

Taking the intersection of (49) and (50) gives

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ c & 0 & -2\,a+2\,b & 0 & 0 & 0 \\ d & 0 & e & -a+2\,b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ f & 0 & g & e & 0 & 2\,b \end{bmatrix} \tag{51}$$

and leads to the following Lie algebra:

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1, \\ &[e_2, e_8] = e_5,\ [e_3, e_5] = e_8,\ [e_4, e_6] = e_7,\ [e_4, e_7] = e_9, \\ &[e_4, e_{10}] = ae_4 + ce_6 + de_7 + fe_9,\ [e_5, e_8] = e_9,\ [e_5, e_{10}] = be_5, \\ &[e_6, e_{10}] = 2(b-a)e_6 + ee_7 + ge_9,\ [e_7, e_{10}] = (2b-a)e_7 + ee_9, \\ &[e_8, e_{10}] = be_8,\ [e_9, e_{10}] = 2be_9. \end{aligned} \tag{52}$$

Algebra (52) contains seven parameters that we shall normalize. First of all, we use a change of basis in which $e_{10}$ is replaced by $e_{10} - ee_4 + de_6 + fe_7$, which has the effect of setting $d, e$ and $f$ to zero.

Now we shall consider the algebra of $R$-constants, which is spanned by

$$\{e_4, e_6, e_7, e_9, e_{10}\}.$$

We shall take the basis in the form $e_9, e_7, e_6, -e_4, e_{10}$ since that will align it with the class of algebras $A_{5.30}, A_{5.31}$ and $A_{5.32}$, which are five-dimensional solvable with $A_{4.1}$ as nilradical. We can obtain these algebras in terms of the parameters $a, b, c, g$ by taking in these three respective cases:

- $a = 1,\ b = \frac{a+1}{2},\ c = g = 0$
- $a = 1,\ b = \frac{3}{2},\ c = -1, g = 0$
- $a = 0,\ b = \frac{1}{2},\ c = 0$

These values have been slightly modified for the ten-dimensional Levi decomposition algebras in subsubsection 12.3.5, so as to avoid unnecessary fractions.

## 9. NR six-dimensional decomposable

According to Section 4 there are seven decomposable nilpotent Lie algebras that have a non-zero semi-simple subalgebra of derivations. We will examine each of these cases in turn.

**9.1.** $NR = A_{5.1} \oplus \mathbb{R}$**.** The Lie algebra brackets are formally the same as in subsection 7.1 except that in addition the $\mathbb{R}$ factor is spanned by $e_9$. The derivation algebra is given by

$$\begin{bmatrix} s_1 & 0 & s_2 & s_3 & s_4 & s_5 \\ 0 & s_1 & s_6 & s_7 & s_8 & s_9 \\ 0 & 0 & s_{10} & 0 & s_{11} & 0 \\ 0 & 0 & 0 & s_{10} & s_{12} & 0 \\ 0 & 0 & 0 & 0 & s_1 - s_{10} & 0 \\ 0 & 0 & s_{13} & s_{14} & s_{15} & s_{16} \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & 0 & 0 & 0 & 0 \\ t_3 & -t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_1 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & -t_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{53}$$

The semi-simple subalgebra occurs in the representation $2D_{\frac{1}{2}}$. The centralizer of the $R$-representation is:

$$\begin{bmatrix} z_4 & 0 & z_5 & 0 & 0 & 0 \\ 0 & z_4 & 0 & z_5 & 0 & 0 \\ z_6 & 0 & z_7 & 0 & 0 & 0 \\ 0 & z_6 & 0 & z_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_8 & z_1 \\ 0 & 0 & 0 & 0 & z_2 & z_3 \end{bmatrix}. \tag{54}$$

The most general way to satisfy Jacobi in a ten-dimensional algebra is as follows:

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_6] = e_6,$
$[e_1, e_7] = -e_7,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = e_6,\ [e_3, e_4] = e_5,$
$[e_3, e_6] = e_7,\ [e_4, e_{10}] = ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_8] = e_4,\ [e_6, e_{10}] = be_6 + ee_4,$
$[e_7, e_8] = e_5,\ [e_7, e_{10}] = be_7 + ee_5,\ [e_8, e_{10}] = de_9 + (a - b)e_8,\ [e_9, e_{10}] = ce_9.$

Replacing $e_{10}$ by $e_{10} - ee_8$, reduces $e$ to zero. Now if $c = 0$ the algebra is decomposable, so we may assume by scaling that $c = 1$. Then replacing $e_8$ by $e_8 + \alpha e_9$, the only effect on the algebra is to change $d$ into $d + (b - a + 1)\alpha$, so we may eliminate $d$, provided $b \neq a - 1$. Of course we must have $a^2 + b^2 \neq 0$, or else the algebra will be decomposable. In the special case when $a = b + 1$, replacing $e_9$ by $\alpha e_9$ we find that $d$ changes to $\frac{d}{\alpha}$ and so we may reduce $d$ to 0 or 1.

**9.2.** $NR = A_{5.3} \oplus \mathbb{R}$**.** The Lie algebra brackets are formally the same as in subsection 7.2 except that, in addition, the $\mathbb{R}$ factor is spanned by $e_9$. The space of derivations is given by:

$$\begin{bmatrix} 3s_1 & 0 & s_2 & s_3 & s_4 & s_5 \\ 0 & 3s_1 & s_6 & s_7 & s_8 & s_9 \\ 0 & 0 & 2s_1 & s_2 & -s_6 & 0 \\ 0 & 0 & 0 & s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_1 & 0 \\ 0 & 0 & 0 & s_{10} & s_{11} & s_{12} \end{bmatrix} + \begin{bmatrix} t_1 & t_2 & 0 & 0 & 0 & 0 \\ t_3 & -t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t_1 & t_3 & 0 \\ 0 & 0 & 0 & t_2 & t_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{55}$$

The semi-simple subalgebra occurs in the $R$-representation $2D_{\frac{1}{2}} \oplus 2D_0$.

The centralizer of the $R$-representation is:

$$\begin{bmatrix} z_1 & 0 & 0 & 0 & z_2 & 0 \\ 0 & z_1 & 0 & z_2 & 0 & 0 \\ 0 & 0 & z_5 & 0 & 0 & z_6 \\ 0 & z_3 & 0 & z_4 & 0 & 0 \\ z_3 & 0 & 0 & 0 & z_4 & 0 \\ 0 & 0 & z_7 & 0 & 0 & z_8 \end{bmatrix}. \tag{56}$$

The most general way to satisfy Jacobi in a ten-dimensional algebra is as follows:
$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_7] = -e_7,$
$[e_1, e_8] = e_8,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = e_8,\ [e_3, e_4] = e_5,\ [e_3, e_8] = e_7,$
$[e_4, e_{10}] = 3ae_4,\ [e_5, e_{10}] = 3ae_5,\ [e_6, e_7] = e_5,\ [e_6, e_8] = e_4,\ [e_6, e_{10}] = 2ae_6,$
$[e_7, e_8] = e_6,\ [e_7, e_{10}] = ce_5 + ae_7,\ [e_8, e_{10}] = ce_4 + ae_8,\ [e_9, e_{10}] = be_9.$

Now if $a = 0$ the algebra is decomposable, so we may assume by scaling that $a = 1$. Then we may replace $e_7$ and $e_8$ by $e_7 - \frac{c}{2}e_5$ and $e_8 - \frac{c}{2}e_4$, respectively, which reduces $c$ to zero. Furthermore we must have $b \neq 0$ or else the algebra will be decomposable.

**9.3.** $NR = A_{5.4} \oplus \mathbb{R}$**.** We take the non-zero brackets as $[e_5, e_7] = e_4,\ [e_6, e_8] = e_4$. The derivation algebra is as follows:

$$\begin{bmatrix} 2s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ 0 & s_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_1 & 0 \\ 0 & s_7 & s_8 & s_9 & s_{10} & s_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & t_3 & t_4 & 0 \\ 0 & t_5 & t_6 & t_4 & t_7 & 0 \\ 0 & t_8 & t_9 & -t_1 & -t_5 & 0 \\ 0 & t_9 & t_10 & -t_2 & -t_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

The semi-simple subalgebra occurs in the representations $D_{\frac{1}{2}} \oplus 4D_0, 2D_{\frac{1}{2}} \oplus 2D_0$ and $D_{\frac{3}{2}} \oplus 2D_0$.

**9.3.1.** $NR = A_{5.4} \oplus \mathbb{R}$**,** $R$**-representation** $D_{\frac{1}{2}} \oplus 4D_0$**.** We take the $R$-representation as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & 0 & t_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_3 & 0 & -t_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{58}$$

The centralizer of the $R$-representation is given by

$$\begin{bmatrix} z_{11} & 0 & z_{10} & 0 & z_9 & z_8 \\ 0 & z_3 & 0 & 0 & 0 & 0 \\ z_7 & 0 & z_6 & 0 & z_5 & z_4 \\ 0 & 0 & 0 & z_3 & 0 & 0 \\ z_2 & 0 & z_1 & 0 & z_{17} & z_{16} \\ z_{15} & 0 & z_{14} & 0 & z_{13} & z_{12} \end{bmatrix}. \tag{59}$$

The intersection of (57) and (59) consists of the following space of matrices:

$$\begin{bmatrix} 2a & 0 & d & 0 & e & f \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & g & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & h & 0 & 2a-b & 0 \\ 0 & 0 & i & 0 & j & c \end{bmatrix} \tag{60}$$

and leads to the following ten-dimensional algebra:

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_7] = -e_7,\ [e_2, e_3] = e_1, \\ &[e_2, e_7] = e_5,\ [e_3, e_5] = e_7,\ [e_4, e_{10}] = 2ae_4,\ [e_5, e_7] = e_4,\ [e_5, e_{10}] = ae_5, \\ &[e_6, e_8] = e_4,\ [e_6, e_{10}] = be_6 + de_4 + he_8 + ie_9,\ [e_7, e_{10}] = ae_7, \\ &[e_8, e_{10}] = (2a-b)e_8 + ee_4 + ge_6 + je_9,\ [e_9, e_{10}] = ce_9 + fe_4. \end{aligned}$$

We replace $e_{10}$ by $e_{10} + ee_6 - de_8$ which has the effect of setting $d$ and $e$ to zero. Now we consider the algebra of $R$-constants, which is spanned by $\{e_4, e_6, e_8, e_9, e_{10}\}$. It is five-dimensional solvable and has a four-dimensional nilradical that is isomorphic to $H \oplus \mathbb{R}$. Furthermore, $\mathrm{ad}(e_{10})$ cannot be nilpotent and its restriction to the $R$-constants is also not nilpotent. The only way for the $R$-constants to be decomposable is, if the restriction of $e_9$ is central and does not appear on the right hand sides of brackets, which would imply that $e = f = i = j = 0$. However, in that case the the ten-dimensional algebra is decomposable. We quote the classification of the indecomposable five-dimensional solvable algebras with nilradical isomorphic to $H + \mathbb{R}$ from [11] and match them case by case with the $R$-constants, so as to obtain ten-dimensional algebras. These brackets have to be appended to the $R$-representation together with $[e_5, e_7] = e_4$, $[e_6, e_8] = e_4$.

**9.3.2.** $NR = A_{5.4} \oplus \mathbb{R}$, **$R$-representation** $2D_{\frac{1}{2}} \oplus 2D_0$**.** The $R$-representation is given by:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 & 0 & 0 \\ 0 & t_3 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t_1 & -t_3 & 0 \\ 0 & 0 & 0 & -t_2 & t_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{61}$$

The centralizer of the $R$-representation is:

$$\begin{bmatrix} z_1 & 0 & 0 & 0 & 0 & z_2 \\ 0 & z_5 & 0 & 0 & -z_6 & 0 \\ 0 & 0 & z_5 & z_6 & 0 & 0 \\ 0 & 0 & z_7 & z_8 & 0 & 0 \\ 0 & -z_7 & 0 & 0 & z_8 & 0 \\ z_3 & 0 & 0 & 0 & 0 & z_4 \end{bmatrix}. \tag{62}$$

The most general way to satisfy the Jacobi identity in a ten-dimensional algebra is as follows:

$$\begin{aligned}&[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_8]=e_8,\ [e_1,e_9]=-e_9,\ [e_2,e_3]=e_1,\\&[e_2,e_9]=e_8,\ [e_3,e_8]=e_9,\ [e_4,e_{10}]=(a+b)e_4,\ [e_5,e_7]=e_4,\ [e_5,e_{10}]=ae_5,\\&[e_6,e_7]=e_5,\ [e_6,e_{10}]=ae_6,\ [e_7,e_{10}]=be_7,\ [e_8,e_{10}]=be_8,\ [e_9,e_{10}]=ce_9+de_4.\end{aligned}$$

If $c=0$, the algebra is decomposable so we may assume by scaling that $c=1$. Then we replace $e_9$ by $e_9+\alpha e_4$. The only effect of this change, is to transform $d$ into $d+(a+b-1)\alpha$ and provided $a+b\neq 1$, we may reduce $d$ to zero. If $a=1$, we may replace $e_9$ by $\alpha e_9$ and as such $d$ changes to $\alpha d$, so that we may assume that $d=0$ or $d=1$.

**9.3.3.** $NR=A_{5.4}\oplus\mathbb{R}, R$**-representation** $D_{\frac{3}{2}}\oplus 2D_0$**.** We start from the nine-dimensional algebra spanned by $\{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8,e_9\}$

$$\begin{aligned}&[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_5]=e_5,\ [e_1,e_6]=-3e_6,\ [e_1,e_7]=-e_7,\\&[e_1,e_8]=3e_8,\ [e_2,e_3]=e_1,\ [e_2,e_5]=3e_8,\ [e_2,e_6]=3e_7,\ [e_2,e_7]=2e_5,\\&[e_3,e_5]=2e_7,\ [e_3,e_7]=e_6,\ [e_3,e_8]=e_5,\ [e_5,e_7]=e_4,\ [e_6,e_8]=e_4.\end{aligned}\tag{63}$$

The centralizer of the $R$-representation is

$$\begin{bmatrix} z_5 & 0 & 0 & 0 & 0 & z_4\\ 0 & z_3 & 0 & 0 & 0 & 0\\ 0 & 0 & z_3 & 0 & 0 & 0\\ 0 & 0 & 0 & z_3 & 0 & 0\\ 0 & 0 & 0 & 0 & z_3 & 0\\ z_2 & 0 & 0 & 0 & 0 & z_1\end{bmatrix}.\tag{64}$$

Reconciling (57) with (64) leads us to append to (63) the following brackets:

$$\begin{aligned}&[e_4,e_{10}]=4ae_4,\ [e_5,e_{10}]=2ae_5,\ [e_6,e_{10}]=2ae_6,\ [e_7,e_{10}]=2ae_7,\\&[e_8,e_{10}]=2ae_8,\ [e_9,e_{10}]=be_9+ce_4.\end{aligned}$$

We obtain a decomposable ten-dimensional algebra if $a=0$ and so we normalize $a$ to $1$. If we replace $e_9$ by $e_9+\alpha e_4$ the only effect on the algebra is to replace $c$ by $c-(b-2)\alpha$. Thus, provided $b\neq 2$, we can eliminate $c$. If $b=2$, then we may replace $e_9$ by $\alpha e_9$, which has the effect of changing $c$ into $\alpha c$. Thus we may assume that $c=0$ or $c=1$.

