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Harmonic Vector Fields on 4-Dimensional Lorentzian Oscillator Groups

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Abstract. We first give the Milnor-type theorem on the 4-dimensional Lorentzian oscillator group. Then we study harmonic sections and the critical point for the energy functional restricted to vector fields of the same length.

Mathematics Subject Classification: 22E25, 53C30, 53C50.

Key Words: Harmonic vector fields, harmonic sections, oscillator group.

1. Introduction

Oscillator groups are the important solvable Lie groups, which have many applications both in geometry and physics. For example, it was shown that, apart from direct extensions with Euclidean groups, oscillator groups are the only non-commutative simple connected solvable Lie groups that admit a bi-invariant Lorentzian metric (see [19]). And, the reductive pairs determined by the homogeneous Lorentzian structures on the four-dimensional oscillator group, equipped with a bi-invariant Lorentzian metric, provide four solutions to the Einstein-Yang-Mills equations (see [8], [18]). Boucetta and Medina investigated the solutions of the generalized classical Yang-Baxter equation and the classical Yang-Baxter equation on a broad class of oscillator Lie algebras [2]. Moreover, all the homogeneous pseudo-Riemannian structures on oscillator groups, equipped with a specific family of left-invariant Lorentzian metrics, were determined in [9]. Besides, the corresponding reductive decompositions and groups of isometries were identified in the four-dimensional case. Onda studied algebraic Ricci solitons on oscillator groups endowed with left-invariant pseudo-Riemannian metrics in [22]. It was known that there exist exactly three left-invariant Lorentzian metrics up to scaling and automorphisms on the three dimensional Heisenberg group. In [14], left-invariant Lorentzian metrics are classified on the direct product of three dimensional Heisenberg group and $n-3$ $(n \geq 4)$ dimensional Euclidean space.

In the Riemannian case, it was proved that parallel vector fields are the only ones that define harmonic maps from a compact Riemannian manifold (M, g) to the tangent bundle (TM, g^s), where g^s denotes the Sasaki metric on TM (see [13] [21]). Furthermore, The critical points of the restricted energy functional $E|_{\mathfrak{X}(M)}$

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are also parallel vector fields [10]. However, in the Lorentzian case, vector fields satisfying harmonicity properties are not necessarily parallel ([4] [3]). Additionally, it is known that a Riemannian manifold admitting a parallel vector field must be locally reducible. This property also holds for a pseudo-Riemannian manifold admitting a parallel vector field that is either space-like or time-like. Recently, harmonic properties of vector fields were investigated on the oscillator groups with neutral signature. However, due to the complexity of metric Lie algebras of four dimensional oscillator groups with neutral signature, there are only three families of metrics, which are considered to analyse the harmonicity of vector fields (see [28]).

Our aim in this paper is to classify 4-dimensional Lorentzian oscillator group, and investigate the existence of the harmonic section and the critical point for the energy functional restricted to vector fields of the same length. This paper is organized as follows: In section 2, we present some basic facts on harmonic properties of vector fields. In section 3, we obtain the Milnor-type theorem on the 4-dimensional Lorentzian oscillator group. In section 4, we study the existence of the harmonic section and the critical point for the energy functional restricted to vector fields of the same length on the 4-dimensional Lorentzian oscillator group.

2. Preliminaries

Let (M, g) be a compact, connected, and oriented pseudo-Riemannian manifold of dimension n. The tangent bundle TM of M can be endowed with the Sasaki metric g^s, as described in [3]. Now, consider a smooth vector field V on M, we can associate with V a corresponding smooth map $V : (M, g) \to (TM, g^s)$. The energy of V is defined as the energy of this smooth map,

$$E(V) = \frac{n}{2}\mathrm{vol}(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 dv. \tag{1}$$

In the non-compact case, the concept of a harmonic map is defined as follows: $V : (M, g) \to (TM, g^s)$ is a harmonic map if it is a critical point for the energy functional. The Euler-Lagrange equations provide a characterization of vector fields V which defines harmonic maps, i.e. the tension field $\tau(V) = \mathrm{tr}(\nabla^2 V)$ must vanish. In [3], it is shown that a smooth vector field V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$\mathrm{tr}[R(\nabla_. V, V).)] = 0 \quad \text{and} \quad \nabla^*\nabla V = 0, \tag{2}$$

where with respect to a pseudo-orthonormal local frame $\{e_1, \ldots, e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, we have

$$\nabla^*\nabla V = \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V). \tag{3}$$

Now, we consider the vertical energy $E^v(V) = \frac{1}{2}\int_M \|\nabla V\|^2 dv$ of V. Then V is said to be a harmonic section if it is a critical point of the vertical energy E^v. The corresponding Euler-Lagrange equations are given by

$$\nabla^*\nabla V = 0. \tag{4}$$

When V is non-compact, the harmonic section is usually defined by the equation (4). And the definition of vector field defining a harmonic map is shown as the equation (2).

Assume ρ be a non-zero real number and $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : \|V\|^2 = \rho\}$. We consider the vector fields $V \in \mathfrak{X}^\rho(M)$. Assume that the vector fields V are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$ which is restricted to vector fields of the same length, the Euler-Lagrange equations of this variational condition show that V is a harmonic vector field if and only if

$$\nabla^*\nabla V \text{ is collinear to } V. \tag{5}$$

This characterization is well known in the Riemannian case (see [1], [26], [27]). If V is not light-like, the same argument applies to the pseudo-Riemannian case [3]. Even if V is a light-like vector field, (5) is still a sufficient condition for V to be a critical point for the energy functional $E|_{\mathfrak{X}^0(M)}$, restricted to light-like vector fields (see [3], Theorem 26). In the non-compact case, condition (5) is usually taken as the definition of critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$.

3. Milnor-type theorem on 4-dimensional Lorentzian oscillator group

In general, for every inner product $\langle\cdot,\cdot\rangle$ of signature (p,q) on low dimensional Lie algebra $\mathfrak{g}$, there exists a basis of $\mathfrak{g}$, which is pseudo-orthonormal with respect to $\langle\cdot,\cdot\rangle$ up to scalar, and the bracket relations among the elements of the basis can be written with relatively small number of parameters. In this section, we classify Lorentzian metrics on 4-dimensional oscillator groups.

The kind of theorem to classify left-invariant pseudo-Riemannian metrics on Lie groups is called a Milnor-type theorem in [12]. Milnor first obtained this kind of theorems for left-invariant Riemannian metrics on all three-dimensional unimodular Lie groups ([20]). Milnor-type theorems were also investigated for left-invariant Riemannian metrics on Lie groups with dimension less than five([6, 11, 15, 16]), and for left-invariant pseudo-Riemannian (Lorentzian) metrics on some Lie groups ([7, 17, 23, 24]).

Definition 3.1. Let G be a simply connected Lie group with the Lie algebra $\mathfrak{g}$ which is spanned by $(2n+2)$-elements:

$$Q, X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z.$$

Lie group G is called oscillator group if the non-vanishing Lie brackets on $\mathfrak{g}$ are shown as follows:

$$[X_i, Y_j] = \delta_{ij} Z, \quad [Q, X_j] = \lambda_j Y_j, \quad [Q, Y_j] = \lambda_j X_j, \qquad 1 \le i, j \le n. \tag{6}$$

Denote G by $G_n(\lambda) = G(\lambda_1, \ldots, \lambda_n)$.

Now, we give the following result, which is the Milnor-type theorem on 4-dimensional Lorentzian oscillator group.

Theorem 3.2. *For the four-dimensional oscillator group $(G_1(1), g)$, there exists a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that the non-vanishing Lie brackets are*

$$[e_1, e_2] = ae_2 + be_3 + ce_4, [e_1, e_3] = de_2 - ae_3 + fe_4, [e_2, e_3] = ke_4, \tag{7}$$

where $a^2 + bd \neq 0$, $k \neq 0$, and the matrix of the left-invariant metric g with respect to the above basis is of the following types:

$$(I)\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad (II)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}, \quad (III)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix},$$

$$(IV)\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad (V)\begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}, \quad (VI)\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Proof. From the definition 3.1, the oscillator group $G_1(1)$ is spanned by the four elements $\{Q, X, Y, Z\}$ with the Lie bracket $[X, Y] = Z, [Q, X] = Y, [Q, Y] = X$. The Lorentzian metric of the oscillator group is non-degenerate or degenerate on the derivative algebras spanned by $\{X, Y, Z\}$.

(1) The metric is non-degenerate on the derivative algebras spanned by $\{X, Y, Z\}$.

