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A Topological Paley-Wiener-Schwartz Theorem for Sections of Homogeneous Vector Bundles on G/K

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Abstract. We study the Fourier transforms for compactly supported distributional sections of complex homogeneous vector bundles on symmetric spaces of non-compact type $X = G/K$. We prove a characterization of their range. In fact, from Delorme's Paley-Wiener theorem for compactly supported smooth functions on a real reductive group of Harish-Chandra class, we deduce topological Paley-Wiener and Paley-Wiener-Schwartz theorems for sections.

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1. Introduction

One of the central theorems of harmonic analysis on $\mathbb{R}^n$ is the so-called Paley-Wiener theorem, named after the two mathematicians Raymond Paley and Norbert Wiener. It describes the image of the Fourier transform of the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support as the space of entire functions on $\mathbb{C}^n$ satisfying some growth condition. The theorem has a counterpart, known as the Paley-Wiener-Schwartz theorem. Here, the smooth functions are replaced by distributions $T \in C_c^{-\infty}(\mathbb{R}^n)$ and the growth condition by a weaker growth condition (e.g. [13], Theorem. 7.3.1).

Both theorems have been generalized to more general Lie groups G and furthermore to some smooth manifolds carrying symmetries. For example, the case of Riemannian symmetric spaces of non-compact type $X = G/K$ was considered by Helgason [9] and Gangolli [8]. They proved a Paley-Wiener theorem for compactly supported K-invariant smooth functions and Helgason [10] even showed it for general compactly supported smooth functions on X. There is also a Paley-Wiener theorem for $K \times K$-finite compactly supported smooth functions on a real reductive Lie group G of Harish-Chandra class due to Arthur [1] and Delorme [6], formulated in terms of the so-called Arthur-Campolli and Delorme conditions, respectively. Delorme even proved a version without the $K \times K$-finiteness. A generalization to K-finite functions on reductive symmetric spaces was presented by van den Ban and Schlichtkrull [2]. Furthermore, later van den Ban and Souaifi [4] proved, without using the proof or validity of any associated Paley-Wiener theorems of Arthur or Delorme, that the two compatibility conditions are equivalent.

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Concerning the Paley-Wiener-Schwartz theorem for distributions on symmetric spaces, we mention Helgason [10] and Eguchi, Hashizume, Okamoto [7]. Moreover, van den Ban and Schlichtkrull [3] also proved a topological Paley-Wiener-Schwartz theorem for K-finite distributions on reductive symmetric spaces.

Our aim is to establish a topological Paley-Wiener theorem for (distributional) sections of homogeneous vector bundles on X using Delorme's intertwining conditions. Thus, starting, in Section 2 with Delorme's Paley-Wiener theorem ([6], Thm. 2) in the setting of van den Ban and Souaifi [4], we will adjust it, in Sections 3 and 4, for our purposes. More precisely, we describe the intertwining conditions for sections and show that they are equivalent to Delorme's ones by using Frobenius-reciprocity (Prop. 4.3 and Thm. 4.5). We consider three levels, (Level 1) refers to Delorme's Paley-Wiener theorem (Thm. 2.7), (Level 2) corresponds to the desired Paley-Wiener theorem for sections (Thm. 5.2) and (Level 3) stands for the Paley-Wiener theorem for 'spherical functions' (Thm. 5.2). For the last, we fix an additional K-representation on the left while a right, not necessarily irreducible, K-type $*$ is fixed by the bundle $\mathbb{E}_* \to X$. The intertwining conditions are much easier to handle (Thm. 4.5) in (Level 3) than in (Level 2), see [19] for examples.

Finally in Section 6, we present our main theorem (Thm. 6.3), a topological Paley-Wiener-Schwartz theorem for distributional sections in both levels (Level 2) and (Level 3). Roughly speaking, it describes the image of the Fourier transform, say in (Level 2), as a space of holomorphic functions on a complex vector space with values in principal series representations satisfying the usual Paley-Wiener-Schwartz growth condition and additional relations between the values as well as the derivatives of these functions at different points that correspond to the same infinitesimal character. For its proof, we use van den Ban and Schlichtkrull's technique [3] as well as Camporesi's Plancherel theorem for sections ([5], Thm. 3.4 and Thm. 4.3).

This paper ends, in Section 7, by analyzing some consequences of this theorem for linear invariant differential operators between sections of homogeneous vector bundles (Prop. 7.2).

The motivation behind this work lies in solvability questions of systems of invariant differential equations on symmetric spaces G/K. These questions forced us to find a manageable description for the image under the Fourier transform of non-K-finite distributional sections of homogeneous vector bundles on G/K, i.e. to establish a Paley-Wiener-Schwartz theorem in (Level 2). However, the complicated nature of the intertwining conditions in (Level 2) makes it necessary to switch from time to time to (Level 3), i.e. to fix an additional K-type. Let us explain the questions in more detail.

Given an invariant differential operator D between sections of homogeneous vector bundles over G/K, we want to establish that the equation $Df = g$ is solvable in smooth sections, provided the given section g satisfies the obvious 'algebraic' integrability conditions: $D'g = 0$ for all invariant differential operators D' satisfying $D' \circ D = 0$. A positive answer to this problem would be the precise analog of the famous Ehrenpreis-Malgrange theorem for *systems* of constant coefficient differential equations on $\mathbb{R}^n$ (see e.g. [13]). The special and easy case of a single equation corresponds to one-dimensional (trivial) vector bundles where the integrability condition disappears. For symmetric spaces of non-compact type, a positive answer in this special case has been established by Helgason using his own Paley-Wiener-Schwartz theorem [10]. Also for many differential operators between higher dimensional vector bundles, a positive answer is known as a consequence of a deep theorem of Kashiwara-Schmid on maximal globalizations of Harish-Chandra modules [14, 15]. Its proof is a tour de force in $\mathcal{D}$-module theory. We wanted to understand to what extent a more classical approach using Fourier analysis is possible, similar as in the proof of the Ehrenpreis-Malgrange theorem [13]. This approach should work by dualizing, i.e. consider compactly supported distributional sections instead of smooth sections and then transfer the problem via Fourier transform to the Paley-Wiener-Schwartz space. In [20], we develop a precise strategy on how to attack this problem in the Paley-Wiener-Schwartz space and show that the strategy works completely for $\mathrm{SL}(2,\mathbb{R})$. In fact, the results of the present paper as well as applications to solvability questions are part of the doctoral dissertation [21] of the second author. For further details, we refer to [21] and the follow-up papers [19, 20].

2. On Delorme's Paley-Wiener theorem

Let G be a real connected semi-simple Lie group with finite center of non-compact type with Lie algebra $\mathfrak{g}$ and $K \subset G$ its maximal compact subgroup with Lie algebra $\mathfrak{k}$. The quotient $X = G/K$ is then a Riemannian symmetric space of non-compact type.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Fix a corresponding minimal parabolic subgroup $P = MAN$ of G with split component $A = \exp(\mathfrak{a})$, nilpotent Lie group N, and $M = Z_K(\mathfrak{a})$ being the centralizer of A in K. Let $(\sigma, E_\sigma) \in \widehat{M}$ be a finite dimensional irreducible representation of M and $\lambda \in \mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}^n$. For fixed $(\sigma,\lambda) \in \widehat{M} \times \mathfrak{a}^*_{\mathbb{C}}$, let $(\sigma_\lambda, E_{\sigma,\lambda})$ be the representation of P on the vector space $E_{\sigma,\lambda} := E_\sigma$ such that we have $\sigma_\lambda(man) = a^{\lambda+\rho}\sigma(m) \in \mathrm{End}(E_{\sigma,\lambda})$ for $m \in M$, $a \in A, n \in N$ and where $\rho \in \mathfrak{a}^*$ is the half sum of the positive roots of $(\mathfrak{g},\mathfrak{a})$, counted with multiplicities. We use the notation a^λ for $e^{\lambda \log(a)}$. Then, the space

$$H^{\sigma,\lambda}_\infty := \left\{ f : G \overset{C^\infty}{\to} E_{\sigma,\lambda} \;\middle|\; \begin{array}{l} f(gman) = a^{-(\lambda+\rho)}\sigma(m)^{-1}(f(g)), \\ g \in G,\ m \in M,\ a \in A,\ n \in N \end{array} \right\}$$

together with the left regular action $(\pi_{\sigma,\lambda}(g)f)(x) := f(g^{-1}x) = (l_g f)(x)$ for $g, x \in G$ and $f \in H^{\sigma,\lambda}_\infty$, is the space of smooth vectors of the principal series representation of G induced from the P-representation σ_λ on $E_{\sigma,\lambda}$ (e.g. [16], p. 168). The restriction map from $H^{\sigma,\lambda}_\infty$ to functions on K is injective by the Iwasawa decomposition $g = \kappa(g)a(g)n(g) \in KAN$ of G. In particular, for $f \in H^{\sigma,\lambda}_\infty$ we have $f(g) = f(\kappa(g)a(g)n(g)) = a(g)^{-(\lambda+\rho)}(f(\kappa(g)))$. This yields the so-called *compact picture* of $H^{\sigma,\lambda}_\infty$ (e.g. [16], p. 168). It has the advantage that the representation space

$$H^\sigma_\infty := \{\varphi : K \overset{C^\infty}{\to} E_\sigma \,|\, \varphi(km) = \sigma(m)^{-1}\varphi(k),\ k \in K, m \in M\} \tag{1}$$

does not depend on λ. Here, H^σ_∞ is equipped with the usual Fréchet topology. From time to time, we need the L^2-norm. The corresponding action in the compact picture, which we denote – by a slight abuse of notation – also by $\pi_{\sigma,\lambda}$, is more involved:

$$(\pi_{\sigma,\lambda}(g)\varphi)(k) = a(g^{-1}k)^{-(\lambda+\rho)}\varphi(\kappa(g^{-1}k)), \quad \varphi \in H^\sigma_\infty. \tag{2}$$

We remark that the representations (σ, E_σ) of M and $(\sigma, E_{\sigma,\lambda})$ of P define homogeneous vector bundles $\mathbb{E}_\sigma$ on K/M and $\mathbb{E}_{\sigma,\lambda}$ on G/P, respectively. Then, we have $H^\sigma_\infty \cong C^\infty(K/M, \mathbb{E}_\sigma)$ and $H^{\sigma,\lambda}_\infty \cong C^\infty(G/P, \mathbb{E}_{\sigma,\lambda})$.

Fourier transform for G in (Level 1)

Let $$C^\infty_c(G) = \bigcup_{r>0} C^\infty_r(G) := \bigcup_{r>0} \{f \in C^\infty(G) \mid supp(f) \in \overline{B}_r(o)\}$$

be the space of compactly supported smooth complex functions on G, where

$$\overline{B}_r(o) := \{g \in G \mid dist_X(gK, o) \leq r\} \subset G$$

denotes the preimage of the closed ball of radius r and center $o = eK$ in X under the projection $G \to X$. Here, $dist_X$ means a fixed G-invariant Riemannian distance on X, and e is the neutral element of G. We equip $C^\infty_r(G)$ with the usual Fréchet topology, thus $C^\infty_c(G)$ is a LF-space.

Definition 2.1. (Fourier transform for G in (Level 1)) Fix $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}^*_\mathbb{C}$, we define the *Fourier transform* of $f \in C^\infty_c(G)$ of (σ, λ) by the operator

$$\mathcal{F}_{\sigma,\lambda}(f) := \pi_{\sigma,\lambda}(f) = \int_G f(g)\pi_{\sigma,\lambda}(g)\, \mathrm{d}g \in \mathrm{End}(H^\sigma_\infty). \qquad \blacksquare$$

We denote in the following by $\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C})$ the space of holomorphic functions in $\mathfrak{a}^*_\mathbb{C}$ and by $\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty))$ the space of maps $\mathfrak{a}^*_\mathbb{C} \ni \lambda \mapsto \phi(\lambda) \in \mathrm{End}(H^\sigma_\infty)$ such that

(1.i) for $\varphi \in H^\sigma_\infty$, the map $\lambda \mapsto \phi(\lambda)\varphi \in H^\sigma_\infty$ is holomorphic function on $\mathfrak{a}^*_\mathbb{C}$ with values in the Fréchet space H^σ_∞.

From Delorme's Lemma 10(ii) in [6], we deduce the following statement.

Proposition 2.2. *The family of maps $f \mapsto (\mathcal{F}_{\sigma,\lambda}(f))_{(\sigma,\lambda)\in\hat{M}\times\mathfrak{a}^*_\mathbb{C}}$ defines a linear map from $C^\infty_c(G)$ into $\prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty))$.* $\blacksquare$

Delorme's Paley-Wiener theorem and intertwining conditions in (Level 1)

We now proceed with the definition of Delorme's Paley-Wiener space ([6], Def. 3). It involves Delorme's intertwining conditions for *derived* versions of H^σ_∞ ([6], Sect. 1.5 and Déf. 3 (4.4)). Van den Ban and Souaifi present a more elegant reformulation of them ([4], Section 4.5, in particular Lemma 4.4 and Proposition 4.5). Our formulation will be very similar to theirs.

Definition 2.3. (m-th derived representation) For $\lambda \in \mathfrak{a}^*_\mathbb{C}$, let Hol_λ be the set of germs at λ of $\mathbb{C}$-valued holomorphic functions $\mu \mapsto f_\mu$ and $m_\lambda \subset \mathrm{Hol}_\lambda$ the maximal ideal of germs vanishing at λ. Denote by $H^\sigma_{[\lambda]}$ the set of germs at λ of H^σ_∞-valued holomorphic functions $\mu \mapsto \psi_\mu \in H^\sigma_\infty$ with G-action

$$(g\psi)_\mu = \pi_{\sigma,\mu}(g)\psi_\mu, \quad g \in G.$$

For $m \in \mathbb{N}_0$, it induces a representation $\pi^{(m)}_{\sigma,\lambda}$ on the space

$$H^{\sigma,\lambda}_{\infty,(m)} := H^\sigma_{[\lambda]}/m^{m+1}_\lambda H^\sigma_{[\lambda]}, \tag{3}$$

which is equipped with the natural Fréchet topology. We call this representation the *m-th derived principal series representation* of G. $\blacksquare$

Here, Hol_λ acts on $H^\sigma_{[\lambda]}$ by pointwise multiplication. Note that the 0-th derived representation is the space of smooth vectors of the principal series G-representation in the compact picture: $H^{\sigma,\lambda}_{\infty,(0)} \cong H^\sigma_\infty$. Intuitively, we can say that $H^{\sigma,\lambda}_{\infty,(m)}$ contains all Taylor polynomials of order m at λ of holomorphic families ψ_μ.

Moreover, $\phi \in \prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty))$ induces an operator $\phi^\sigma_{\lambda,m}$ on $H^{\sigma,\lambda}_{\infty,(m)}$ by $\phi^\sigma_{\lambda,m}([\psi_\mu]) := [\phi^\sigma_\mu \psi_\mu]$, where $\psi_\mu \in H^\sigma_{[\lambda]}$ represents the corresponding class in $H^{\sigma,\lambda}_{\infty,(m)}$. The following definition turns out to be equivalent to Delorme's intertwining condition ([6], Déf. 3 (4.4)).

Definition 2.4. (Delorme's intertwining condition in (Level 1)) Let Ξ be the set of all 3-tuples (σ, λ, m) with $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}^*_\mathbb{C}$ and $m \in \mathbb{N}_0$. Consider the m-th derived G-representation $H^{\sigma,\lambda}_{\infty,(m)}$ defined in (3). For every finite sequence $\xi = (\xi_1, \xi_2, \ldots, \xi_s) \in \Xi^s$, $s \in \mathbb{N}$, we define the G-representation

$$H_\xi := \bigoplus_{i=1}^{s} H^{\sigma_i,\lambda_i}_{\infty,(m_i)}.$$

We consider proper closed G-subrepresentations $W \subsetneq H_\xi$.

Such a pair (ξ, W) with $\xi \in \Xi^s$ and $W \subset H_\xi$ as above, is called an *intertwining datum.* Every function $\phi \in \prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty))$ induces an element

$$(\phi^{\sigma_i}_{\lambda_i,m_i})_{i=1,\ldots,s} =: \phi_\xi \in \bigoplus_{i=1}^{s} \mathrm{End}(H^{\sigma_i,\lambda_i}_{\infty,(m_i)}) \subset \mathrm{End}(H_\xi).$$

(D.a) We say that ϕ satisfies *Delorme's intertwining condition*, if $\phi_\xi(W) \subseteq W$ for every intertwining datum (ξ, W). ■

Next, we define Delorme's Paley-Wiener space ([6], Déf. 3). We denote by $\mathcal{U}(\mathfrak{k})$ the universal enveloping algebra of the complexification of $\mathfrak{k}$. Note that our fixed Riemannian metric on $X = G/K$ corresponds to an Ad-invariant bilinear form on $\mathfrak{g}$, which is definite on $\mathfrak{k}$ and $\mathfrak{p}$. Therefore, we get a norm $|\cdot|$ on $\mathfrak{b}^*_\mathbb{C}$ for each subspace $\mathfrak{b} \subset \mathfrak{k}$ or $\mathfrak{b} \subset \mathfrak{p}$.