**9.4.** $NR=A_{4.1}\oplus\mathbb{R}^2$**.** We take the non-zero brackets as $[e_5,e_7]=e_4,\ [e_6,e_7]=e_5$. The derivation algebra is:

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6\\ 0 & s_7 & s_2 & s_8 & 0 & 0\\ 0 & 0 & -s_1+2s_7 & s_9 & 0 & 0\\ 0 & 0 & 0 & s_1-s_7 & 0 & 0\\ 0 & 0 & s_{10} & s_{11} & s_{12} & 0\\ 0 & 0 & s_{13} & s_{14} & 0 & s_{12}\end{bmatrix}+\begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&t_1&t_2\\0&0&0&0&t_3&-t_1\end{bmatrix}.\tag{65}$$

The most general way to satisfy Jacobi in a ten-dimensional algebra is:

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_8] = e_8,\ [e_1, e_9] = -e_9,\ [e_2, e_3] = e_1,\ [e_2, e_9] = e_8,$
$[e_3, e_8] = e_9,\ [e_4, e_{10}] = ae_4,\ [e_5, e_7] = e_4,\ [e_5, e_{10}] = be_5 + de_4,\ [e_6, e_7] = e_5,$
$[e_6, e_{10}] = ee_4 + de_5 + (2b - a)e_6,\ [e_7, e_{10}] = (a - b)e_7 + he_6 + ge_5 + fe_4,$
$[e_8, e_{10}] = ce_8,\ [e_9, e_{10}] = ce_9.$

We normalize the coefficients $a, b, c, d, e, f, g, h$ as follows. First of all, we make a transformation in which $e_{10}$ is replaced by $e_{10} + fe_5 + ge_6 - de_7$ which has the effect of setting $d, f$ and $g$ to zero.

Now we consider the algebra of $R$-constants spanned by $\{e_4, e_5, e_6, e_7, e_{10}\}$, which is five-dimensional solvable having $A_{4.1}$ as nilradical provided $a$ and $b$ are not both zero. These algebras correspond to $A_{5.30}$, $A_{5.31}$ and $A_{5.32}$ and the $R$-constants may be put into these forms without affecting the remaining brackets in the ten-dimensional algebra. As such the case $e = h = 0$ corresponds to $A_{5.30}$. We assume that $a \neq b$ and hence we may divide $e_{10}$ by $a - b$ and then rename $\frac{b}{a-b}$ as $a$ so as to obtain $A_{5.30}$. For $A_{5.31}$ we have $a = 3,\ b = 2,\ e = 0,\ h = 1$. When $a = b \neq 0$, by scaling we may assume $a = b = 1$ and $e = 0,\ h = 1$ gives $A_{5.32}$.

If $a = b = 0$, then we cannot have $e = h = 0$, or else the ten-dimensional algebra will be decomposable. If $h \neq 0$ and $e = 0$, we may assume that $h = 1$ and the $R$-constants is isomorphic to $A_{5.2}$. If $e \neq 0$ and $h = 0$, we may assume that $e = \pm 1$ and the $R$-constants is isomorphic to $A_{5.5}$. If $e \neq 0$ and $h \neq 0$, we may assume that $e = \pm 1$ and $h = 1$ and the $R$-constants is isomorphic to $A_{5.6}$. These normalizations do not involve $e_{10}$, so can assume that $c = 1$ in these cases.

**9.5.** $NR = H \oplus H$**.** We take the non-zero brackets as $[e_5, e_6] = e_4,\ [e_8, e_9] = e_7$. The algebra of derivations is

$$\begin{bmatrix} 2s_1 & s_2 & s_3 & 0 & s_4 & s_5 \\ 0 & s_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & s_6 & s_7 & 2s_8 & s_9 & s_{10} \\ 0 & 0 & 0 & 0 & s_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 & 0 & 0 \\ 0 & t_3 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_4 & t_5 \\ 0 & 0 & 0 & 0 & t_6 & -t_4 \end{bmatrix}. \tag{66}$$

The semi-simple subalgebra of the derivation algebra consists of the direct sum of two copies of $\mathfrak{sl}(2,\mathbb{R})$ in the standard representation. The associated subalgebras that are isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ appear either as $2D_{\frac{1}{2}}$ or $D_{\frac{1}{2}} \oplus 4D_0$. The latter case occurs in two ways that are clearly isomorphic, so we shall assume that $\mathfrak{sl}(2,\mathbb{R})$ comprises the first factor in the decomposition.

**9.5.1.** $NR = H \oplus H, R$**-representation** $2D_{\frac{1}{2}} \oplus 2D_0$ In this case the centralizer of the $R$-representation is

$$\begin{bmatrix} z_1 & 0 & 0 & z_2 & 0 & 0 \\ 0 & z_3 & 0 & 0 & z_4 & 0 \\ 0 & 0 & z_3 & 0 & 0 & z_4 \\ z_5 & 0 & 0 & z_6 & 0 & 0 \\ 0 & z_7 & 0 & 0 & z_8 & 0 \\ 0 & 0 & z_7 & 0 & 0 & z_8 \end{bmatrix}. \tag{67}$$

The intersection of (66) and (67) is two-dimensional diagonal and leads immediately to the ten-dimensional algebra.

**9.5.2.** $NR = H \oplus H, R$**-representation** $D_{\frac{1}{2}} \oplus 4D_0$**.** The centralizer of the $R$-representation is

$$\begin{bmatrix} z_6 & 0 & 0 & z_5 & z_4 & z_3 \\ 0 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 \\ z_1 & 0 & 0 & z_{17} & z_{16} & z_{15} \\ z_{14} & 0 & 0 & z_{13} & z_{12} & z_{11} \\ z_{10} & 0 & 0 & z_9 & z_8 & z_7 \end{bmatrix}. \tag{68}$$

The most general possible ten-dimensional algebra is given by:

$$\begin{aligned}
&[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_2, e_3] = e_1, \\
&[e_2, e_6] = e_5,\ [e_3, e_5] = e_6,\ [e_4, e_{10}] = 2ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5, \\
&[e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = (b + c)e_7,\ [e_8, e_9] = e_7,\ [e_8, e_{10}] = de_4 + he_7 + be_8 + ge_9, \\
&[e_9, e_{10}] = ee_4 + ie_7 + fe_8 + ce_9.
\end{aligned}$$

However, replacing $e_{10}$ by $e_{10} + ie_8 - he_9$, has the effect of setting $h$ and $i$ to zero. We note also that $a = 0$ implies that the algebra is decomposable, so we may suppose that $a \neq 0$.

Now we consider the algebra of $R$-constants, which is spanned by $\{e_4, e_7, e_8, e_9, e_{10}\}$. Then $e_7, e_8, e_9$ span a subalgebra isomorphic to $H$ and $e_4, e_{10}$ span a subalgebra isomorphic to $A_{2.1}$; accordingly, if the $R$-constants were to be decomposable, it would have to split as $H \oplus A_{2.1}$, but in that case we would have $b = c = d = e = f = g = 0$ and the ten-dimensional algebra would be decomposable.

We note that the $R$-constants is five-dimensional indecomposable and has a four-dimensional nilradical that is isomorphic to $H + \mathbb{R}$. Furthermore, the restriction of $\mathrm{ad}\, e_{10}$ to the $R$-constants cannot be nilpotent. Thus it must be one of the algebras in the range $A_{5.19} - A_{5.29}$; however, not every such algebra occurs.

**9.6.** $NR = H \oplus \mathbb{R}^3$**.** The algebra of derivations is

$$\begin{bmatrix} 2s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ 0 & s_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & s_7 & s_8 & s_{13} & 0 & 0 \\ 0 & s_9 & s_{10} & 0 & s_{13} & 0 \\ 0 & s_{11} & s_{12} & 0 & 0 & s_{13} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_1 & t_2 & 0 & 0 & 0 \\ 0 & t_3 & -t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_4 & t_5 & t_6 \\ 0 & 0 & 0 & t_7 & t_8 & t_9 \\ 0 & 0 & 0 & t_{10} & t_{11} & -t_4 - t_8 \end{bmatrix}. \tag{69}$$

The semi-simple subalgebra is isomorphic to block diagonal $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$. There are five ways to obtain an $R$-representation isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. The $R$-representation can occur as $D_{\frac{1}{2}} \oplus 4D_0$ (two ways), $3D_0 \oplus D_1$, $2D_0 \oplus 2D_{\frac{1}{2}}$, $D_0 \oplus D_{\frac{1}{2}} \oplus D_1$. We shall consider each of these cases in turn.

**9.6.1.** $NR = H \oplus \mathbb{R}^3$, $R$**-representation**$= D_0 \oplus D_{\frac{1}{2}} \oplus 3D_0$. Here we have $D_{\frac{1}{2}}$ acting on the $H$-factor. The centralizer of the $R$-representation is given by:

$$\begin{bmatrix} z_6 & 0 & 0 & z_5 & z_4 & z_3 \\ 0 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 \\ z_1 & 0 & 0 & z_{17} & z_{16} & z_{15} \\ z_{14} & 0 & 0 & z_{13} & z_{12} & z_{11} \\ z_{10} & 0 & 0 & z_9 & z_8 & z_7 \end{bmatrix}. \tag{70}$$

Reconciling (70) with (69) leads to the algebra:

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_2, e_3] = e_1, \\ &[e_2, e_6] = e_5,\ [e_3, e_5] = e_6,\ [e_4, e_{10}] = 2ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5, \\ &[e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = be_4 + ee_7 + he_8 + ke_9, \\ &[e_8, e_{10}] = ce_4 + fe_7 + ie_8 + me_9,\ [e_9, e_{10}] = de_4 + ge_7 + je_8 + ne_9. \end{aligned} \tag{71}$$

We may assume in (71) that $a = 1$, otherwise, if $a = 0$, the algebra is decomposable. If we consider the subalgebra of $R$-constants spanned by $\{e_6, e_7, e_8, e_9, e_{10}\}$ we see that it is a five-dimensional solvable algebra with a four-dimensional abelian nilradical. As such, we can adapt the classification of such algebras [11] to normalize our ten-dimensional Levi decomposition algebra; not every algebra in the range $A_{5.7} - A_{5.18}$ occurs. In fact $A_{5.8}, A_{5.10}, A_{5.14}, A_{5.17}$ and $A_{5.18}$ do not occur at all. Furthermore, we choose a normalization so that the first bracket reads $[e_4, e_{10}] = 2e_4$ rather than $[e_4, e_{10}] = e_4$.

**9.6.2.** $NR = H \oplus \mathbb{R}^3$, $R$**-representation**$= 3D_0 \oplus D_{\frac{1}{2}} \oplus D_0$. Here we have $D_{\frac{1}{2}}$ acting on the $\mathbb{R}^3$-factor. The centralizer of the $R$-representation is given by:

$$\begin{bmatrix} z_6 & z_5 & z_4 & 0 & 0 & z_3 \\ z_2 & z_1 & z_{17} & 0 & 0 & z_{16} \\ z_{15} & z_{14} & z_{13} & 0 & 0 & z_{12} \\ 0 & 0 & 0 & z_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & z_{11} & 0 \\ z_{10} & z_9 & z_8 & 0 & 0 & z_7 \end{bmatrix}. \tag{72}$$

Reconciling (72) with (69) leads to the algebra

$$\begin{aligned} &[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1, \\ &[e_2, e_8] = e_7,\ [e_3, e_7] = e_8,\ [e_4, e_{10}] = (a + d)e_4,\ [e_5, e_6] = e_4, \\ &[e_5, e_{10}] = ee_4 + ae_5 + ce_6 + he_9,\ [e_6, e_{10}] = fe_4 + be_5 + de_6 + +ie_9, \\ &[e_7, e_{10}] = je_7,\ [e_8, e_{10}] = je_8,\ [e_9, e_{10}] = ge_4 + ke_9. \end{aligned} \tag{73}$$

We can remove $e$ and $f$ by replacing $e_{10}$ by $e_{10} + fe_5 - ee_6$. If $j = 0$, the algebra is decomposable, so we shall agree, by scaling, that $j = 1$.

If we consider the subalgebra of $R$-constants spanned by $\{e_4, e_5, e_6, e_9, e_{10}\}$ we see that it is a five-dimensional solvable algebra with a four-dimensional nilradical isomorphic to $H \oplus \mathbb{R}$. As such, we can adapt the classification of such algebras [11] to normalize our ten-dimensional Levi decomposition algebra; every algebra in the range $A_{5.19} - A_{5.29}$ occurs.

**9.6.3.** $NR = H \oplus \mathbb{R}^3, R$**-representation**$= 3D_0 \oplus D_1$**.** The most general possible ten-dimensional algebra is given by:

$$\begin{aligned}&[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_7] = 2e_7,\ [e_1, e_9] = -2e_9,\ [e_2, e_3] = e_1,\\&[e_2, e_8] = 2e_7,\ [e_2, e_9] = e_8,\ [e_3, e_7] = e_8,\ [e_3, e_8] = 2e_9,\ [e_4, e_{10}] = (a+d)e_4,\\&[e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + ce_6 + fe_4,\ [e_6, e_{10}] = be_5 + de_6 + ge_4,\\&[e_7, e_{10}] = ee_7,\ [e_8, e_{10}] = ee_8,\ [e_9, e_{10}] = ee_9.\end{aligned} \tag{74}$$

If we replace $e_{10}$ by $e_{10} + ge_5 - fe_6$ then we eliminate $f$ and $g$. If $e = 0$ then the algebra is decomposable so we shall set $e$ to 1 by scaling.

Now we consider the subalgebra of $R$-constants spanned by $\{e_4, e_5, e_6, e_{10}\}$. It is a four-dimensional solvable algebra whose nilradical contains $H$. It cannot be decomposable as $H \oplus \mathbb{R}$, otherwise we would have $a = b = c = d = 0$ and the ten-dimensional Levi decomposition algebra would be decomposable. Accordingly, it must be one of $A_{4.7-4.11}$ in [11]. Actually, we distinguish just three cases here, since $A_{4.8}$ and $A_{4.10}$ are limiting cases of $A_{4.9}$ and $A_{4.11}$, respectively.

**9.6.4.** $NR = H \oplus \mathbb{R}^3, R$**-representation**$= 2D_0 \oplus 2D_{\frac{1}{2}} \approx D_0 \oplus 2D_{\frac{1}{2}} \oplus D_0$**.** The centralizer of the $R$-representation is given by:

$$\begin{bmatrix} z_8 & 0 & 0 & 0 & 0 & z_7 \\ 0 & z_6 & 0 & z_5 & 0 & 0 \\ 0 & 0 & z_6 & 0 & z_5 & 0 \\ 0 & z_4 & 0 & z_3 & 0 & 0 \\ 0 & 0 & z_4 & 0 & z_3 & 0 \\ z_2 & 0 & 0 & 0 & 0 & z_1 \end{bmatrix}. \tag{75}$$

Reconciling (75) with (69) leads to the algebra:

$$\begin{aligned}&[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_1, e_7] = e_7,\\&[e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,\ [e_2, e_8] = e_7,\ [e_3, e_5] = e_6,\ [e_3, e_7] = e_8,\\&[e_4, e_{10}] = 2ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + de_7,\ [e_6, e_{10}] = ae_6 + de_8,\\&[e_7, e_{10}] = be_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = ce_4 + ee_9.\end{aligned}$$

If $e = 0$ the algebra is decomposable, so by scaling we assume that $e = 2$. If we replace $e_9$ by $e_9 + \alpha e_4$, then $c$ changes to $c + 2\alpha(a-1)$ and hence we can eliminate $c$ provided $a \neq 1$. If we replace $e_5$ and $e_6$ by $e_5 + \beta e_7$ and $e_6 + \beta e_8$, then $d$ changes to $d - \beta(a-b)$ and hence we can eliminate $d$ provided $a \neq b$. If $a \neq 1$ and $a \neq b$ we may assume $c = d = 0$.