Case 1. The metric on the derivative algebras spanned by $\{X, Y, Z\}$ is Riemannian. Let $\{X, Y, Z\} = \{e_2, e_3, e_4\}$ is the orthonormal basis with $[e_2, e_3] = ke_4$ where $k \neq 0$. Then we take the orthogonal complement e_1 of the Lie algebra spanned by $\{e_2, e_3, e_4\}$. We know that e_1 is spanned by $\{Q, X, Y, Z\}$. It is easy to see that the following equation holds

$$ade_1(e_2, e_3, e_4) = (e_2, e_3, e_4)\begin{pmatrix} a & d & 0 \\ b & v & 0 \\ c & f & 0 \end{pmatrix}.$$

We can get $v = -a$ by the Jacobi identity $[e_1, [e_2, e_3]] + [e_2, [e_3, e_1]] + [e_3, [e_2, e_1]] = 0$. Thus we can get the equations:

$$[e_1, e_2] = ae_2 + be_3 + ce_4, [e_1, e_3] = de_2 - ae_3 + fe_4, [e_2, e_3] = ke_4,$$

where $a^2 + bd \neq 0, k \neq 0$. The Lorentzian metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (I).

Case 2. The metric on the derivative algebras spanned by $\{X, Y, Z\}$ is Lorentzian. We have the following results by the classification of three-dimensional unimodular Lorentzian Lie groups (see [5]).

(a) $\mathfrak{g}_3 : [e_2, e_3] = -\gamma e_4, [e_2, e_4] = 0, [e_3, e_4] = 0$. The frame field $\{e_2, e_3, e_4\}$ with e_4 time-like is pseudo-orthonormal. It is easy to see that there is the same Lie bracket as case 1 and the metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (III).

(b) $\mathfrak{g}_3 : [e_2, e_3] = 0, [e_2, e_4] = 0, [e_3, e_4] = \alpha e_2$. The frame field $\{e_2, e_3, e_4\}$ with e_4 time-like is pseudo-orthonormal. Rename e_3, e_4, e_2 to e_2, e_3, e_4, respectively. Then we can get the algebra $\mathfrak{g}$ has the same Lie bracket as (a) by direct calculation. And the metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (II).

(c) $\mathfrak{g}_4 : [e_2, e_3] = -e_3 + e_4, [e_2, e_4] = -e_3 + e_4, [e_3, e_4] = 0$. The frame field $\{e_2, e_3, e_4\}$ with e_4 time-like is pseudo-orthonormal.

Take $\frac{-e_3 - e_4}{\sqrt{2}}, \frac{-e_3 + e_4}{\sqrt{2}}$ and rename them e_3, e_4, respectively.

Then we can get the algebra $\mathfrak{g}$ has the same Lie bracket as (a) by direct calculation.

And the metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (IV).

(d) $\mathfrak{g}_4 : [e_2, e_3] = -e_3 - e_4, [e_2, e_4] = e_3 + e_4, [e_3, e_4] = 0$. The frame field $\{e_2, e_3, e_4\}$ with e_4 time-like is pseudo-orthonormal.

Take $\frac{e_3 - e_4}{\sqrt{2}}, \frac{e_3 + e_4}{\sqrt{2}}$ and rename them e_3, e_4, respectively.

Then we can get the algebra $\mathfrak{g}$ has the same Lie bracket as (a) by direct calculation. And the metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (IV).

(2) When the metric is degenerate on the derivative algebras spanned by $\{X, Y, Z\}$.

Case 1. The metric is degenerate with respect to e_4. Because the algebra $\mathfrak{g}$ is degenerate with respect to e_4, it is obvious that $g(e_4, e_4) = 0$. Then we can assume $\{e_1, e_4\}$ spans the Lorentzian subspace, and the orthogonal complement of the Lorentzian subspace is spanned by $\{e_2, e_3\}$. We can obtain the algebra $\mathfrak{g}$ has the same Lie bracket as (a) by direct calculation. And the metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (V).

Case 2. The metric is non-degenerate with respect to e_4. That is, $g(e_4, e_4) = 1$. Therefore, we can assume $\{e_1, e_2\}$ spans the Lorentzian subspace, and then the orthogonal complement of the Lorentzian subspace is spanned by $\{e_3, e_4\}$. We can get the algebra $\mathfrak{g}$ has the same Lie bracket as (a) by direct calculation. And the metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has the matrix of type (VI). ∎

Remark 3.3. In fact, we can make $c = 0$ in the Theorem 3.2 except type (V). The details can be found in section 4.

4. The harmonic vector of 4-dimensional Lorentzian oscillator group

Now, we study the harmonic properties of vector fields of 4-dimensional Lorentzian oscillator groups case by case.

4.1. The metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has a matrix of type (I).

By the equations (7) and $2\langle\nabla_X Y, Z\rangle = \langle[X, Y], Z\rangle - \langle[Y, Z], X\rangle + \langle[Z, X], Y\rangle$, we have the Levi-Civita connections of the 4-dimensional Lorentzian oscillator group $G_1(1)$ as follows:

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, && \nabla_{e_1}e_2 = \tfrac{1}{2}(b-d)e_3 + \tfrac{1}{2}ce_4,\\
&\nabla_{e_1}e_3 = \tfrac{1}{2}(-b+d)e_2 + \tfrac{1}{2}fe_4, && \nabla_{e_1}e_4 = -\tfrac{1}{2}ce_2 - \tfrac{1}{2}fe_3,\\
&\nabla_{e_2}e_1 = -ae_2 - \tfrac{1}{2}(b+d)e_3 - \tfrac{1}{2}ce_4, && \nabla_{e_2}e_2 = -ae_1,\\
&\nabla_{e_2}e_3 = -\tfrac{1}{2}(b+d)e_1 + \tfrac{1}{2}ke_4, && \nabla_{e_2}e_4 = -\tfrac{1}{2}ce_1 - \tfrac{1}{2}ke_3,\\
&\nabla_{e_3}e_1 = -\tfrac{1}{2}(b+d)e_2 + ae_3 - \tfrac{1}{2}fe_4, && \nabla_{e_3}e_2 = -\tfrac{1}{2}(b+d)e_1 - \tfrac{1}{2}ke_4,\\
&\nabla_{e_3}e_3 = ae_1, && \nabla_{e_3}e_4 = -\tfrac{1}{2}fe_1 + \tfrac{1}{2}ke_2,\\
&\nabla_{e_4}e_1 = -\tfrac{1}{2}ce_2 - \tfrac{1}{2}fe_3, && \nabla_{e_4}e_2 = -\tfrac{1}{2}ce_1 - \tfrac{1}{2}ke_3,\\
&\nabla_{e_4}e_3 = -\tfrac{1}{2}fe_1 + \tfrac{1}{2}ke_2, && \nabla_{e_4}e_4 = 0.
\end{aligned}$$

Let $V = xe_1 + ye_2 + ze_3 + te_4$. Then by the formula (3), we have

$$\begin{aligned}
\nabla^*\nabla V &= \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V) \\
&= \frac{1}{2}(4a^2x + b^2x + c^2x + 2bdx + d^2x + f^2x + fky - ckz)e_1 \\
&+\frac{1}{2}(act + dft - fkx + 2a^2y + b^2y + c^2y + d^2y - k^2y + cfz)e_2 \\
&+\frac{1}{2}(bct - aft + ckx + cfy + 2a^2z + b^2z + d^2z + f^2z - k^2z)e_3 \\
&+\frac{1}{2}(c^2t + f^2t - k^2t + acy + dfy + bcz - afz)e_4.
\end{aligned}$$

Thus, V is a harmonic section if and only if

$$\begin{cases}
4a^2x + b^2x + c^2x + 2bdx + d^2x + f^2x + fky - ckz = 0, \\
act + dft - fkx + 2a^2y + b^2y + c^2y + d^2y - k^2y + cfz = 0, \\
bct - aft + ckx + cfy + 2a^2z + b^2z + d^2z + f^2z - k^2z = 0, \\
c^2t + f^2t - k^2t + acy + dfy + bcz - afz = 0.
\end{cases} \tag{8}$$

First we can assume that $c = 0$ in the equation (7). In fact, let

$$e_1' = e_1 - me_3 - \frac{m^2}{2}e_2, \quad e_2' = e_2, \quad e_3' = e_3 + me_2, \quad e_4 = e_4',$$

we have $$[e_1', e_2'] = ae_2 + be_3 + ce_4 + mke_4 = a'e_2' + b'e_3' + c'e_4'.$$