Definition 2.5. (Paley-Wiener space in (Level 1)) For $r > 0$, *Delorme's Paley-Wiener space* is the vector space

$$PW_r(G) := \left\{ \phi \in \prod_{\sigma\in\widehat{M}} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty)) \;\middle|\; \begin{array}{l} \phi \text{ satisfies the growth condition} \\ (1.ii)_r \text{ below and } (D.a) \end{array} \right\}. \quad (4)$$

Here,

$(1.\mathrm{ii})_r$ for all $Y_1, Y_2 \in \mathcal{U}(\mathfrak{k})$, $(\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}^*_\mathbb{C}$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,Y_1,Y_2} > 0$ such that

$$\|\pi_{\sigma,\lambda}(Y_1)\phi(\sigma,\lambda)\pi_{\sigma,\lambda}(Y_2)\| \leq C_{r,N,Y_1,Y_2}(1 + |\Lambda_\sigma|^2 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|}$$

for $\phi \in \mathrm{End}(H^\sigma_\infty)$, where Λ_σ is the highest weight of σ and $\|\cdot\|$ is the operator norm on H^σ_∞ with respect to the L^2-norm of H^σ_∞. ■

Notice that due to Lemma 10(i) in [6], the space $PW_r(G)$ is equipped with seminorms

$$\|\phi\|_{r,N,Y_1,Y_2} := \sup_{(\sigma,\lambda)\in\widehat{M}\times\mathfrak{a}_{\mathbb{C}}^*} (1+|\Lambda_\sigma|^2+|\lambda|^2)^N e^{-r|\mathrm{Re}(\lambda)|}\|\pi_{\sigma,\lambda}(Y_1)\phi(\sigma,\lambda)\pi_{\sigma,\lambda}(Y_2)\|_{H_\infty^\sigma},$$

$\phi \in PW_r(G)$, is a Fréchet space. Furthermore, the intertwining condition $(D.a)$ in Def. 2.5 is a special case of van den Ban's and Souaifi's one ([4], Cor. 4.7 and Prop. 4.10.). The small difference is, that instead of the defined m-th derived representations $H^{\sigma,\lambda}_{\infty,(m)}$ (3), they consider

$$H^\sigma_{[\lambda],E} := H^\sigma_{[\lambda]} \otimes_{\mathrm{Hol}_\lambda} E,$$

where E is a finite dimensional Hol_λ-module. By the following proposition, this leads to equivalent intertwining conditions.

Proposition 2.6. *With the previous notations, let $(\sigma,\lambda)\in\widehat{M}\times\mathfrak{a}_{\mathbb{C}}^*$. Then, for $E=\mathrm{Hol}_\lambda/m_\lambda^{m+1}$, we have $H^\sigma_{[\lambda],E}\cong H^{\sigma,\lambda}_{\infty,(m)}$.*
Moreover, for any finite dimensional Hol_λ-module E, there exist $m_1,\dots,m_s\in\mathbb{N}_0$ such that $H^\sigma_{[\lambda],E}$ is a quotient of $H^{\sigma,\lambda}_{\infty,(m_1)}\oplus\cdots\oplus H^{\sigma,\lambda}_{\infty,(m_s)}$.

Proof. Consider a (commutative) ring R with neutral element 1, a R-module M and $I\subset R$ an ideal. Then, we have the following isomorphism

$$M\otimes_R R/I \cong M/IM.$$

In fact, by an algebraic computation, one can easily show that the two R-maps

$$\begin{array}{ccc} \alpha: M\otimes_R R/I\to M/IM & \text{and} & \beta: M/IM\to M\otimes_R R/I \\ \alpha(m\otimes[r]) := [rm] & & \beta([m]) := m\otimes[1] \end{array}$$

are well-defined and inverse to each other. Here, $[\cdot]$ denotes the class in the corresponding quotient. For $m\in\mathbb{N}_0$ and $R=\mathrm{Hol}_\lambda$, consider its ideal $m_\lambda^{m+1}\subset\mathrm{Hol}_\lambda$. Take $E=\mathrm{Hol}_\lambda/m_\lambda^{m+1}$ and $M=H^\sigma_{[\lambda]}$, then

$$H^\sigma_{[\lambda]}\otimes_{\mathrm{Hol}_\lambda} E\cong H^\sigma_{[\lambda]}/m_\lambda^{m+1}H^\sigma_{[\lambda]} =: H^{\sigma,\lambda}_{\infty,(m)}.$$

Moreover, an ideal $\mathcal{I}$ in Hol_λ is cofinite if and only if there exists $m\in\mathbb{N}_0$ such that $m_\lambda^{m+1}\subset\mathcal{I}$ (see e.g. [4], Lemma 2.1).
Thus, for some $s\in\mathbb{N}$ and finitely many cofinite ideals $m_\lambda^{m_1+1},\dots,m_\lambda^{m_s+1}$ of Hol_λ, we have that a finite dimensional Hol_λ-module E is a quotient of the direct sum

$$\mathrm{Hol}_\lambda/m_\lambda^{m_1+1}\oplus\mathrm{Hol}_\lambda/m_\lambda^{m_2+1}\oplus\cdots\oplus\mathrm{Hol}_\lambda/m_\lambda^{m_s+1}.$$

Hence, the map $$H^{\sigma,\lambda}_{\infty,(m_1)}\oplus\cdots\oplus H^{\sigma,\lambda}_{\infty,(m_s)}\longrightarrow H^\sigma_{[\lambda],E}$$
is surjective and the result follows. ■

Now, we can formulate Delorme's Paley-Wiener theorem.

Theorem 2.7. (Paley-Wiener Theorem, [6], Thm. 2) *For $r>0$, the Fourier transform $\mathcal{F}$ given by*

$$C_r^\infty(G)\ni f\mapsto(\mathcal{F}_{\sigma,\lambda}(f))_{(\sigma,\lambda)\in\widehat{M}\times\mathfrak{a}_{\mathbb{C}}^*}\in PW_r(G)$$

is a topological isomorphism between the two Fréchet spaces $C_r^\infty(G)$ and $PW_r(G)$. ■

Remark 2.8. Delorme formulated the Paley-Wiener Thm. 2.7 in terms of all cuspidal parabolic subgroups. By Casselman's subrepresentation theorem (e.g. [24], Thm. 3.8.3.), it is clear that it remains true if we restrict to the minimal parabolic subgroup P (compare [4], Lem. 4.4).

3. Fourier transforms for (distributional) sections and its properties

Let (τ, E_τ) be a finite dimensional, not necessarily irreducible, representation of K. We obtain a homogeneous vector bundle $\mathbb{E}_\tau$ over X, whose space $C^\infty(X, \mathbb{E}_\tau)$ of smooth sections is identified with the following space:

$$C^\infty(X, \mathbb{E}_\tau) \cong \{f : G \xrightarrow{C^\infty} E_\tau \mid f(gk) = \tau^{-1}(k)(f(g)), \forall g \in, k \in K\}.$$

The group G acts on $C^\infty(X, \mathbb{E}_\tau)$ by left translations, $(g \cdot f)(g') = f(g^{-1}g')$, where $g, g' \in G$. We have the following G-isomorphism:

$$C^\infty(X, \mathbb{E}_\tau) \cong [C^\infty(G) \otimes E_\tau]^K.$$

Here, K-invariants are taken with respect to representation $r \otimes \tau$, where r denotes the right regular representation on $C^\infty(G)$. By taking the topological linear dual of $C^\infty(X, \mathbb{E}_{\tilde{\tau}})$, where $(\tilde{\tau}, E_{\tilde{\tau}})$ is the dual of the representation (τ, E_τ), we obtain the space of compactly supported distributional sections of E_τ:

$$C_c^{-\infty}(X, \mathbb{E}_\tau) := (C^\infty(X, \mathbb{E}_{\tilde{\tau}}))'. \tag{5}$$

We have

$$C_c^{-\infty}(X, \mathbb{E}_\tau) = \bigcup_{r \geq 0} C_r^{-\infty}(X, \mathbb{E}_\tau) := \bigcup_{r \geq 0} \{T \in C^{-\infty}(X, \mathbb{E}_\tau) \mid supp(T) \in \overline{B}_r(o)\}$$

and the inclusion $C_c^\infty(X, \mathbb{E}_\tau) \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$.

Fourier transform in (Level 2)

We want to study the map induced on the space $[C_c^\infty(G) \otimes E_\tau]^K \cong C_c^\infty(X, \mathbb{E}_\tau)$ by the Fourier transform $\mathcal{F} : C_c^\infty(G) \longrightarrow \bigcup_{r \geq 0} PW_r(G)$ (see Thm. 2.7):

$$\sum_{i=1}^{d_\tau} f_i \otimes v_i \mapsto \sum_{i=1}^{d_\tau} \mathcal{F}(f_i) \otimes v_i, \quad f_i \in C_c^\infty(G),$$

where d_τ denotes the dimension of E_τ and $v_i, i \in \{1, \cdots, d_\tau\}$, is a basis of E_τ. For $r > 0$, one deduces from Thm. 2.7 that

$$C_r^\infty(X, \mathbb{E}_\tau) \cong [C_r^\infty(G) \otimes E_\tau]^K \overset{\text{Thm. 2.7}}{\cong} [PW_r(G) \otimes E_\tau]^K.$$

The goal is to make $[PW_r(G) \otimes E_\tau]^K$ more explicit and, moreover, to do the same study for distributional sections $C_c^{-\infty}(X, \mathbb{E}_\tau)$. For this, let us study the map

$$\begin{aligned} C_r^\infty(X, \mathbb{E}_\tau) \ni f \quad &\hat{=} \quad \sum_{i=1}^{d_\tau} f_i \otimes v_i \in [C_r^\infty(G) \otimes E_\tau]^K \\ &\mapsto \quad \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i) \otimes v_i \in [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K \cong H_\infty^\sigma \otimes \mathrm{Hom}_K(H_\infty^\sigma, E_\tau). \end{aligned}$$

Bringing the Frobenius reciprocity into play, gives us a better description of the space $\mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$. Namely, we have

$$\begin{aligned} \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) &\overset{Frob}{\cong} \mathrm{Hom}_M(E_\sigma, E_\tau) \text{ defined by} \\ \langle Frob(S)w, \tilde{v}\rangle &= \langle w, S^*\tilde{v}(e)\rangle, \quad w \in E_\sigma,\ \tilde{v} \in E_{\tilde{\tau}}. \end{aligned} \tag{6}$$

Here, $S^* : E_{\tilde{\tau}} \to H^{\tilde{\sigma}}_\infty$ is the adjoint of S. Let us next compute the inverse of $Frob$.

Lemma 3.1. ([18], Lemma 2.12) *Let $s \in \mathrm{Hom}_M(E_\sigma, E_\tau)$ and $f \in H^\sigma_\infty$. Then, we have*

$$Frob^{-1}(s)(f) = \int_K \tau(k) s f(k)\,\mathrm{d}k. \qquad \blacksquare$$

The dual of $Frob$ is given by

$$\begin{aligned} \mathrm{Hom}_K(E_\tau, H^\sigma_\infty) &\overset{\widetilde{Frob}}{\cong} \mathrm{Hom}_M(E_\tau, E_\sigma) \\ \widetilde{Frob}(T)(v) &= T(v)(e),\ v \in E_\tau \end{aligned} \tag{7}$$

and for $t \in \mathrm{Hom}_M(E_\tau, E_\sigma)$ and $v \in E_\tau$, the inverse of $\widetilde{Frob}$ is

$$\widetilde{Frob}^{-1}(t)(v)(k) = t(\tau(k^{-1})v), k \in K. \tag{8}$$

Coming back to our previous computation, we get

$$\begin{aligned} [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K &\cong H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) \overset{Frob}{\cong} H^\sigma_\infty \otimes \mathrm{Hom}_M(E_\sigma, E_\tau) \\ &\overset{(1)}{\cong} C^\infty(K, E_\sigma \otimes \mathrm{Hom}_M(E_\sigma, E_\tau))^M \cong H^{\tau|_M(\sigma)}_\infty, \end{aligned} \tag{9}$$

where $\tau|_M(\sigma)$ is the σ-isotypic component of $\tau|_M$. Here, as τ is restricted to M, it is generally no more irreducible and splits into a finite direct sum of the form $\tau|_M = \bigoplus_{\sigma\in\widehat{M}} m(\sigma,\tau)\sigma$, where $m(\sigma,\tau) = \dim(\mathrm{Hom}_M(E_\sigma, E_\tau)) \geq 0$ is the multiplicity of σ in $\tau|_M$. Now by taking the algebraic direct sum over all $\sigma \in \widehat{M}$, where only finitely many of them appear, we obtain

$$\begin{aligned} \bigoplus_{\sigma\in\widehat{M}} [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K &\overset{(9)}{\cong} \bigoplus_{\sigma\in\widehat{M}} H^{\tau|_M(\sigma)}_\infty \cong H^{\tau|_M}_\infty \\ &:= \{f : K \overset{C^\infty}{\to} E_\tau \mid f(km) = \tau(m)^{-1} f(k)\}, \end{aligned}$$

which can be viewed as the principal series representations corresponding to $\tau|_M$.

Definition 3.2. (Fourier transform for sections of homogeneous vector bundles in (Level 2)) Let $g = \kappa(g)a(g)n(g) \in KAN = G$ be the Iwasawa decomposition. For fixed $\lambda \in \mathfrak{a}^*_\mathbb{C}$ and $k \in K$, we define the function $e^\tau_{\lambda,k}$ by

$$\begin{aligned} e^\tau_{\lambda,k} : G &\to \mathrm{End}(E_\tau) \cong E_{\tilde{\tau}} \otimes E_\tau \\ g &\mapsto e^\tau_{\lambda,k}(g) := \tau(\kappa(g^{-1}k))^{-1} a(g^{-1}k)^{-(\lambda+\rho)}. \end{aligned} \tag{10}$$

(a) For $f \in C^\infty_c(X, \mathbb{E}_\tau)$, the *Fourier transformation* is given by

$$\mathcal{F}_\tau f(\lambda, k) = \int_G e^\tau_{\lambda,k}(g) f(g)\,\mathrm{d}g = \int_{G/K} e^\tau_{\lambda,k}(g) f(g)\,\mathrm{d}g \in E_\tau, \tag{11}$$

where the last equality makes sense since the integrand is right K-invariant.

(b) The *Fourier transform for a distributional section* $T\in C_c^{-\infty}(X,\mathbb{E}_\tau)$ is defined by

$$\mathcal{F}_\tau T(\lambda,k):=\langle T,e^\tau_{\lambda,k}\rangle\in E_\tau,\ (\lambda,k)\in\mathfrak{a}^*_{\mathbb{C}}\times K/M.$$

Here we view $e^\tau_{\lambda,k}$ as an element of $C^\infty(X,\mathbb{E}_{\tilde\tau})\otimes E_\tau$, and

$$\langle\cdot,\cdot\rangle : C_c^{-\infty}(X,\mathbb{E}_\tau)\times(C^\infty(X,\mathbb{E}_{\tilde\tau})\otimes E_\tau)\to E_\tau$$

is the natural pairing. ■

Note that the Fourier transform for sections has already been introduced and studied by Camporesi ([5], (3.18)). It is a direct generalization of Helgason's Fourier transform for $E_\tau=\mathbb{C}$. It is not difficult to see that $\lambda\mapsto\mathcal{F}_\tau f(\lambda,\cdot)$ and $\lambda\mapsto\mathcal{F}_\tau T(\lambda,\cdot)$ are in $\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}},H_\infty^{\tau|_M})$. Observe that, for $k\in K$ and $g\in G$, we have, by definition

$$e^\tau_{\lambda,k}(g)=l_k(e^\tau_{\lambda,1}(g))=e^\tau_{\lambda,1}(k^{-1}g). \tag{12}$$

The function $e^\tau_{\lambda,k}$ in Def. 3.2 can be seen as the analogue of the exponential function in the definition of Fourier transform in the Euclidean case $\mathbb{R}^n$. It has some interesting properties. Note that for fixed $k\in K$, $e^\tau_{\lambda,k}(g)$ is an entire function on $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$, since $a(g^{-1}k)^{-(\lambda+\rho)}$ is an entire function on $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$.