If we scale $e_9$ by $\beta$ and $e_4$ by $\alpha^2$ and simultaneously $e_5$ and $e_6$ by $\alpha$, then $c$ and $d$ change to $\frac{\beta c}{\alpha^2}$ and $\alpha d$. Hence we can always reduce $c$ and $d$ to the values 0 or 1. Thus if $a = 1, b \neq 1$ then we may assume that $c = 0$ or $c = 1$ and $d = 0$. If $a \neq 1$ and $a = b$ then we may assume that $c = 0$ and $d = 0$ or $d = 1$. If $a = b = 1$ then we may assume that $c = 0$ or $c = 1$ and $d = 0$ or $d = 1$.

**9.6.5.** $NR = H \oplus \mathbb{R}^3, R$**-representation**$= D_0 \oplus D_{\frac{1}{2}} \oplus D_1$**.** The centralizer of the $R$-representation is

$$\begin{bmatrix} z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_3 \end{bmatrix} \tag{76}$$

and immediately gives the Lie algebra.

**9.7.** $NR = \mathbb{R}^6$**.** According to Section 4 there are ten $R$-representations that need to be considered.

**9.7.1.** $NR = \mathbb{R}^6, R$**-representation** $= D_{\frac{5}{2}}$**.** We obtain the Lie algebra immediately in view of Theorem 5.11 in [1].

**9.7.2.** $NR = \mathbb{R}^6, R$**-representation** $= 2D_1$**.** The centralizer of the $R$-representation is four-dimensional and the normalization of the algebra effectively reduces to finding the Jordan form of a $2 \times 2$ matrix.

**9.7.3.** $NR = \mathbb{R}^6, R$**-representation** $= D_2 \oplus D_0$**.** We assume that the $R$-constants are spanned by $\{e_9, e_{10}\}$ and they must satisfy $[e_9, e_{10}] = ae_9$ for some non-zero $a$. The centralizer of the $R$-representation is two-dimensional and immediately gives the Lie algebra.

**9.7.4.** $NR = \mathbb{R}^6, R$**-representation** $= D_1 \oplus D_{\frac{1}{2}} \oplus D_0$ The centralizer of the $R$-representation is three-dimensional diagonal and we easily obtain the ten-dimensional Lie algebra.

**9.7.5.** $NR = \mathbb{R}^6, R$**-representation** $= 3D_{\frac{1}{2}}$**.** Besides the $R$-representation the brackets involving $e_{10}$ are:

$$[e_4, e_{10}] = ae_4 + be_6 + ce_8,\ [e_5, e_{10}] = ae_5 + be_7 + ce_9,\ [e_6, e_{10}] = de_4 + ee_6 + e_8 f,$$
$$[e_7, e_{10}] = de_5 + ee_7 + fe_9,\ [e_8, e_{10}] = ge_4 + he_6 + je_8,\ [e_9, e_{10}] = ge_5 + he_7 + je_9.$$

We can normalize the central $6 \times 6$ block of $\mathrm{ad}\, e_{10}$ by conjugating by a matrix of the form

$$\begin{bmatrix} z_9 & 0 & z_8 & 0 & z_7 & 0 \\ 0 & z_9 & 0 & z_8 & 0 & z_7 \\ z_6 & 0 & z_5 & 0 & z_4 & 0 \\ 0 & z_6 & 0 & z_5 & 0 & z_4 \\ z_3 & 0 & z_2 & 0 & z_1 & 0 \\ 0 & z_3 & 0 & z_2 & 0 & z_1 \end{bmatrix},$$

so formally, we are really looking at the Jordan normal form of a $3 \times 3$ together with a scaling. As such we obtain the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} (a = 1 \,\mathrm{or}\, b = 1), \begin{bmatrix} a & 0 & 0 \\ 0 & b & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

**9.7.6.** $NR = \mathbb{R}^6$, $R$**-representation**$= 2D_{\frac{1}{2}} \oplus 2D_0$. Besides the $R$-representation, the brackets involving $e_{10}$ are:

$$[e_4, e_{10}] = ae_4 + be_6,\ [e_5, e_{10}] = ae_5 + be_7,\ [e_6, e_{10}] = ce_4 + de_6,$$
$$[e_7, e_{10}] = ce_5 + de_7,\ [e_8, e_{10}] = ee_8 + fe_9,\ [e_9, e_{10}] = ge_8 + he_9.$$

The normalization reduces, effectively, to putting the matrices $\left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} e & g \\ f & h \end{smallmatrix}\right]$ into simultaneous Jordan normal form and one overall scaling.

**9.7.7.** $NR = \mathbb{R}^6$, $R$**-representation**$= D_{\frac{3}{2}} \oplus D_{\frac{1}{2}}$. Besides the $R$-representation, the brackets involving $e_{10}$ are:

$$[e_4, e_{10}] = ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = ae_7,$$
$$[e_8, e_{10}] = be_8,\ [e_9, e_{10}] = be_9.$$

We can scale $a$ or $b$ to $1$.

**9.7.8.** $NR = \mathbb{R}^6$, $R$**-representation**$= D_{\frac{3}{2}} \oplus 2D_0$. Besides the $R$-representation, the brackets involving $e_{10}$ are:

$$[e_4, e_{10}] = ee_4,\ [e_5, e_{10}] = ee_5,\ [e_6, e_{10}] = ee_6,\ [e_7, e_{10}] = ee_7,$$
$$[e_8, e_{10}] = ae_8 + be_9,\ [e_9, e_{10}] = ce_8 + de_9.$$

The subalgebra of $R$-constants is spanned by $\{e_8, e_9, e_{10}\}$. It cannot be decomposable, or else the ten-dimensional algebra will also be decomposable. If $e = 0$ the ten-dimensional algebra is decomposable, so by scaling, we assume that $e = 1$. We find the ten-dimensional algebras by using the classification in [11], including the case $A_{3.1} = H$.

**9.7.9.** $NR = \mathbb{R}^6$, $R$**-representation**$= D_1 \oplus 3D_0$. Besides the $R$-representation, the brackets involving $e_{10}$ are:

$$[e_4, e_{10}] = je_4,\ [e_5, e_{10}] = je_5,\ [e_6, e_{10}] = je_6,\ [e_7, e_{10}] = ae_7 + de_8 + ge_9,$$
$$[e_8, e_{10}] = be_7 + ee_8 + he_9,\ [e_9, e_{10}] = ce_7 + fe_8 + ie_9.$$

The subalgebra of $R$-constants is spanned by $\{e_7, e_8, e_9, e_{10}\}$. If $j = 0$ the ten-dimensional algebra is decomposable, so by scaling, we assume that $j = 1$. We see that the $R$-constants is a four-dimensional solvable algebra, whose nilradical is three-dimensional abelian. It cannot be decomposable, or else the ten-dimensional algebra will also be decomposable. Thus we can find the ten-dimensional algebras by using the classification in [11], including the case $A_{4.1}$.

**9.7.10.** $NR = \mathbb{R}^6$, $R$**-representation** $= D_{\frac{1}{2}} \oplus 4D_0$. Besides the $R$-representation, the brackets involving $e_{10}$ are:

$$[e_4, e_{10}] = te_4,\ [e_5, e_{10}] = te_5,\ [e_6, e_{10}] = ae_6 + be_7 + ce_8 + de_9,$$
$$[e_7, e_{10}] = ee_6 + fe_7 + ge_8 + he_9,\ [e_8, e_{10}] = ie_6 + je_7 + ke_8 + me_9,$$
$$[e_9, e_{10}] = pe_6 + qe_7 + re_8 + se_9.$$

The algebra of $R$-constants spanned by $\{e_6, e_7, e_8, e_9, e_{10}\}$ comprises a five-dimensional indecomposble solvable algebra with a four-dimensional abelian nilradical. This class of Lie algebras is well documented as $A_{5.7} - A_{5.18}$ in [11].

## 10. NR seven dimension indecomposable

As a result of Section 4, we have to investigate eleven indecomposable nilpotent Lie algebras. In most of these cases, the semi-simple subalgebra of derivations is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. As such it is a relatively simple matter to extend the nilpotent Lie algebra by three dimensions and the corresponding $R$-representation can be read off from the Lie brackets. Hence in these cases we are content to simply give the Lie brackets without further comment. Instead we focus attention on the few cases where there are multiple $R$-representations.

**10.1.1.** 7.17 **Heisenberg.** We take the algebra in the form

$$[e_4,e_7]=e_{10},\ [e_5,e_8]=e_{10},\ [e_6,e_9]=e_{10}.$$

Its semi-simple space of derivations is isomorphic to $\mathfrak{sp}(6)$. We can find $D_{\frac{1}{2}}\oplus 5D_0$, $2D_{\frac{1}{2}}\oplus 3D_0$, $3D_{\frac{1}{2}}\oplus D_0$, $2D_1\oplus D_0$, $D_{\frac{3}{2}}\oplus 3D_0$, $D_{\frac{3}{2}}\oplus D_{\frac{1}{2}}\oplus D_0$, $D_{\frac{5}{2}}\oplus D_0$ as $R$-representations of $\mathfrak{sl}(2,\mathbb{R})$.

**10.1.2.** $NR=7.37A$ **Anti-Heisenberg.** We take the algebra in the form

$$[e_4,e_5]=e_8,\ [e_4,e_6]=e_9,\ [e_4,e_7]=e_{10}.$$

Its space of derivations is: (77)

$$\begin{bmatrix} s_1 & 0 & 0 & 0 & 0 & 0 & 0\\ s_3 & s_2 & 0 & 0 & 0 & 0 & 0\\ s_4 & 0 & s_2 & 0 & 0 & 0 & 0\\ s_5 & 0 & 0 & s_2 & 0 & 0 & 0\\ s_6 & s_7 & s_8 & s_9 & s_1+s_2 & 0 & 0\\ s_{10} & s_{11} & s_{12} & s_{13} & 0 & s_1+s_2 & 0\\ s_{14} & s_{15} & s_{16} & s_{17} & 0 & 0 & s_1+s_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & t_1 & t_2 & t_3 & 0 & 0 & 0\\ 0 & t_4 & t_5 & t_6 & 0 & 0 & 0\\ 0 & t_7 & t_8 & -t_1-t_5 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & t_1 & t_2 & t_3\\ 0 & 0 & 0 & 0 & t_4 & t_5 & t_6\\ 0 & 0 & 0 & 0 & t_7 & t_8 & -t_1-t_5 \end{bmatrix}$$

The semi-simple subalgebra of derivations is $2\mathfrak{sl}(3)$. The $R$-representation can appear as $2D_{\frac{1}{2}}\oplus 3D_0$ or $2D_1\oplus D_0$.

**10.1.3.** $7.37D$. We take the algebra in the form

$$[e_4,e_5]=e_9,\ [e_4,e_6]=e_8,\ [e_5,e_7]=e_{10},\ [e_6,e_7]=e_9.$$

Its semi-simple space of derivations is

$$\begin{bmatrix} t_1+t_4 & 0 & t_2 & t_5 & 0 & 0 & 0\\ 0 & -t_1-t_4 & t_6 & t_3 & 0 & 0 & 0\\ -t_3 & -t_5 & -t_1+t_4 & 0 & 0 & 0 & 0\\ -t_6 & -t_2 & 0 & t_1-t_4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2t_4 & -t_5 & 0\\ 0 & 0 & 0 & 0 & 2t_6 & 0 & -2t_5\\ 0 & 0 & 0 & 0 & 0 & t_6 & -2t_4 \end{bmatrix} \tag{78}$$

and leads to $R$-representations $2D_{\frac{1}{2}}\oplus 3D_0$ and $2D_{\frac{1}{2}}\oplus D_1$.

## 11. NR seven dimensional decomposable

**11.1.1.** $NR = H \oplus \mathbb{R}^4$**.** We take the bracket for $H$ as $[e_5, e_6] = e_4$ and a basis for the $\mathbb{R}^4$ summand as $\{e_7, e_8, e_9, e_{10}\}$. The semi-simple subalgebra of derivations of $NR = H \oplus \mathbb{R}^4$ is isomorphic to the direct sum of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(4,\mathbb{R})$, coming from each summand. In order to obtain an indecomposable ten-dimensional Levi decomposition algebra, we have to use an $R$-representation of $\mathfrak{sl}(2,\mathbb{R})$ that correlates representations in each of the summands. Moreover, the representations in the second summand can contain no $D_0$ factor, or else the ten-dimensional algebra will be decomposable. We have to use $D_{\frac{1}{2}}$ in the first summand and we can takes its "diagonal" representation with either of $2D_{\frac{1}{2}}$ or $D_{\frac{3}{2}}$.

**11.1.2.** $NR = A_{5.1} \oplus \mathbb{R}^2$**.** The semi-simple subalgebra of derivations of $NR = A_{5.1} \oplus \mathbb{R}^2$ is the direct sum of two copies of $\mathfrak{sl}(2,\mathbb{R})$, each copy coming from one of the two factors, the first as a $2D_{\frac{1}{2}}$ and the second as a $D_{\frac{1}{2}}$ representation. The only way to obtain an indecomposable ten-dimensional Levi decomposition algebra, is to take the diagonal representation, which gives $3D_{\frac{1}{2}}$.

**11.1.3.** $NR = A_{5.3} \oplus \mathbb{R}^2$**.** Similar to $NR = A_{5.1} \oplus \mathbb{R}^2$.

**11.1.4.** $NR = A_{5.4} \oplus \mathbb{R}^2$**.** Again the semi-simple subalgebra of derivations of $NR = A_{5.4} \oplus \mathbb{R}^2$ is a block diagonal sum of $\mathfrak{sp}(4)$ and $\mathfrak{sl}(2,\mathbb{R})$. We obtain representations by taking diagonal representations coming from $NR = A_{5.4}$, there are three such, and $D_{\frac{1}{2}}$ from $\mathbb{R}^2$.