Here $c' = c + mk$. It is obvious that there exists m which makes $c' = 0$. The system of equations (8) can be written to $M_1X = 0$ where $X = (t, x, y, z)^T$. Then the determinant $Det(M_1)$ of the coefficient matrix M_1 is:

$$\begin{aligned}
&- f^2(4a^2 + b^2 + 2bd + d^2 + f^2)(2a^4 + 3a^2b^2 + b^4 + a^2d^2 + b^2d^2 + 2a^2f^2 + b^2f^2) \\
&+ (16a^6 + 20a^4b^2 + 8a^2b^4 + b^6 + 8a^4bd + 8a^2b^3d + 2b^5d + 20a^4d^2 \\
&+ 16a^2b^2d^2 + 3b^4d^2 + 8a^2bd^3 + 4b^3d^3 + 8a^2d^4 + 3b^2d^4 + 2bd^5 + d^6 \\
&+ 24a^4f^2 + 21a^2b^2f^2 + 4b^4f^2 + 10a^2bdf^2 + 6b^3df^2 + 17a^2d^2f^2 \\
&+ 7b^2d^2f^2 + 4bd^3f^2 + 3d^4f^2 + 8a^2f^4 + 3b^2f^4 + 2bdf^4 + 2d^2f^4)k^2 \\
&+ (-16a^4 - 12a^2b^2 - 2b^4 - 8a^2bd - 4b^3d - 12a^2d^2 - 4b^2d^2 - 4bd^3 \\
&- 2d^4 - 10a^2f^2 - 3b^2f^2 - 4bdf^2 - 3d^2f^2)k^4 + (4a^2 + b^2 + 2bd + d^2)k^6 \\
&\triangleq\ a_0 + a_2k^2 + a_4k^4 + a_6k^6.
\end{aligned}$$

If $a_6 \neq 0$, we can see that, $Det(M_1) < 0$ when $k^2 = 0$, and $Det(M_1) = +\infty$ when $k^2 = +\infty$. Therefore, $\exists k$ s.t. $Det(M_1) = 0$, i.e. there is a non-trivial solution to the system of equations (8).

If $a_6 = 0$, i.e. $a = 0, d = -b$, we have

$$Det(M_1) = -b^2f^2(f^2 - 2k^2)(2b^2 + f^2 - k^2).$$

It is easy to see that there is a non-trivial solution to equations (8) if $f = 0$, or $f^2 - 2k^2 = 0$ or $2b^2 + f^2 - k^2 = 0$. Due to the above three cases, we have the following theorem.

Theorem 4.1. *For the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (I), then there exists a harmonic section on $G_1 = G_1(1)$ if $a \neq 0$ or $b+d \neq 0$. If $a = 0$ and $b+d = 0$, then V is a harmonic section if and only if*

(1) $V = xe_1$ *when* $f = 0$, *or*

(2) $V = -\frac{\sqrt{f^2}}{2\sqrt{2}b}te_1 + \frac{f}{2b}te_2 + te_4$ *when* $k = \sqrt{\frac{f^2}{2}}$, *or*

(3) $V = \frac{\sqrt{f^2}}{2\sqrt{2}b}te_1 + \frac{f}{2b}te_2 + te_4$ *when* $k = -\sqrt{\frac{f^2}{2}}$, *or*

(4) $V = ze_3$ *when* $k^2 = 2b^2 + f^2$.

Furthermore, we can obtain the result on the critical point for the energy functional restricted to the vector field of the same length.

Proposition 4.2. *Let V denote a left-invariant vector field on the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (I). For arbitrary real numbers a, b, d, f, there exists $k \neq 0$ such that G_1 has a critical point for the energy functional restricted to vector fields of the same length.*

Proof. If $G_1(1)$ is a critical point for the energy functional restricted to vector fields of the same length, we have the equations as follows.

$$\begin{cases} 4a^2x + b^2x + c^2x + 2bdx + d^2x + f^2x + fky - ckz = 2\lambda x, \\ act + dft - fkx + 2a^2y + b^2y + c^2y + d^2y - k^2y + cfz = 2\lambda y, \\ bct - aft + ckx + cfy + 2a^2z + b^2z + d^2z + f^2z - k^2z = 2\lambda z, \\ c^2t + f^2t - k^2t + acy + dfy + bcz - afz = 2\lambda t. \end{cases} \tag{9}$$

The system of equations (9) can be written to $M_1'X = 0$ where $X = (t, x, y, z)^T$. Then the determinant $Det(M_1')$ of the coefficient matrix M_1' is:

$$\begin{aligned}
&- f^2(4a^2 + b^2 + 2bd + d^2 + f^2)(2a^4 + 3a^2b^2 + b^4 + a^2d^2 + b^2d^2 + 2a^2f^2 + b^2f^2) \\
&+ (16a^6 + 20a^4b^2 + 8a^2b^4 + b^6 + 8a^4bd + 8a^2b^3d + 2b^5d + 20a^4d^2 \\
&+ 16a^2b^2d^2 + 3b^4d^2 + 8a^2bd^3 + 4b^3d^3 + 8a^2d^4 + 3b^2d^4 + 2bd^5 + d^6 \\
&+ 24a^4f^2 + 21a^2b^2f^2 + 4b^4f^2 + 10a^2bdf^2 + 6b^3df^2 + 17a^2d^2f^2 \\
&+ 7b^2d^2f^2 + 4bd^3f^2 + 3d^4f^2 + 8a^2f^4 + 3b^2f^4 + 2bdf^4 + 2d^2f^4)k^2 \\
&+ (-16a^4 - 12a^2b^2 - 2b^4 - 8a^2bd - 4b^3d - 12a^2d^2 - 4b^2d^2 - 4bd^3 \\
&- 2d^4 - 10a^2f^2 - 3b^2f^2 - 4bdf^2 - 3d^2f^2)k^4 + (4a^2 + b^2 + 2bd + d^2)k^6 \\
&+ (32a^6 + 40a^4b^2 + 16a^2b^4 + 2b^6 + 16a^4bd + 16a^2b^3d + 4b^5d + 40a^4d^2 \\
&+ 32a^2b^2d^2 + 6b^4d^2 + 16a^2bd^3 + 8b^3d^3 + 16a^2d^4 + 6b^2d^4 + 4bd^5 + 2d^6 \\
&+ 52a^4f^2 + 48a^2b^2f^2 + 10b^4f^2 + 20a^2bdf^2 + 12b^3df^2 + 36a^2d^2f^2 \\
&+ 16b^2d^2f^2 + 8bd^3f^2 + 6d^4f^2 + 22a^2f^4 + 10b^2f^4 + 4bdf^4 + 6d^2f^4 \\
&+ 2f^6 - 72a^4k^2 - 56a^2b^2k^2 - 10b^4k^2 - 32a^2bdk^2 - 16b^3dk^2 \\
&- 56a^2d^2k^2 - 20b^2d^2k^2 - 16bd^3k^2 - 10d^4k^2 - 54a^2f^2k^2 - 20b^2f^2k^2 \\
&- 16bdf^2k^2 - 18d^2f^2k^2 - 6f^4k^2 + 32a^2k^4 + 10b^2k^4 + 12bdk^4 + 10d^2k^4 + 6f^2k^4 - 2k^6)\lambda
\end{aligned}$$

$$
\begin{aligned}
&(-80a^4-64a^2b^2-12b^4-32a^2bd-16b^3d-64a^2d^2-24b^2d^2-16bd^3\\
&-12d^4-68a^2f^2-28b^2f^2-16bdf^2-24d^2f^2-12f^4+80a^2k^2\\
&+28b^2k^2+24bdk^2+28d^2k^2+24f^2k^2-12k^4)\lambda^2\\
&+(64a^2+24b^2+16bd+24d^2+24f^2-24k^2)\lambda^3-16\lambda^4\\
&\triangleq\ Det(M_1)+s_1\lambda+s_2\lambda^2+s_3\lambda^3+s_4\lambda^4.
\end{aligned}
$$

By the above discussion about harmonic section, it is easy to see that for arbitrary real numbers a,b,d,f, there is $k\neq 0$ such that $Det(M_1)>0$. Because $s_4=-16<0$, we can see that $\exists\lambda$, s.t. $Det(M_1')=0$, i.e. there is a non-trivial solution to the system of equations (9). ■

4.2. The metric g with respect to the basis $\{e_1,e_2,e_3,e_4\}$ has a matrix of type (II).

Levi-Civita connection of the 4-dimensional Lorentzian oscillator group $G_1(1)$ is given as follows:

$$
\begin{aligned}
&\nabla_{e_1}e_1=0, &&\nabla_{e_1}e_2=\tfrac{1}{2}(b+d)e_3+\tfrac{1}{2}ce_4,\\
&\nabla_{e_1}e_3=\tfrac{1}{2}(b+d)e_2+\tfrac{1}{2}fe_4, &&\nabla_{e_1}e_4=-\tfrac{1}{2}ce_2+\tfrac{1}{2}fe_3,\\
&\nabla_{e_2}e_1=-ae_2-\tfrac{1}{2}(b-d)e_3-\tfrac{1}{2}ce_4, &&\nabla_{e_2}e_2=ae_1,\\
&\nabla_{e_2}e_3=\tfrac{1}{2}(-b+d)e_1+\tfrac{1}{2}ke_4, &&\nabla_{e_2}e_4=\tfrac{1}{2}ce_1+\tfrac{1}{2}ke_3,\\
&\nabla_{e_3}e_1=\tfrac{1}{2}(b-d)e_2+ae_3-\tfrac{1}{2}fe_4, &&\nabla_{e_3}e_2=\tfrac{1}{2}(-b+d)e_1-\tfrac{1}{2}ke_4,\\
&\nabla_{e_3}e_3=ae_1, &&\nabla_{e_3}e_4=\tfrac{1}{2}fe_1+\tfrac{1}{2}ke_2,\\
&\nabla_{e_4}e_1=-\tfrac{1}{2}ce_2+\tfrac{1}{2}fe_3, &&\nabla_{e_4}e_2=\tfrac{1}{2}ce_1+\tfrac{1}{2}ke_3,\\
&\nabla_{e_4}e_3=\tfrac{1}{2}fe_1+\tfrac{1}{2}ke_2, &&\nabla_{e_4}e_4=0.
\end{aligned}
$$