Proposition 3.3. *Let $\tau\in\widehat{K}$, $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ and $k\in K$. Then, we have*

$$e^\tau_{\lambda,k}(hg)=e^\tau_{\lambda,\kappa(h^{-1}k)}(g)a(h^{-1}k)^{-(\lambda+\rho)},\quad g,h\in G. \tag{13}$$

Proof. Let $h,g\in G=KAN$, then by Iwasawa decomposition, we obtain

$$\begin{aligned} hg=h\kappa(g)a(g)n(g) &= \kappa(h\kappa(g))\,a(h\kappa(g))\,n(h\kappa(g))\,a(g)\,n(g)\\ &= \underbrace{\kappa(h\kappa(g))}_{\in K}\ \underbrace{a(h\kappa(g))\,a(g)}_{\in A}\ \underbrace{n(h\kappa(g))\,n(g)}_{\in N}. \end{aligned}$$

In other words, we have $\kappa(hg)=\kappa(h\kappa(g)a(g)n(g))=\kappa(h\kappa(g))$, and $a(hg)=a(h\kappa(g)a(g)n(g))=a(h\kappa(g))a(g)$. Hence,

$$\begin{aligned} e^\tau_{\lambda,k}(hg) &\overset{(10)}{=} \tau(\kappa(g^{-1}h^{-1}k))^{-1}a(g^{-1}h^{-1}k)^{-(\lambda+\rho)}\\ &= \tau(\kappa(g^{-1}\kappa(h^{-1}k)))^{-1}a(g^{-1}\kappa(h^{-1}k))^{-(\lambda+\rho)}a(h^{-1}k)^{-(\lambda+\rho)}\\ &\overset{(10)}{=} e^\tau_{\lambda,\kappa(h^{-1}k)}(g)a(h^{-1}k)^{-(\lambda+\rho)}. \end{aligned}$$ ■

Fourier transform in (Level 3) and its properties

Now consider an additional finite dimensional K-representation $\gamma:K\to GL(E_\gamma)$. The Fourier transform in Def. 3.2 induces a mapping

$$\mathrm{Hom}_K(E_\gamma,C_c^\infty(X,\mathbb{E}_\tau))\longrightarrow\mathrm{Hom}_K(E_\gamma,\mathrm{Hol}(\mathfrak{a}^*_{\mathbb{C}},H_\infty^{\tau|_M})). \tag{14}$$

The left hand side of (14) can be identified with a space of functions with values in $\mathrm{Hom}(E_\gamma,E_\tau)$, the (γ,τ)-spherical functions:

$$\begin{aligned} &\mathrm{Hom}_K(E_\gamma,C_c^\infty(X,\mathbb{E}_\tau))\cong C_c^\infty(G,\gamma,\tau)\\ :=\{f:G\to\mathrm{Hom}(E_\gamma,E_\tau)\mid f(k_1gk_2)&=\tau(k_2)^{-1}f(g)\gamma(k_1)^{-1},\ \forall k_1,k_2\in K\}. \end{aligned}$$

For the right hand side of the mapping (14), we use the Frobenius reciprocity between K and M by evaluating at $k=1$. In consequence we obtain the space of functions $\{\phi : \mathfrak{a}_{\mathbb{C}}^* \to \mathrm{Hom}_M(E_\gamma, E_\tau)\}$. This motivates the definition of the Fourier transformation for (γ,τ)-spherical functions.

Definition 3.4. (Fourier transform in (Level 3)) With the previous notations, the *Fourier transformation* for $f \in C_c^\infty(G,\gamma,\tau)$ is given by

$$
{}_\gamma\mathcal{F}_\tau f(\lambda) := \int_G e_{\lambda,1}^\tau(g) f(g)\,\mathrm{d}g \in \mathrm{Hom}(E_\gamma, E_\tau), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*. \tag{15}
$$

Similar, the *Fourier transformation for a distribution*

$$
T \in C_c^{-\infty}(G,\gamma,\tau) \cong \mathrm{Hom}_K(E_\gamma, C_c^{-\infty}(X,\mathbb{E}_\tau))
$$

is defined by
$$
{}_\gamma\mathcal{F}_\tau T(\lambda) := \langle T, e_{\lambda,1}^\tau \rangle \in \mathrm{Hom}(E_\gamma, E_\tau).
$$

Again, we view $e_{\lambda,k}^\tau$ as an element of $C^\infty(X,\mathbb{E}_{\tilde{\tau}}) \otimes E_\tau$, and

$$
\langle\cdot,\cdot\rangle : \mathrm{Hom}_K(E_\gamma, C_c^{-\infty}(X,\mathbb{E}_\tau)) \times (C^\infty(X,\mathbb{E}_{\tilde{\tau}}) \otimes E_\tau) \to \mathrm{Hom}(E_\gamma, E_\tau)
$$

is the natural pairing. ■

Observe that for $f \in C_c^\infty(G,\gamma,\tau)$ and $m \in M$

$$
\tau(m)\,{}_\gamma\mathcal{F}_\tau f(\lambda) = \int_G e_{\lambda,1}^\tau(mg) f(g)\,\mathrm{d}g = \int_G e_{\lambda,1}^\tau(g) f(m^{-1}g)\,\mathrm{d}g = {}_\gamma\mathcal{F}_\tau f(\lambda)\gamma(m).
$$

Thus ${}_\gamma\mathcal{F}_\tau f(\lambda) \in \mathrm{Hom}_M(E_\gamma, E_\tau)$. The same holds for distributions. Let us consider now the convolution of $f \in C_c^\infty(X,\mathbb{E}_\gamma)$ by a (γ,τ)-spherical function $\varphi \in C_c^\infty(G,\gamma,\tau)$, which is defined by

$$
(f * \varphi)(g) := \int_G \varphi(x^{-1}g) f(x)\,\mathrm{d}x = \int_G \varphi(xg) f(x^{-1})\,\mathrm{d}x, \quad g \in G. \tag{16}
$$

By considering the corresponding Fourier transform, we obtain the following result, which is analogous to Helgason's Lemma 1.4 in ([11], Chap. 3).

Proposition 3.5. *With the notations above, we then have that*

$$
\mathcal{F}_\tau(f * \varphi)(\lambda,k) = {}_\gamma\mathcal{F}_\tau\varphi(\lambda)\mathcal{F}_\gamma f(\lambda,k), \quad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, k \in K.
$$

Proof. For $(\lambda,k) \in \mathfrak{a}_{\mathbb{C}}^* \times K$, we compute

$$
\begin{aligned}
\mathcal{F}_\tau(f * \varphi)(\lambda,k) &\overset{(16)}{=} \int_{G\times G} e_{\lambda,k}^\tau(g)\varphi(\underbrace{x^{-1}g}_{=:h}) f(x)\,\mathrm{d}x\,\mathrm{d}g \\
&\overset{\text{Fubini's thm.}}{=} \int_G \Big(\int_G e_{\lambda,k}^\tau(xh)\varphi(h)\,\mathrm{d}h\Big) f(x)\,\mathrm{d}x \\
&\overset{\text{Prop. 3.3}}{=} \int_G \Big(\int_G e_{\lambda,\kappa(x^{-1}k)}^\tau(h) a(x^{-1}k)^{-(\lambda+\rho)}\varphi(h)\,\mathrm{d}h\Big) f(x)\,\mathrm{d}x \\
&\overset{(12)}{=} \int_G \Big(\int_G e_{\lambda,1}^\tau(\underbrace{\kappa(x^{-1}k)^{-1}h}_{=:g})\varphi(h)\,\mathrm{d}h\Big) a(x^{-1}k)^{-(\lambda+\rho)} f(x)\,\mathrm{d}x
\end{aligned}
$$

$$= \int_G \Big(\int_G e^{\tau}_{\lambda,1}(g)\varphi(\kappa(x^{-1}k)g)\,\mathrm{d}g \Big) a(x^{-1}k)^{-(\lambda+\rho)} f(x)\,\mathrm{d}x$$
$$= \int_G \Big(\int_G e^{\tau}_{\lambda,1}(g)\varphi(g)\,\mathrm{d}g \Big) \gamma(\kappa(x^{-1}k))^{-1} a(x^{-1}k)^{-(\lambda+\rho)} f(x)\,\mathrm{d}x$$
$$= {}_{\gamma}\mathcal{F}_{\tau}\varphi(\lambda)\mathcal{F}_{\gamma}f(\lambda,k). \qquad \blacksquare$$

Remark 3.6. (a) In a similar way, one can define the left convolution by a scalar valued-function $\varphi \in C_c^\infty(G)$. In fact, we know that, for $f \in C_c^\infty(X, \mathbb{E}_\tau)$ and $g \in G$, we have

$$\mathcal{F}_\tau(l_g f)(\lambda,k) = \int_G e^{\tau}_{\lambda,k}(x) l_g f(x)\,\mathrm{d}x = \int_G e^{\tau}_{\lambda,k}(gh) f(h)\,\mathrm{d}h$$
$$\overset{(13)}{=} a(g^{-1}k)^{-(\lambda+\rho)} \int_G e^{\tau}_{\lambda,\kappa(g^{-1}k)}(h) f(h)\,\mathrm{d}h$$
$$\overset{(2)}{=} (\pi_{\tau,\lambda}(g)\mathcal{F}_\tau f(\lambda,\cdot))(k).$$

Hence, we can deduce for $\varphi \in C_c^\infty(G)$

$$\mathcal{F}_\tau(\varphi * f)(\lambda,k) = (\pi_{\tau,\lambda}(\varphi)\mathcal{F}_\tau f(\lambda,\cdot))(k). \tag{17}$$

(b) Analogously as for smooth compactly supported sections (16), we define the convolution for distributional sections $T \in C_c^{-\infty}(X, \mathbb{E}_\gamma)$ by

$$(T * \varphi)(g) := \langle T, l_g \varphi^\vee \rangle, \quad g \in G, \varphi \in C_c^\infty(G,\gamma,\tau).$$

Then, Prop. 3.5 holds for distributions as well. $\blacksquare$

Now, for $\epsilon > 0$, take a K-conjugation invariant open neighbourhood $U_\epsilon \subset B_\epsilon(e)$ in G so that $\bigcap_{\epsilon>0} U_\epsilon = \{e\}$, and for $0 < \epsilon_1 < \epsilon_2$, we have $U_{\epsilon_1} \subset U_{\epsilon_2}$. Consider a scalar-valued positive function $\tilde{\eta}_\epsilon \in C_c^\infty(G)$ with compact support contained in U_ϵ satisfying

$$\int_{U_\epsilon} \tilde{\eta}_\epsilon(g)\,\mathrm{d}g = 1. \tag{18}$$

Let us construct from this an endomorphism-valued function $\eta_\epsilon \in C^\infty(G,\tau,\tau)$ by

$$\eta_\epsilon(g) := \int_{K\times K} \tilde{\eta}_\epsilon(k_1 g k_2)\tau(k_2 k_1)\,\mathrm{d}k_1\,\mathrm{d}k_2, \quad g \in G. \tag{19}$$

Then, we get the following observation.

Corollary 3.7. *For each $\epsilon > 0$, let $\eta_\epsilon \in C_c^\infty(G,\tau,\tau)$ be the (τ,τ)-spherical endomorphism function given by* (19)*. Then, its Fourier transform ${}_{\tau}\mathcal{F}_{\tau}\eta_\epsilon$ converges uniformly on compact subsets of $\mathfrak{a}^*_{\mathbb{C}}$ to the identity map:*

$${}_{\tau}\mathcal{F}_{\tau}\eta_\epsilon(\lambda) \to \mathrm{Id}_{E_\tau},$$

when $\epsilon \to 0$. Moreover, for all $\epsilon_0 > 0, c > 0$ the family ${}_{\tau}\mathcal{F}_{\tau}\eta_\epsilon$, $\epsilon \le \epsilon_0$, is uniformly bounded in the strip $|\mathrm{Re}(\lambda)| \le c$.

Proof. Consider $\eta_\epsilon \in C^\infty(G,\tau,\tau)$, then for $g\in G$:

$$\eta_\epsilon(g) = \int_K\int_K \tilde{\eta}_\epsilon(k_1gk_2)\tau(k_2k_1)\,\mathrm{d}k_1\,\mathrm{d}k_2 = \int_K\int_K \tilde{\eta}_\epsilon(k_1glk_1^{-1})\tau(l)\,\mathrm{d}k_1\,\mathrm{d}l$$
$$= \int_K \overline{\eta}_\epsilon(gl)\tau(l)\,\mathrm{d}l,$$

where we did a change of variable and set $\overline{\eta}_\epsilon(g) := \int_K \tilde{\eta}_\epsilon(k_1gk_1^{-1})\,dk_1$. Here, $\tilde{\eta}_\epsilon$ is as above (18). By computing its Fourier transform, we obtain, for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$

$${}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda) \overset{(15)}{=} \int_G e^\tau_{\lambda,1}(g)\eta_\epsilon(g)\,\mathrm{d}g = \int_G\Big(\int_K e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(gl)\tau(l)\,dl\Big)\,\mathrm{d}g$$
$$= \int_G\Big(\int_K e^\tau_{\lambda,1}(gl)\overline{\eta}_\epsilon(gl)\,dl\Big)\,\mathrm{d}g = \int_G e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(g)\,\mathrm{d}g = \int_{U_\epsilon} e^\tau_{\lambda,1}(g)\overline{\eta}_\epsilon(g)\,\mathrm{d}g.$$

We see that ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ is uniformly bounded on strips by (18) and since $e^\tau_{\lambda,1}(g)$ is so for $g\in U_{\epsilon_0}$. Moreover, the last integral is equal to $\int_{U_\epsilon}\overline{\eta}_\epsilon(g)(e^\tau_{\lambda,1}(g)-\mathrm{Id}_{E_\tau})\,\mathrm{d}g+\mathrm{Id}_{E_\tau}$. Now, consider a compact set $C\subset\mathfrak{a}^*_{\mathbb{C}}$ and $\delta>0$. Then there exists $\epsilon>0$ such that

$$|e^\tau_{\lambda,1}(g)-\mathrm{Id}_{E_\tau}|<\delta \text{ for } g\in U_\epsilon, \lambda\in C.$$

Thus, this implies that ${}_\tau\mathcal{F}_\tau\eta_\epsilon$ converges uniformly on compact sets to Id_{E_τ}, when ϵ converges to 0. ∎

Furthermore, consider a non-zero linear G-invariant differential operator between sections of homogeneous vector bundles

$$D: C^\infty(X,\mathbb{E}_\tau)\longrightarrow C^\infty(X,\mathbb{E}_\gamma). \tag{20}$$

Invariant means that $D(l_gf)=l_g(Df)$, for all $g\in G$ and $f\in C^\infty(X,\mathbb{E}_\tau)$. Denote by $\mathcal{D}_G(\mathbb{E}_\tau,\mathbb{E}_\gamma)$ the vector space of all these G-invariant differential operators on sections. We get the following relation.

Proposition 3.8. *Let $Q\in\mathcal{D}_G(\mathbb{E}_{\tilde{\tau}},\mathbb{E}_{\tilde{\gamma}})$ be an invariant linear differential operator. Then, we have*

$$(Q\otimes\mathrm{Id}_{E_\tau})e^\tau_{\lambda,k} = ((Q\otimes\mathrm{Id}_{E_\tau})e^\tau_{\lambda,1})(1)\circ e^\gamma_{\lambda,k},\quad \lambda\in\mathfrak{a}^*_{\mathbb{C}}, k\in K. \tag{21}$$

Here we view $e^\tau_{\lambda,k}$ as an element of $C^\infty(X,E_{\tilde{\tau}})\otimes E_\tau$ and $\circ$ denotes the natural composition $\mathrm{Hom}(E_\gamma,E_\tau)\times\mathrm{End}(E_\gamma)\to\mathrm{Hom}(E_\gamma,E_\tau)$.