## 12. Indecomposable Levi decomposition Lie algebras of dimension ten with semi-simple factor isomorphic to $\mathfrak{sl}(2,\mathbb{R})$

In this final Section we summarize our results and provide a complete enumeration of the isomorphism classes of indecomposable ten-dimensional Levi decomposition algebra for which the semi-simple Levi factor is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

### 12.1. NR five-dimensional indecomposable

**12.1.1.** $NR = A_{5.1}, S \ltimes NR = L_{8.14}$, **$R$-rep.** $2D_{\frac{1}{2}} \oplus D_0$**, $R$-const.** $\{e_8, e_9, e_{10}\}$

- $A_{2.1} \oplus \mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,$
  $[e_1, e_6] = e_6,\ [e_1, e_7] = -e_7,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = e_6,$
  $[e_3, e_4] = e_5,\ [e_3, e_6] = e_7,\ [e_4, e_9] = e_4,\ [e_5, e_9] = e_5,\ [e_6, e_8] = e_4,$
  $[e_6, e_{10}] = e_6,\ [e_7, e_8] = e_5,\ [e_7, e_{10}] = e_7,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = -e_8.$

**12.1.2.** $NR = A_{5.4}$, $S \ltimes NR = L_{8.6}$, **$R$-rep.** $D_{\frac{1}{2}} \oplus 3D_0$**, $R$-const.** $\{e_6, e_7, e_8, e_9, e_{10}\}$

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,$$
$$[e_2, e_5] = e_4,\ [e_3, e_4] = e_5,\ [e_5, e_7] = e_4,\ [e_6, e_8] = e_4$$

- $A_{5.36} : [e_4, e_9] = 2e_4,\ [e_5, e_9] = e_5,\ [e_6, e_9] = 2e_6,\ [e_6, e_{10}] = -e_6,$
  $[e_7, e_9] = e_7,\ [e_8, e_{10}] = e_8$
- $A_{5.37} : [e_4, e_9] = 2e_4,\ [e_5, e_9] = e_5,\ [e_6, e_9] = e_6,\ [e_6, e_{10}] = -e_8,$
  $[e_7, e_9] = e_7,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = e_6.$

**12.1.3.** $NR = A_{5.4}, S \ltimes NR = L_{8.13}, R$**-rep.** $2D_{\frac{1}{2}} \oplus D_0$**,** $R$**-const.** $\{e_4, e_9, e_{10}\}$

- $A_{2.1} \oplus \mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = e_6,$
$[e_1, e_7] = -e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_7] = e_5,\ [e_2, e_8] = e_6,$
$[e_3, e_5] = e_7,\ [e_3, e_6] = e_8,\ [e_4, e_9] = 2e_4,\ [e_5, e_7] = e_4,\ [e_5, e_9] = e_5,$
$[e_5, e_{10}] = e_6,\ [e_6, e_8] = e_4,\ [e_6, e_9] = e_6,\ [e_6, e_{10}] = -e_5,\ [e_7, e_9] = e_7,$
$[e_7, e_{10}] = e_8,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = -e_7, [e_9, e_{10}] = \epsilon e_4, (\epsilon = 0, \pm 1)$

### 12.2. NR five-dimensional decomposable

**12.2.1.** $NR = H \oplus \mathbb{R}^2, S \ltimes NR = L^*_{8.23}, R$**-rep.** $2D_{\frac{1}{2}} \oplus D_0, R$**-const.** $\{e_4, e_9, e_{10}\}$

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,$$
$$[e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,$$
$$[e_2, e_8] = e_7,\ [e_3, e_5] = e_6,\ [e_3, e_7] = e_8,\ [e_5, e_6] = e_4.$$

- $A_{2.1} \oplus \mathbb{R} : [e_4, e_9] = 2e_4,\ [e_5, e_9] = e_5,\ [e_6, e_9] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8.$
- $A_{2.1} \oplus \mathbb{R} : [e_4, e_9] = 2e_4,\ [e_4, e_{10}] = 2ae_4,\ [e_5, e_9] = e_5,\ [e_5, e_{10}] = ae_5 + e_7,$
$[e_6, e_9] = e_6,\ [e_6, e_{10}] = ae_6 + e_8,\ [e_7, e_9] = e_7,\ [e_7, e_{10}] = be_7,\ [e_8, e_9] = e_8,$
$[e_8, e_{10}] = be_8\ (a = 1\ or\ b = 1).$

**12.2.2.** $NR = H \oplus \mathbb{R}^2, S \ltimes NR = L_{6.2} \oplus \mathbb{R}^2, R$**-rep.** $D_{\frac{1}{2}} \oplus 3D_0, D_{\frac{1}{2}}$ **acting on** $H, R$**-const.** $\{e_4, e_7, e_8, e_9, e_{10}\}$

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_2, e_3] = e_1,$$
$$[e_2, e_6] = e_5,\ [e_3, e_5] = e_6,\ [e_5, e_6] = e_4.$$

- $A_{5.33} : [e_4, e_9] = 2e_4, [e_5, e_9] = e_5, [e_6, e_9] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_9] = ae_8,$
$[e_8, e_{10}] = be_8\ (a^2 + b^2 \neq 0)$
- $A_{5.34} : [e_4, e_9] = 2ae_4,\ [e_4, e_{10}] = 2e_4,\ [e_5, e_9] = ae_5, [e_5, e_{10}] = e_5,$
$[e_6, e_9] = ae_6,\ [e_6, e_{10}] = e_6,\ [e_7, e_9] = e_7,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = e_7$
- $A_{5.35} : [e_4, e_9] = 2ae_4,\ [e_4, e_{10}] = 2be_4,\ [e_5, e_9] = ae_5,\ [e_5, e_{10}] = e_5,$
$[e_6, e_9] = ae_6,\ [e_6, e_{10}] = e_6,\ [e_7, e_9] = e_7,\ [e_7, e_{10}] = -e_8,$
$[e_8, e_9] = e_8,\ [e_8, e_{10}] = e_7\ (a^2 + b^2 \neq 0)$
- $A_{5.38} : [e_4, e_9] = 2e_4,\ [e_5, e_9] = e_5, [e_6, e_9] = e_6,\ [e_7, e_{10}] = e_7,\ [e_9, e_{10}] = e_8$
- $A_{5.39} : [e_4, e_9] = 2e_4,\ [e_5, e_9] = e_5,\ [e_6, e_9] = e_6,\ [e_7, e_9] = e_7, [e_7, e_{10}] = e_8,$
$[e_8, e_9] = e_8,\ [e_8, e_{10}] = -e_7,\ [e_9, e_{10}] = e_4.$

**12.2.3.** $NR = H \oplus \mathbb{R}^2, S \ltimes NR = L_{5.1} \oplus H, R$**-rep.** $D_{\frac{1}{2}} \oplus 3D_0, D_{\frac{1}{2}}$ **acting on** $\mathbb{R}^2$**,** $R$**-const.** $\{e_6, e_7, e_8, e_9, e_{10}\}$

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,$$
$$[e_2, e_8] = e_7,\ [e_3, e_7] = e_8,\ [e_5, e_6] = e_4$$

- $A_{5.36} : [e_4, e_9] = e_4,\ [e_5, e_6] = e_4,\ [e_5, e_9] = e_5,\ [e_5, e_{10}] = -e_5,\ [e_6, e_{10}] = e_6,$
$[e_7, e_9] = ae_7,\ [e_7, e_{10}] = be_7,\ [e_8, e_9] = ae_8,\ [e_8, e_{10}] = be_8,\ (a^2 + b^2 \neq 0)$
- $A_{5.37} : [e_4, e_9] = 2e_4,\ [e_5, e_6] = e_4,\ [e_5, e_9] = e_5,\ [e_6, e_9] = e_6,$
$[e_5, e_{10}] = -e_6,\ [e_6, e_{10}] = e_5,\ [e_7, e_9] = ae_7,\ [e_7, e_{10}] = be_7,\ [e_8, e_9] = ae_8,$
$[e_8, e_{10}] = be_8\ (a^2 + b^2 \neq 0)$

**12.2.4.** $NR = \mathbb{R}^5, S \rtimes NR = L_{8.22}, R$**-rep.** $D_{\frac{1}{2}} \oplus D_1, R$**-const.** $\{e_9, e_{10}\}$
$A_{2.1} \oplus \mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_6] = 2e_6,$
$[e_1, e_8] = -2e_8,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = 2e_6,\ [e_2, e_8] = e_7,\ [e_3, e_4] = e_5,$
$[e_3, e_6] = e_7,\ [e_3, e_7] = 2e_8,\ [e_4, e_9] = e_4,\ [e_5, e_9] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,$
$[e_8, e_{10}] = e_8.$

**12.2.5.** $NR = \mathbb{R}^5 S \rtimes NR = L_{7.7} \oplus \mathbb{R}, R$**-rep.** $2D_{\frac{1}{2}} \oplus D_0, R$**-const.** $\{e_8, e_9, e_{10}\}$
$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_6] = e_6,$
$[e_1, e_7] = -e_7,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = e_6,\ [e_3, e_4] = e_5,\ [e_3, e_6] = e_7$

- $A_{2.1} \oplus \mathbb{R}$ or $A_{3.1} : [e_4, e_9] = e_4,\ [e_5, e_9] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,$ $[e_8, e_9] = ae_8,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = \epsilon e_8\ (a^2 + b^2 \neq 0,\ \epsilon = 0$ or $\epsilon = 1)$
- $A_{2.1} \oplus \mathbb{R}$ or $A_{3.1} : [e_4, e_9] = e_4,\ [e_4, e_{10}] = e_6,\ [e_5, e_9] = e_5,\ [e_5, e_{10}] = e_7,$ $[e_6, e_9] = e_6,\ [e_6, e_{10}] = -e_4,\ [e_7, e_9] = e_7,\ [e_7, e_{10}] = -e_5,\ [e_8, e_9] = ae_8,$ $[e_8, e_{10}] = be_8,\ [e_9, e_{10}] = \epsilon e_8\ (a^2 + b^2 \neq 0,\ \epsilon = 0$ or $\epsilon = 1)$
- $A_{2.1} \oplus \mathbb{R}$ or $A_{3.1} : [e_4, e_9] = e_4,\ [e_5, e_9] = e_5,\ [e_6, e_9] = e_4 + e_6,\ [e_6, e_{10}] = e_4,$ $[e_7, e_9] = e_5 + e_7,\ [e_7, e_{10}] = e_5,\ [e_8, e_9] = ae_8,\ [e_8, e_{10}] = e_8,$ $[e_9, e_{10}] = \epsilon e_8\ (\epsilon = 0$ or $1).$

**12.2.6.** $NR = \mathbb{R}^5, S \rtimes NR = L_{6.4} \oplus \mathbb{R}^2, R$**-rep.** $D_1 \oplus 2D_0, R$**-const.** $\{e_7, e_8, e_9, e_{10}\}$
$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6,\ [e_2, e_3] = e_1,$
$[e_2, e_5] = 2e_4,\ [e_2, e_6] = e_5,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6.$

- $A_{2.1} \oplus A_{2.1} : [e_4, e_9] = ae_4,\ [e_4, e_{10}] = be_4,\ [e_5, e_9] = ae_5,\ [e_5, e_{10}] = be_5,$ $[e_6, e_9] = ae_6,\ [e_6, e_{10}] = be_6,\ [e_7, e_9] = e_7,\ [e_8, e_{10}] = e_8\ (ab \neq 0)$
- $A_{4.12} : [e_4, e_9] = ae_4,\ [e_4, e_{10}] = be_4,\ [e_5, e_9] = ae_5,\ [e_5, e_{10}] = be_5,$ $[e_6, e_9] = ae_6,\ [e_6, e_{10}] = be_6,\ [e_7, e_9] = e_7,\ [e_7, e_{10}] = -e_8, [e_8, e_9] = e_8,$ $[e_8, e_{10}] = e_7\ (a^2 + b^2 \neq 0)$
- $A_{3.3} \oplus \mathbb{R} : [e_4, e_9] = ae_4,\ [e_4, e_{10}] = e_4,\ [e_5, e_9] = ae_5,\ [e_5, e_{10}] = e_5,$ $[e_6, e_9] = ae_6,\ [e_6, e_{10}] = e_6,\ [e_7, e_9] = e_7,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = e_7$

**12.2.7.** $NR = \mathbb{R}^5, S \rtimes NR = L_{5.1} \oplus \mathbb{R}^3, R$**-rep.** $D_{\frac{1}{2}} \oplus 3D_0, R$**-const.** $\{e_6, e_7, e_8, e_9, e_{10}\}$
$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,$
$[e_3, e_4] = e_5$

- $A_{5.19_{a=1,b=1}} : [e_4, e_9] = e_4,\ [e_5, e_9] = ae_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,$ $[e_8, e_9] = e_6,\ [e_8, e_{10}] = e_8\ (a \neq 0)$
- $A_{5.27} : [e_4, e_9] = ae_4,\ [e_5, e_9] = ae_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6 + e_7,$ $[e_8, e_9] = e_6,\ [e_8, e_{10}] = e_7 + e_8\ (a \neq 0)$
- $A_{5.28_{a=1}} : [e_4, e_9] = ae_4,\ [e_5, e_9] = ae_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,$ $[e_8, e_9] = e_6,\ [e_8, e_{10}] = e_7 + e_8\ (a \neq 0)$
- $A_{5.32} : [e_4, e_9] = ae_4,\ [e_5, e_9] = ae_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_9] = e_6,$ $[e_7, e_{10}] = e_7,\ [e_8, e_9] = e_7,\ [e_8, e_{10}] = be_6 + e_8\ (a \neq 0).$
- $A_{5.33} : [e_4, e_9] = ae_4, [e_5, e_9] = ae_5, [e_6, e_9] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_9] = be_8,$ $[e_8, e_{10}] = ce_8\ (abc \neq 0)$

- $A_{5.34}: [e_4, e_9] = ae_4,\ [e_4, e_{10}] = ae_4,\ [e_6, e_9] = be_6,\ [e_6, e_{10}] = e_6,$ $[e_7, e_9] = e_7,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = e_7\ (a \neq 0)$
- $A_{5.35}: [e_4, e_9] = ae_4,\ [e_5, e_9] = ae_5,\ [e_6, e_9] = be_6,\ [e_6, e_{10}] = ce_6,$ $[e_7, e_9] = e_7,\ [e_7, e_{10}] = -e_8,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = e_7\ (b^2 + c^2 \neq 0)$
- $A_{5.38}: [e_4, e_9] = ae_4,\ [e_5, e_9] = ae_5, [e_6, e_9] = e_6,\ [e_7, e_{10}] = e_7,$ $[e_9, e_{10}] = e_8\ (a \neq 0)$
- $A_{5.39}: [e_4, e_9] = ae_4,\ [e_5, e_9] = ae_5,\ [e_7, e_{10}] = -e_7,\ [e_7, e_9] = e_7,$ $[e_7, e_{10}] = e_6,\ [e_8, e_9] = e_8,\ [e_8, e_{10}] = -e_7,\ [e_9, e_{10}] = e_8.$

### 12.3. NR six-dimensional indecomposable

**12.3.1.** $NR = A_{6.3}, S \rtimes NR = L_{9.40}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$, $R$**-const.** $\{e_6, e_9, e_{10}\}$
$A_{2.1} \oplus \mathbb{R}$ or $A_{3.5}: [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,$
$[e_1, e_7] = -e_7,\ [e_1, e_8] = e_8,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = -e_8,\ [e_3, e_4] = e_5,$
$[e_3, e_8] = -e_7,\ [e_4, e_5] = e_9,\ [e_4, e_6] = -e_8,\ [e_4, e_{10}] = ae_4,\ [e_5, e_6] = e_7,$
$[e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = be_6 + ce_9,\ [e_7, e_{10}] = (a + b)e_7,\ [e_8, e_{10}] = (a + b)e_8,$
$[e_9, e_{10}] = 2ae_9\ (a = 1,\ b \neq 2,\ c = 0\ \mathit{or}\ a \neq 1,\ b = 2,\ c = 0\ \mathit{or}\ a = 1,\ b = 2,\ c = 1).$