Let $V=xe_1+ye_2+ze_3+te_4$, then by the formula (3) we can get the result.

$$
\begin{aligned}
\nabla^*\nabla V &= \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V-\nabla_{\nabla_{e_i}e_i}V)\\
&= \frac{1}{2}(-4a^2x+b^2x-c^2x-2bdx+d^2x+f^2x+fky+ckz)e_1\\
&\quad+\frac{1}{2}(-act+dft+fkx-2a^2y+b^2y-c^2y+d^2y+k^2y-cfz)e_2\\
&\quad+\frac{1}{2}(-bct-aft-ckx+cfy-2a^2z+b^2z+d^2z+f^2z+k^2z)e_3\\
&\quad+\frac{1}{2}(-c^2t+f^2t+k^2t-acy+dfy+bcz+afz)e_4.
\end{aligned}
$$

Thus, V is a harmonic section if and only if

$$
\begin{cases}
-4a^2x+b^2x-c^2x-2bdx+d^2x+f^2x+fky+ckz=0,\\
-act+dft+fkx-2a^2y+b^2y-c^2y+d^2y+k^2y-cfz=0,\\
-bct-aft-ckx+cfy-2a^2z+b^2z+d^2z+f^2z+k^2z=0,\\
-c^2t+f^2t+k^2t-acy+dfy+bcz+afz=0.
\end{cases}
\tag{10}
$$

It is the same as type (I) that we can get $c=0$ in the equation (7). The system of equations (10) can be written to $M_2X=0$ where $X=(t,x,y,z)^T$. Then the determinant $Det(M_2)$ of the coefficient matrix M_2 is:

$$
\begin{aligned}
&- (b^2+d^2+f^2+k^2)((b-d)^2(b^2f^2+b^2k^2+d^2k^2+f^2k^2+k^4)+b^2f^4+b^2f^2k^2+d^2f^2k^2)\\
&+ (7b^4f^2 - 6b^3df^2 + 8b^2d^2f^2 - 2bd^3f^2 + d^4f^2 + 9b^2f^4 - 4bdf^4 + 3d^2f^4 + 2f^6\\
&+ 8b^4k^2 - 8b^3dk^2 + 16b^2d^2k^2 - 8bd^3k^2 + 8d^4k^2 + 21b^2f^2k^2 - 10bdf^2k^2\\
&+ 17d^2f^2k^2 + 8f^4k^2 + 12b^2k^4 - 8bdk^4 + 12d^2k^4 + 10f^2k^4 + 4k^6)a^2\\
&(-14b^2f^2 + 4bdf^2 - 6d^2f^2 - 10f^4 - 20b^2k^2 + 8bdk^2 - 20d^2k^2\\
&- 24f^2k^2 - 16k^4)a^4 + (8f^2 + 16k^2)a^6\\
&\triangleq\ m_0 + m_2a^2 + m_4a^4 + m_6a^6.
\end{aligned}
$$

It is easy to see that $m_6 > 0$, $m_0 \leq 0$. When $m_0 < 0$, it is obvious that there exists a^2, s.t. $Det(M_2) = 0$. When $m_0 = 0$, we can get $b = d = 0$ or $b = d,\ f = 0$. If $b = d = 0$, we have

$$
\begin{aligned}
Det(M_2) &= 2a^2(a^2 - f^2)(4a^2 - f^2)f^2 + (16a^6 - 24a^4f^2 + 8a^2f^4)k^2\\
&\quad +(-16a^4 + 10a^2f^2)k^4 + 4a^2k^6\\
&\triangleq a_0 + a_2k^2 + a_4k^4 + a_6k^6.
\end{aligned}
$$

We know that there exists f^2, s.t. $a_0 < 0$. Therefore there exists f^2 satisfying $Det(M_2) = 0$, i.e. there is a non-trivial solution to the system of equations (10).

If $b = d,\ f = 0$, we have $Det(M_2) = 4a^2k^2(-2a^2 + 2b^2 + k^2)^2$.

It is easy to see that there is a non-trivial solution to the system of equations (10) if $k^2 = 2a^2 - 2b^2$ or $a = 0$. Based on the above two cases, we have the following theorem.

Theorem 4.3. *For the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (II), then there exists a harmonic section on $G_1 = G_1(1)$ if $b \neq d$, $b = d \neq 0$ or $f \neq 0$. If $b = d$ and $f = 0$, then V is a harmonic section if and only if*

(1) $V = ye_2 + ze_3$ *when* $k^2 = 2a^2 - 2b^2$, *or*

(2) $V = xe_1$ *when* $a = 0$.

Besides, we have the result about the critical point for the energy functional restricted to vector fields of the same length.

Proposition 4.4. *Let V denote a left-invariant vector field on the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (II). For arbitrary real numbers b, d, f and $k \neq 0$, there exists a such that G_1 has a critical point for the energy functional restricted to vector fields of the same length.*

Proof. If $G_1(1)$ is a critical point for the energy functional restricted to vector fields of the same length, we have the equations as follows.

$$
\begin{cases}
-4a^2x + b^2x - c^2x - 2bdx + d^2x + f^2x + fky + ckz = 2\lambda x,\\
-act + dft + fkx - 2a^2y + b^2y - c^2y + d^2y + k^2y - cfz = 2\lambda y,\\
-bct - aft - ckx + cfy - 2a^2z + b^2z + d^2z + f^2z + k^2z = 2\lambda z,\\
c^2t + f^2t + k^2t - acy + dfy + bcz + afz = 2\lambda t.
\end{cases}
\tag{11}
$$

The system of equations (11) can be written to $M_2'X = 0$ where $X = (t, x, y, z)^T$.

Then the determinant $Det(M_2')$ of the coefficient matrix M_2' is:

$$
\begin{aligned}
&-(b^2+d^2+f^2+k^2)((b-d)^2(b^2f^2+b^2k^2+d^2k^2+f^2k^2+k^4)\\
&+b^2f^4+b^2f^2k^2+d^2f^2k^2)\\
&+(7b^4f^2-6b^3df^2+8b^2d^2f^2-2bd^3f^2+d^4f^2+9b^2f^4-4bdf^4\\
&+3d^2f^4+2f^6+8b^4k^2-8b^3dk^2+16b^2d^2k^2-8bd^3k^2+8d^4k^2\\
&+21b^2f^2k^2-10bdf^2k^2+17d^2f^2k^2+8f^4k^2+12b^2k^4-8bdk^4\\
&+12d^2k^4+10f^2k^4+4k^6)a^2\\
&(-14b^2f^2+4bdf^2-6d^2f^2-10f^4-20b^2k^2+8bdk^2-20d^2k^2\\
&-24f^2k^2-16k^4)a^4+(8f^2+16k^2)a^6\\
&+(-32a^6+40a^4b^2-16a^2b^4+2b^6-16a^4bd+16a^2b^3d-4b^5d\\
&+40a^4d^2-32a^2b^2d^2+6b^4d^2+16a^2bd^3-8b^3d^3-16a^2d^4+6b^2d^4\\
&-4bd^5+2d^6+52a^4f^2-48a^2b^2f^2+10b^4f^2+20a^2bdf^2-12b^3df^2\\
&-36a^2d^2f^2+16b^2d^2f^2-8bd^3f^2+6d^4f^2-22a^2f^4+10b^2f^4\\
&-4bdf^4+6d^2f^4+2f^6+72a^4k^2-56a^2b^2k^2+10b^4k^2+32a^2bdk^2\\
&-16b^3dk^2-56a^2d^2k^2+20b^2d^2k^2-16bd^3k^2+10d^4k^2-54a^2f^2k^2\\
&+20b^2f^2k^2-16bdf^2k^2+18d^2f^2k^2+6f^4k^2-32a^2k^4+10b^2k^4\\
&-12bdk^4+10d^2k^4+6f^2k^4+2k^6)\lambda\\
&+(-80a^4+64a^2b^2-12b^4-32a^2bd+16b^3d+64a^2d^2-24b^2d^2\\
&+16bd^3-12d^4+68a^2f^2-28b^2f^2+16bdf^2-24d^2f^2-12f^4\\
&+80a^2k^2-28b^2k^2+24bdk^2-28d^2k^2-24f^2k^2-12k^4)\lambda^2\\
&+(-64a^2+24b^2-16bd+24d^2+24f^2+24k^2)\lambda^3-16\lambda^4\\
&\triangleq\ Det(M_2)+s_1\lambda+s_2\lambda^2+s_3\lambda^3+s_4\lambda^4.
\end{aligned}
$$