Proof. Let us first consider the case $k=1$. We then have

$$e^\tau_{\lambda,1}(nak_1) = a^{\lambda+\rho}\tau(k_1) = a^{\lambda+\rho}e^\tau_{\lambda,1}(k_1),\quad n\in N, a\in A, k_1\in K. \tag{22}$$

In particular, for $n_1a_1\in NA$ and $g=nak_1\in NAK=G$

$$l_{(n_1a_1)^{-1}}e^\tau_{\lambda,1}(g) = e^\tau_{\lambda,1}(n_1a_1nak_1) = e^\tau_{\lambda,1}(n_1(a_1na_1^{-1})a_1ak_1)$$
$$\overset{(22)}{=} a_1^{\lambda+\rho}a^{\lambda+\rho}\tau(k_1) = a_1^{\lambda+\rho}e^\tau_{\lambda,1}(g).$$

Hence, since Q is linear and G-invariant, we obtain that

$$l_{(n_1a_1)^{-1}}(Q\otimes\mathrm{Id}_{E_\tau})e^\tau_{\lambda,1}(g) = (Q\otimes\mathrm{Id}_{E_\tau})(l_{(n_1a_1)^{-1}}e^\tau_{\lambda,1})(g)$$
$$= (Q\otimes\mathrm{Id}_{E_\tau})(a_1^{\lambda+\rho}e^\tau_{\lambda,1})(g) = a_1^{\lambda+\rho}(Q\otimes\mathrm{Id}_{E_\tau})e^\tau_{\lambda,1}(g) \tag{23}$$

and by setting $g = 1$, we have

$$(Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1}(n_1a_1) \overset{(23)}{=} a_1^{\lambda+\rho}(Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1}(1). \tag{24}$$

Therefore, since $e^\tau_{\lambda,1} \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}) \otimes E_\tau \subset C^\infty(G, \mathrm{End}(E_\tau))$, we have that

$$(Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1} \in C^\infty(X, \mathbb{E}_{\tilde{\gamma}}) \otimes E_\tau \subset C^\infty(G, \mathrm{Hom}(E_\gamma, E_\tau)).$$

Therefore, for $g = n_1a_1k_2 \in G$, we can conclude that

$$\begin{aligned}(Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1}(n_1a_1k_2) &= (Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1}(n_1a_1)\gamma(k_2) \\ &\overset{(24)}{=} a_1^{\lambda+\rho}((Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1})(1)\gamma(k_2) \\ &\overset{(22)}{=} ((Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1})(1) \circ e^\gamma_{\lambda,1}(n_1a_1k_2). \end{aligned} \tag{25}$$

Now for general $k \in K$, we observe that $e^\tau_{\lambda,k} = l_k e^\tau_{\lambda,1}$. Hence

$$\begin{aligned}(Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,k} = (Q \otimes \mathrm{Id}_{E_\tau})(l_k e^\tau_{\lambda,1}) &\overset{(25)}{=} l_k((Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1})(1) \circ e^\gamma_{\lambda,1} \\ &= ((Q \otimes \mathrm{Id}_{E_\tau})e^\tau_{\lambda,1})(1) \circ e^\gamma_{\lambda,k}.\end{aligned}$$

Thus, we get the desired result. ■

4. Delorme's intertwining conditions and some examples

We study Delorme's intertwining conditions $(D.a)$ in Def. 2.4 and determine the intertwining conditions in (Level 2) and (Level 3) induced by them. To do this, we first need some preparations. In the previous Sect. 3, we have seen the identification (9). Let us now take a closer look. Using the Frobenius reciprocity (6), we shall define a linear map

$$I : \bigoplus_{\sigma \in \hat{M}} H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) \longrightarrow H^{\tau|_M}_\infty$$

on each summand separately. Let $\alpha = \sum_{i=1}^{m(\tau,\sigma)} \alpha_i \otimes S_i \in H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$, where $S_i, i = 1, \ldots, m(\tau, \sigma)$, runs through a basis of $\mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$.
Set $s_i := Frob(S_i) \in \mathrm{Hom}_M(E_\sigma, E_\tau)$ and define

$$I(\alpha) := d_\sigma \sum_{i=1}^{m(\tau,\sigma)} s_i \circ \alpha_i.$$

Hence, on each summand, I is the composition of the following isomorphisms

$$H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau) \overset{1 \otimes Frob}{\longrightarrow} H^\sigma_\infty \otimes \mathrm{Hom}_M(E_\sigma, E_\tau) \overset{d_\sigma \cdot m}{\longrightarrow} H^{\tau(\sigma)}_\infty,$$

where m is induced by the natural multiplication map

$$E_\sigma \otimes \mathrm{Hom}_M(E_\sigma, E_\tau) \overset{\sim}{\longrightarrow} E_\tau(\sigma).$$

Summing up over σ, we see that I is an isomorphism. For $T \in \mathrm{Hom}_K(E_\tau, H^\sigma_\infty)$, let

$$\langle \alpha, T \rangle := \sum_{i=1}^{m(\tau,\sigma)} \mathrm{Tr}_\tau(S_i \circ T)\alpha_i \in H^\sigma_\infty. \tag{26}$$

We now consider $\mathrm{End}(H^\sigma_\infty)\otimes E_\tau$ equipped with the K-action

$$k(\phi\otimes v) := (\phi\circ l_{k^{-1}})\otimes \tau(k)v.$$

Then, there is a natural identification $[\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K \cong H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$, which gives rise to an isomorphism

$$j : \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K \to \bigoplus_{\sigma\in\hat{M}} H^\sigma_\infty \otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau).$$

We define the map
$$J : \bigoplus_{\sigma\in\hat{M}} [\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K \longrightarrow H^{\tau|_M}_\infty \tag{27}$$

by setting $J := I\circ j$. In addition, for $\beta = \sum_{i=1}^{d_\tau} \beta_i \otimes v_i \in [\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K$ and $T\in \mathrm{Hom}_K(E_\tau, H^\sigma_\infty)$, let

$$\langle \beta, T\rangle := \sum_{i=1}^{d_\tau} (\beta_i \circ T)(v_i) \in H^\sigma_\infty. \tag{28}$$

We check that $\langle \beta, T\rangle = \langle j(\beta), T\rangle$, where the pairing on the right hand side is given by (26). For two vector spaces A, B and $a\in A, \tilde{b}\in \mathrm{Hom}(B,\mathbb{C})$, we denote by $a\cdot \tilde{b} \in \mathrm{Hom}(B,A)$ the one dimensional operator

$$B\ni b\mapsto \tilde{b}(b)a.$$

The operator $\beta_i\in \mathrm{End}(H^\sigma_\infty)$ is finite dimensional. Hence it can be written in the form $\beta_i = \sum_j \alpha_{ij}\cdot\tilde{\alpha}_{ij}$ for some $\alpha_{ij}\in H^\sigma_\infty$, $\tilde{\alpha}_{ij}\in \mathrm{Hom}(H^\sigma_\infty,\mathbb{C})$. Therefore we have $j(\beta) = \sum_{i,j} \alpha_{ij}\otimes (v_i\cdot \tilde{\alpha}_{ij})$ and thus

$$\begin{aligned}\langle j(\beta), T\rangle &= \sum_{i,j} \mathrm{Tr}_\tau((v_i\cdot\tilde{\alpha}_{ij})\circ T)\alpha_{ij} = \sum_{i,j} \tilde{\alpha}_{ij}(T(v_i))\alpha_{ij}\\ &= \sum_{i,j} \alpha_{ij}\cdot\tilde{\alpha}_{ij}(T(v_i)) = \sum_i (\beta_i\circ T)(v_i) = \langle \beta, T\rangle.\end{aligned}$$

This justifies our slightly abusive use of the same symbol $\langle\cdot,\cdot\rangle$ for two different pairings. Recall the definition (7) of the dual Frobenius reciprocity.

Proposition 4.1. *Let $f := \sum_{i=1}^{d_\tau} f_i\otimes v_i \in C^\infty_c(X,\mathbb{E}_\tau)$. We shortly write $\mathcal{F}_{\sigma,\lambda}(f)$ for $\sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i)\otimes v_i \in [\mathrm{End}(H^\sigma_\infty)\otimes E_\tau]^K$. Then, for every $T\in \mathrm{Hom}_K(E_\tau, H^\sigma_\infty)$, $t = \widetilde{Frob}(T) \in \mathrm{Hom}_M(E_\tau, E_\sigma)$ and $\alpha \in H^\sigma_\infty\otimes \mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$, we obtain*

1. $\langle \alpha, T\rangle = t\circ (I(\alpha))$,
2. $\langle \mathcal{F}_{\sigma,\lambda}(f), T\rangle = t\circ \mathcal{F}_\tau f(\lambda,\cdot) \in H^\sigma_\infty$, *for* $\lambda\in\mathfrak{a}^*_\mathbb{C}$,
3. $\mathcal{F}_\tau f(\lambda,\cdot) = J(\bigoplus_{\sigma\in\widehat{M}} \mathcal{F}_{\sigma,\lambda}(f))$, *for* $\lambda\in\mathfrak{a}^*_\mathbb{C}$.

Proof. (1) It is sufficient to prove it for only one summand in α, hence let $\alpha = \alpha_1\otimes S$. For $T\in \mathrm{Hom}_K(E_\tau, H^\sigma_\infty), t = \widetilde{Frob}(T), S\in \mathrm{Hom}_K(H^\sigma_\infty, E_\tau)$ and $s = Frob(S)$, we thus obtain

$$\begin{aligned}\langle \alpha, T\rangle &= \mathrm{Tr}_\tau(S\circ T)\alpha_1 \overset{\text{Lem. 3.1\&(8)}}{=} \mathrm{Tr}_\tau(Frob^{-1}(s)\circ \widetilde{Frob}^{-1}(t))\alpha_1\\ &= \mathrm{Tr}_\tau\Big(\int_K \tau(k)s\circ t\tau(k^{-1})\,\mathrm{d}k\Big)\alpha_1 = \mathrm{Tr}_\tau(s\circ t)\alpha_1 = \mathrm{Tr}_\sigma(t\circ s)\alpha_1.\end{aligned}$$

Since $\sigma \in \widehat{M}$ is irreducible and $t \circ s \in \mathrm{End}_M(E_\sigma)$, by Schur's lemma, we have that $t \circ s = \lambda \cdot \mathrm{Id}_{E_\sigma}$, for some $\lambda \in \mathbb{C}$ and thus $\mathrm{Tr}_\sigma(t \circ s) = d_\sigma \lambda$. Hence we obtain $\langle \alpha, T\rangle = (t \circ s) \circ (d_\sigma \alpha_1) = t \circ (I(\alpha))$.

(2) By computation, we get

$$\langle \mathcal{F}_{\sigma,\lambda}(f), T\rangle = \sum_{i=1}^{d_\tau} (\mathcal{F}_{\sigma,\lambda}(f_i) \circ T)(v_i) \overset{(8)}{=} \sum_{i=1}^{d_\tau} \mathcal{F}_{\sigma,\lambda}(f_i)(t\tau(\cdot)^{-1}(v_i))$$

$$\overset{\text{Def. 2.1}}{=} \sum_{i=1}^{d_\tau} \int_G f_i(g) \pi_{\sigma,\lambda}(g)(t\tau(\cdot)^{-1}(v_i))\,\mathrm{d}g = \sum_{i=1}^{d_\tau} \int_G f_i(g)(\pi_{\sigma,\lambda}(g)\varphi_i)\,\mathrm{d}g.$$

In the last line, we set $\varphi_i(k) := t \circ \tau(k^{-1})(v_i)$, for $k \in K$. Fix $k \in K$, by applying (2), we have $(\pi_{\sigma,\lambda}(g)\varphi_i)(k) = a(g^{-1}k)^{-(\lambda+\rho)}\varphi_i(\kappa(g^{-1}k))$.

Thus, the above is equal to

$$\sum_{i=1}^{d_\tau} \int_G f_i(g) t \circ \tau(\kappa(g^{-1}k))^{-1} a(g^{-1}k)^{-(\lambda+\rho)} v_i \,\mathrm{d}g = \sum_{i=1}^{d_\tau} \int_G f_i(g) t \circ e^\tau_{\lambda,k}(g) v_i \,\mathrm{d}g$$

$$= t \circ \int_G e^\tau_{\lambda,k}(g) \sum_{i=1}^{d_\tau} f_i(g) v_i \,\mathrm{d}g = t \circ \int_G e^\tau_{\lambda,k}(g) f(g)\, dg = t \circ \mathcal{F}_\tau f(\lambda, k).$$

(3) We rewrite (1) in the following way:

(1') $\mathrm{Tr}_\tau(I^{-1}(\beta) \circ T) = t \circ \beta,\ \beta \in H_\infty^{\tau|_M},\ T \in \mathrm{Hom}_K(E_\tau, H^\sigma_\infty)$.

Here $I^{-1}(\beta) \circ T \in H^\sigma_\infty \otimes \mathrm{End}_K(E_\tau)$ and Tr_τ is taken in the second component. Similarly (2) gives (2'):

(2') $\mathrm{Tr}_\tau(j(\mathcal{F}_{\sigma,\lambda}(f)) \circ T) = t \circ \mathcal{F}_\tau f(\lambda, \cdot)$.

We obtain

$$\mathrm{Tr}_\tau(I^{-1}(\mathcal{F}_\tau f(\lambda,\cdot)) \circ T) \overset{(1')}{=} t \circ \mathcal{F}_\tau f(\lambda, \cdot) \overset{(2')}{=} \mathrm{Tr}_\tau(j(\mathcal{F}_{\sigma,\lambda}(f)) \circ T).$$

The pairing $\mathrm{Hom}_K(H^\sigma_\infty, E_\tau) \times \mathrm{Hom}_K(E_\tau, H^\sigma_\infty) \ni (S,T) \mapsto Tr_\tau(S \circ T) \in \mathbb{C}$ is non-degenerate. We conclude that $I^{-1}(\mathcal{F}_\tau f(\lambda,\cdot)) = j(\bigoplus_{\sigma \in \widehat{M}} \mathcal{F}_{\sigma,\lambda}(f))$. This gives (3). ∎

We first study what happens to Delorme's intertwining condition $(D.a)$ if we tensor it with E_τ and take K-invariants.

Definition 4.2. (1) We say that a function

$$\phi \in \prod_{\sigma \in \widehat{M}} [\mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty)) \otimes E_\tau]^K \cong \bigoplus_{\sigma \subset \tau|_M} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, [\mathrm{End}(H^\sigma_\infty) \otimes E_\tau]^K)$$

satisfies the *intertwining condition*, if for each $\tilde{v} \in E_{\tilde{\tau}}$

$$\langle \phi, \tilde{v}\rangle_\tau \in \prod_{\sigma \in \widehat{M}} \mathrm{Hol}(\mathfrak{a}^*_\mathbb{C}, \mathrm{End}(H^\sigma_\infty))$$

satisfies the intertwining condition in Definition 2.4.

Proposition 4.3. *Let (ξ, W) be an intertwining datum defined in Definition* 2.4. *For any $\phi \in \prod_{\sigma\in\widehat{M}}[\mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{End}(H_\infty^\sigma)) \otimes E_\tau]^K$, let $\phi_\xi \in [\mathrm{End}(H_\xi) \otimes E_\tau]^K$ be defined by*

$$\langle \phi_\xi, \tilde{v} \rangle_\tau := (\langle \phi, \tilde{v} \rangle_\tau)_\xi \in \mathrm{End}(H_\xi), \quad \tilde{v} \in E_{\tilde{\tau}}.$$

Moreover, for any $\beta \in [\mathrm{End}(H_\xi) \otimes E_\tau]^K$ and $T \in \mathrm{Hom}(E_\tau, H_\xi)$, let $\langle \beta, T \rangle$ be defined similarly to the one in Eq. (28).

(*D*.1) *Then, ϕ satisfies the intertwining condition* (1) *of Definition* 4.2 *if, and only if, for each intertwining datum (ξ, W) and $T \in \mathrm{Hom}_K(E_\tau, W) \subset \mathrm{Hom}_K(E_\tau, H_\xi)$, the induced element $\phi_\xi \in [\mathrm{End}(H_\xi) \otimes E_\tau]^K$ satisfies*

$$\langle \phi_\xi, T \rangle \in W.$$

Proof. Let $\phi_\xi = \sum_{i=1}^{d_\tau} f_i \otimes v_i \in [\mathrm{End}(H_\xi) \otimes E_\tau]^K$. We have to show

$$\begin{aligned} & f_i(W) \subset W, \quad i = 1, \ldots, d_\tau \\ \iff & \langle \phi_\xi, T \rangle = \sum_{i=1}^{d_\tau} f_i \circ T(v_i) \in W, \quad \forall T \in \mathrm{Hom}_K(E_\tau, W). \end{aligned} \tag{29}$$

The right implication is obvious. For the left, we remark that by K-invariance of ϕ_ξ the endomorphism f_i vanishes on all K-isotypic components $H_\xi(\gamma)$ with $E_\tau(\gamma) = 0$. Thus $f_i(W) \subset W$ if $f_i \circ T \in \mathrm{Hom}_K(E_\tau, W)$, for all $T \in \mathrm{Hom}_K(E_\tau, W)$. Let us fix $T_1 \in \mathrm{Hom}_K(E_\tau, W)$. It suffices to show that under the assumption of the right hand side of (29), we have

$$f_i \circ T_1(v_j) \in W, \quad \forall i, j = 1, \ldots, d_\tau.$$

Consider the mapping $A_{ij} \in \mathrm{End}(E_\tau)$ sending v_i to v_j and v_k to 0, $k \neq i$. Then, for all $i, j \in \{1, \ldots, d_\tau\}$, we have

$$f_i \circ T_1(v_j) = \langle \phi_\xi, T_1 \circ A_{ij} \rangle = \langle \phi_\xi, p_K(T_1 \circ A_{ij}) \rangle, \tag{30}$$

where $p_K : \mathrm{Hom}(E_\tau, W) \to \mathrm{Hom}_K(E_\tau, W)$ is the natural projection to K-invariants. Since $p_K(T_1 \circ A_{ij}) \in \mathrm{Hom}_K(E_\tau, W)$, we obtain that $f_i \circ T_1(v_j) \in W$ by assumption and (30). ■

Next, we state the intertwining condition in (Level 2) and (Level 3) induced from Delorme's intertwining condition (D.a), more precisely from (1) in Definition 4.2.