**12.3.2.** $NR = A_{6.3}, S \rtimes NR = L_{9.62} \oplus \mathbb{R}$, $R$**-rep.** $2D_1$, $R$**-const.** $\{e_{10}\}$
$\mathbb{R}: [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6,$
$[e_1, e_7] = -2e_7,\ [e_1, e_9] = 2e_9,\ [e_2, e_5] = 2e_4,\ [e_2, e_6] = e_5,\ [e_2, e_7] = -2e_8,$
$[e_2, e_8] = -e_9,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_3, e_8] = -e_7,\ [e_3, e_9] = -2e_8,$
$[e_4, e_{10}] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = 2e_7,\ [e_8, e_{10}] = 2e_8,\ [e_9, e_{10}] = 2e_9.$

**12.3.3.** $NR = A_{6.4}, S \rtimes NR = L_{9.41}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$, $R$**-const.** $\{e_8, e_9, e_{10}\}$
$A_{2.1} \oplus \mathbb{R}$ *or* $A_{3.5}: [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,$
$[e_1, e_6] = -e_6,\ [e_1, e_7] = e_7,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_6] = -e_7,\ [e_3, e_4] = e_5,$
$[e_3, e_7] = -e_6,\ [e_4, e_5] = e_8,\ [e_4, e_6] = e_9,\ [e_4, e_{10}] = ae_4 + ce_7,\ [e_5, e_7] = e_9,$
$[e_5, e_{10}] = ae_5 - ce_6,\ [e_6, e_{10}] = be_6,\ [e_7, e_{10}] = be_7,\ [e_8, e_{10}] = 2ae_8 - 2ce_9,$
$[e_9, e_{10}] = (a + b)e_9\ (a \neq b,\ a = 1\ \mathit{or}\ b = 1,\ c = 0,\ \mathit{or}\ a = b = 1,\ c = 0, 1).$

**12.3.4.** $NR = A_{6.5}, S \rtimes NR = L_{9.42}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$**,** $R$**-const.** $\{e_8, e_9, e_{10}\}$
$A_{3.6}$ or $A_{3.7}: [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = e_5,$
$[e_1, e_6] = -e_6,\ [e_1, e_7] = -e_7,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_4,\ [e_2, e_7] = -e_5,$
$[e_3, e_4] = e_6,\ [e_3, e_5] = -e_7,\ e_4, e_6] = e_8,\ [e_4, e_7] = e_9,\ [e_4, e_{10}] = ae_4 - be_5,$
$[e_5, e_6] = -e_9,\ [e_5, e_7] = e_8,\ [e_5, e_{10}] = ae_5 + be_4,\ [e_6, e_{10}] = ae_6 + be_7,$
$[e_7, e_{10}] = ae_7 - be_6,\ [e_8, e_{10}] = 2ae_8 + 2be_9,\ [e_9, e_{10}] = 2ae_9 - 2be_8\ (a{=}1\ \mathit{or}\ b{=}1).$

**12.3.5.** $NR = A_{6.12}, S \rtimes NR = L_{9.12}$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 4D_0$, $R$**-const.** $\{e_4, e_6, e_7, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,$
$[e_2, e_8] = e_5,\ [e_3, e_5] = e_8,\ [e_4, e_6] = e_7,\ [e_4, e_7] = e_9,$

- $A_{5.30}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = (a + 1)e_5,\ [e_6, e_{10}] = 2(a - 1)e_6,$ $[e_7, e_{10}] = 2ae_7,\ [e_8, e_{10}] = (a + 1)e_8,\ [e_9, e_{10}] = 2(a + 1)e_9.$
- $A_{5.31}: [e_4, e_{10}] = 4e_4 - 2e_6,\ [e_5, e_8] = e_9,\ [e_5, e_{10}] = 3e_5,\ [e_6, e_{10}] = -2e_6 + ae_9,$ $[e_7, e_{10}] = 2e_7,\ [e_8, e_{10}] = 3e_8,\ [e_9, e_{10}] = 6e_9.$
- $A_{5.32}: [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = 2e_6 + ae_9,\ [e_7, e_{10}] = 2e_7,\ [e_8, e_{10}] = e_8,$ $[e_9, e_{10}] = 2e_9.$

### 12.4. NR six-dimensional decomposable

**12.4.1.** $NR = A_{5.1} \oplus \mathbb{R}, S \rtimes NR = L_{8.14} \oplus \mathbb{R}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$, $R$**-const.** $\{e_8, e_9, e_{10}\}$
$A_{2.1} \oplus \mathbb{R}$, $A_{3.1}$ or $A_{3.5}$ : $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_1, e_6] = e_6$, $[e_1, e_7] = -e_7$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_2, e_7] = e_6$, $[e_3, e_4] = e_5$, $[e_3, e_6] = e_7$, $[e_4, e_{10}] = ae_4$, $[e_5, e_{10}] = ae_5$, $[e_6, e_8] = e_4$, $[e_6, e_{10}] = be_6$, $[e_7, e_8] = e_5$, $[e_7, e_{10}] = be_7$, $[e_8, e_{10}] = de_9 + (a-b)e_8$, $[e_9, e_{10}] = e_9$ ($a^2 + b^2 \neq 0$, $d = 0$ or $b = a - 1$, $d = 1$).

**12.4.2.** $NR = A_{5.3} \oplus \mathbb{R}, S \rtimes NR = L_{8.15} \oplus \mathbb{R}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$, $R$**-const.** $\{e_6, e_9, e_{10}\}$
$A_{3.5}$ : $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_1, e_7] = -e_7$, $[e_1, e_8] = e_8$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_2, e_7] = e_8$, $[e_3, e_4] = e_5$, $[e_3, e_8] = e_7$, $[e_4, e_{10}] = 3e_4$, $[e_5, e_{10}] = 3e_5$, $[e_6, e_7] = e_5$, $[e_6, e_8] = e_4$, $[e_6, e_{10}] = 2e_6$, $[e_7, e_8] = e_6$, $[e_7, e_{10}] = e_7$, $[e_8, e_{10}] = e_8$, $[e_9, e_{10}] = ae_9$ ($a \neq 0$).

**12.4.3.** $NR = A_{5.4} \oplus \mathbb{R}$, $S \rtimes NR = L^*_{8.23} \oplus \mathbb{R}$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 4D_0$, $R$**-const.** $\{e_4, e_6, e_8, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_5] = e_5$, $[e_1, e_7] = -e_7$, $[e_2, e_3] = e_1$, $[e_2, e_7] = e_5$, $[e_3, e_5] = e_7$, $[e_5, e_7] = e_4$, $[e_6, e_8] = e_4$

- $A_{5.19}$ : $[e_4, e_{10}] = 2ae_4, [e_5, e_{10}] = ae_5$, $[e_6, e_{10}] = e_6$, $[e_7, e_{10}] = ae_7$, $[e_8, e_{10}] = (2a-1)e_8$, $[e_9, e_{10}] = be_9$ ($b \neq 0$)
- $A_{5.20}$ : $[e_4, e_{10}] = 2ae_4$, $[e_5, e_{10}] = ae_5$, $[e_6, e_{10}] = e_6$, $[e_7, e_{10}] = ae_7$, $[e_8, e_{10}] = (2a-1)e_8$, $[e_9, e_{10}] = 2ae_9 + e_4$
- $A_{5.21}$ : $[e_4, e_{10}] = 2e_4$, $[e_5, e_{10}] = e_5$, $[e_6, e_{10}] = e_6 + e_8$, $[e_7, e_{10}] = e_7$, $[e_8, e_{10}] = e_8 + e_9$, $[e_9, e_{10}] = e_9$
- $A_{5.22}$ : $[e_6, e_{10}] = e_8$, $[e_9, e_{10}] = e_9$
- $A_{5.23}$ : $[e_4, e_{10}] = 2e_4$, $[e_5, e_{10}] = e_5$, $[e_6, e_{10}] = e_6 + e_8$, $[e_7, e_{10}] = e_7$, $[e_8, e_{10}] = e_8$, $[e_9, e_{10}] = ae_9$ ($a \neq 0$)
- $A_{5.24}$ : $[e_4, e_{10}] = 2e_4$, $[e_5, e_{10}] = e_5$, $[e_6, e_{10}] = e_6 + e_8$, $[e_7, e_{10}] = e_7$, $[e_8, e_{10}] = e_8$, $[e_9, e_{10}] = \pm e_4 + 2e_9$
- $A_{5.25}$ : $[e_4, e_{10}] = 2ae_4$, $[e_5, e_{10}] = ae_5$, $[e_6, e_{10}] = ae_6 + e_8$, $[e_7, e_{10}] = ae_7$, $[e_8, e_{10}] = ae_8 - e_6$, $[e_9, e_{10}] = be_9$ ($b \neq 0$)
- $A_{5.26}$ : $[e_4, e_{10}] = 2ae_4$, $[e_5, e_{10}] = ae_5$, $[e_6, e_{10}] = ae_6 + e_8$, $[e_7, e_{10}] = ae_7$, $[e_8, e_{10}] = ae_8 - e_6$, $[e_9, e_{10}] = 2ae_9 \pm e_4$
- $A_{5.27}$ : $[e_4, e_{10}] = 2e_4$, $[e_5, e_{10}] = e_5$, $[e_7, e_{10}] = e_7$, $[e_8, e_{10}] = 2e_8 + e_9$, $[e_9, e_{10}] = e_4 + 2e_9$
- $A_{5.28}$ : $[e_4, e_{10}] = 2ae_4$, $[e_5, e_{10}] = ae_5$, $[e_6, e_{10}] = (2a-1)e_6$, $[e_7, e_{10}] = ae_7$, $[e_8, e_{10}] = e_8 + e_9$, $[e_9, e_{10}] = e_9$
- $A_{5.29}$ : $[e_4, e_{10}] = 2e_4$, $[e_5, e_{10}] = e_5$, $[e_6, e_{10}] = 2e_6$, $[e_7, e_{10}] = e_7$, $[e_8, e_{10}] = e_9$

**12.4.4.** $NR = A_{5.4} \oplus \mathbb{R}, S \rtimes NR = L_{8.13} \oplus \mathbb{R}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$, $R$**-const.** $\{e_4, e_9, e_{10}\}$
$A_{2.1} \oplus \mathbb{R}, A_{3.1}$ *or* $A_{3.5}$ : $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_5] = e_5$, $[e_1, e_6] = -e_6$, $[e_1, e_7] = -e_7$, $[e_1, e_8] = e_8$, $[e_2, e_3] = e_1$, $[e_2, e_6] = e_5$, $[e_2, e_7] = -e_8$, $[e_3, e_5] = e_6$, $[e_3, e_8] = -e_7$, $[e_4, e_{10}] = (a+b)e_4$, $[e_5, e_7] = e_4$, $[e_5, e_{10}] = ae_5$, $[e_6, e_8] = e_4$,

$[e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = be_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = ce_4 + e_9,$
$((a + b - 1)(a^2 + b^2) \neq 0,\ c = 0\ \textit{or}\ a + b = 1,\ c = 0, 1)$

**12.4.5.** $NR = A_{5.4} \oplus \mathbb{R}, S \rtimes NR = L_{8.19} \oplus \mathbb{R}$, $R$**-rep.** $D_{\frac{3}{2}} \oplus 2D_0$, $R$**-const.** $\{e_4, e_9, e_{10}\}$
$A_{3.2}$ *or* $A_{3.5}$ : $[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -3e_6,$
$[e_1, e_7] = -e_7,\ [e_1, e_8] = 3e_8,\ [e_2, e_3] = e_1,\ [e_2, e_5] = 3e_8,\ [e_2, e_6] = 3e_7,$
$[e_2, e_7] = 2e_5,\ [e_3, e_5] = 2e_7,\ [e_3, e_7] = e_6,\ [e_3, e_8] = e_5,\ [e_4, e_{10}] = 2e_4,\ [e_5, e_7] = e_4,$
$[e_5, e_{10}] = e_5,\ [e_6, e_8] = e_4,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,$
$[e_9, e_{10}] = ae_9 + be_4,\ (a \neq 0, 2, b = 0\ \textit{or}\ a = 2, b = 0, 1)$

**12.4.6.** $NR = A_{4.1} \oplus \mathbb{R}^2, S \rtimes NR = L_{5.1} \oplus \mathbb{R}^4$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 4D_0$,
$R$**-const.** $\{e_4, e_5, e_6, e_7, e_{10}\}$
$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_8] = e_8,\ [e_1, e_9] = -e_9,$
$[e_2, e_3] = e_1,\ [e_2, e_9] = e_8,\ [e_3, e_8] = e_9.$

- $A_{5.2}$ : $[e_5, e_7] = e_4,\ [e_6, e_7] = e_5,\ [e_7, e_{10}] = e_6,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = e_9.$
- $A_{5.5}$ : $[e_5, e_7] = e_4,\ [e_6, e_7] = e_5,\ [e_6, e_{10}] = \pm e_6,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = e_9.$
- $A_{5.6}$ : $[e_5, e_7] = e_4,\ [e_6, e_7] = e_5,\ [e_6, e_{10}] = \pm e_6,\ [e_7, e_{10}] = e_6,$
  $[e_8, e_{10}] = e_8,\ [e_9, e_{10}] = e_9.$
- $A_{5.30}$ : $[e_4, e_{10}] = (a + 1)e_4,\ [e_5, e_7] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_7] = e_5,$
  $[e_6, e_{10}] = (a - 1)e_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = be_9\ (b \neq 0).$
- $A_{5.31}$ : $[e_4, e_{10}] = 3e_4,\ [e_5, e_7] = e_4,\ [e_5, e_{10}] = 2e_5,\ [e_6, e_7] = e_5,$
  $[e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6 + e_7,\ [e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = ae_9\ (a \neq 0).$
- $A_{5.32}$ : $[e_4, e_{10}] = e_4,\ [e_5, e_7] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_7] = e_5,$
  $[e_6, e_{10}] = \pm e_4 + e_6,\ [e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = ae_9\ (a \neq 0).$

**12.4.7.** $NR = H \oplus H, S \rtimes NR = L_{9.37}$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 2D_0$, $R$**-const.** $\{e_4, e_7, e_{10}\}$
$A_{3.5}$ : $[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_1, e_8] = e_8,$
$[e_1, e_9] = -e_9,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,\ [e_2, e_9] = e_8,\ [e_3, e_5] = e_6,$
$[e_3, e_8] = e_9,\ [e_4, e_{10}] = 2e_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,$
$[e_7, e_{10}] = 2ae_7,\ [e_8, e_9] = e_7,\ [e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = ae_9\ (a \neq 0).$

**12.4.8.** $NR = H \oplus H, S \rtimes NR = L_{6.2} \oplus H$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 4D_0$,
$R$**-const.** $\{e_4, e_7, e_8, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_2, e_3] = e_1,$
$[e_2, e_6] = e_5,\ [e_3, e_5] = e_6, [e_5, e_6] = e_4,\ [e_8, e_9] = e_7$