By the above discussion about harmonic section and the same as the discussion of type (I), we can know that for arbitrary real numbers b, d, f and $k \neq 0$, there is a such that $Det(M_2) > 0$. Because $s_4 = -16 < 0$, there exists λ such that $Det(M_2') = 0$, i.e. there is a non-trivial solution to the system of equations (11). ■

4.3. The metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has a matrix of type (III).

The Levi-Civita connections of the 4-dimensional Lorentzian oscillator group $G_1(1)$ are given as follows:

$$
\begin{aligned}
&\nabla_{e_1}e_1=0, && \nabla_{e_1}e_2=\tfrac{1}{2}(b-d)e_3+\tfrac{1}{2}ce_4,\\
&\nabla_{e_1}e_3=\tfrac{1}{2}(-b+d)e_2+\tfrac{1}{2}fe_4, && \nabla_{e_1}e_4=\tfrac{1}{2}ce_2+\tfrac{1}{2}fe_3,\\
&\nabla_{e_2}e_1=-ae_2-\tfrac{1}{2}(b+d)e_3-\tfrac{1}{2}ce_4, && \nabla_{e_2}e_2=ae_1,\\
&\nabla_{e_2}e_3=\tfrac{1}{2}(b+d)e_1+\tfrac{1}{2}ke_4, && \nabla_{e_2}e_4=-\tfrac{1}{2}ce_1+\tfrac{1}{2}ke_3,\\
&\nabla_{e_3}e_1=-\tfrac{1}{2}(b+d)e_2+ae_3-\tfrac{1}{2}fe_4, && \nabla_{e_3}e_2=\tfrac{1}{2}(b+d)e_1-\tfrac{1}{2}ke_4,\\
&\nabla_{e_3}e_3=-ae_1, && \nabla_{e_3}e_4=-\tfrac{1}{2}fe_1-\tfrac{1}{2}ke_2,\\
&\nabla_{e_4}e_1=\tfrac{1}{2}ce_2+\tfrac{1}{2}fe_3, && \nabla_{e_4}e_2=-\tfrac{1}{2}ce_1+\tfrac{1}{2}ke_3,\\
&\nabla_{e_4}e_3=-\tfrac{1}{2}fe_1-\tfrac{1}{2}ke_2, && \nabla_{e_4}e_4=0.
\end{aligned}
$$

Let $V = xe_1 + ye_2 + ze_3 + te_4$, then by the formula (3) we can get the result.

$$\begin{aligned}\nabla^*\nabla V &= \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V)\\ &= \frac{1}{2}(-4a^2x - b^2x + c^2x - 2bdx - d^2x + f^2x + fky - ckz)e_1\\ &\quad+\frac{1}{2}(act + dft + fkx - 2a^2y - b^2y + c^2y - d^2y + k^2y + cfz)e_2\\ &\quad+\frac{1}{2}(bct - aft - ckx + cfy - 2a^2z - b^2z - d^2z + f^2z + k^2z)e_3\\ &\quad+\frac{1}{2}(c^2t + f^2t + k^2t - acy - dfy - bcz + afz)e_4.\end{aligned}$$

Thus, V is a harmonic section if and only if

$$\begin{cases}-4a^2x - b^2x + c^2x - 2bdx - d^2x + f^2x + fky - ckz = 0,\\ act + dft + fkx - 2a^2y - b^2y + c^2y - d^2y + k^2y + cfz = 0,\\ bct - aft - ckx + cfy - 2a^2z - b^2z - d^2z + f^2z + k^2z = 0,\\ c^2t + f^2t + k^2t - acy - dfy - bcz + afz = 0.\end{cases} \tag{12}$$

It is the same as (I) that we can get $c = 0$ in the equation (7). The system of equations (12) can be written to $M_3X = 0$ where $X = (t, x, y, z)^T$. Then the determinant $Det(M_3)$ of the coefficient matrix M_3 is:

$$\begin{aligned}&f^2(-4a^2 - b^2 - 2bd - d^2 + f^2)(-2a^4 - 3a^2b^2 - b^4 - a^2d^2 - b^2d^2 + 2a^2f^2 + b^2f^2)\\ &+ (16a^6 + 20a^4b^2 + 8a^2b^4 + b^6 + 8a^4bd + 8a^2b^3d + 2b^5d + 20a^4d^2\\ &+ 16a^2b^2d^2 + 3b^4d^2 + 8a^2bd^3 + 4b^3d^3 + 8a^2d^4 + 3b^2d^4 + 2bd^5 + d^6\\ &- 24a^4f^2 - 21a^2b^2f^2 - 4b^4f^2 - 10a^2bdf^2 - 6b^3df^2 - 17a^2d^2f^2\\ &- 7b^2d^2f^2 - 4bd^3f^2 - 3d^4f^2 + 8a^2f^4 + 3b^2f^4 + 2bdf^4 + 2d^2f^4)k^2\\ &+ (-16a^4 - 12a^2b^2 - 2b^4 - 8a^2bd - 4b^3d - 12a^2d^2 - 4b^2d^2 - 4bd^3\\ &- 2d^4 + 10a^2f^2 + 3b^2f^2 + 4bdf^2 + 3d^2f^2)k^4 + (4a^2 + b^2 + 2bd + d^2)k^6\\ &\triangleq\ a_0 + a_2k^2 + a_4k^4 + a_6k^6.\end{aligned}$$

If $a_6 \neq 0$, it is easy to get that $a_0 = 0$ when

$$f^2 = 0,\ 4a^2 + b^2 + 2bd + d^2 \ \text{ or } \ \frac{2a^4 + 3a^2b^2 + b^4 + a^2d^2 + b^2d^2}{2a^2 + b^2}.$$

Therefore there exists $f^2 > 0$, s.t. $a_0 < 0$, i.e. there is a non-trivial solution to the system of equations (12). If $a_6 = 0$, i.e. $a = 0, d = -b$, we have

$$Det(M_3) = -b^2f^2(2b^2 - f^2 - k^2)(f^2 + 2k^2).$$

It is easy to see the conclusion that there is a non-trivial solution to the system of equations (12) if $f = 0$ or $2b^2 - f^2 - k^2 = 0$. Thus, we have the following theorem.

Theorem 4.5. *For the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (III), then there exists a harmonic section on $G_1 = G_1(1)$ if $a \neq 0$ or $b + d \neq 0$. If $a = 0$ and $b + d = 0$, then V is a harmonic section if and only if*

(1) $V = xe_1$ *when* $f = 0$, *or*

(2) $V = ze_3$ *when* $k^2 = 2b^2 - f^2$.

For the critical point for the energy functional restricted to vector fields of the same length, we have the result as follows.

Proposition 4.6. *Let V denote a left-invariant vector field on the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (III). For arbitrary real numbers a, b, d, f, there exists $k \neq 0$ such that G_1 has a critical point for the energy functional restricted to vector fields of the same length.*

Proof. If $G_1(1)$ is a critical point for the energy functional restricted to vector fields of the same length, we have the equations as follows.

$$\begin{cases} -4a^2x - b^2x + c^2x - 2bdx - d^2x + f^2x + fky - ckz = 2\lambda x, \\ act + dft + fkx - 2a^2y - b^2y + c^2y - d^2y + k^2y + cfz = 2\lambda y, \\ bct - aft - ckx + cfy - 2a^2z - b^2z - d^2z + f^2z + k^2z = 2\lambda z, \\ c^2t + f^2t + k^2t - acy - dfy - bcz + afz = 2\lambda t. \end{cases} \tag{13}$$