Definition 4.4. (Intertwining conditions in (Level 2) and (Level 3))

(2) Consider the map J defined in (27). We say that a function $\psi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M})$ satisfies the intertwining condition, if

$$J^{-1}\psi \in \bigoplus_{\sigma \subset \tau|_M} \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, [\mathrm{End}(H_\infty^\sigma) \otimes E_\tau]^K)$$

satisfies the intertwining condition (1) in Def. 4.2.

(3) Let γ be a second finite dimensional representation of K. We say that a function $\varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))$ satisfies the intertwining condition, if for all $w \in E_\gamma$, the element $\varphi^w \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M})$ defined by

$$\varphi^w(\lambda, k) := \varphi(\lambda)\gamma(k^{-1})w$$

satisfies the above intertwining condition (2).

We now want to make the intertwining conditions more explicit. Let us first introduce some notation. Define $H_{\infty,(m)}^{\tau|_M,\lambda}$ as in (3) with σ replaced by $\tau|_M$. For two M-representations acting on E_1, E_2, respectively, let $\mathrm{Hol}_\lambda(\mathrm{Hom}_M(E_1, E_2))$ be the set of germs at λ of holomorphic functions with values in $\mathrm{Hom}_M(E_1, E_2)$. We define for $m \in \mathbb{N}$

$$\mathrm{Hom}_M(E_1, E_2)_{(m)}^\lambda := \mathrm{Hol}_\lambda(\mathrm{Hom}_M(E_1, E_2))/m_\lambda^{m+1}\,\mathrm{Hol}_\lambda(\mathrm{Hom}_M(E_1, E_2)).$$

In particular, we will need the spaces $\mathrm{Hom}_M(E_\tau, E_\sigma)_{(m)}^\lambda$ and $\mathrm{Hom}_M(E_\gamma, E_\tau)_{(m)}^\lambda$. Let (ξ, W) be an intertwining datum with

$$\xi = ((\sigma_1, \lambda_1, m_1), (\sigma_2, \lambda_2, m_2), \ldots, (\sigma_s, \lambda_s, m_s)), s \in \mathbb{N}.$$

The Frobenius reciprocity $\widetilde{Frob}^{-1} : \mathrm{Hom}_M(E_\tau, E_\sigma) \xrightarrow{\sim} \mathrm{Hom}_K(E_\tau, H_\infty^\sigma)$ (see (8)) induces an isomorphism $\mathrm{Hom}_M(E_\tau, E_\sigma)_{(m)}^\lambda \to \mathrm{Hom}_K(E_\tau, H_{\infty,(m)}^\sigma)$ and therefore an isomorphism

$$\bigoplus_{i=1}^s \mathrm{Hom}_M(E_\tau, E_{\sigma_i})_{(m_i)}^{\lambda_i} \to \mathrm{Hom}_K\Big(E_\tau, \bigoplus_{i=1}^s H_{\infty,(m_i)}^{\sigma_i}\Big) = \mathrm{Hom}_K(E_\tau, H_\xi)$$

also denoted by $\widetilde{Frob}^{-1}$. Then, we set

$$\begin{aligned} D_W^\tau &:= \Big\{t \in \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\tau, E_{\sigma_i})_{(m_i)}^{\lambda_i} \mid \widetilde{Frob}^{-1}(t) \in \mathrm{Hom}_K(E_\tau, W) \subset \mathrm{Hom}_K(E_\tau, H_\xi)\Big\} \\ &\subset \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\tau, E_{\sigma_i})_{(m_i)}^{\lambda_i}. \end{aligned} \tag{31}$$

Write by $\overline{\Xi}$ the set of all 2-tuples (λ, m) with $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $m \in \mathbb{N}_0$. We define the map

$$\Xi \longrightarrow \overline{\Xi}, \quad \xi = (\sigma, \lambda, m) \mapsto \overline{\xi} = (\lambda, m).$$

For $s \in \mathbb{N}$ and $\xi \in \Xi^s$, we have the corresponding element $\overline{\xi} \in \overline{\Xi}^s$. As in Definition 2.4, $\psi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M})$ and $\overline{\xi} \in \overline{\Xi}^s$ induces an element $\psi_{\overline{\xi}} \in \bigoplus_{i=1}^s H_{\infty,(m_i)}^{\tau|_M,\lambda_i} =: H_{\overline{\xi}}^{\tau|_M}$. Similarly, $\varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))$ induces

$$\varphi_{\overline{\xi}} \in \bigoplus_{i=1}^s \mathrm{Hom}_M(E_\gamma, E_\tau)_{(m_i)}^{\lambda_i} =: H_{\overline{\xi}}^{\gamma,\tau}.$$

Theorem 4.5 (Intertwining conditions in Levels 2 and 3)**.** *With the notations above, we have:*

$(D.2)$ (Level 2) $\psi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M})$ *satisfies the intertwining condition* (2) *of Definition* 4.4 *if, and only if, for each intertwining datum* (ξ, W)*, the induced element* $\psi_{\overline{\xi}} \in H_{\overline{\xi}}^{\tau|_M}$ *satisfies, for each non-zero* $t = (t_1, t_2, \ldots, t_s) \in D_W^\tau$

$$t \circ \psi_{\overline{\xi}} = (t_1 \circ \psi_1, \ldots, t_s \circ \psi_s) \in W.$$

$(D.3)$ (Level 3) $\varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))$ *satisfies the intertwining condition* (3) *of Definition* 4.4 *if, and only if, for each intertwining datum* (ξ, W)*, the induced element* $\varphi_{\overline{\xi}} \in H_{\overline{\xi}}^{\gamma,\tau}$ *satisfies, for each non-zero* $t = (t_1, t_2, \ldots, t_s) \in D_W^\tau$

$$t \circ \varphi_{\overline{\xi}} = (t_1 \circ \varphi_1, \ldots, t_s \circ \varphi_s) \in D_W^\gamma.$$

Proof. For the first assertion, we have to check, in view of Proposition 4.3, the equivalence of the condition $(D.1)$ for $J^{-1}\psi$ and the condition $(D.2)$ for ψ. This follows by Proposition 4.1(1). Indeed, by construction of D_W^τ, we know that $\widetilde{Frob}^{-1} : D_W^\tau \to \mathrm{Hom}_K(E_\tau, W)$ is an isomorphism and

$$\langle J^{-1}\psi, \widetilde{Frob}^{-1}(t) \rangle = (t_1 \circ \psi_1, t_2 \circ \psi_2, \ldots, t_s \circ \psi_s) \in H_\xi.$$

For the second assertion, we need equivalence of $(D.2)$ for φ^w for all $w \in E_\gamma$ and $(D.3)$ for φ, see Def. 4.4. Note that $\varphi^w(\lambda, \cdot) = \widetilde{Frob}^{-1}(\varphi(\lambda))(w)$. Hence $\varphi_{\overline{\xi}}^w = \widetilde{Frob}^{-1}(\varphi_{\overline{\xi}}(\lambda))(w)$. We conclude that

$$t \circ \varphi_{\overline{\xi}}^w = \Big(t \circ \widetilde{Frob}^{-1}(\varphi_{\overline{\xi}})\Big)(w) = \widetilde{Frob}^{-1}(t \circ \varphi_{\overline{\xi}})(w).$$

We obtain that $t \circ \varphi_{\overline{\xi}}^w \in W$ for all $w \in E_\tau$ if and only if $t \circ \varphi_{\overline{\xi}} \in D_W^\gamma$. ∎

Example 4.6. (a) Consider $s = 1$ and $m = 0$. Let $\xi := (\sigma, \lambda, 0) \in \Xi$ and $W \subset H_\infty^{\sigma,\lambda}$ a closed G-invariant subspace. Consider $D_W^\tau \subset \mathrm{Hom}_M(E_\tau, E_\sigma)$ as in Thm. 4.5. The corresponding intertwining conditions read as follows:

(D.2a) (Level 2) $\psi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, H_\infty^{\tau|_M,\cdot})$ satisfies $t \circ \psi(\lambda, \cdot) \in W$ for each $t \in D_W^\tau$.

(D.3a) (Level 3) $\varphi \in \mathrm{Hol}(\mathfrak{a}_{\mathbb{C}}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))$ satisfies $t \circ \varphi(\lambda) \in D_W^\gamma$ for each $t \in D_W^\tau$.

(b) Consider now $s = 2$ and $m_1 = m_2 = 0$. Let $L : H_\infty^{\sigma_1,\lambda_1} \longrightarrow H_\infty^{\sigma_2,\lambda_2}$ be an intertwining operator between the two principal series representations. Let $\xi := ((\sigma_1, \lambda_1, 0), (\sigma_2, \lambda_2, 0)) \in \Xi^2$ and $W = graph(L) \subset H_\infty^{\sigma_1,\lambda_1} \oplus H_\infty^{\sigma_2,\lambda_2}$. Moreover, define $l^\tau : \mathrm{Hom}_M(E_\tau, E_{\sigma_1}) \longrightarrow \mathrm{Hom}_M(E_\tau, E_{\sigma_2})$ by

$$l^\tau(t)(v) = L(t\tau(\cdot)^{-1}v)(e)$$

for $v \in E_\tau$ and $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$. Then

$$\begin{aligned} D_W^\tau = \{(t_1, t_2) \mid t_2 = l^\tau(t_1)\} &= \{(t, l^\tau(t)) \mid t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})\} \\ &\subset \mathrm{Hom}_M(E_\tau, E_{\sigma_1}) \oplus \mathrm{Hom}_M(E_\tau, E_{\sigma_2}). \end{aligned}$$

In this situation, we have the following intertwining conditions:

(D2.b) (Level 2) For each $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$ and $\psi \in \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|_M,\cdot})$ we have

$$L(t \circ \psi(\lambda_1, \cdot)) = l^\tau(t) \circ \psi(\lambda_2, \cdot).$$

(D3.b) (Level 3) For each $t \in \mathrm{Hom}_M(E_\tau, E_{\sigma_1})$ and $\varphi \in \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, \mathrm{Hom}_M(E_\gamma, E_\tau))$ we get

$$l^\gamma(t \circ \varphi(\lambda_1)) = l^\tau(t) \circ \varphi(\lambda_2).$$

5. Topological Paley-Wiener theorem for sections

The Paley-Wiener space for sections of homogeneous vector bundles is defined as follows.

Definition 5.1. (Paley-Wiener space for sections in (Level 2) and (Level 3))

(a) For $r > 0$, let $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ be the space of sections $\psi \in C^\infty(\mathfrak{a}_\mathbb{C}^* \times K/M, \mathbb{E}_{\tau|_M})$ such that

(2.i) the section ψ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$, i.e. $\psi \in \mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*, H_\infty^{\tau|_M})$.

(2.ii)$_r$ (growth condition) for all $Y \in \mathcal{U}(\mathfrak{k})$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,Y} > 0$ such that

$$\|l_Y \psi(\lambda, k)\|_{E_\tau} \le C_{r,N,Y}(1 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|}, \quad k \in K,$$

where $\|\cdot\|_{E_\tau}$ denotes the norm on the finite dimensional vector space E_τ (for convenience, we often denote it by $|\cdot|$).

(2.iii) (intertwining condition) (D.2) from Theorem 4.5.

(b) By considering an additional K-representation γ, let ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$ be the space of functions $\mathfrak{a}_\mathbb{C}^* \ni \lambda \mapsto \varphi(\lambda) \in \mathrm{Hom}_M(E_\gamma, E_\tau)$ be such that

(3.i) the function φ is holomorphic in $\lambda \in \mathfrak{a}_\mathbb{C}^*$.

(3.ii)$_r$ (growth condition) for all $N \in \mathbb{N}_0$, there exists a constant $C_{r,N} > 0$ such that

$$\|\varphi(\lambda)\|_{\mathrm{op}} \le C_{r,N}(1 + |\lambda|^2)^{-N} e^{r|\mathrm{Re}(\lambda)|},$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm on $\mathrm{Hom}_M(E_\gamma, E_\tau)$.

(3.iii) (intertwining condition) (D.3) from Theorem 4.5. ■

The inequalities provide semi-norms $\|\cdot\|_{r,-N,Y}$ (resp. $\|\cdot\|_{r,-N}$) on the space $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$) and make the vector space $PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. ${}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$) a Fréchet space, compare e.g. Lemma 10 of Delorme [6].

Combining Delorme's Paley-Wiener Thm. 2.7 with the above identifications and observations, we obtain a Paley-Wiener theorem in (Level 2) and (Level 3).

Theorem 5.2. (Topological Paley-Wiener theorem for sections in (Level 2) and (Level 3)) *Let (τ, E_τ) be a K-representation with associated homogeneous vector bundle $\mathbb{E}_\tau$. For $r > 0$, the Fourier transform*

$$C_r^\infty(X, \mathbb{E}_\tau) \ni \psi \mapsto \mathcal{F}_\tau(\psi) \in PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$$

is a topological isomorphism.

Moreover, for an additional K-representation (γ, E_γ), the Fourier transform

$$C_r^\infty(G,\gamma,\tau) \ni \varphi \mapsto {}_\gamma\mathcal{F}_\tau(\varphi) \in {}_\gamma PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$$

is a topological isomorphism. ■

Furthermore, the Paley-Wiener space $PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ is defined as

$$PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M) := \bigcup_{r>0} PW_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M),$$

similar for ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*)$. Equip $PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ and ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*)$ with the *inductive limit topology* (compare the next Sect. 6). Then, by the above result (Thm. 5.2), we also have a linear topological isomorphism from $C_c^\infty(X,\mathbb{E}_\tau)$ (resp. $C_c^\infty(G,\gamma,\tau)$) onto $PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ (resp. ${}_\gamma PW_\tau(\mathfrak{a}_\mathbb{C}^*)$).

6. On topological Paley-Wiener-Schwartz theorem for sections and its proof

Spaces of distributional sections and their topology

In (5), we already introduced $C_c^{-\infty}(X,\mathbb{E}_\tau)$ as the topological linear dual of $C^\infty(X,\mathbb{E}_{\tilde\tau})$. We provide $C_c^{-\infty}(X,\mathbb{E}_\tau)$ with the *strong dual topology.* Actually, we know that $C^\infty(X,\mathbb{E}_{\tilde\tau})$ is a Fréchet space with semi-norms

$$\|h\|_{\Omega,Y} := \sup_{g\in\Omega} |l_Y h(g)|, \quad h \in C^\infty(X,\mathbb{E}_{\tilde\tau}), \tag{32}$$

where $Y \in \mathcal{U}(\mathfrak{g})$ and Ω is a compact subset of G. A subset $B \subset C^\infty(X,\mathbb{E}_{\tilde\tau})$ is called bounded, if for each compact $\Omega \subset G$ and $Y \in \mathcal{U}(\mathfrak{g})$ there exists a constant $C_{\Omega,Y} > 0$ such that $\sup_{\varphi\in B}\|\varphi\|_{\Omega,Y} \le C_{\Omega,Y}$. Shortly, every semi-norm is bounded on B.

The strong dual topology on $C_c^{-\infty}(X,\mathbb{E}_\tau)$ is the locally convex topology given by the semi-norm system

$$p_B(T) := \sup_{\varphi\in B} |T(\varphi)|, \quad T \in C_c^{-\infty}(X,\mathbb{E}_\tau), \tag{33}$$

where B belongs to the family of all bounded subsets of $C^\infty(X,\mathbb{E}_{\tilde\tau})$. Similarly, we equip $C^{-\infty}(X,\mathbb{E}_\tau) = (C_c^\infty(X,\mathbb{E}_{\tilde\tau}))'$ with the strong dual topology. As an immediate consequence of these dualities, the topologies on $C_c^{-\infty}(X,\mathbb{E}_\tau)$ and $C^{-\infty}(X,\mathbb{E}_\tau)$ induce the same topology on the space of distributions supported in a fixed compact subset Ω of G ([3], Section 14). For example, one can take $\Omega = \overline{B}_r(o)$.

