

The Schur Multiplier of Some Finite Multiplicative Lie Algebras

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Communicated by A. Fialowski

Abstract. We classify all the multiplicative Lie algebra structures on symmetric groups and alternating groups. We also determine the Schur multiplier and the Lie exterior square of some finite multiplicative Lie algebras. For a non abelian simple group K with trivial Schur multiplier, we see that the Schur multiplier of multiplicative Lie algebra K is trivial and the Lie exterior square of K is isomorphic to K with improper multiplicative Lie algebra structure.

Mathematics Subject Classification: 15A75, 19C09, 20F12.

Key Words: Multiplicative Lie algebra, Schur multiplier, exterior square, commutator.

1. Introduction

The Schur multiplier of a group was introduced by Schur in 1904 during his works on projective representations of groups [14]. Given a group K , the Schur multiplier of K is the second cohomology group $H^2(K, \mathbb{C}^*)$, where K acts trivially on \mathbb{C}^* . Several attempts have been done to characterize the structure of the Schur multiplier for some classes of groups, for example, the dihedral group, metacyclic group, alternating group, quaternion group, and many other groups [6]. It has been a powerful tool in the classification of finite simple groups, and finite p -groups [5, 6, 9]. It is worth to describe the structure of the Schur multiplier for a given group.

The finite dimensional Lie algebra analogue to the Schur multiplier was developed in [2, 3]. Again, the classification of finite dimensional nilpotent Lie algebras according to the dimension of its Schur multiplier has been a line of investigation.

In 1993, G. J. Ellis [4] introduced an interesting algebraic structure called “multiplicative Lie algebra” which is a generalization of groups as well as of Lie algebras. In [12], F. Point and P. Wantiez introduced the concept of nilpotency for multiplicative Lie algebras and proved many nilpotency results which are known for groups and Lie algebras. In 2021, M. S. Pandey et al. [10] discussed another notion of solvable and nilpotent multiplicative Lie algebra with the help of a multiplicative Lie center and Lie commutator. To see more structural properties of multiplicative Lie algebra, in [1] Bak et al. constructed and studied two homology theories for multiplicative Lie algebras which are already known for groups and Lie algebras. The authors introduced and studied Schur multiplier in terms of homology. In 2018, Lal and Upadhyay studied the theory of extensions of multiplicative Lie algebras and introduced the Schur multiplier as the second cohomology of multiplicative Lie

algebras [7]. The authors also introduced the concept of the Lie exterior square of multiplicative Lie algebras and consequently, they described the Schur multiplier of a multiplicative Lie algebra as the group of non-trivial relations satisfied by the Lie product. Schur multiplier was also expressed in terms of presentations.

It is an interesting problem to characterize the structure of the Schur multiplier for multiplicative Lie algebras. Note that, given a group K , there is always a trivial multiplicative Lie algebra structure on K . Every non abelian group holds at least two multiplicative Lie algebra structures: one is trivial and the other one is given by a commutator. In [11], Pandey et al. determined all possible multiplicative Lie algebra structures for certain finite groups. In this paper, we give a complete classification (upto isomorphism) of multiplicative Lie algebra structures on symmetric groups S_n and the alternating groups A_n , for every n . We also find the Schur multipliers and the Lie exterior squares of certain classes of finite multiplicative Lie algebras given in [11]. At the end of the article, we give a table having the Schur multiplier of some precise multiplicative Lie algebras.

2. Preliminaries

In this section we recall some basics of the Schur multiplier of a group from [6], multiplicative Lie algebras, and second Lie cohomology from [7].

Definition 2.1. Let K be a multiplicative group and H be an abelian multiplicative group. We say that K acts on H if for each $x \in K$ and $a \in H$, there is a unique element of H denoted by a^x or ax , such that

1. $(ab)^x = a^x b^x$, for all $a, b \in H$ and $x \in K$;
2. $(a^x)^y = a^{xy}$, for all $a \in H$ and $x, y \in K$;
3. $a^1 = a$, for all $a \in H$.