- $A_{5.19}$ : $[e_4, e_{10}] = 2ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = (b + 1)e_7,$
  $[e_8, e_9] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = be_9,\ (ab \neq 0)$
- $A_{5.21}$ : $[e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = 2e_7,\ [e_8, e_9] = e_7,$
  $[e_8, e_{10}] = e_8 + e_9,\ [e_9, e_{10}] = e_4 + e_9$
- $A_{5.22}$ : $[e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_8, e_9] = e_7,\ [e_8, e_{10}] = e_9$

- $A_{5.23}: [e_4, e_{10}] = 2ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = 2e_7,$ $[e_8, e_9] = e_7,\ [e_8, e_{10}] = e_8 + e_9,\ [e_9, e_{10}] = e_9\ (a \neq 0)$
- $A_{5.25}: [e_4, e_{10}] = 2ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = 2be_7,$ $[e_8, e_9] = e_7,\ [e_8, e_{10}] = be_8 + e_9,\ [e_9, e_{10}] = be_9 - e_8\ (a \neq 0)$
- $A_{5.28}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_8, e_9] = e_7,$ $[e_7, e_{10}] = (a+1)e_7,\ [e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = e_4 + e_9$
- $A_{5.29}: [e_7, e_{10}] = e_7,\ [e_8, e_9] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = e_4$

**12.4.9.** $NR = H \oplus \mathbb{R}^3, S \ltimes NR = L_{6.2} \oplus \mathbb{R}^3$, **$R$-rep.** $D_0 \oplus D_{\frac{1}{2}} \oplus 3D_0$, **$R$-const.** $\{e_4, e_7, e_8, e_9, e_{10}\}$

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_2, e_3] = e_1,$$
$$[e_2, e_6] = e_5,\ [e_3, e_5] = e_6,\ [e_5, e_6] = e_4$$

- $A_{5.7}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = be_8,$ $[e_9, e_{10}] = ce_9,\ (abc \neq 0, -1 \leq c \leq b \leq a \leq 1)$
- $A_{5.9}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_4 + 2e_7,$ $[e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = be_9\ (0 \neq b \leq a)$
- $A_{5.11}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_4 + 2e_7,$ $[e_8, e_{10}] = e_7 + 2e_8,\ [e_9, e_{10}] = ae_9,\ (a \neq 0)$
- $A_{5.12}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_4 + 2e_7,$ $[e_8, e_{10}] = e_7 + 2e_8,\ [e_9, e_{10}] = e_8 + 2e_9$
- $A_{5.13}: [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = ae_7, [e_8, e_{10}] = be_8 - ce_9,\ [e_9, e_{10}] = ce_9 + be_8,$ $(ac \neq 0)$
- $A_{5.15}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_4 + 2e_7,$ $[e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = ae_9 + e_8\ (|a| \leq 1)$
- $A_{5.16}: [e_4, e_{10}] = 2e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_4 + 2e_7,$ $[e_8, e_{10}] = ae_8 - be_9,\ [e_9, e_{10}] = ae_9 + be_8,\ (b \neq 0)$

**12.4.10.** $NR = H \oplus \mathbb{R}^3, S \ltimes NR = L_{5.1} \oplus H \oplus \mathbb{R}$, **$R$-rep.** $3D_0 \oplus D_{\frac{1}{2}} \oplus D_0$, **$R$-const.** $\{e_4, e_5, e_6, e_9, e_{10}\}$

$$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,$$
$$[e_2, e_8] = e_7,\ [e_3, e_7] = e_8,\ [e_5, e_6] = e_4$$

- $A_{5.19}: [e_4, e_{10}] = (a+b)e_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = be_6,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ce_9,\ (c \neq 0)$
- $A_{5.20}: [e_4, e_{10}] = (a+b)e_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = be_6,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = (a+b)e_9 + e_4$
- $A_{5.21}: [e_4, e_{10}] = 2ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + e_6,\ [e_6, e_{10}] = ae_6 + e_7,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ae_9\ (a \neq 0)$
- $A_{5.22}: [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_6,\ [e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,$ $[e_9, e_{10}] = ae_9\ (a \neq 0)$

- $A_{5.23}: [e_4, e_{10}] = 2ae_4, [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + e_6,\ [e_6, e_{10}] = ae_6,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = be_9\ (b \neq 0)$
- $A_{5.24}: [e_4, e_{10}] = 2ae_4, [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + e_6,\ [e_6, e_{10}] = ae_6,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = 2ae_9 + \epsilon e_4$
- $A_{5.25}: [e_4, e_{10}] = 2ae_4, [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + be_6,\ [e_6, e_{10}] = ae_6 - be_5,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ce_9\ (c \neq 0)$
- $A_{5.26}: [e_4, e_{10}] = 2ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5 + be_6,\ [e_6, e_{10}] = ae_6 - be_5,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = 2ae_9 \pm e_4$
- $A_{5.27}: [e_4, e_{10}] = ae_4,\ [e_5, e_6] = e_4,\ [e_6, e_{10}] = ae_6 + e_9,\ [e_7, e_{10}] = e_7,$ $[e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ae_9 + e_4]\ (a \neq 0)$
- $A_{5.28}: [e_4, e_{10}] = (a + b)e_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = be_6 + e_9,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = be_9,\ (ab \neq 0)$
- $A_{5.29}: [e_4, e_{10}] = ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = e_9,\ [e_7, e_{10}] = e_7,$ $[e_8, e_{10}] = e_8\ (a \neq 0)$

**12.4.11.** $NR = H \oplus \mathbb{R}^3$, $S \rtimes NR = L_{6.4} \oplus H$, $R$**-rep.** $3D_0 \oplus D_1$, $R$**-const.** $\{e_4, e_5, e_6, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_7] = 2e_7,\ [e_1, e_9] = -2e_9,\ [e_2, e_3] = e_1,$

$[e_2, e_8] = 2e_7,\ [e_2, e_9] = e_8,\ [e_3, e_7] = e_8,\ [e_3, e_8] = 2e_9,\ [e_5, e_6] = e_4$

- $A_{4.7}: [e_4, e_{10}] = 2ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = e_5 + ae_6,\ (a \neq 0)$
- $A_{4.8/9}: [e_4, e_{10}] = (a + b)e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = be_6],\ (b \neq 0)$
- $A_{4.10/11}: [e_4, e_{10}] = 2ae_4, [e_5, e_{10}] = ae_5 - be_6,\ [e_6, e_{10}] = be_5 + ae_6,\ (b \neq 0).$

**12.4.12.** $NR = H \oplus \mathbb{R}^3, S \rtimes NR = L^*_{8.23} \oplus \mathbb{R}$, $R$**-rep.** $2D_0 \oplus 2D_{\frac{1}{2}}$, $R$**-const.** $\{e_4, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,$

$[e_1, e_7] = e_7,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,$

$[e_2, e_8] = e_7,\ [e_3, e_5] = e_6,\ [e_3, e_7] = e_8$

- $A_{2.1} \oplus \mathbb{R}$ *or* $A_{3,5}: [e_4, e_{10}] = 2ae_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,$ $[e_7, e_{10}] = be_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = 2e_9\ ((a-1)(a-b) \neq 0)$
- $A_{3.2}$ *or* $A_{3,3}: [e_4, e_{10}] = 2e_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6,$ $[e_7, e_{10}] = be_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = \epsilon e_4 + 2e_9\ (b \neq 1,\ \epsilon = 0, 1)$
- $A_{3.2}$ *or* $A_{3,3}: [e_4, e_{10}] = 2e_4,\ [e_5, e_6] = e_4,\ [e_5, e_{10}] = e_5 + \delta de_7,\ [e_6, e_{10}] = e_6 + \delta e_8,$ $[e_7, e_{10}] = e_7,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = \epsilon e_4 + 2e_9\ (\delta = 0,\ 1,\ \epsilon = 0, 1).$

**12.4.13.** $NR = H \oplus \mathbb{R}^3, S \rtimes NR = L_{9.58}$, $R$**-rep.** $D_0 \oplus D_{\frac{1}{2}} \oplus D_1$, $R$**-const.** $\{e_4, e_{10}\}$

$\mathbb{R}^2$ *or* $A_{2.1}: [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,$

$[e_1, e_7] = 2e_7,\ [e_1, e_9] = -2e_9,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,\ [e_2, e_8] = 2e_7,$

$[e_2, e_9] = e_8,\ [e_3, e_5] = e_6,\ [e_3, e_7] = e_8,\ [e_3, e_8] = 2e_9,\ [e_4, e_{10}] = 2ae_4,$

$[e_5, e_6] = e_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,\ [e_7, e_{10}] = be_7,\ [e_8, e_{10}] = be_8,$

$[e_9, e_{10}] = be_9\ (a = 1\ \ or\ \ b = 1).$

**12.4.14.** $NR = \mathbb{R}^6$, $S \rtimes NR = L_{9.59}$, $R$**-rep.** $D_{\frac{5}{2}}$, $R$**-const.** $\{e_{10}\}$

$\mathbb{R}: [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 5e_4,\ [e_1, e_5] = 3e_5,$

$[e_1, e_6] = e_6,\ [e_1, e_7] = -e_7,\ [e_1, e_8] = -3e_8,\ [e_1, e_9] = -5e_9,\ [e_2, e_3] = e_1,$

$[e_2,e_5]=5e_4,\ [e_2,e_6]=4e_5,\ [e_2,e_7]=3e_6,\ [e_2,e_8]=2e_7,\ [e_2,e_9]=e_8,$
$[e_3,e_4]=e_5,\ [e_3,e_5]=2e_6,\ [e_3,e_6]=3e_7,\ [e_3,e_7]=4e_8,\ [e_3,e_8]=5e_9,$
$[e_4,e_{10}]=e_4,\ [e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=e_7,\ [e_8,e_{10}]=e_8,\ [e_9,e_{10}]=e_9$

**12.4.15.** $NR=\mathbb{R}^6, S\rtimes NR=L_{9.61}$, **$R$-rep.** $2D_1$, **$R$-const.** $\{e_{10}\}$
$\mathbb{R}:[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=2e_4,\ [e_1,e_6]=-2e_6,\ [e_1,e_7]=2e_7,$
$[e_1,e_9]=-2e_9,\ [e_2,e_3]=e_1,\ [e_2,e_5]=2e_4,\ [e_2,e_6]=e_5,\ [e_2,e_8]=2e_7,$
$[e_2,e_9]=e_8,\ [e_3,e_4]=e_5,\ [e_3,e_5]=2e_6,\ [e_3,e_7]=e_8,\ [e_3,e_8]=2e_9,$
$[e_4,e_{10}]=ae_4+ce_7,\ [e_5,e_{10}]=ae_5+ce_8,\ [e_6,e_{10}]=ae_6+ce_9,$
$[e_7,e_{10}]=be_4+de_7,\ [e_8,e_{10}]=be_5+de_8,\ [e_9,e_{10}]=be_6+de_9$
$(a=1,\ b=c=0$ *or* $d=a,\ b=1,\ c=-1,$ *or* $a=b=d=1,\ c=0)$

**12.4.16.** $NR=\mathbb{R}^6, S\rtimes NR=L_{8.21}\oplus\mathbb{R}$, **$R$-rep.** $D_2\oplus D_0$, **$R$-const.** $\{e_9,e_{10}\}$
$A_{2,1}:[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=4e_4,\ [e_1,e_5]=2e_5,\ [e_1,e_7]=-2e_7,$
$[e_1,e_8]=-4e_8,\ [e_2,e_3]=e_1,\ [e_2,e_5]=4e_4,\ [e_2,e_6]=3e_5,\ [e_2,e_7]=2e_6,$
$[e_2,e_8]=e_7,\ [e_3,e_4]=e_5,\ [e_3,e_5]=2e_6,\ [e_3,e_6]=3e_7,\ [e_3,e_7]=4e_8,$
$[e_4,e_{10}]=e_4,\ [e_5,e_{10}]=e_5,\ [e_6,e_{10}]=e_6,\ [e_7,e_{10}]=e_7,\ [e_8,e_{10}]=e_8,$
$[e_9,e_{10}]=ae_9\ (a\neq 0)$

**12.4.17.** $NR=\mathbb{R}^6, S\rtimes NR=L_{8.22}\oplus\mathbb{R}$, **$R$-rep.** $D_1\oplus D_{\frac{1}{2}}\oplus D_0$, **$R$-const.** $\{e_9,e_{10}\}$
$A_{2,1}:[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=2e_4,\ [e_1,e_6]=-2e_6,\ [e_1,e_7]=e_7,$
$[e_1,e_8]=-e_8,\ [e_2,e_3]=e_1,\ [e_2,e_5]=2e_4,\ [e_2,e_6]=e_5,\ [e_2,e_8]=e_7,$
$[e_3,e_4]=e_5,\ [e_3,e_5]=2e_6,\ [e_3,e_7]=e_8,\ [e_4,e_{10}]=ae_4,\ [e_5,e_{10}]=ae_5,$
$[e_6,e_{10}]=ae_6,\ [e_7,e_{10}]=be_7,\ [e_8,e_{10}]=be_8,\ [e_9,e_{10}]=e_9,\ (a^2+b^2\neq 0)$

**12.4.18.** $NR=\mathbb{R}^6,\ S\rtimes NR=L_{9.63}$, **$R$-rep.** $3D_{\frac{1}{2}}$, **$R$-const.** $\{e_{10}\}$
$[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=e_4,\ [e_1,e_5]=-e_5,\ [e_1,e_6]=e_6,$
$[e_1,e_7]=-e_7,\ [e_1,e_8]=e_8,\ [e_1,e_9]=-e_9,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_4,$
$[e_2,e_7]=e_6,\ [e_2,e_9]=e_8,\ [e_3,e_4]=e_5,\ [e_3,e_6]=e_7,\ [e_3,e_8]=e_9,$
$[e_4,e_{10}]=ae_4+be_6+ce_8,\ [e_5,e_{10}]=ae_5+be_7+ce_9,$
$[e_6,e_{10}]=de_4+ee_6+e_8f,\ [e_7,e_{10}]=de_5+ee_7+fe_9,$
$[e_8,e_{10}]=ge_4+he_6+ie_8,\ [e_9,e_{10}]=ge_5+he_7+ie_9$

- $\mathbb{R}:a=1,\ b=c=d=f=g=h=0$
- $\mathbb{R}:a=d=e=h=i=1,\ b=c=f=g=0$
- $\mathbb{R}:a=e,\ d=1,\ i=b,\ b=c=f=g=h=0$
- $\mathbb{R}:a=a,\ b=0,\ e=i=b,\ f=-1,h=1,\ c=d=g=0$