A subset $B' \subset C_c^{-\infty}(X,\mathbb{E}_\tau)$ is bounded in the strong dual topology if for each bounded $B \subset C^\infty(X,\mathbb{E}_{\tilde\tau})$, we have

$$\sup_{T\in B'} p_B(T) = \sup_{T\in B',\varphi\in B} |T(\varphi)| < \infty. \tag{34}$$

Since, by Schaefer ([22], Cor. 1.6, p. 127), all such sets B' are equicontinuous, this means that there exists a continuous semi-norm p on $C^\infty(X,\mathbb{E}_{\tilde\tau})$ and a constant $C > 0$ such that

$$B' \subset \{T \in C_c^{-\infty}(X,\mathbb{E}_\tau) \mid |T(\varphi)| \le Cp(\varphi), \forall\varphi \in C^\infty(X,\mathbb{E}_{\tilde\tau})\}.$$

Consider a basis $Y_1,\dots,Y_n$ of $\mathfrak{g}$. Then, for a multi-index $\alpha\in\mathbb{N}_0^n$, we define $Y_\alpha := Y_1^{\alpha_1}\cdots Y_n^{\alpha_n}\in\mathcal{U}(\mathfrak{g})$. We may assume that the semi-norm p has the form

$$p(\varphi)=\sum_{|\alpha|\le m}\|\varphi\|_{\Omega,\alpha}\overset{(32)}{=}\sum_{|\alpha|\le m}\sup_{g\in\Omega}|l_{Y_\alpha}\varphi(g)|,\qquad \varphi\in C^\infty(X,\mathbb{E}_{\tilde\tau})\tag{35}$$

for some $m\in\mathbb{N}_0$ and compact $\Omega\subset G$.

It is interesting to notice that $C^\infty(X,\mathbb{E}_{\tilde\tau})$ is a reflexive Fréchet space, even a Montel space, that is, it is reflexive, and a subset is bounded if, and only if, it is relatively compact ([22], p. 147).

Thus, since $C_c^{-\infty}(X,\mathbb{E}_\tau)$ is the strong dual space of a Montel space $C^\infty(X,\mathbb{E}_{\tilde\tau})$, we can deduce by Schaefer's Cor. 1 in ([22], p. 154) that $C_c^{-\infty}(X,\mathbb{E}_\tau)$ is a bornological space, that is a locally convex space on which each semi-norm p_B, which is bounded on bounded subsets, is continuous ([22], Chap. 2.8, p. 61).

This observation leads us to the following general result, which will play an important role in the proof of the Paley-Wiener-Schwartz theorem. For bornological spaces, bounded linear maps are continuous ([22], Thm. 8.3, p. 62), hence, we obtain the following.

Lemma 6.1. *Let W be any locally convex topological vector space and consider a linear map*

$$A: C_c^{-\infty}(X,\mathbb{E}_\tau)\to W.$$

Then, A is continuous if and only if $A(B')$ is bounded in W for every bounded subset $B'\subset C_c^{-\infty}(X,\mathbb{E}_\tau)$. ■

Let $Y_1,\dots,Y_l$ be a basis of the algebra $\mathfrak{k}$. For a multi-index $\alpha\in\mathbb{N}_0^l$, we consider $Y_\alpha := Y_1^{\alpha_1}\cdots Y_l^{\alpha_l}\in\mathcal{U}(\mathfrak{k})$. Now we are in the position to define the Paley-Wiener-Schwartz space for sections.

Definition 6.2. (Paley-Wiener-Schwartz space for sections in (Level 2) and (Level 3))

(a) For $r>0$, let $PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^*\times K/M)$ be the space of sections $\psi\in C^\infty(\mathfrak{a}_\mathbb{C}^*\times K/M,\mathbb{E}_{\tau|_M})$ such that

(2.*i*) the section ψ is holomorphic in $\lambda\in\mathfrak{a}_\mathbb{C}^*$, i.e. $\psi\in\mathrm{Hol}(\mathfrak{a}_\mathbb{C}^*,H_\infty^{\tau|_M})$;

$(2.iis)_r$ (growth condition) there exists $N\in\mathbb{N}_0$ such that for all multi-indices $\alpha\in\mathbb{N}_0^l$ we have

$$\|l_{Y_\alpha}\psi(\lambda,k)\|_{E_\tau}\le C_{r,N,\alpha}(1+|\lambda|^2)^{N+\frac{|\alpha|}{2}}e^{r|\mathrm{Re}(\lambda)|},\quad k\in K$$

for some constant $C_{r,N,\alpha}$.

(2.*iii*) (intertwining condition) (D.2) from Thm. 4.5.

(b) For an additional K-representation γ, let ${}_\gamma PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^*)$ be the space of functions $\mathfrak{a}_\mathbb{C}^*\ni\lambda\mapsto\varphi(\lambda)\in\mathrm{Hom}_M(E_\gamma,E_\tau)$ be such that

(3.*i*) the function φ is holomorphic in $\lambda\in\mathfrak{a}_\mathbb{C}^*$.

$(3.iis)_r$ (growth condition) there exist $N\in\mathbb{N}_0$ and a positive constant $C_{r,N}$ such that $\|\varphi(\lambda)\|_{\mathrm{op}}\le C_{r,N}(1+|\lambda|^2)^N e^{r|\mathrm{Re}(\lambda)|}$.

(3.*iii*) (intertwining condition) (D.3) from Theorem 4.5.

For all $r \geq 0$ and $N \in \mathbb{N}_0$, we consider

$$PWS_{\tau,r,N} := \{\psi \in PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \mid \|\psi\|_{r,N,\alpha} < \infty, \alpha \in \mathbb{N}_0^l\}, \quad \text{where}$$

$$\|\psi\|_{r,N,\alpha} := \sup_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*,\, k \in K/M} (1+|\lambda|^2)^{-(N+\frac{|\alpha|}{2})} e^{-r|\mathrm{Re}(\lambda)|} \|l_{Y_\alpha} \psi(\lambda, k)\|_{E_\tau}, \quad \alpha \in \mathbb{N}_0^l,\ k \in K.$$

For fixed r, N, the semi-norms $\|\cdot\|_{r,N,\alpha}$ give $PWS_{\tau,r,N}$ the structure of a Fréchet space. We set

$$PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M) := \bigcup_{r\geq 0} \bigcup_{N \in \mathbb{N}_0} PWS_{\tau,r,N}$$

and equip it with the *locally convex inductive limit topology.* It is the finest locally convex topology on $PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ such that all the embeddings

$$PWS_{\tau,r,N} \overset{i_{r,N}}{\hookrightarrow} PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$$

are continuous. Furthermore, this topology is characterized by the following property. A linear map

$$A : PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \to W,$$

where W is any locally convex space, is continuous if, and only if, all compositions

$$PWS_{\tau,r,N} \overset{i_{r,N}}{\hookrightarrow} PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \xrightarrow{A} W$$

are continuous, i.e., $A \circ i_{r,N}$ are continuous. Exactly the same procedure can be done for ${}_\gamma PWS_\tau(\mathfrak{a}_{\mathbb{C}}^*)$.

We are now in the position to state the main theorem.

Theorem 6.3. (Topological Paley-Wiener-Schwartz theorem for sections)

(a) *Let (τ, E_τ) be a K-representation with associated homogeneous vector bundle $\mathbb{E}_\tau$. Then, for each $r \geq 0$, the Fourier transform $\mathcal{F}_\tau$ is a linear bijection between the two spaces $C_r^{-\infty}(X, \mathbb{E}_\tau)$ and the Paley-Wiener-Schwartz space $PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$. Moreover, it is a linear topological isomorphism from $C_c^{-\infty}(X, \mathbb{E}_\tau)$ onto $PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$.*

(b) *Similarly, if we consider an additional K-representation (γ, E_γ) then the Fourier transform ${}_\gamma\mathcal{F}_\tau$ is a linear bijection between the two spaces $C_r^{-\infty}(G, \gamma, \tau)$ and ${}_\gamma PWS_{\tau,r}(\mathfrak{a}_{\mathbb{C}}^*)$, for each $r \geq 0$, and a linear topological isomorphism from $C_c^{-\infty}(G, \gamma, \tau)$ onto ${}_\gamma PWS_\tau(\mathfrak{a}_{\mathbb{C}}^*)$.*

Remark 6.4. Delorme proved in his paper [6] the Paley-Wiener(-Schwartz) theorem in (Level 1) for the Hecke algebra

$$\mathcal{H}(G, K) := C_{r=0}^{-\infty}(G)_K \cong \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{k})} C^\infty(K)_K, \tag{36}$$

which consists of all $K \times K$-finite distributions on G supported by $K \subset G$.

Harish-Chandra inversion and Plancherel Theorem for sections

In order to prove Theorem 6.3, we need the Harish-Chandra Plancherel inversion formula for sections of homogeneous vector bundles.

Theorem 6.5. (Plancherel Theorem for sections, [5], Thm. 3.4 and Thm. 4.3) *Let $\mathcal{Q}$ be a complete set of representatives of association classes of cuspidal parabolic subgroups $Q = M_Q A_Q N_Q$ of G with $Q \supset P = MAN$ and $A_Q \subset A$. We have $\mathfrak{a}^* = \mathfrak{a}_Q^* \oplus \mathfrak{a}_{M_Q}^*$. Then, there exists a finite set $A_Q^\tau \subset \mathfrak{a}_{M_Q}^* \subset \mathfrak{a}^*$, and for $\nu \in A_Q^\tau$ there exists an analytic function of at most polynomial growth*

$$\mu_\nu^Q : i\mathfrak{a}_Q^* \longrightarrow \mathrm{End}_M(E_\tau)$$

such that for each $f \in C_c^\infty(X, \mathbb{E}_\tau)$, we have

$$f(e) = \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \tau(k)\mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\,\mathrm{d}k\,\mathrm{d}\lambda. \qquad \blacksquare$$

Note that $A_P^\tau = \{0\}$.

Corollary 6.6. *With the notations above, let $f \in C_c^\infty(X, \mathbb{E}_\tau)$ and $\varphi \in C_c^\infty(X, \mathbb{E}_{\tilde\tau})$. Then*

$$\int_G \langle \varphi(g), f(g)\rangle_\tau \,\mathrm{d}g$$

$$= \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde\tau}(\varphi)(-\nu-\lambda, k), \mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\rangle_\tau \,\mathrm{d}k\,\mathrm{d}\lambda. \tag{37}$$

Proof. Let $\{\tilde v_i, i = 1, \ldots, d_\tau\}$ be a basis of $E_{\tilde\tau}$. We write $\varphi = \sum_{i=1}^{d_\tau} \varphi_i \cdot \tilde v_i$ with $\varphi_i \in C_c^\infty(G)$. For $h \in C_c^\infty(G)$, we set $h^\vee(g) := h(g^{-1})$. Then

$$\int_G \langle \varphi(g), f(g)\rangle \,\mathrm{d}g = \sum_{i=1}^{d_\tau} \langle (\varphi_i^\vee * f)(e), \tilde v_i\rangle,$$

where we used the usual convolution defined in (16). Note that $h * f = l(h)f$, where l is the (left) regular representation on $C_c^\infty(X, \mathbb{E}_\tau)$. By the G-equivariance of the Fourier transform, we have by (17): $\mathcal{F}_\tau(h*f)(\lambda, k) = \pi_{\tau,\lambda}(h)(\mathcal{F}_\tau(f)(\lambda,\cdot))(k)$. By applying Theorem 6.5, we obtain for all $i \in \{1, \ldots, d_\tau\}$

$$\langle \tilde v_i, (\varphi_i^\vee * f)(e)\rangle = \sum_{Q,\nu} \int_{i\mathfrak{a}_Q^*} \int_K \langle \tilde v_i, \tau(k)\mu_\nu^Q(\lambda)\pi_{\tau,\nu+\lambda}(\varphi_i^\vee)(\mathcal{F}_\tau(f)(\nu+\lambda,\cdot))(k)\rangle \,\mathrm{d}k\,\mathrm{d}\lambda.$$

Using that μ_ν^Q commutes with $\pi_{\tau,\nu+\lambda}$ and that integration over K gives a G-equivariant pairing between $H_\infty^{\tau,\nu+\lambda}$ and $H_\infty^{\tilde\tau,-(\nu+\lambda)}$, we obtain that the K-integral above equals

$$\begin{aligned} &\int_K \langle \tilde\tau(k^{-1})\tilde v_i,\ \ \pi_{\tau,\nu+\lambda}(\varphi_i^\vee)\mu_\nu^Q(\lambda)(\mathcal{F}_\tau(f)(\nu+\lambda,\cdot))(k)\rangle \,\mathrm{d}k \\ &\qquad = \int_K \langle (\pi_{\tilde\tau,-(\nu+\lambda)}(\varphi_i)\tilde\tau(\cdot)^{-1}\tilde v_i)(k),\ \ \mu_\nu^Q(\lambda)\mathcal{F}_\tau(f)(\nu+\lambda, k)\rangle \,\mathrm{d}k. \end{aligned}$$

Now

$$\begin{aligned} (\pi_{\tilde\tau,-(\nu+\lambda)}(\varphi_i)\tilde\tau(\cdot)^{-1}\tilde v_i)(k) &= \int_G \varphi_i(g)\tilde\tau(\kappa(g^{-1}k))^{-1}a(g^{-1}k)^{\nu+\lambda-\rho}\tilde v_i \,\mathrm{d}g \\ &= \int_G \varphi_i(g)e^{\tilde\tau}_{-(\nu+\lambda),k}(g)\tilde v_i \,\mathrm{d}g. \end{aligned}$$

The sum over all i equals $\mathcal{F}_{\tilde{\tau}}(\varphi)(-(\nu+\lambda),k)$. Combining all the previous formulas, we obtain the corollary. ■

Proof of the topological Paley-Wiener-Schwartz Theorem 6.3

For $r \geq 0$, let us first provide the bijection between the vector spaces $C_r^{-\infty}(X,\mathbb{E}_\tau)$ and $PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$.

Proposition 6.7. *Consider a K-representation (τ, E_τ).*

(a) *Let $T \in C_c^{-\infty}(X,\mathbb{E}_\tau)$ such that $\mathcal{F}_\tau(T)=0$, then $T=0$.*

(b) *For $r \geq 0$ and $\tilde{T} \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$, there exists $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$ such that $\tilde{T} = \mathcal{F}_\tau(T)$.*

(c) *For $r \geq 0$ and $T \in C_r^{-\infty}(X,\mathbb{E}_\tau)$, we have $\mathcal{F}_\tau(T) \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$.*

Proof. For each $\epsilon > 0$, consider $\eta_\epsilon \in C^\infty(G,\tau,\tau)$ with compact support in the closed ball $\overline{B}_\epsilon(e)$ as in Eq. (19). Let $T \in C_c^{-\infty}(X,\mathbb{E}_\tau)$ be a distribution, then

$$T_\epsilon := T * \eta_\epsilon \in C_c^\infty(X,\mathbb{E}_\tau),$$

see Remark 3.6(b). Moreover, by using the same arguments as in the proof of Cor. 3.7, we have that $T_\epsilon \overset{\epsilon\to 0}{\longrightarrow} T$ (weakly). Hence, by the Paley-Wiener Theorem 5.2, this implies that $\mathcal{F}_\tau(T_\epsilon) \in PW_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$. Note that $\mathcal{F}_\tau(T_\epsilon)$ is holomorphic on $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and it satisfies the conditions $(2.i)$ and $(2.ii)_r$ of Definition 5.1. Furthermore, by Remark 3.6(b) after Proposition 3.5, we have

$$\mathcal{F}_\tau(T_\epsilon)(\lambda,k) = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\mathcal{F}_\tau(T)(\lambda,k), \quad (\lambda,k) \in \mathfrak{a}_\mathbb{C}^* \times K/M. \tag{38}$$

Due to Corollary 3.7, ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ converges uniformly on compact subsets of $\mathfrak{a}_\mathbb{C}^*$ to the identity map, whenever ϵ tends to 0. Hence, $\lim_{\epsilon\to 0}\mathcal{F}_\tau(T_\epsilon) = \mathcal{F}_\tau(T)$ uniformly on compact sets of $\mathfrak{a}_\mathbb{C}^*$.

(a) Now assume that $\mathcal{F}_\tau(T)=0$. By (38), we have that $\mathcal{F}_\tau(T_\epsilon)=0$. By applying the Paley-Wiener Thm. 5.2, this implies that $T_\epsilon = 0$. Hence, since $T_\epsilon \overset{\epsilon\to 0}{\longrightarrow} T$ weakly, we have that $T=0$.