A function $f : K^n \rightarrow H$ is called an n -cochain of K in H . The set of all n -cochains, written as $C^n(K, H)$, becomes an abelian group under the multiplication. For $n = 0$, we put $C^0(K, H) = H$. The formula

$$d_{n+1}(f(x_1, \dots, x_{n+1})) = f(x_2, \dots, x_{n+1}) \times \prod_{i=1}^n f(x_1, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_{n+1})^{(-1)^i} \times f(x_1, \dots, x_n)^{(-1)^{n+1}} x_{n+1}$$

determines a group homomorphism $d_{n+1} : C^n(K, H) \rightarrow C^{n+1}(K, H)$.

Thus, we have the following sequence of abelian groups:

$$\dots C^{m-1}(K, H) \xrightarrow{d_n} C^m(K, H) \xrightarrow{d_{n+1}} C^{m+1}(K, H) \dots$$

with the composition $d_{n+1} \circ d_n : C^{n-1}(K, H) \rightarrow C^{n+1}(K, H)$ is zero. So we have $\text{Im}(d_n) \subseteq \ker(d_{n+1})$. Denote $Z^n(K, H) = \ker(d_{n+1})$ and $B^n(K, H) = \text{Im}(d_n)$. The elements of $Z^n(K, H)$ and $B^n(K, H)$ are called n -cocycles and n -coboundaries, respectively.

The n -th cohomology group $H^n(K, H)$ of K with coefficients in H is defined by:

$$H^n(K, H) = \frac{Z^n(K, H)}{B^n(K, H)}, n \geq 1. \quad \blacksquare$$

Definition 2.2. Let K be a finite group. Suppose K acts on \mathbb{C}^* trivially. Then, the second cohomology group $H^2(K, \mathbb{C}^*)$ is called the Schur multiplier of K and is denoted by $M(K)$.

Now we state the following results related to the Schur multiplier of a group.

Proposition 2.3. $M(K) = \{1\}$, if K is a finite cyclic group.

Proposition 2.4. Let K_1 and K_2 be finite abelian groups. Then

$$M(K_1 \times K_2) \cong M(K_1) \times M(K_2) \times \text{Hom}(K_1, K_2).$$

Definition 2.5. Let K be a group. Then the non-abelian exterior square $K \wedge K$ of K is a group generated by the elements of the set $\{a \wedge b : a, b \in K\}$ satisfying the following conditions for all $a, b, c \in K$:

1. $a \wedge a = 1$;
2. $ab \wedge c = ({}^a b \wedge {}^a c)(a \wedge c)$;
3. $a \wedge bc = (a \wedge b)({}^b a \wedge {}^b c)$. \blacksquare

Now, we have an epimorphism $\chi : K \wedge K \rightarrow [K, K]$ defined by $\chi(g \wedge h) = [g, h]$ and the kernel of χ is the Schur multiplier $M(K)$ of K , that is, we have the short exact sequence

$$1 \rightarrow M(K) \rightarrow K \wedge K \xrightarrow{\chi} [K, K] \rightarrow 1. \quad (1)$$

Definition 2.6. A multiplicative Lie algebra is a triple (K, \cdot, \star) , where K is a set, \cdot and \star are two binary operations on K such that (K, \cdot) is a group (need not be abelian) and for all $x, y, z \in K$, the following identities hold:

1. $x \star x = 1$
2. $x \star (y \cdot z) = (x \star y) \cdot {}^y (x \star z)$
3. $(x \cdot y) \star z = {}^x (y \star z) \cdot (x \star z)$
4. $((x \star y) \star {}^y z)((y \star z) \star {}^z x)((z \star x) \star {}^x y) = 1$
5. ${}^z (x \star y) = (z x \star {}^z y)$

where ${}^x y$ denotes xyx^{-1} . We say that \star is a multiplicative Lie algebra structure on the group K .

Proposition 2.7. [[4]] Let (K, \cdot, \star) be a multiplicative Lie algebra. Then the following further identities hold:

1. $(1 \star x) = 1 = (x \star 1)$ for all $x \in K$.
2. $(x \star y) \cdot (y \star x) = 1$ for all $x, y \in K$.
3. $({}^{x \star y})(u \star v) = [{}^{x, y}] (u \star v)$ for all $x, y, u, v \in K$.
4. $[(x \star y), z] = ([x, y] \star z)$ for all $x, y, z \in K$.
5. $x^{-1} \star y = {}^{x^{-1}} ((x \star y)^{-1})$ and $x \star y^{-1} = {}^{y^{-1}} ((x \star y)^{-1})$ for all $x, y \in K$.