**12.4.19.** $NR=\mathbb{R}^6, S\rtimes NR=L_{7.7}\oplus\mathbb{R}^2$, **$R$-rep.** $2D_{\frac{1}{2}}\oplus 2D_0$, **$R$-const.** $\{e_8,e_9,e_{10}\}$
$[e_1,e_2]=2e_2,\ [e_1,e_3]=-2e_3,\ [e_1,e_4]=e_4,\ [e_1,e_5]=-e_5,\ [e_1,e_6]=e_6,$
$[e_1,e_7]=-e_7,\ [e_2,e_3]=e_1,\ [e_2,e_5]=e_4,\ [e_2,e_7]=e_6,\ [e_3,e_4]=e_5,$
$[e_3,e_6]=e_7,\ [e_4,e_{10}]=ae_4+be_6,\ [e_5,e_{10}]=ae_5+be_7,\ [e_6,e_{10}]=ce_4+de_6,$
$[e_7,e_{10}]=ce_5+de_7,\ [e_8,e_{10}]=ee_8+fe_9,\ [e_9,e_{10}]=ge_8+he_9$

- $A_{3.5} : e = 1, h \neq 0, b = c = f = g = 0$
- $A_{3.5} : c = 1, b = 0, d = a, f = 0, g = 0,\ (eh \neq 0)$
- $A_{3.5} : c = 1, b = -1, d = a, f = 0, g = 0,\ (eh \neq 0)$
- $A_{3.1}$ or $A_{3.2} : a = 1, b = 0, c = 0, f = 0, g = 1, h = e$
- $A_{3.1}$ or $A_{3.2} : c = 1, b = 0, d = a, f = 0, g = 1, h = e\,(a = 1$ or $e = 1)$
- $A_{3.1}$ or $A_{3.2} : c = 1, b = -1, d = a, f = 0, g = 1, h = e\,(a = 1$ or $e = 1)$
- $A_{3.6}$ or $A_{3.7} : c = 0, b = 0, h = e, g = 1, f = -1, (a = 1$ or $d = 1)$
- $A_{3.6}$ or $A_{3.7} : b = 0, c = 1, d = a, f = -1, g = 1$
- $A_{3.6}$ or $A_{3.7} : b = -c, d = a, h = e, f = -1, g = 1, (c \neq 0)$

**12.4.20.** $NR = \mathbb{R}^6, S \ltimes NR = L_{9.60}$, $R$**-rep.** $D_{\frac{3}{2}} \oplus D_{\frac{1}{2}}$, $R$**-const.** $\{e_{10}\}$

$\mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 3e_4,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,$
$[e_1, e_7] = -3e_7,\ [e_1, e_8] = e_8,\ [e_1, e_9] = -e_9,\ [e_2, e_3] = e_1,\ [e_2, e_5] = 3e_4,$
$[e_2, e_6] = 2e_5,\ [e_2, e_7] = e_6,\ [e_2, e_9] = e_8,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,$
$[e_3, e_6] = 3e_7,\ [e_3, e_8] = e_9,\ [e_4, e_{10}] = ae_4,\ [e_5, e_{10}] = ae_5,\ [e_6, e_{10}] = ae_6,$
$[e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = be_9,\ (a = 1\ or\ b = 1).$

**12.4.21.** $NR = \mathbb{R}^6$, $S \ltimes NR = L_{8.20} \oplus \mathbb{R}$, $R$**-rep.** $D_{\frac{3}{2}} \oplus 2D_0$, $R$**-const.** $\{e_8, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = 3e_4,\ [e_1, e_5] = e_5,$
$[e_1, e_6] = -e_6,\ [e_1, e_7] = -3e_7,\ [e_2, e_5] = 3e_4,\ [e_2, e_6] = 2e_5,\ [e_2, e_7] = e_6,$
$[e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_3, e_6] = 3e_7,\ [e_4, e_{10}] = e_4,\ [e_5, e_{10}] = e_5,$
$[e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_7.$

- $A_{3.1} : [e_9, e_{10}] = ae_8,\ (a \neq 0)$
- $A_{3.2} : [e_8, e_{10}] = ae_8, [e_9, e_{10}] = e_8 + be_9,\ (ab \neq 0)$
- $A_{3.3/4/5} : [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ae_9,\ (a \neq 0)$
- $A_{3.6/7} : [e_8, e_{10}] = ae_8 + be_9,\ [e_9, e_{10}] = -be_8 + ae_9,\ (ab \neq 0)$.

**12.4.22.** $NR = \mathbb{R}^6$, $S \ltimes NR = L_{7.5} \oplus \mathbb{R}^2$, $R$**-rep.** $D_1 \oplus 3D_0$,
$R$**-const.** $\{e_7, e_8, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6,$
$[e_2, e_5] = 2e_4,\ [e_2, e_6] = e_5,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_4, e_{10}] = e_4,$
$[e_5, e_{10}] = e_5,\ [e_6, e_{10}] = e_6.$

- $A_{4.1} : [e_8, e_{10}] = ae_7,\ [e_9, e_{10}] = ae_8\ (a \neq 0)$
- $A_{4.2} : [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = e_8 + be_9\ (ab \neq 0)$
- $A_{4.3} : [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = ae_9,\ (a \neq 0)$
- $A_{4.4} : [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = e_7 + ae_8, [e_9, e_{10}] = e_8 + ae_9\ (a \neq 0)$
- $A_{4.5} : [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = ce_9\ (abc \neq 0)$
- $A_{4.6} : [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = be_8 - ce_9,\ [e_9, e_{10}] = ce_8 + be_9\ (ac \neq 0)$

**12.4.23.** $NR = \mathbb{R}^6$, $S \rtimes NR = L_{6.3} \oplus \mathbb{R}^3$, $R$**-representation** $D_{\frac{1}{2}} \oplus 4D_0$, $R$**-const.** $\{e_6, e_7, e_8, e_9, e_{10}\}$

$[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_2, e_3] = e_1,$
$[e_2, e_5] = e_4,\ [e_3, e_4] = e_5,\ [e_4, e_{10}] = ae_4,\ [e_5, e_{10}] = ae_5,\ (a \neq 0)$

- $A_{5.7} : [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = ae_7,\ [e_8, e_{10}] = be_8,\ [e_9, e_{10}] = ce_9,$
  $(abc \neq 0,\ -1 \leq c \leq b \leq a \leq 1)$
- $A_{5.8} : [e_7, e_{10}] = e_6,\ [e_8, e_{10}] = e_8,\ [e_9, e_{10}] = ae_9\ (0 < |a| \leq 1)$
- $A_{5.9} : [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6+e_7,\ [e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = be_9\ (0 \neq b \leq a)$
- $A_{5.10} : [e_7, e_{10}] = e_6,\ [e_8, e_{10}] = e_7,\ [e_9, e_{10}] = e_9$
- $A_{5.11} : [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6+e_7,\ [e_8, e_{10}] = e_7+e_8,\ [e_9, e_{10}] = ae_9\ (a \neq 0)$
- $A_{5.12} : [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6 + e_7,\ [e_8, e_{10}] = e_7 + e_8,\ [e_9, e_{10}] = e_8 + e_9$
- $A_{5.13} : [e_1, e_5] = e_1,\ [e_2, e_5] = ae_2,\ [e_3, e_5] = be_3 - ce_4,\ [e_4, e_5] = be_4 + ce_3$
  $(ac \neq 0,\ |a| \leq 1)$
- $A_{5.14} : [e_7, e_{10}] = e_6,\ [e_8, e_{10}] = ae_8 - e_9,\ [e_9, e_{10}] = ae_9 + e_8$
- $A_{5.15} : [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6 + e_7,\ [e_8, e_{10}] = ae_8,\ [e_9, e_{10}] = ae_9 + e_8$
  $(|a| \leq 1)$
- $A_{5.16} : [e_6, e_{10}] = e_6,\ [e_7, e_{10}] = e_6 + e_7,\ [e_8, e_{10}] = ae_8 - be_9,$
  $[e_9, e_{10}] = ae_9 + be_8\ (b \neq 0)$
- $A_{5.17} : [e_6, e_{10}] = ae_6 - e_7,\ [e_7, e_{10}] = ae_7 + e_6,\ [e_8, e_{10}] = be_8 - ce_9,$
  $[e_9, e_{10}] = ce_9 + be_8\ (c \neq 0)$
- $A_{5.18} : [e_6, e_{10}] = ae_6 - e_7,\ [e_7, e_{10}] = ae_7 + e_6,\ [e_8, e_{10}] = ae_8 + e_6 - e_9,$
  $[e_9, e_{10}] = ae_9 + e_7 + e_8,\ (a \geq 0).$

## 12.5. NR seven-dimensional indecomposable

**12.5.1.** $NR = 7.17$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 5D_0$, $R$**-const.** $\{e_6, e_7, e_8, e_9, e_{10}\}$

$A_{5.4} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_2, e_3] = e_1,$
$[e_2, e_5] = e_4,\ [e_3, e_4] = e_5,\ [e_4, e_5] = e_{10},\ [e_6, e_7] = e_{10},\ [e_8, e_9] = e_{10}.$

**12.5.2.** $NR = 7.17$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 3D_0$, $R$**-const.** $\{e_6, e_9, e_{10}\}$

$A_{3.1} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_7] = -e_7,$
$[e_1, e_8] = e_8,\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,\ [e_2, e_7] = -e_8,\ [e_3, e_4] = e_5,\ [e_3, e_8] = -e_7,$
$[e_4, e_7] = e_{10},\ [e_5, e_8] = e_{10},\ [e_6, e_9] = e_{10}$

**12.5.3.** $NR = 7.17$, $R$**-rep.** $3D_{\frac{1}{2}} \oplus D_0$, $R$**-const.** $\{e_{10}\}$

$\mathbb{R} : [e_1, e_2]=2e_2,\ [e_1, e_3]=-2e_3,\ [e_1, e_4]=e_4,\ [e_1, e_5]=e_5,\ [e_1, e_6]=e_6,\ [e_1, e_7]=-e_7,$
$[e_1, e_8] = -e_8,\ [e_1, e_9] = -e_9,\ [e_2, e_3] = e_1,\ [e_2, e_7] = e_4,\ [e_2, e_8] = e_5,\ [e_2, e_9] = e_6,$
$[e_3, e_4] = e_7,\ [e_3, e_5] = e_8,\ [e_3, e_6] = e_9,\ [e_4, e_7] = e_{10},\ [e_5, e_8] = e_{10},\ [e_6, e_9] = e_{10}$

**12.5.4.** $NR = 7.17$, $R$**-rep.** $2D_1 \oplus D_0$, $R$**-const.** $\{e_{10}\}$

$\mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6,\ [e_1, e_7] = -2e_7,$
$[e_1, e_9] = 2e_9,\ [e_2, e_3] = e_1,\ [e_2, e_5] = 2e_4,\ [e_2, e_6] = e_5,\ [e_2, e_7] = -2e_8,$
$[e_2, e_8] = -e_9,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,\ [e_3, e_8] = -e_7,\ [e_3, e_9] = -2e_8,$
$[e_4, e_7] = e_{10},\ [e_5, e_8] = e_{10},\ [e_6, e_9] = e_{10}$

**12.5.5.** $NR = 7.17, R$-**rep.** $D_{\frac{3}{2}} \oplus 3D_0, R$-**const.** $\{e_6, e_9, e_{10}\}$
$A_{3.1} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -3e_5,\ [e_1, e_7] = -e_7,$
$[e_1, e_8] = 3e_8,\ [e_2, e_3] = e_1,\ [e_2, e_4] = 3e_8,\ [e_2, e_5] = 3e_7,\ [e_2, e_7] = 2e_4,\ [e_3, e_4] = 2e_7,$
$[e_3, e_7] = e_5,\ [e_3, e_8] = e_4,\ [e_4, e_7] = e_{10},\ [e_5, e_8] = e_{10},\ [e_6, e_9] = e_{10}$

**12.5.6.** $NR = 7.17, R$-**rep.** $D_{\frac{3}{2}} \oplus D_{\frac{1}{2}} \oplus D_0, R$-**const.** $\{e_{10}\}$
$\mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -3e_5,\ [e_1, e_6] = e_6,$
$[e_1, e_7] = -e_7,\ [e_1, e_8] = 3e_8,\ [e_1, e_9] = -e_9,\ [e_2, e_3] = e_1,\ [e_2, e_4] = 3e_8,$
$[e_2, e_5] = 3e_7,\ [e_2, e_7] = 2e_4,\ [e_2, e_9] = e_6,\ [e_3, e_4] = 2e_7,\ [e_3, e_6] = e_9,$
$[e_3, e_7] = e_5,\ [e_3, e_8] = e_4,\ [e_4, e_7] = e_{10},\ [e_5, e_8] = e_{10},\ [e_6, e_9] = e_{10}$

**12.5.7.** $NR = 7.17, R$-**rep.** $D_{\frac{5}{2}} \oplus D_0, R$-**const.,** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -3e_5,\ [e_1, e_6] = 5e_6,$
$[e_1, e_7] = -e_7,\ [e_1, e_8] = 3e_8,\ [e_1, e_9] = -5e_9,\ [e_2, e_3] = e_1,\ [e_2, e_4] = 2e_8,\ [e_2, e_5] = 2e_7,$
$[e_2, e_7] = 3e_4,\ [e_2, e_8] = 5e_6,\ [e_2, e_9] = 5e_5,\ [e_3, e_4] = 3e_7,\ [e_3, e_5] = e_9,\ [e_3, e_6] = e_8,$
$[e_3, e_7] = 4e_5,\ [e_3, e_8] = 4e_4,\ [e_4, e_7] = e_{10},\ [e_5, e_8] = e_{10},\ [e_6, e_9] = e_{10}$

**12.5.8.** $NR = 7.37A, R$-**rep.** $2D_{\frac{1}{2}} \oplus 3D_0, R$-**const.** $\{e_4, e_7, e_{10}\}$
$H = A_{3.1} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_1, e_8] = e_8,$
$[e_1, e_9] = -e_9,\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,\ [e_2, e_9] = e_8,\ [e_3, e_5] = e_6,\ [e_3, e_8] = e_9,$
$[e_4, e_5] = e_8,\ [e_4, e_6] = e_9,\ [e_4, e_7] = e_{10}$

**12.5.9.** $NR = 7.37A, R$-**rep.** $2D_1 \oplus D_0, R$-**const.** $\{e_4\}$
$\mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = 2e_5,\ [e_1, e_7] = -2e_7,\ [e_1, e_8] = 2e_8,$
$[e_1, e_{10}] = -2e_{10},\ [e_2, e_3] = e_1,\ [e_2, e_6] = 2e_5,\ [e_2, e_7] = e_6,\ [e_2, e_9] = 2e_8,$
$[e_2, e_{10}] = e_9,\ [e_3, e_5] = e_6,\ [e_3, e_6] = 2e_7,\ [e_3, e_8] = e_9,\ [e_3, e_9] = 2e_{10},$
$[e_4, e_5] = e_8,\ [e_4, e_6] = e_9,\ [e_4, e_7] = e_{10}$

**12.5.10.** $NR = 7.37D, R$-**rep.** $2D_{\frac{1}{2}} \oplus 3D_0, R$-**const.** $\{e_8, e_9, e_{10}\}$
$\mathbb{R}^3 : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = -e_4,\ [e_1, e_5] = e_5,\ [e_1, e_6] = e_6,$
$[e_1, e_7] = -e_7,\ [e_2, e_3] = e_1,\ [e_2, e_4] = e_6,\ [e_2, e_7] = -e_5,\ [e_3, e_5] = -e_7,$
$[e_3, e_6] = e_4,\ [e_4, e_5] = e_9,\ [e_4, e_6] = e_8,\ [e_5, e_7] = e_{10},\ [e_6, e_7] = e_9]$