(b) Consider $\psi \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$. For each $\epsilon > 0$ and $h \in C_c^\infty(X,\mathbb{E}_{\tilde{\tau}})$, let T_ϵ be the functional given by

$$T_\epsilon(h) := \sum_{Q\in\mathcal{Q}}\sum_{\nu\in A_Q^\tau}\int_{i\mathfrak{a}_Q^*}\int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu-\lambda,k) \\ \mu_\nu^Q(\lambda)\,{}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\nu+\lambda)\psi(\nu+\lambda,k)\rangle\,\mathrm{d}k\,\mathrm{d}\lambda \tag{39}$$

with the notations introduced in Theorem 6.5. Since $\mathrm{supp}(\eta_\epsilon) \subset \overline{B}_\epsilon(e)$, by the second part of Theorem 5.2 the function ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)$ satisfies the growth condition $(3.ii)_\epsilon$ of Definition 5.1. On the other hand, ψ satisfies the growth condition $(2.iis)_r$ of Definiition 6.2. This implies that for each multi-index $\alpha \in \mathbb{N}_0^l$ and $N \in \mathbb{N}_0$, there exists a constant $C_{r,N,\alpha} > 0$ such that

$$|l_{Y_\alpha}\,{}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\psi(\lambda,k)| \leq C_{r,N,\alpha}(1+|\lambda|^2)^{-N}e^{(r+\epsilon)|\mathrm{Re}(\lambda)|}, \quad (\lambda,k) \in \mathfrak{a}_\mathbb{C}^* \times K. \tag{40}$$

In addition, for each intertwining datum (ξ, W), the induced element

$$({}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi)_{\overline{\xi}} \in H_{\xi}^{\tau|_M}$$

satisfies the intertwining condition $(2.iii)$ of Definition 5.1. In fact, going over to the truncated Taylor series is multiplicative: $({}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi)_{\overline{\xi}} = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}}\psi_{\overline{\xi}}$. For $t \in D_W^\tau$, we have $t \circ {}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}} \in D_W^\tau$ by the intertwining condition $(3.iii)$, and since $\psi \in PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ satisfies $(2.ii)$, we obtain

$$t \circ ({}_\tau\mathcal{F}_\tau(\eta_\epsilon)_{\overline{\xi}} \circ \psi_{\overline{\xi}}) \in W.$$

Together with (40), we see immediately that ${}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi \in PW_{\tau,r+\epsilon}(\mathfrak{a}_\mathbb{C}^* \times K/M)$. Therefore, we can conclude by the Paley-Wiener Theorem 5.2, that there exists a unique function $f_\epsilon \in C_{r+\epsilon}^\infty(X, \mathbb{E}_\tau)$ such that

$$\mathcal{F}_\tau(f_\epsilon) = {}_\tau\mathcal{F}_\tau(\eta_\epsilon)\psi.$$

On the other hand, by (39) and Corollary 6.6, we have $T_\epsilon = f_\epsilon$. In particular, $\operatorname{supp}(T_\epsilon) \subset \overline{B}_{r+\epsilon}(o)$. Now we take $\lim_{\epsilon\to 0}$ in (39). Using Corollary 3.7 and that $\mathcal{F}_{\tilde{\tau}}(h)$ is rapidly decreasing as well as $\mu_\nu^Q, \psi(\nu + \cdot)$ are slowly increasing in $i\mathfrak{a}_Q^*$, we can interchange the limit and the integral. We obtain

$$T_\epsilon(h) \overset{\epsilon\to 0}{\longrightarrow} T(h), \quad \text{where}$$

$$T(h) := \sum_{Q\in\mathcal{Q}} \sum_{\nu\in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu-\lambda, k),\ \mu_\nu^Q(\lambda)\psi(\nu+\lambda, k)\rangle \,\mathrm{d}k \,\mathrm{d}\lambda \tag{41}$$

with $\operatorname{supp}(T) \subset \overline{B}_r(o)$. Note that T is continuous on $C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$. Since T is compactly supported, we can set $h := e_{\lambda,k}^\tau$. In conclusion, we have found a distribution $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$ such that

$$\begin{aligned}\mathcal{F}_\tau(T)(\lambda, k) = T(e_{\lambda,k}^\tau) &\overset{(6)}{=} \lim_{\epsilon\to 0} T_\epsilon(e_{\lambda,k}^\tau) = \lim_{\epsilon\to 0} \mathcal{F}_\tau(f_\epsilon)(\lambda, k)\\ &= \lim_{\epsilon\to 0} {}_\tau\mathcal{F}_\tau(\eta_\epsilon)(\lambda)\psi(\lambda, k) = \psi(\lambda, k), \quad \lambda \in \mathfrak{a}_\mathbb{C}^*, k \in K.\end{aligned}$$

(c) Let us check that for $r \geq 0$, $\mathcal{F}_\tau(T) \in PWS_{\tau,r}(\mathfrak{a}_\mathbb{C}^* \times K/M)$. This means that we need to verify that the Fourier transform of $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$ satisfies the conditions $(2.i), (2.iis)_r, (2.iii)$ of Definition 6.2.

The condition $(2.i)$ is immediate. Concerning the intertwining condition $(2.iii)$, in order to show that for each intertwining datum (ξ, W) and $t \in D_W^\tau$, we have

$$t \circ (\mathcal{F}_\tau(T))_{\overline{\xi}} \in W \subseteq H_\xi,$$

we will use a similar convolution argument as above, except that now we are interested in left convolution instead of right convolution. For each $\epsilon > 0$, let $\delta_\epsilon \in C_\epsilon^\infty(G)$ be such that $\lim_{\epsilon\to 0} \delta_\epsilon = \delta_0$. Therefore, $\lim_{\epsilon\to 0} \delta_\epsilon * T = T$, for $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$. Moreover, for all Fréchet representations (π, H) of G and $v \in H$, we have $\pi(\delta_\epsilon)v \overset{\epsilon\to 0}{\longrightarrow} v$. By taking the Fourier transform of $\delta_\epsilon * T \in C_r^\infty(X, \mathbb{E}_\tau)$, we first prove that for each intertwining datum (ξ, W) and $t \in D_W^\tau$:

$$\lim_{\epsilon\to 0} t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}} \in W.$$

We denote by π_ξ and $\pi_{\overline{\xi}}$ the natural G-representations on H_ξ and $H_{\overline{\xi}}^{\tau|_M}$, respectively. By the Paley-Wiener Theorem 5.2, we already know that

$$t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}} \in W.$$

Then, we have $W \ni t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}} \overset{(17)}{=} t \circ (\pi_{\tau,\cdot}(\delta_\epsilon)\mathcal{F}_\tau(T))_{\overline{\xi}} = t \circ (\pi_{\overline{\xi}}(\delta_\epsilon)\mathcal{F}_\tau(T)_{\overline{\xi}})$. For $\overline{t} \in \mathrm{Hom}_K(E_\tau, E_\sigma)$, we have $\overline{t} \circ \pi_{\tau,\lambda}(\delta_\epsilon) = \pi_{\sigma,\lambda}(\delta_\epsilon) \circ \overline{t}$. This implies for $t \in \bigoplus_{i=1}^s \mathrm{Hom}_K(E_\tau, E_{\sigma_i})_{(m_i)}^{\lambda_i}$ that $t \circ \pi_{\overline{\xi}}(\delta_\epsilon) = \pi_\xi(\delta_\epsilon) \circ t$, and we obtain

$$t \circ \mathcal{F}_\tau(\delta_\epsilon * T)_{\overline{\xi}} = \pi_\xi(\delta_\epsilon)(t \circ \mathcal{F}_\tau(T)_{\overline{\xi}}).$$

Hence, by taking $\epsilon \to 0$ and since W is closed, we obtain that $t \circ (\mathcal{F}_\tau(T))_{\overline{\xi}} \in W$.

It remains to check that $\mathcal{F}_\tau(T)$ satisfies the slow growth condition $(2.iis)_r$. Fix $r \geq 0$. We need to show that there exists $N \in \mathbb{N}_0$ such that for each multi-index α

$$|l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda, k)| \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N + \frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|}$$

for some constant $C_{r,N,\alpha} > 0$. Note that $l_{Y_\alpha}\mathcal{F}_\tau(T) = \mathcal{F}_\tau(l_{Y_\alpha}T)$. Let $m \in \mathbb{N}_0$ be the order of the distribution $T \in C_r^{-\infty}(X, \mathbb{E}_\tau)$. Write $X_\beta \in \mathcal{U}(\mathfrak{n})$ and $H_\gamma \in \mathcal{U}(\mathfrak{a})$ for all multi-indices β, γ. Since $G/K \cong NA$ and $\mathcal{U}(\mathfrak{n} \oplus \mathfrak{a}) \cong \mathcal{U}(\mathfrak{n})\mathcal{U}(\mathfrak{a})$, then, there exists a constant $C > 0$ such that

$$|T(h)| \leq C \sum_{|\beta|+|\gamma| \leq m} \sup_{g \in \overline{B}_r(o)} |(l_{X_\beta}(l_{H_\gamma}h))(g)|, \quad h \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}). \tag{42}$$

Next, we want to apply it to $h = e_{\lambda,1}^\tau$. We observe that

$$l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda, k) = \mathcal{F}_\tau(l_{Y_\alpha}T)(\lambda, k) = (l_{Y_\alpha}T)(e_{\lambda,k}^\tau) \overset{(12)}{=} (l_{Y_\alpha}T)(l_k e_{\lambda,1}^\tau) = (l_{k^{-1}}l_{Y_\alpha}T)(e_{\lambda,1}^\tau).$$

Moreover, since $l_{k^{-1}}l_{Y_\alpha}T$ is a distribution of order $m + |\alpha|$, applying (42) to $(l_{k^{-1}}l_{Y_\alpha}T)(h)$ instead of $T(h)$, we obtain

$$\sup_{k \in K} |(l_{k^{-1}}l_{Y_\alpha}T)(h)| \leq C' \sum_{|\beta|+|\gamma| \leq m+|\alpha|} \sup_{g \in \overline{B}_r(o)} |(l_{X_\beta}(l_{H_\gamma}h))(g)|, \quad h \in C^\infty(X, \mathbb{E}_{\tilde{\tau}}).$$

In fact, since K is compact and operates continuously on $C_c^{-\infty}(X, \mathbb{E}_\tau)$, the constant C' can be chosen independently of $k \in K$. Moreover, $e_{\lambda,1}^\tau$ is annihilated by each l_{X_β} for $\beta \neq 0$ and it is an eigenfunction of each l_{H_γ} with eigenvalue a polynomial in $\lambda \in \mathfrak{a}_\mathbb{C}^*$ of degree $\leq |\gamma|$, i.e.,

$$|l_{Y_\alpha}\mathcal{F}_\tau(T)(\lambda, k)| = |(l_{k^{-1}}l_{Y_\alpha}T)(e_{\lambda,1}^\tau)| \leq C_{r,N,\alpha}(1 + |\lambda|^2)^{N + \frac{|\alpha|}{2}} e^{r|\mathrm{Re}(\lambda)|} \tag{43}$$

for $N \geq \frac{m}{2}$. This completes the proof. ■

Consequently, by (6), the *inverse Fourier transform* of $\psi \in PWS_\tau(\mathfrak{a}_\mathbb{C}^* \times K/M)$ for a test function $h \in C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$ is given by

$$\langle \mathcal{F}_\tau^{-1}(\psi), h \rangle := \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(h)(-\nu - \lambda, k),\ \mu_\nu^Q(\lambda)\psi(\nu + \lambda, k) \rangle \,\mathrm{d}k \,\mathrm{d}\lambda.$$

Finally, we discuss the topology on the image space by which the Fourier transform becomes a topological isomorphism.

Lemma 6.8.

(a) *The Fourier transform* $\mathcal{F}_\tau : C_c^{-\infty}(X, \mathbb{E}_\tau) \longrightarrow PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ *is continuous.*

(b) *The following inverse Fourier transform is continuous:*

$$\mathcal{F}_\tau^{-1} : PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \longrightarrow C_c^{-\infty}(X, \mathbb{E}_\tau). \tag{44}$$

Proof. (a) We will show that for each bounded $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$, there exist $r \geq 0$ and $N \in \mathbb{N}_0$ such that $\mathcal{F}_\tau(B')$ is contained as a bounded set in $PWS_{\tau,r,N}$.

Since $PWS_{\tau,r,N} \hookrightarrow PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$ is continuous, by definition of inductive limit, then the set $\mathcal{F}_\tau(B')$ is also bounded in $PWS_\tau(\mathfrak{a}_{\mathbb{C}}^* \times K/M)$. By Lem. 6.1, we will have that $\mathcal{F}_\tau$ is continuous.

Now let $B' \subset C_c^{-\infty}(X, \mathbb{E}_\tau)$ be bounded. Since B' is equicontinuous and because of (35), there exist $r \geq 0, m \in \mathbb{N}_0$ and a constant $C > 0$ such that

$$\sup_{T \in B'} p_B(T) = \sup_{T \in B',\, \varphi \in B} |T(\varphi)| \leq C \sum_{|\alpha| \leq m} \sup_{g \in \overline{B}_r(o)} |l_{Y_\alpha} \varphi(g)|, \quad B \subset C^\infty(X, \mathbb{E}_{\tilde{\tau}}). \tag{45}$$

Here, Y_α refers to a basis $Y_1, \ldots, Y_n$ of G. Note that (45) just means that (42) holds uniformly for all $T \in B'$. The same argument that shows (42)$\Rightarrow$(43), in the proof of Prop. 6.7 (c), now gives that for $N \geq \frac{m}{2}$ there exists a constant $C_{r,N,\alpha}$ such that

$$\|\mathcal{F}_\tau(T)\|_{r,N,\alpha} \leq C_{r,N,\alpha}, \quad T \in B',$$

i.e., $\mathcal{F}_\tau(B') \subset PWS_{\tau,r,N}$ is bounded. Hence the Fourier transform is continuous.

(b) It suffices to show that for all $r \geq 0$ and $N \in \mathbb{N}_0$,

$$\mathcal{F}_\tau^{-1} : PWS_{\tau,r,N}(\mathfrak{a}_{\mathbb{C}}^* \times K/M) \longrightarrow C^{-\infty}(X, \mathbb{E}_\tau) \tag{46}$$

is continuous. Indeed, by the construction of the inductive limit topology and the remark between (33) and (34), as well as using $\mathcal{F}_\tau^{-1}(PWS_{\tau,r,N}) \subset C_r^{-\infty}(X, \mathbb{E}_\tau)$ assertion (46) implies assertion (44).

Fix $r \geq 0$ and $N \in \mathbb{N}_0$. To establish (46), it suffices to show that for every bounded $\tilde{B} \subset C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, we have

$$p_{\tilde{B}}(\mathcal{F}_\tau^{-1}(\psi)) \leq C \|\psi\|_{r,N,0}, \quad \psi \in PWS_{\tau,r,N},$$

where $p_{\tilde{B}}(\cdot)$ is the seminorm (33) and C is a positive constant. Since $\tilde{B}$ is bounded subset in $C_c^\infty(X, \mathbb{E}_{\tilde{\tau}})$, there exists $R \geq 0$ so that the support of any $\varphi \in \tilde{B}$ is in $\overline{B}_R(o)$. Thus, for $\psi \in PWS_{\tau,r,N}$, we have that

$$\begin{aligned}
& p_{\tilde{B}}(\mathcal{F}_\tau^{-1}(\psi)) \\
\overset{(33)}{=} \; & \sup_{\varphi \in \tilde{B}} |\langle \mathcal{F}_\tau^{-1}(\psi), \varphi \rangle| \\
\overset{(37)}{=} \; & \sup_{\varphi \in \tilde{B}} \Big| \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k) \,,\, \mu_\nu^Q(\lambda) \psi(\nu + \lambda, k) \rangle \, \mathrm{d}k \, \mathrm{d}\lambda \Big| \\
\leq \; & \sup_{\varphi \in \tilde{B}} \sum_{Q \in \mathcal{Q}} \sum_{\nu \in A_Q^\tau} \int_{i\mathfrak{a}_Q^*} \int_K \Big| \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu - \lambda, k) \,,\, \mu_\nu^Q(\lambda) \psi(\nu + \lambda, k) \rangle \Big| \, \mathrm{d}k \, \mathrm{d}\lambda.
\end{aligned}$$

Fix now $Q \in \mathcal{Q}$ and $\nu \in A_Q^\tau$. Set

$$d_{Q,\nu} := \sup_{\varphi \in \tilde{B}} \int_{i\mathfrak{a}_Q^*} \int_K \Big| \langle \mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k) \ , \ \mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)\rangle \Big| \, \mathrm{d}k \, \mathrm{d}\lambda.$$

It suffices to show that $d_{Q,\nu} \leq C \|\psi\|_{r,N,0}$. We have

$$\begin{aligned} d_{Q,\nu} &\leq \sup_{\varphi \in \tilde{B}} \int_{i\mathfrak{a}_Q^*} \int_K (1+|\nu{+}\lambda|^2)^{-d_Q} (1+|\nu{+}\lambda|^2)^{d_Q} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\qquad\qquad \times |\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)| \, \mathrm{d}k \, \mathrm{d}\lambda \\ &\leq C \sup_{\substack{\varphi \in \tilde{B} \\ k \in K, \lambda \in i\mathfrak{a}_Q^*}} (1+|\nu{+}\lambda|^2)^{d_Q} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \, |\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)|, \end{aligned}$$

where $d_Q \in \mathbb{N}_0$ depends on the dimension of $i\mathfrak{a}_Q^*$ such that

$$C := \int_{i\mathfrak{a}_Q^*} (1+|\nu{+}\lambda|^2)^{-d_Q} \, \mathrm{d}\lambda < \infty.$$