By the universal property of $K \wedge K$, there is a unique group homomorphism

$$\phi : K \wedge K \longrightarrow K$$

given by $\phi(a \wedge b) = a \star b$ whose image is $K \star K$ and we have the short exact sequence

$$1 \longrightarrow \ker(\phi) \longrightarrow K \wedge K \xrightarrow{\phi} K \star K \longrightarrow 1. \quad (2)$$

Definition 2.8. A multiplicative Lie algebra structure \star on a group K is said to be a *proper multiplicative Lie algebra structure* if $a \star b \neq 1$ for some pair of elements $a, b \in K$ and also $a \star b \neq [a, b]$ for some pair of elements $a, b \in K$. We term a group K as a *Lie simple group* if there is no proper multiplicative Lie algebra structure on K . We say the multiplicative Lie algebra structure \star is *trivial* if $a \star b = 1$ for every $a, b \in K$ and is *improper* if $a \star b = [a, b]$ for every $a, b \in K$. ■

Definition 2.9. Let (G, \cdot) be a group. Two multiplicative Lie algebra operations \star_1 and \star_2 defined on G are said to be *distinct* if and only if there is no multiplicative Lie algebra isomorphism from (G, \cdot, \star_1) to (G, \cdot, \star_2) .

Proposition 2.10. [7] *Let K be a group with trivial Schur multiplier. Then any multiplicative Lie algebra structure on K is uniquely determined by a K -equivariant homomorphism from $[K, K]$ to K .*

Definition 2.11. A *multiplicative Lie 2-cocycle* of a multiplicative Lie algebra K with coefficients in an abelian group H with trivial Lie product is a pair (f, h) , where $f \in Z^2(K, H)$ is a group 2-cocycle of K with coefficient in the trivial K -module H , and h is a map from $K \times K$ to H satisfying the equations:

1. $h(x, 1) = h(1, x) = h(x, x) = 1$ for all $x \in K$.
2. $h(x, yz) = h(x, y)h(x, z)f(y^{-1}, y)^{-1}f(y, x \star z)f(y(x \star z), y^{-1})f(x \star y, y(x \star z))$ for all $x, y, z \in K$.
3. $h(xy, z) = h(y, z)h(x, z)f(x^{-1}, x)^{-1}f(x, y \star z)f(x(y \star z), x^{-1})f(x(y \star z), (x \star z))$ for all $x, y, z \in K$.
4. $h(y \star x, xz)h(x \star z, zy)h(z \star y, yx)f((y \star x) \star^x z, (x \star z) \star^z y)f((y \star x) \star^x z) \cdot (x \star z) \star^z y, (z \star y) \star^y x) = 1$ for all $x, y, z \in K$.
5. $h(zx, zy) = h(x, y)f(z, x \star y)f(z^{-1}, z)^{-1}f(z(x \star y), z^{-1})$ for all $x, y, z \in K$.

The set $Z_{ML}^2(K, H)$ of all multiplicative Lie 2-cocycles of K with coefficient in H is an abelian group with respect to the coordinate wise operation given by

$$(f, h) \cdot (f', h') = (ff', hh').$$

Given any identity preserving map g from K to H , the pair $(\delta g, g^*)$ of maps from $K \times K$ to H given by

$$\delta g(x, y) = g(y)g(xy)^{-1}g(x) \quad \text{and} \quad g^*(x, y) = g(x \star y)^{-1}$$

is a member of $Z_{ML}^2(K, H)$.

Let $MAP(K, H)$ denote the group of identity preserving maps from K to H .

We have a group homomorphism

$$\chi : MAP(K, H) \longrightarrow Z_{ML}^2(K, H) \text{ given by } \chi(g) = (\delta g, g^*).$$

The image of χ is called the *group of multiplicative Lie 2-coboundaries* of K with coefficient in H and is denoted by $B_{ML}^2(K, H)$. The quotient group $\frac{Z_{ML}^2(K, H)}{B_{ML}^2(K, H)}$ is called the *second Lie cohomology* of K with coefficient in H , and is denoted by $H_{ML}^2(K, H)$.