The system of equations (13) can be written to $M_3'X = 0$ where $X = (t, x, y, z)^T$. Then the determinant $Det(M_3')$ of the coefficient matrix M_3' is:

$$\begin{aligned}
& f^2(-4a^2 - b^2 - 2bd - d^2 + f^2)(-2a^4 - 3a^2b^2 - b^4 - a^2d^2 - b^2d^2 + 2a^2f^2 + b^2f^2) \\
& + (16a^6 + 20a^4b^2 + 8a^2b^4 + b^6 + 8a^4bd + 8a^2b^3d + 2b^5d + 20a^4d^2 \\
& + 16a^2b^2d^2 + 3b^4d^2 + 8a^2bd^3 + 4b^3d^3 + 8a^2d^4 + 3b^2d^4 + 2bd^5 + d^6 \\
& - 24a^4f^2 - 21a^2b^2f^2 - 4b^4f^2 - 10a^2bdf^2 - 6b^3df^2 - 17a^2d^2f^2 \\
& - 7b^2d^2f^2 - 4bd^3f^2 - 3d^4f^2 + 8a^2f^4 + 3b^2f^4 + 2bdf^4 + 2d^2f^4)k^2 \\
& + (-16a^4 - 12a^2b^2 - 2b^4 - 8a^2bd - 4b^3d - 12a^2d^2 - 4b^2d^2 - 4bd^3 \\
& - 2d^4 + 10a^2f^2 + 3b^2f^2 + 4bdf^2 + 3d^2f^2)k^4 + (4a^2 + b^2 + 2bd + d^2)k^6 \\
& (-32a^6 - 40a^4b^2 - 16a^2b^4 - 2b^6 - 16a^4bd - 16a^2b^3d - 4b^5d - 40a^4d^2 \\
& - 32a^2b^2d^2 - 6b^4d^2 - 16a^2bd^3 - 8b^3d^3 - 16a^2d^4 - 6b^2d^4 - 4bd^5 - 2d^6 \\
& + 52a^4f^2 + 48a^2b^2f^2 + 10b^4f^2 + 20a^2bdf^2 + 12b^3df^2 + 36a^2d^2f^2 \\
& + 16b^2d^2f^2 + 8bd^3f^2 + 6d^4f^2 - 22a^2f^4 - 10b^2f^4 - 4bdf^4 - 6d^2f^4 \\
& + 2f^6 + 72a^4k^2 + 56a^2b^2k^2 + 10b^4k^2 + 32a^2bdk^2 + 16b^3dk^2 \\
& + 56a^2d^2k^2 + 20b^2d^2k^2 + 16bd^3k^2 + 10d^4k^2 - 54a^2f^2k^2 - 20b^2f^2k^2 \\
& - 16bdf^2k^2 - 18d^2f^2k^2 + 6f^4k^2 - 32a^2k^4 - 10b^2k^4 - 12bdk^4 \\
& - 10d^2k^4 + 6f^2k^4 + 2k^6)\lambda \\
& + (-80a^4 - 64a^2b^2 - 12b^4 - 32a^2bd - 16b^3d - 64a^2d^2 - 24b^2d^2 - 16bd^3 \\
& - 12d^4 + 68a^2f^2 + 28b^2f^2 + 16bdf^2 + 24d^2f^2 - 12f^4 + 80a^2k^2 + 28b^2k^2 \\
& + 24bdk^2 + 28d^2k^2 - 24f^2k^2 - 12k^4)\lambda^2 \\
& + (-64a^2 - 24b^2 - 16bd - 24d^2 + 24f^2 + 24k^2)\lambda^3 - 16\lambda^4 \\
& \triangleq \; Det(M_3) + s_1\lambda + s_2\lambda^2 + s_3\lambda^3 + s_4\lambda^4.
\end{aligned}$$

By the above discussion about harmonic section and the same as the discussion of type (I), we can see that there exists λ such that $Det(M_3') = 0$, i.e. there is a non-trivial solution to the system of equations (13). ■

4.4. The metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has a matrix of type (IV).

The Levi-Civita connections of the 4-dimensional Lorentzian oscillator group $G_1(1)$ are given as follows:

$$\begin{array}{ll}
\nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = \frac{1}{2}be_3 + \frac{1}{2}(c-d)e_4, \\
\nabla_{e_1}e_3 = \frac{1}{2}(-c+d)e_2 - \frac{1}{2}ae_3, & \nabla_{e_1}e_4 = -\frac{1}{2}be_2 + \frac{1}{2}ae_4, \\
\nabla_{e_2}e_1 = -ae_2 - \frac{1}{2}be_3 - \frac{1}{2}(c+d)e_4, & \nabla_{e_2}e_2 = ae_1, \\
\nabla_{e_2}e_3 = \frac{1}{2}(c+d)e_1, & \nabla_{e_2}e_4 = \frac{1}{2}be_1, \\
\nabla_{e_3}e_1 = -\frac{1}{2}(c+d)e_2 + \frac{1}{2}ae_3 - fe_4, & \nabla_{e_3}e_2 = \frac{1}{2}(c+d)e_1 - ke_4, \\
\nabla_{e_3}e_3 = fe_1 + ke_2, & \nabla_{e_3}e_4 = -\frac{1}{2}ae_1, \\
\nabla_{e_4}e_1 = -\frac{1}{2}be_2 + \frac{1}{2}ae_4, & \nabla_{e_4}e_2 = \frac{1}{2}be_1, \\
\nabla_{e_4}e_3 = -\frac{1}{2}ae_1, & \nabla_{e_4}e_4 = 0.
\end{array}$$

Let $V = xe_1 + ye_2 + ze_3 + te_4$ and write

$$e_1' = e_1,\ e_2 = e_2',\ e_3' = \frac{e_3 + e_4}{\sqrt{2}},\ e_4' = \frac{e_3 - e_4}{\sqrt{2}}.$$

Then the metric g with respect to the basis $\{e_1', e_2', e_3', e_4'\}$ has the matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

By the bilinearity of $\nabla_{e_i}\nabla_{e_j}V$ and $\nabla_{\nabla_{e_i}e_j}V$, and (3), we have:

$$\begin{aligned}
\nabla^*\nabla V &= \sum_i \varepsilon_i(\nabla_{e_i'}\nabla_{e_i'}V - \nabla_{\nabla_{e_i'}e_i'}V) \\
&= (\nabla_{e_1'}\nabla_{e_1'}V - \nabla_{\nabla_{e_1'}e_1'}V) + (\nabla_{e_2'}\nabla_{e_2'}V - \nabla_{\nabla_{e_2'}e_2'}V) \\
&\quad +(\nabla_{e_3'}\nabla_{e_3'}V - \nabla_{\nabla_{e_3'}e_3'}V) - (\nabla_{e_4'}\nabla_{e_4'}V - \nabla_{\nabla_{e_4'}e_4'}V) \\
&= (\nabla_{e_1}\nabla_{e_1}V - \nabla_{\nabla_{e_1}e_1}V) + (\nabla_{e_2}\nabla_{e_2}V - \nabla_{\nabla_{e_2}e_2}V) \\
&\quad (\nabla_{e_3}\nabla_{e_4}V - \nabla_{\nabla_{e_3}e_4}V) + (\nabla_{e_4}\nabla_{e_3}V - \nabla_{\nabla_{e_4}e_3}V) \\
&= \sum_{i,j} \varepsilon_{ij}(\nabla_{e_i}\nabla_{e_j}V - \nabla_{\nabla_{e_i}e_j}V),
\end{aligned}$$

where $\varepsilon_{ij} = g(e_i, e_j)$. Furthermore,

$$\begin{aligned}
\nabla^*\nabla V &= \frac{1}{2}(-3a^2x - 2bcx - 2bdx + bkz)e_1 \\
&\quad +\frac{1}{2}(-abt - 2a^2y - 2bcy - adz - bfz)e_2 \\
&\quad +\frac{1}{2}(-b^2t - aby - bcz)e_3 \\
&\quad +\frac{1}{2}(-bct + bkx - ady - bfy - c^2z - d^2z + 2afz)e_4.
\end{aligned}$$

Thus, V is a harmonic section if and only if

$$\begin{cases} -3a^2x - 2bcx - 2bdx + bkz = 0, \\ -abt - 2a^2y - 2bcy - adz - bfz = 0, \\ -b^2t - aby - bcz = 0, \\ -bct + bkx - ady - bfy - c^2z - d^2z + 2afz = 0. \end{cases} \tag{14}$$

It is the same as (I) that we can get $c = 0$ in the equation (7). By the third equation in the system (14), we can see that b is possibly equal to 0. If $b = 0$, then by direct calculation, we can get the following solutions.

(1) $x = 0$, $y = -\frac{d}{2a}z$ if $f = \frac{d^2}{4a}$;

(2) $x = 0$, $y = 0$, $z = 0$, if $f \neq \frac{d^2}{4a}$.

If we have $b \neq 0$, the system of equations (14) can be written as $M_4X = 0$ where $X = (t, x, y, z)^T$. Then the determinant $Det(M_4)$ of the coefficient matrix M_4 is:

$$\begin{aligned} & - b^2(6a^5f + 10a^3dfb + (4ad^2f + 3a^2f^2 + a^2k^2)b^2 + 2b^3df^2) \\ \triangleq \ & -b^2(a_0 + a_1b + a_2b^2 + a_3b^3). \end{aligned}$$

By calculations, when $d \neq 0$, there exists f, s.t. $Det(M_4) = 0$, i.e. there is a non-trivial solution to the system of equations (14). And by direct calculations, when $d = 0$, we have the following non-trivial solutions.