For some positive constant N and growth constant $m \in \mathbb{N}_0$, we get

$$\begin{aligned} d_{Q,\nu} &\leq C \sup_{\substack{\varphi \in \tilde{B} \\ k \in K, \lambda \in i\mathfrak{a}_Q^*}} (1+|\nu{+}\lambda|^2)^{d_Q+N+m} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\quad \times \sup_{k \in K, \lambda \in i\mathfrak{a}_Q^*} (1+|\nu{+}\lambda|^2)^{-(N+m)} |\mu_\nu^Q(\lambda)\psi(\nu+\lambda,k)| \\ &\leq C' \sup_{\substack{\varphi \in \tilde{B} \\ k \in K, \lambda \in i\mathfrak{a}_Q^*}} (1+|\nu{+}\lambda|^2)^{d_Q+N+m} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\quad \times \sup_{k \in K, \lambda \in i\mathfrak{a}_{\mathbb{C}}^*} (1+|\nu+\lambda|^2)^{-N} |\psi(\nu+\lambda,k)|, \end{aligned}$$

where $\|\mu_\nu^Q(\lambda)\|_{\mathrm{op}} \leq C'(1+|\nu{+}\lambda|^2)^m$ of at most polynomial growth of $m \in \mathbb{N}_0$. We incorporate the constants $e^{-r|\nu|}$ and $e^{-R|\nu|}$ into the estimate and obtain

$$\begin{aligned} d_{Q,\nu} &\leq C'' \sup_{\substack{\varphi \in \tilde{B} \\ k \in K, \lambda \in i\mathfrak{a}_Q^*}} e^{-R|\nu|} (1+|\nu{+}\lambda|^2)^{d_Q+N+m} |\mathcal{F}_{\tilde{\tau}}(\varphi)(-\nu-\lambda,k)| \\ &\quad \times \sup_{k \in K, \lambda \in i\mathfrak{a}_Q^*} e^{-r|\nu|} (1+|\nu{+}\lambda|^2)^{-N} |\psi(\nu+\lambda,k)| \\ &= C'' \sup_{\varphi \in \tilde{B}} \|\mathcal{F}_{\tilde{\tau}}(\varphi)\|_{R,-(d_Q+N+m),0} \|\psi\|_{r,N,0}. \end{aligned}$$

By the Paley-Wiener Theorem 5.2, $\mathcal{F}_{\tilde{\tau}}$ is continuous, thus

$$\sup_{\varphi \in \tilde{B}} \|\mathcal{F}_{\tilde{\tau}}(\varphi)\|_{R,-(d_Q+N+m),0} < \widetilde{C} < \infty.$$

Hence, $d_{Q,\nu} \leq C''' \|\psi\|_{r,N,0}$ and hence the inverse Fourier transform is continuous. ∎

End of the proof of Theorem 6.3. That the Fourier transform $\mathcal{F}_\tau$ is a linear isomorphism is an outcome of Proposition 6.7. The continuity and topology statement results from Lemma 6.8.

We establish the (topological) isomorphisms in (Level 3) by treating $C_c^{-\infty}(G,\gamma,\tau)$ instead of $C_c^{-\infty}(X,\mathbb{E}_\tau)$ in a complete analogous way. Alternatively, one can derive Theorem 6.3(b) by applying $\mathrm{Hom}_K(E_\gamma,\cdot)$ to Theorem 6.3(a), see the discussion in Section 3. ∎

7. Invariant differential operators and the Fourier range

We consider the vector space of distributional sections $C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau) := C^{-\infty}_{r=0}(X,\mathbb{E}_\tau)$ supported at the origin $o = eK \in X$. Since $g{\cdot}o \neq o$, G does not act on $C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)$, but K as well as $\mathfrak{g}$ do, thus $C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)$ is a $(\mathfrak{g},K)$-module (e.g. [24], 3.3.1). Moreover, it is generated by the so-called *vector-valued Dirac delta-distributions* δ_v at $v\in E_\tau$:

$$\delta_v(f) = \langle v, f(e)\rangle_\tau, \quad f\in C^\infty(X,\mathbb{E}_{\tilde\tau}),$$

where $\langle\cdot,\cdot\rangle_\tau$ denotes the pairing between E_τ and $E_{\tilde\tau}$. In particular, we have the following identification:

$$\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} E_\tau \overset{\beta}{\cong} C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)$$

given by $\beta(Z\otimes v)(f) := \langle r_Z f(e), v\rangle_\tau$, for $Z\in\mathcal{U}(\mathfrak{g}), v\in E_\tau, f\in C^\infty(X,\mathbb{E}_{\tilde\tau})$, with actions $Y(Z\otimes v) = YZ\otimes v$, and $k(Z\otimes v) = \mathrm{Ad}(k)Z\otimes\tau(k)v$, for $Y\in\mathfrak{g}$ (or $\mathcal{U}(\mathfrak{g})$), $k\in K$.

In addition, every invariant differential operator $D\in\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)$ may be viewed as a linear map between these spaces $D : C^{-\infty}_{\{o\}}(X,\mathbb{E}_\gamma)\longrightarrow C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)$. This map defines an element

$$H_D\in\mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau))\cong[C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)\otimes E_{\tilde\gamma}]^K$$

given by $$H_D(v) := D(\delta_v)\in C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau),\quad v\in E_\gamma, \delta_v\in C^{-\infty}_{\{o\}}(X,\mathbb{E}_\gamma). \tag{47}$$

In other words, for $f\in C^\infty(X,\mathbb{E}_\tau)$,

$$\langle H_D(v), f\rangle_\tau \overset{(47)}{=} \langle\delta_v, D^t(f)\rangle_\gamma = \langle v, D^t(f)(1)\rangle_\gamma, \tag{48}$$

where $D^t\in\mathcal{D}_G(\mathbb{E}_{\tilde\tau},\mathbb{E}_{\tilde\gamma})$ is the adjoint operator of D. Since the graded space of both filtered spaces $\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)$ and $\mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau))$ is isomorphic to $[S(\mathfrak{p})\otimes\mathrm{Hom}(E_\gamma,E_\tau)]^K$, we have the following isomorphism:

$$\begin{array}{ccc}\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau) & \xrightarrow{\sim} & \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau))\\ D & \mapsto & H_D.\end{array}$$

Here, $S(\mathfrak{p})$ denotes the symmetric algebra of the complexification of $\mathfrak{p}\subset\mathfrak{g}$. Consequently, we have $\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)\cong C^{-\infty}_{r=0}(G,\gamma,\tau)$. Hence, we can apply the Fourier transform in (Level 3) to invariant differential operators. Then, the Paley-Wiener-Schwartz Thm. 6.3 (b) tells us that

$${}_\gamma\mathcal{F}_\tau(\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)) = {}_\gamma PWS_{\tau,0}(\mathfrak{a}^*_{\mathbb{C}}).$$

Using that holomorphic functions of polynomial growth are polynomials, we can deduce the following result.

Proposition 7.1. *With the notations above, we have*

$$ {}_\gamma\mathcal{F}_\tau(\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)) = \{P\in \mathrm{Pol}(\mathfrak{a}^*_{\mathbb{C}}, \mathrm{Hom}_M(E_\gamma,E_\tau)) \mid P \textit{ satisfies } (3.iii) \textit{ of Def. } 6.2\}. \ \blacksquare $$

Thus, provided one has a good understanding of the intertwining condition $(3.iii)$, one can determine $\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)$. In simple cases, that is worked out in [19].

The converse holds by van den Ban's and Souaifi's Lemma 5.3 and Corollary 5.4 in [4]. Strictly speaking, these results are in terms of the Hecke algebra (36). But the $(\gamma,\tilde\tau)$-isotypic component $\mathcal{H}(G,K)(\gamma\otimes\tilde\tau)$ of the Hecke algebra is exactly $\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)\otimes \mathrm{Hom}(E_\tau,E_\gamma)$: By Peter-Weyl's theorem, we have $\mathrm{Hom}_K(E_{\tilde\tau},C^\infty(K)_K)\cong E_\tau$, therefore

$$ \begin{aligned} \mathcal{H}(G,K)(\gamma\otimes\tilde\tau) &\cong \mathrm{Hom}_{K\times K}(E_\gamma\otimes E_{\tilde\tau}, \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} C^\infty(K)_K)\otimes (E_\gamma\otimes E_{\tilde\tau}) \\ &\cong \mathrm{Hom}_K(E_\gamma, \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{k})} \mathrm{Hom}_K(E_{\tilde\tau}, C^\infty(K)_K))\otimes \mathrm{Hom}(E_\tau,E_\gamma) \\ &\cong \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau))\otimes \mathrm{Hom}(E_\tau,E_\gamma) \\ &\cong \mathcal{D}_G(E_\gamma,\mathbb{E}_\tau)\otimes \mathrm{Hom}(E_\tau,E_\gamma). \end{aligned} $$

In other words, given all invariant differential operators $D\in\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)$, one can determine explicitly the intertwining condition $(3.iii)$ and the corresponding Paley-Wiener space.

Moreover, we remark that the isomorphism in Prop. 7.1 can also be described more algebraically as a Harish-Chandra type homomorphism, we refer to ([18], p. 4) or ([21], Section 2.1) for more details.

In addition, we also have the following result.

Proposition 7.2. *Let $D\in\mathcal{D}_G(\mathbb{E}_\gamma,\mathbb{E}_\tau)$ be an invariant linear differential operator. For $f\in C^{\pm\infty}_c(X,\mathbb{E}_\gamma)$, we then have that*

$$ \mathcal{F}_\tau(Df)(\lambda,k) = {}_\gamma\mathcal{F}_\tau(H_D)(\lambda)\mathcal{F}_\gamma(f)(\lambda,k), \qquad \lambda\in\mathfrak{a}^*_{\mathbb{C}}, k\in K, \tag{49} $$

where ${}_\gamma\mathcal{F}_\tau(H_D)\in\mathrm{Pol}(\mathfrak{a}^*_{\mathbb{C}},\mathrm{Hom}_M(E_\gamma,E_\tau))$ is a polynomial in $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$ with values in $\mathrm{Hom}_M(E_\gamma,E_\tau)$.

Proof. We know that the Fourier transform of a distribution

$$ H_D\in \mathrm{Hom}_K(E_\gamma, C^{-\infty}_{\{o\}}(X,\mathbb{E}_\tau)) $$

is defined by ${}_\gamma\mathcal{F}_\tau(H_D)(\lambda)(v) = \langle H_D(v), e^\tau_{\lambda,1}\rangle$, for $v\in E_\gamma$, where $e^\tau_{\lambda,1}\in C^\infty(G,\tau,\tilde\tau)$. Hence, by (48), we obtain for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$,

$$ {}_\gamma\mathcal{F}_\tau(H_D)(\lambda)(v) = \langle H_D(v), e^\tau_{\lambda,1}\rangle_\tau \overset{(48)}{=} \langle v, D^t(e^\tau_{\lambda,1})(1)\rangle_\gamma = (D^t(e^\tau_{\lambda,1})(1))v. \tag{50} $$

Now, by considering a function $f\in C^\infty_c(X,\mathbb{E}_\gamma)$, we conclude, via 'partial integration', that (49) holds. In fact

$$ \begin{aligned} \mathcal{F}_\tau(Df)(\lambda,k) &= \int_G e^\tau_{\lambda,k}(g)(Df)(g)\,\mathrm{d}g \overset{\text{def. of } D^t}{=} \int_G (D^t e^\tau_{\lambda,k})(g) f(g)\,\mathrm{d}g \\ &\overset{(21)}{=} \int_G (D^t e^\tau_{\lambda,1})(1)\circ e^\gamma_{\lambda,k}(g) f(g)\,\mathrm{d}g = (D^t e^\tau_{\lambda,1})(1)\mathcal{F}_\gamma(f)(\lambda,k) \\ &\overset{(50)}{=} {}_\gamma\mathcal{F}_\tau(H_D)(\lambda)\mathcal{F}_\gamma(f)(\lambda,k). \end{aligned} $$

The same computation remains true for $f \in C_c^{-\infty}(X, \mathbb{E}_\gamma)$, by using the pairing $\langle \cdot, \cdot \rangle$ instead of the integration. ∎

Remark 7.3. Consider an additional not necessarily irreducible K-representation (δ, E_δ). Then, for $D_1 \in \mathcal{D}_G(\mathbb{E}_\tau, \mathbb{E}_\delta)$ and $D_2 \in \mathcal{D}_G(\mathbb{E}_\gamma, \mathbb{E}_\tau)$, Proposition 7.2 implies that

$$ {}_\gamma\mathcal{F}_\delta(H_{D_1 \circ D_2})(\lambda) = {}_\tau\mathcal{F}_\delta(H_{D_1})(\lambda) \circ {}_\gamma\mathcal{F}_\tau(H_{D_2})(\lambda), \ \lambda \in \mathfrak{a}_{\mathbb{C}}^*. $$

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References

[1] J. Arthur: *A Paley-Wiener theorem for real reductive groups*, Acta Math. 150 (1983) 1–89.

[2] E. P. van den Ban, H. Schlichtkrull: *A Paley-Wiener theorem for reductive symmetric spaces*, Ann. Math. 164/3 (2006) 879–909.

[3] E. P. van den Ban, H. Schlichtkrull: *A Paley-Wiener theorem for distributions on reductive symmetric spaces*, J. Inst. Math. Jussieu 6/4 (2006) 557–577.

[4] E. P. van den Ban, S. Souaifi: *A comparison of Paley-Wiener theorems*, J. Reine Angew. Math. (Crelles Journal) 695 (2014) 99–149.

[5] R. Camporesi: *The Helgason Fourier transform for homogeneous vector bundles over Riemannian symmetric spaces*, Pacific J. Appl. Math. 179/2 (1997) 263–300.

[6] P. Delorme: *Sur le théorème de Paley-Wiener d'Arthur*, Ann. Math. 162/2 (2005) 987–1029.

[7] M. Eguchi, M. Hashizume, K. Okamoto: *The Paley-Wiener Theorem for distributions on symmetric spaces*, Hiroshima Math. J. 3 (1973) 109–120.

[8] R. Gangolli: *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. Math. 93 (1971) 159–165.

[9] S. Helgason: *An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces*, Math. Ann. 165 (1966) 297–308.

[10] S. Helgason: *Paley-Wiener theorems and surjectivity of invariant differential operators on symmetric spaces and Lie groups*, Bull. Amer. Math. Soc. 79 (1973) 129–132.

[11] S. Helgason: *Geometric Analysis on Symmetric Spaces*, American Mathematical Society, Providence (1994).

[12] S. Helgason: *Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators and Spherical Functions*, American Mathematical Society, Providence (2000).

[13] L. Hörmander: *The Analysis of Linear Partial Differential Operators I*, Springer, Berlin (1983).

[14] M. Kashiwara: *Equivariant derived category and representation of real semisimple Lie Groups*, in: *Representation Theory and Complex Analysis*, Lecture Notes in Mathematics Vol. 1931, Springer, Berlin (2008) 137–234.

[15] M. Kashiwara, W. Schmid: *Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups*, in: *Lie Theory and Geometry*, Progress in Mathematics Vol. 123, Birkhäuser, Boston (1994) 457–488.

[16] A. W. Knapp: *Representation Theory of Semisimple Groups. On Overview Based on Examples*, Princeton University Press, Princeton (1986).

[17] A. W. Knapp: *Lie Groups Beyond an Introduction*, 2nd ed., Birkhäuser, Basel (2002).

[18] M. Olbrich: *Die Poisson-Tranformation für homogene Vektorbündel*, Doctoral Thesis, Humboldt Universität, Berlin (1995).

[19] M. Olbrich, G. Palmirotta: *Delorme's intertwining conditions for sections of homogeneous vector bundles on two- and three-dimensional hyperbolic spaces*, Ann. Global Anal. Geom. 63/1 (2023), art. no. 9, 31p.

[20] M. Olbrich, G. Palmirotta: *Solvability of systems of invariant differential equations on $\mathbb{H}^2$ and beyond*, arXiv: 2206.01835 (2022).

[21] G. Palmirotta: *Solvability of Systems of Invariant Differential Equations on Symmetric Spaces G/K*, Doctoral Dissertation, University of Luxembourg (2021).

[22] H. H. Schaefer: *Topological Vector Spaces*, Graduate Texts in Mathematics Vol. 3, Springer, Berlin (1971).

[23] F. Trèves: *Topological Vector Spaces, Distributions, and Kernels*, Dover Publications, Mineola (1967).

[24] N. R. Wallach: *Real Reductive Groups I*, Academic Press, New York (1988).

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