Now, we have a natural projection homomorphism $\tilde{p} : H_{ML}^2(K, H) \longrightarrow H^2(K, H)$ defined by

$$\tilde{p}((f, h) + B_{ML}^2(K, H)) = f + B^2(K, H).$$

Now onwards, we denote $(f, h) + B_{ML}^2(K, H)$ and $f + B^2(K, H)$ by (\bar{f}, h) and \bar{f} , respectively. ■

Now we have the following results:

Proposition 2.12. ([7], Proposition 3.3) *Let H and K be abelian groups with trivial multiplicative Lie algebra structures. Then*

$$H_{ML}^2(K, H) \cong H^2(K, H) \times Hom(\wedge^2 K, H).$$

Proposition 2.13. ([7], Proposition 3.4) *The kernel of the homomorphism*

$$\tilde{p} : H_{ML}^2(K, H) \longrightarrow H^2(K, H)$$

is isomorphic to $Hom(\frac{\wedge^2 K}{J}, H)$, where $\wedge^2 K$ is the non abelian exterior square of K , and J is the subgroup of $\wedge^2 K$ generated by

$$\{((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y)\}.$$

Remark 2.14. We have an exact sequence

$$1 \longrightarrow Hom\left(\frac{\wedge^2 K}{J}, H\right) \longrightarrow H_{ML}^2(K, H) \xrightarrow{\tilde{p}} H^2(K, H).$$

Corollary 2.15. [[7], Corollary 3.5] *Let K be a group with trivial Schur multiplier and improper multiplicative Lie algebra structure. Then*

$$H_{ML}^2(K, H) \cong H^2(K, H) \times Hom(\wedge^2 K, H).$$

Definition 2.16. Let K be a finite multiplicative Lie algebra. The group $H_{ML}^2(K, \mathbb{C}^*)$ is called the Schur multiplier of K , and is denoted by $\tilde{M}(K)$.

Definition 2.17. Let (K, \cdot, \star) be a multiplicative Lie algebra. Consider the multiplicative Lie algebra $K \wedge^L K$ generated by the set

$$\{a \wedge b | a, b \in K\} \cup \{[a, b]_0 | a, b \in K\}$$

of formal symbols subject to the following relations:

1. $1 \wedge a = a \wedge 1 = a \wedge a = [1, a]_0 = [a, 1]_0 = [a, a]_0 = 1$
2. $(a \wedge b)(b \wedge a) = 1 = [a, b]_0[b, a]_0$
3. $(a \wedge bc)^{-1}(a \wedge b)(b \wedge a \wedge c) = 1 = [a, bc]_0^{-1}[a, b]_0[b, a, c]_0$

4. $(ab \wedge c)^{-1}(ab \wedge^a c)(a \wedge c) = 1 = [ab, c]_0^{-1}[ab, {}^a c]_0[a, c]_0$
5. $[a \wedge b, u \wedge v] = [[a, b]_0, (u \wedge v)] = [a, b]_0 \tilde{\star} (u \wedge v) = [a, b]_0 \tilde{\star} [u, v]_0$
6. $(a \wedge b)({}^c b \wedge^c a) = ({}^{ab} a^{-1} \wedge^a c)(a \wedge c)$
7. $[a, b]_0[{}^c b, {}^c a]_0 = [{}^{ab} a^{-1}, {}^a c]_0[a, c]_0$
8. $[[a, b]_0, [u, v]_0] = [[a, b], [u, v]]_0$
9. $(a \wedge b) \tilde{\star} (u \wedge v) = (a \star b) \wedge (u \star v)$ where $a, b, c, u, v \in K$.

The multiplicative Lie algebra $K \wedge^L K$ will be called *Lie exterior square* of K . Proposition 2.7 ensures the existence of a unique homomorphism χ from $K \wedge^L K$ to K given by $\chi(a \wedge b) = a \star b$ and $\chi([a, b]_0) = [a, b]$. We have the following short exact sequence of multiplicative Lie algebras for a finite multiplicative Lie algebra K :

$$1 \rightarrow \tilde{M}(K) \rightarrow K \wedge^L K \rightarrow (K \star K)[K, K] \rightarrow 1. \quad (3)$$

3. Schur multiplier and Lie exterior square of a multiplicative Lie algebra

We start the section with a classification of all the multiplicative Lie algebra structures on symmetric groups and alternating groups.