(1) $x = -\frac{k}{3f}y$, $z = -\sqrt{\frac{a(3f^2+k^2)}{-6f}}y$, $t = -\frac{\sqrt{3f^2+k^2}}{\sqrt{-6af}}y$ if $b = \frac{\sqrt{-6a^3f}}{\sqrt{3f^2+k^2}}$;

(2) $x = -\frac{k}{3f}y$, $z = \sqrt{\frac{a(3f^2+k^2)}{-6f}}y$, $t = \frac{\sqrt{3f^2+k^2}}{\sqrt{-6af}}y$ if $b = -\frac{\sqrt{-6a^3f}}{\sqrt{3f^2+k^2}}$.

Theorem 4.7. *For the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (IV), then there exists a harmonic section on $G_1 = G_1(1)$ if $b \neq 0$ and $d \neq 0$. If $b = 0$ or $d = 0$, then V is a harmonic section if and only if*

(1) $V = -\frac{d}{2a}ze_2 + ze_3 + te_4$ *when* $b = 0$, $f = \frac{d^2}{4a}$, *or*

(2) $V = te_4$ *when* $b = 0$, $f \neq \frac{d^2}{4a}$, *or*

(3) $V = -\frac{k}{3f}ye_1 + ye_2 - \sqrt{\frac{a(3f^2+k^2)}{-6f}}ye_3 - \frac{\sqrt{3f^2+k^2}}{\sqrt{-6af}}ye_4$

when $d = 0$, $b = \frac{\sqrt{-6a^3f}}{\sqrt{3f^2+k^2}}$, *or*

(4) $V = -\frac{k}{3f}ye_1 + ye_2 + \sqrt{\frac{a(3f^2+k^2)}{-6f}}ye_3 + \frac{\sqrt{3f^2+k^2}}{\sqrt{-6af}}ye_4$

when $d = 0$, $b = -\frac{\sqrt{-6a^3f}}{\sqrt{3f^2+k^2}}$.

With respect to the critical point for the energy functional restricted to vector fields of the same length, we have the following:

Proposition 4.8. *Let V denote a left-invariant vector field on the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (IV). For arbitrary real numbers a, d, f and $k \neq 0$, there exists b such that G_1 has a critical point for the energy functional restricted to vector fields of the same length.*

Proof. If $G_1(1)$ is a critical point for the energy functional restricted to vector fields of the same length, we have the equations as follows.

$$\begin{cases} -3a^2x - 2bcx - 2bdx + bkz = 2\lambda x, \\ -abt - 2a^2y - 2bcy - adz - bfz = 2\lambda y, \\ -b^2t - aby - bcz = 2\lambda z, \\ -bct + bkx - ady - bfy - c^2z - d^2z + 2afz = 2\lambda t. \end{cases} \tag{15}$$

The system of equations (15) can be written to $M_4'X = 0$ where $X = (t, x, y, z)^T$. Then the determinant $Det(M_4')$ of the coefficient matrix M_4' is:

$$\begin{aligned} & b^2(6a^5f + 10a^3dfb + (4ad^2f + 3a^2f^2 + a^2k^2)b^2 + 2b^3df^2) \\ & + (12a^4bd + 14a^2b^2d^2 + 4b^3d^3 - 4a^3b^2f - 4ab^3df - 2b^4f^2 \\ & - 2b^4k^2)\lambda + (-24a^4 - 8a^2bd + 4b^2d^2)\lambda^2 \\ & + (-40a^2 - 16bd)\lambda^3 - 16\lambda^4 \\ & \triangleq \ Det(M_4) + s_1\lambda + s_2\lambda^2 + s_3\lambda^3 + s_4\lambda^4. \end{aligned}$$

It is easy to see that for arbitrary real numbers a, d, f and $k \neq 0$, there is b such that $Det(M_4) > 0$. Because $s_4 = -16 < 0$, we can see that there exists λ, s.t. $Det(M_4') = 0$, i.e. there is a non-trivial solution to the system of equations (15). ∎

4.5. The metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has a matrix of type (V).

The Levi-Civita connections of the 4-dimensional Lorentzian oscillator group $G_1(1)$ are given as follows:

$$\begin{aligned} &\nabla_{e_1}e_1 = -ce_2 - fe_3, && \nabla_{e_1}e_2 = \tfrac{1}{2}(b - d - k)e_3 + ce_4, \\ &\nabla_{e_1}e_3 = \tfrac{1}{2}(-b + d + k)e_2 + fe_4, && \nabla_{e_1}e_4 = 0, \\ &\nabla_{e_2}e_1 = -ae_2 - \tfrac{1}{2}(b + d + k)e_3, && \nabla_{e_2}e_2 = ae_4, \\ &\nabla_{e_2}e_3 = \tfrac{1}{2}(b + d + k)e_4, && \nabla_{e_2}e_4 = 0, \\ &\nabla_{e_3}e_1 = -\tfrac{1}{2}(b + d - k)e_2 + ae_3, && \nabla_{e_3}e_2 = \tfrac{1}{2}(b + d - k)e_4, \\ &\nabla_{e_3}e_3 = -ae_4, && \nabla_{e_3}e_4 = 0, \\ &\nabla_{e_4}e_1 = \nabla_{e_4}e_2 = 0, && \nabla_{e_4}e_3 = \nabla_{e_4}e_4 = 0. \end{aligned}$$

Let $V = xe_1 + ye_2 + ze_3 + te_4$ and write

$$e_1' = \frac{e_1 + e_4}{\sqrt{2}}, \ e_2' = e_2, \ e_3' = e_3 e_4' = \frac{e_1 - e_4}{\sqrt{2}}.$$

By calculation, the metric g with respect to the basis $\{e_1', e_2', e_3', e_4'\}$ has the matrix

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

It is the same as type (IV) that we can get the equation

$$\nabla^*\nabla V = \sum_{i,j} \varepsilon_{ij}(\nabla_{e_i}\nabla_{e_j}V - \nabla_{\nabla_{e_i}e_j}V),$$

where $\varepsilon_{ij} = g(e_i, e_j)$. Furthermore,

$$\nabla^*\nabla V = \left(-2a^2 - \frac{1}{2}k^2 - \frac{1}{2}(b+d)^2\right) xe_4.$$

Theorem 4.9. *Let V denote a left-invariant vector field on the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (V). Then V is a harmonic section if and only if $V = ye_2 + ze_3 + te_4$, $\forall y, z, t \in \mathbb{R}$;*

Remark 4.10. G_1 with the metric of type (V) has a critical point for the energy functional restricted to vector fields of the same length if and only if $\lambda = 0$, i.e. V is a harmonic section.

4.6. The metric g with respect to the basis $\{e_1, e_2, e_3, e_4\}$ has a matrix of type (VI).

The Levi-Civita connections of the 4-dimensional Lorentzian oscillator group $G_1(1)$ are given as follows:

$$\begin{array}{ll}
\nabla_{e_1}e_1 = -ae_1 - de_3, & \nabla_{e_1}e_2 = ae_2 + \frac{1}{2}be_3 + \frac{1}{2}ce_4, \\
\nabla_{e_1}e_3 = -\frac{1}{2}be_1 + de_2 + \frac{1}{2}fe_4, & \nabla_{e_1}e_4 = -\frac{1}{2}ce_1 - \frac{1}{2}fe_3, \\
\nabla_{e_2}e_1 = -\frac{1}{2}be_3 - \frac{1}{2}ce_4, & \nabla_{e_2}e_2 = 0, \\
\nabla_{e_2}e_3 = \frac{1}{2}be_2 + \frac{1}{2}ke_4, & \nabla_{e_2}e_4 = \frac{1}{2}ce_2 - \frac{1}{2}ke_3, \\
\nabla_{e_3}e_1 = -\frac{1}{2}be_1 + ae_3 - \frac{1}{2}fe_4, & \nabla_{e_3}e_2 = \frac{1}{2}be_2 - \frac{1}{2}ke_4, \\
\nabla_{e_3}e_3 = -ae_2, & \nabla_{e_3}e_4 = \frac{1}{2}ke_1 + \frac{1}{2}fe_2, \\
\nabla_{e_4}e_1 = -\frac{1}{2}ce_1 - \frac{1}{2}fe_3, & \nabla_{e_4}e_2 = \frac{1}{2}ce_2 - \frac{1}{2}ke_3, \\
\nabla_{e_4}e_3 = \frac{1}{2}ke_1 + \frac{1}{2}fe_2, & \nabla_{e_4}e_4 = 0.
\end{array}$$

Let $V = xe_1 + ye_2 + ze_3 + te_4$ and write

$$e_1' = \frac{e_1 + e_2}{\sqrt{2}},\ e_2' = \frac{e_1 - e_2}{\sqrt{2}},\ e_3' = e_3,\ e_4' = e_4.$$

By calculation, the metric g with respect to the basis $\{e_1', e_2', e_3', e_4'\}$ has the matrix

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

It is the same as type (IV) that we can get the equation

$$\nabla^*\nabla V = \sum_{i,j} \varepsilon_{ij}(\nabla_{e_i}\nabla_{e_j}V - \nabla_{\nabla_{e_i}e_j}V),$$

where $\varepsilon_{ij} = g(e_i, e_j)$.