**12.5.11.** $NR = 7.37D, R$-**rep.** $2D_{\frac{1}{2}} \oplus D_1, R$-**const.** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_6] = e_6,$
$[e_1, e_7] = -e_7,\ [e_1, e_8] = 2e_8,\ [e_1, e_{10}] = -2e_{10},\ [e_2, e_3] = e_1, [e_2, e_5] = e_6,$
$[e_2, e_7] = -e_4,\ [e_2, e_9] = e_8,\ [e_2, e_{10}] = 2e_9,\ [e_3, e_4] = -e_7,\ [e_3, e_6] = e_5,$
$[e_3, e_8] = 2e_9,\ [e_3, e_9] = e_{10},\ [e_4, e_5] = e_9,\ [e_4, e_6] = e_8,\ [e_5, e_7] = e_{10},\ [e_6, e_7] = e_9$

**12.5.12.** $NR = 7.27A, R$-**rep.** $D_{\frac{1}{2}} \oplus 5D_0, R$-**const.** $\{e_4, e_5, e_7, e_9, e_{10}\}$
$A_{5.1} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_6] = e_6,\ [e_1, e_8] = -e_8,\ [e_2, e_3] = e_1,$
$[e_2, e_8] = e_6,\ [e_3, e_6] = e_8,\ [e_4, e_5] = e_9,\ [e_4, e_7] = e_{10},\ [e_6, e_8] = e_{10}$

**12.5.13.** $NR = 7.27B, R$-**rep.** $2D_{\frac{1}{2}} \oplus D_1, R$-**const.** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_1, e_6] = 2e_6,$
$[e_1, e_8] = -2e_8,\ [e_1, e_9] = e_9,\ [e_1, e_{10}] = -e_{10},\ [e_2, e_3] = e_1,\ [e_2, e_5] = e_4,$
$[e_2, e_7] = 2e_6,\ [e_2, e_8] = e_7,\ [e_2, e_{10}] = e_9,\ [e_3, e_4] = e_5,\ [e_3, e_6] = e_7,\ [e_3, e_7] = 2e_8,$
$[e_3, e_9] = e_{10},\ [e_4, e_7] = e_9,\ [e_4, e_8] = e_{10},\ [e_5, e_6] = -e_9,\ [e_5, e_7] = -e_{10}$

**12.5.14.** $NR = 7.37C,\ R$-**rep.** $2\,D_{\frac{1}{2}} \oplus 3D_0, R$-**const.** $\{e_4, e_5, e_8\}$
$A_{3.1} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_6] = e_6,\ [e_1, e_7] = -e_7,\ [e_1, e_9] = e_9,$

$[e_1, e_{10}] = -e_{10}$, $[e_2, e_3] = e_1$, $[e_2, e_7] = e_6$, $[e_2, e_{10}] = e_9$, $[e_3, e_6] = e_7$, $[e_3, e_9] = e_{10}$, $[e_4, e_5] = e_8$, $[e_5, e_6] = e_9$, $[e_5, e_7] = e_{10}$, $[e_6, e_7] = e_8$

**12.5.15.** $NR = 7.157$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 5D_0$, $R$**-const.** $\{e_4, e_5, e_6, e_7, e_{10}\}$
$A_{5.5} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_8] = e_8$, $[e_1, e_9] = -e_9$, $[e_2, e_3] = e_1$, $[e_2, e_9] = e_8$, $[e_3, e_8] = e_9$, $[e_4, e_5] = e_6$, $[e_4, e_6] = e_{10}$, $[e_5, e_7] = e_{10}$, $[e_8, e_9] = e_{10}$

**12.5.16.** $NR = 7.247A$, $R$**-rep.** $3D_{\frac{1}{2}} \oplus D_0$, $R$**-const.** $\{e_4\}$
$\mathbb{R} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_5] = e_5$, $[e_1, e_6] = -e_6$, $[e_1, e_7] = e_7$, $[e_1, e_8] = -e_8$, $[e_1, e_9] = e_9$, $[e_1, e_{10}] = -e_{10}$, $[e_2, e_3] = e_1$, $[e_2, e_6] = e_5$, $[e_2, e_8] = e_7$, $[e_2, e_{10}] = e_9$, $[e_3, e_5] = e_6$, $[e_3, e_7] = e_8$, $[e_3, e_9] = e_{10}$, $[e_4, e_5] = e_7$, $[e_4, e_6] = e_8$, $[e_4, e_7] = e_9$, $[e_4, e_8] = e_{10}$

**12.5.17.** $NR = 7.257K$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 5D_0$, $R$**-const.** $\{e_4, e_5, e_6, e_9, e_{10}\}$
$A_{5.3} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_7] = e_7$, $[e_1, e_8] = -e_8$, $[e_2, e_3] = e_1$, $[e_2, e_8] = e_7$, $[e_3, e_7] = e_8$, $[e_4, e_5] = e_6$, $[e_4, e_6] = e_{10}$, $[e_5, e_6] = e_9$, $[e_7, e_8] = 2e_{10}$

**12.5.18.** $NR = 7.1457A$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 5D_0$, $R$**-const.** $\{e_4, e_5, e_6, e_7, e_{10}\}$
$A_{5.3} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_8] = e_8$, $[e_1, e_9] = -e_9$, $[e_2, e_3] = e_1$, $[e_2, e_9] = e_8$, $[e_3, e_8] = e_9$, $[e_4, e_5] = e_6$, $[e_4, e_6] = e_7$, $[e_4, e_7] = e_{10}$, $[e_8, e_9] = 2e_{10}$

**12.5.19.** $NR = 7.1457B$, $R$**-rep.** $D_{\frac{1}{2}} \oplus 5D_0$, $R$**-const.** $\{e_4, e_5, e_6, e_7, e_{10}\}$
$A_{5.6} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_8] = e_8$, $[e_1, e_9] = -e_9$, $[e_2, e_3] = e_1$, $[e_2, e_9] = e_8$, $[e_3, e_8] = e_9$, $[e_4, e_5] = e_6$, $[e_4, e_6] = e_7$, $[e_4, e_7] = 2e_{10}$, $[e_5, e_6] = 2e_{10}$, $[e_8, e_9] = 2e_{10}$

### 12.6. NR seven-dimensional decomposable

**12.6.1.** $NR = H \oplus \mathbb{R}^4$, $R$**-rep.** $3D_{\frac{1}{2}} \oplus D_0$, $R$**-const.** $\{e_4\}$
$\mathbb{R} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_5] = e_5$, $[e_1, e_6] = -e_6$, $[e_1, e_7] = e_7$, $[e_1, e_8] = -e_8$, $[e_1, e_9] = e_9$, $[e_1, e_{10}] = -e_{10}$, $[e_2, e_3] = e_1$, $[e_2, e_6] = e_5$, $[e_2, e_8] = e_7$, $[e_2, e_{10}] = e_9$, $[e_3, e_5] = e_6$, $[e_3, e_7] = e_8$, $[e_3, e_9] = e_{10}$, $[e_5, e_6] = e_4$

**12.6.2.** $NR = H \oplus \mathbb{R}^4$, $R$**-rep.** $D_{\frac{1}{2}} \oplus D_{\frac{3}{2}} \oplus D_0$, $R$**-const.** $\{e_4\}$
$\mathbb{R} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_5] = e_5$, $[e_1, e_6] = -e_6$, $[e_1, e_7] = 3e_7$, $[e_1, e_8] = e_8$, $[e_1, e_9] = -e_9$, $[e_1, e_{10}] = -3e_{10}$, $[e_2, e_3] = e_1$, $[e_2, e_6] = e_5$, $[e_2, e_8] = 3e_7$, $[e_2, e_9] = 2e_8$, $[e_2, e_{10}] = e_9$, $[e_3, e_5] = e_6$, $[e_3, e_7] = e_8$, $[e_3, e_8] = 2e_9$, $[e_3, e_9] = 3e_{10}$, $[e_5, e_6] = e_4$

**12.6.3.** $NR = A_{5,1} \oplus \mathbb{R}^2$, $R$**-rep.** $3D_{\frac{1}{2}} \oplus D_0$, $R$**-const.** $\{e_8\}$
$\mathbb{R} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_1, e_6] = e_6$, $[e_1, e_7] = -e_7$, $[e_1, e_9] = e_9$, $[e_1, e_{10}] = -e_{10}$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_2, e_7] = e_6$, $[e_2, e_{10}] = e_9$, $[e_3, e_4] = e_5$, $[e_3, e_6] = e_7$, $[e_3, e_9] = e_{10}$, $[e_6, e_8] = e_4$, $[e_7, e_8] = e_5$.

**12.6.4.** $NR = A_{5,3} \oplus \mathbb{R}^2$, $R$**-rep.** $3D_{\frac{1}{2}}$, $R$**-const.** $\{e_6\}$
$\mathbb{R} : [e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_1, e_4] = e_4$, $[e_1, e_5] = -e_5$, $[e_1, e_7] = -e_7$, $[e_1, e_8] = e_8$, $[e_1, e_9] = e_9$, $[e_1, e_{10}] = -e_{10}$, $[e_2, e_3] = e_1$, $[e_2, e_5] = e_4$, $[e_2, e_7] = e_8$, $[e_2, e_{10}] = e_9$, $[e_3, e_4] = e_5$, $[e_3, e_8] = e_7$, $[e_3, e_9] = e_{10}$, $[e_6, e_7] = e_5$, $[e_6, e_8] = e_4$, $[e_7, e_8] = e_6$

**12.6.5.** $NR = A_{5,4} \oplus \mathbb{R}^2$, $R$**-rep.** $3D_{\frac{1}{2}} \oplus D_0$, $R$**-const.** $\{e_4\}$
$\mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -e_6,\ [e_1, e_7] = -e_7,$
$[e_1, e_8] = e_8,\ [e_1, e_9] = e_9,\ [e_1, e_{10}] = -e_{10},\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_5,\ [e_2, e_7] = -e_8,$
$[e_2, e_{10}] = e_9,\ [e_3, e_5] = e_6,\ [e_3, e_8] = -e_7,\ [e_3, e_9] = e_{10},\ [e_5, e_7] = e_4,\ [e_6, e_8] = e_4.$

**12.6.6.** $A_{5,4} \oplus \mathbb{R}^2$, $R$**-rep.** $D_{\frac{3}{2}} \oplus D_{\frac{1}{2}} \oplus D_0$, $R$**-const.** $\{e_8\}$
$\mathbb{R} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 3e_4,\ [e_1, e_5] = e_5,\ [e_1, e_6] = -3e_6,$
$[e_1, e_7] = -e_7,\ [e_2, e_3] = e_1,\ [e_2, e_5] = 3e_4,\ [e_2, e_6] = e_7,\ [e_2, e_7] = 2e_5,\ [e_3, e_4] = e_5,$
$[e_3, e_5] = 2e_7,\ [e_3, e_7] = 3e_6,\ [e_4, e_6] = e_8,\ [e_5, e_7] = -3e_8,\ [e_1, e_9] = e_9,$
$[e_1, e_{10}] = -e_{10},\ [e_2, e_{10}] = e_9,\ [e_3, e_9] = e_{10}$

**12.6.7.** $A_{5,4} \oplus \mathbb{R}^2$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus 3D_0$, $R$**-const.** $\{e_5, e_7, e_8\}$
$A_{3,1} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_6] = -e_6,\ [e_1, e_9] = e_9,$
$[e_1, e_{10}] = -e_{10},\ [e_2, e_3] = e_1,\ [e_2, e_6] = e_4,\ [e_2, e_{10}] = e_9,\ [e_3, e_4] = e_6,\ [e_3, e_9] = e_{10},$
$[e_4, e_6] = e_8,\ [e_5, e_7] = e_8$

## 12.7. $NR = \mathbb{R}^7$

**12.7.1.** $NR = \mathbb{R}^7$, $R$**-rep.** $D_3$, $R$**-const.** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = 6e_4,\ [e_1, e_5] = 4e_5,$
$[e_1, e_6] = 2e_6,\ [e_1, e_8] = -2e_8,\ [e_1, e_9] = -4e_9,\ [e_1, e_{10}] = -6e_{10},\ [e_2, e_5] = 6e_4,$
$[e_2, e_6] = 5e_5,\ [e_2, e_7] = 4e_6,\ [e_2, e_8] = 3e_7,\ [e_2, e_9] = 2e_8,\ [e_2, e_{10}] = e_9,\ [e_3, e_4] = e_5,$
$[e_3, e_5] = 2e_6,\ [e_3, e_6] = 3e_7,\ [e_3, e_7] = 4e_8,\ [e_3, e_8] = 5e_9,\ [e_3, e_9] = 6e_{10}$

**12.7.2.** $NR = \mathbb{R}^7$, $R$**-rep.** $2D_{\frac{1}{2}} \oplus D_1$, $R$**-const.** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,$
$[e_2, e_5] = e_4,\ [e_3, e_4] = e_5,\ [e_1, e_6] = 2e_6,\ [e_1, e_8] = -2e_8,\ [e_2, e_7] = 2e_6,\ [e_2, e_8] = e_7,$
$[e_3, e_6] = e_7,\ [e_3, e_7] = 2e_8,\ [e_1, e_9] = e_9,\ [e_1, e_{10}] = -e_{10},\ [e_2, e_{10}] = e_9,\ [e_3, e_9] = e_{10}$

**12.7.3.** $NR = \mathbb{R}^7$, $R$**-rep.** $D_{\frac{1}{2}} \oplus D_2$, $R$**-const.** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = -e_5,\ [e_2, e_5] = e_4,$
$[e_3, e_4] = e_5,\ [e_1, e_6] = 4e_6,\ [e_1, e_7] = 2e_7,\ [e_1, e_9] = -2e_9,\ [e_1, e_{10}] = -4e_{10},$
$[e_2, e_3] = e_1,\ [e_2, e_7] = 4e_6,\ [e_2, e_8] = 3e_7,\ [e_2, e_9] = 2e_8,\ [e_2, e_{10}] = e_9,\ [e_3, e_6] = e_7,$
$[e_3, e_7] = 2e_8,\ [e_3, e_8] = 3e_9,\ [e_3, e_9] = 4e_{10}$

**12.7.4.** $NR = \mathbb{R}^7$, $R$**-rep.** $D_1 \oplus D_{\frac{3}{2}}$, $R$**-const.** $\{0\}$
$\{0\} : [e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_1, e_4] = 2e_4,\ [e_1, e_6] = -2e_6,\ [e_1, e_7] = 3e_7,$
$[e_1, e_8] = e_8,\ [e_1, e_9] = -e_9,\ [e_1, e_{10}] = -3e_{10},\ [e_2, e_3] = e_1,\ [e_2, e_5] = 2e_4,$
$[e_2, e_6] = e_5,\ [e_2, e_8] = 3e_7,\ [e_2, e_9] = 2e_8,\ [e_2, e_{10}] = e_9,\ [e_3, e_4] = e_5,\ [e_3, e_5] = 2e_6,$
$[e_3, e_7] = e_8,\ [e_3, e_8] = 2e_9,\ [e_3, e_9] = 3e_{10}$

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