Furthermore,

$$\begin{aligned}\nabla^*\nabla V &= \frac{1}{2}(b^2x + c^2x - fkx - k^2y - ckz)e_1\\ &+\frac{1}{2}(act - dkt - 2a^2x - 2bdx - f^2x + b^2y + c^2y - fky + cfz)e_2\\ &+\frac{1}{2}(bct + akt + cfx - cky + b^2z - 2fkz)e_3\\ &+\frac{1}{2}(c^2t - 2fkt + acx - dkx + bcz + akz)e_4.\end{aligned}$$

Thus, V is a harmonic section if and only if

$$\begin{cases} b^2x + c^2x - fkx - k^2y - ckz = 0\\ act - dkt - 2a^2x - 2bdx - f^2x + b^2y + c^2y - fky + cfz = 0\\ bct + akt + cfx - cky + b^2z - 2fkz = 0\\ c^2t - 2fkt + acx - dkx + bcz + akz = 0.\end{cases} \tag{16}$$

It is the same as (I) that we can get when $c = 0$ in equation (7). The system of equations (16) can be written as $M_6X = 0$ where $X = (x, y, z, t)^T$. Then the determinant $Det(M_6)$ of the coefficient matrix M_6 is:

$$\begin{aligned}&-2b^6fk + (-a^2b^4 + 8b^4f^2)k^2 + (6a^2b^2f + 4b^3df - 8b^2f^3)k^3\\ &+ (2a^4 + 2a^2bd - b^2d^2 - 8a^2f^2 - 8bdf^2)k^4 + 2d^2fk^5\\ &\triangleq\ k(a_0 + a_1k + a_2k^2 + a_3k^3 + a_4k^4).\end{aligned}$$

When $d, f \neq 0$, it is easy to get that $a_0a_4 \leq 0$. If $a_0a_4 = 0$, i.e. $b = 0$ we can obtain the following non-trivial solutions

$$y = \frac{d^2f^2}{a^4 - 4a^2f^2}x, \quad z = \frac{ad}{a^2 - 4f^2}x, \quad t = \frac{2df}{a^2 - 4f^2}x.$$

of equations (16) if $k = \dfrac{-a^4 + 4a^2f^2}{d^2f}$.

If $a_0a_4 < 0$, it is obvious that there exists $k \neq 0$, s.t. $Det(M_6) = 0$, i.e. there is a non-trivial solution to the system of equations (16). And by direct calculations, when $f = 0$, there are the following non-trivial solutions

(1) $y = \dfrac{2a^4 + 2a^2bd - b^2d^2}{a^2b^2}x, \ z = \dfrac{d}{a}x, \ t = \dfrac{d\sqrt{2a^4 + 2a^2bd - b^2d^2}}{\sqrt{a^6}}x$

if $k = -\dfrac{\sqrt{a^2}b^2}{\sqrt{2a^4 + 2a^2bd - b^2d^2}}$;

(2) $y = \dfrac{2a^4 + a^2bd - b^2d^2}{a^2b^2}x, \ z = \dfrac{d}{a}x, \ t = -\dfrac{d\sqrt{2a^4 + 2a^2bd - b^2d^2}}{\sqrt{2a^6}}x$

if $k = \dfrac{\sqrt{a^2}b^2}{\sqrt{2a^4 + 2a^2bd - b^2d^2}}$.

when $d = 0$, there are the following non-trivial solutions

(1) $y = \dfrac{2a^4 + a^2f^2 + a^2f\sqrt{2a^2 + f^2}}{b^2(a^2 + f^2 + f\sqrt{2a^2 + f^2})}x, \ z = 0, \ t = 0$ if $k = -\dfrac{b^2}{2a^2}(f + \sqrt{2a^2 + f^2})$;

(2) $y = \dfrac{2a^4 + a^2f^2 - a^2f\sqrt{2a^2 + f^2}}{b^2(a^2 + f^2 - f\sqrt{2a^2 + f^2})}x, \ z = 0, \ t = 0$ if $k = -\dfrac{b^2}{2a^2}(f - \sqrt{2a^2 + f^2})$.

Theorem 4.11. *For the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (VI), then there exists a harmonic section on $G_1 = G_1(1)$ if $b, d, f \neq 0$. If b, d or $f = 0$, then V is a harmonic section if and only if*

(1) $V = xe_1 + \dfrac{d^2 f^2}{a^4 - 4a^2 f^2} xe_2 + \dfrac{ad}{a^2 - 4f^2} xe_3 + \dfrac{2df}{a^2 - 4f^2} xe_4$

when $b = 0$, $k = \dfrac{-a^4 + 4a^2 f^2}{d^2 f}$, *or*

(2) $V = xe_1 + \dfrac{2a^4 + a^2 f^2 + a^2 f\sqrt{2a^2 + f^2}}{b^2(a^2 + f^2 + f\sqrt{2a^2 + f^2})} xe_2$

when $d = 0$, $k = -\dfrac{b^2}{2a^2}(f + \sqrt{2a^2 + f^2})$, *or*

(3) $V = xe_1 + \dfrac{2a^4 + a^2 f^2 - a^2 f\sqrt{2a^2 + f^2}}{b^2(a^2 + f^2 - f\sqrt{2a^2 + f^2})} xe_2$

when $d = 0$, $k = -\dfrac{b^2}{2a^2}(f - \sqrt{2a^2 + f^2})$, *or*

(4) $V = xe_1 + \dfrac{2a^4 + 2a^2 bd - b^2 d^2}{a^2 b^2} xe_2 + \dfrac{d}{a} xe_3 + \dfrac{d\sqrt{2a^4 + 2a^2 bd - b^2 d^2}}{\sqrt{a^6}} xe_4$

when $f = 0$, $k = -\dfrac{\sqrt{a^2} b^2}{\sqrt{2a^4 + 2a^2 bd - b^2 d^2}}$, *or*

(5) $V = xe_1 + \dfrac{2a^4 + a^2 bd - b^2 d^2}{a^2 b^2} xe_2 + \dfrac{d}{a} xe_3 - \dfrac{d\sqrt{2a^4 + 2a^2 bd - b^2 d^2}}{\sqrt{2a^6}} xe_4$

when $f = 0$, $k = \dfrac{\sqrt{a^2} b^2}{\sqrt{2a^4 + 2a^2 bd - b^2 d^2}}$.

As for the critical point for the energy functional restricted to vector fields of the same length, there is the following result.

Proposition 4.12. *Let V denote a left-invariant vector field on the Lorentzian oscillator group $G_1 = G_1(1)$ with the metric of type (VI). For arbitrary real numbers a, b, d, f, there exists $k \neq 0$ such that G_1 has a critical point for the energy functional restricted to vector fields of the same length.*

Proof. If $G_1(1)$ is a critical point for the energy functional restricted to vector fields of the same length, we have the equations as follows.

$$\begin{cases} b^2 x + c^2 x - fkx - k^2 y - ckz = 2\lambda x \\ act - dkt - 2a^2 x - 2bdx - f^2 x + b^2 y + c^2 y - fky + cfz = 2\lambda y \\ bct + akt + cfx - cky + b^2 z - 2fkz = 2\lambda z \\ c^2 t - 2fkt + acx - dkx + bcz + akz = 2\lambda t. \end{cases} \tag{17}$$

The system of equations (17) can be written as $M_6' X = 0$ where $X = (x, y, z, t)^T$. Then the determinant $Det(M_6')$ of the coefficient matrix M_6' is:

$$\begin{aligned} &- 2b^6 fk + (-a^2 b^4 + 8b^4 f^2)k^2 + (6a^2 b^2 f + 4b^3 df - 8b^2 f^3)k^3 + (2a^4 + 2a^2 bd \\ &- b^2 d^2 - 8a^2 f^2 - 8bdf^2)k^4 + 2d^2 fk^5(-2b^6 + 20b^4 fk + 8a^2 b^2 k^2 + 4b^3 dk^2 \\ &- 40b^2 f^2 k^2 - 20a^2 fk^3 - 16bdfk^3 + 16f^3 k^3 + 2d^2 k^4)\lambda + (12b^4 - 56b^2 fk \\ &- 12a^2 k^2 - 8bdk^2 + 48f^2 k^2)\lambda^2 + (-24b^2 + 48fk)\lambda^3 + 16\lambda^4 \\ &\triangleq Det(M_6) + s_1\lambda + s_2\lambda^2 + s_3\lambda^3 + s_4\lambda^4. \end{aligned}$$

By the above discussion about harmonic section, it is easy to see that for arbitrary real numbers a, b, d, f, there is $k \neq 0$ such that $Det(M_6) < 0$. Because $s_4 = 16 > 0$, we can see that there exists λ, s.t. $Det(M_6') = 0$, i.e. there is a non-trivial solution to the system of equations (17). ■

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