Theorem 3.1. *The symmetric group S_n is Lie simple for every $n \in \mathbb{N}$.*

Proof. For $n \leq 3$, it is clear. Assume $n \geq 4$. Let \star be a non trivial multiplicative Lie algebra structure on S_n . Then we know that $S_n \star S_n = A_n$ ([11], Theorem 2.9 (3)). Since for all $a, b \in S_n$, $(a \star b)^{-1}[a, b] \in Z(S_n \star S_n)$ and $Z(A_n) = 1$, $a \star b = [a, b]$ for all $a, b \in S_n$. This shows that \star must be improper. Hence, S_n is Lie simple. ■

Theorem 3.2. *The alternating group (A_n, \cdot) is Lie simple for $n \neq 4$ but the alternating group (A_4, \cdot) has three multiplicative Lie algebra structure up to isomorphism.*

Proof. For $n \neq 4$, (A_n, \cdot) is Lie simple ([11], Theorem 2.9 (2)). Now, consider

$$A_4 = \langle a, b \mid a^3 = b^2 = 1, aba = ba^2b \rangle = \{1, a, a^2, b, ab, ba, a^2b, ba^2, aba, bab, a^2ba, aba^2\},$$

where $a = (123)$ and $b = (12)(34)$. Since $A_4 \star A_4 = V_4$ ([11], Theorem 2.9 (2)), we have the following four possibilities of multiplicative Lie algebra structures on A_4 :

1. $a \star_1 b = 1$, that is, $x \star_1 y = 1$, for all $x, y \in A_4$ (trivial multiplicative Lie algebra structure);
2. $a \star_2 b = [a, b] = a^2ba$, that is, $x \star_2 y = [x, y]$, for all $x, y \in A_4$ (improper multiplicative Lie algebra structure);
3. $a \star_3 b = b$;
4. $a \star_4 b = aba^2$.

Clearly, (A_4, \cdot, \star_1) and (A_4, \cdot, \star_i) are not isomorphic for every $i = 2, 3, 4$.

Claim 1: (A_4, \cdot, \star_2) and (A_4, \cdot, \star_3) are not isomorphic.

Let $\phi : (A_4, \cdot, \star_3) \rightarrow (A_4, \cdot, \star_2)$ be a multiplicative Lie algebra homomorphism. Then

$$\begin{aligned} \phi(a \star_3 b) &= \phi(a) \star_2 \phi(b) \Rightarrow \phi(b) = [\phi(a), \phi(b)] \\ &\Rightarrow \phi(b) = \phi(a)\phi(b)\phi(a)^{-1}\phi(b) \Rightarrow \phi(b) = 1. \end{aligned}$$

Thus, ϕ can not be injective. Therefore, (A_4, \cdot, \star_2) and (A_4, \cdot, \star_3) are not isomorphic.

Claim 2: (A_4, \cdot, \star_2) and (A_4, \cdot, \star_4) are not isomorphic.

Let $\psi : (A_4, \cdot, \star_4) \rightarrow (A_4, \cdot, \star_2)$ be a multiplicative Lie algebra homomorphism. Then

$$\begin{aligned} \psi(a \star_4 b) &= \psi(a) \star_2 \psi(b) \Rightarrow \psi(aba^2) = [\psi(a), \psi(b)] \\ &\Rightarrow \psi(a)\psi(b)\psi(a)^2 = \psi(a)\psi(b)\psi(a)^{-1}\psi(b) \Rightarrow \psi(b) = 1. \end{aligned}$$

Thus, ψ can not be injective. Therefore, (A_4, \cdot, \star_2) and (A_4, \cdot, \star_4) are not isomorphic.

Claim 3: (A_4, \cdot, \star_3) and (A_4, \cdot, \star_4) are isomorphic.

Consider the group homomorphism from $\chi : (A_4, \cdot, \star_3) \rightarrow (A_4, \cdot, \star_4)$ defined by $\chi(a) = a^2, \chi(b) = b$. Clearly, χ is bijective.

Now, we have $\chi(a \star_3 b) = \chi(b) = b$. On the other hand, $\chi(a) \star_4 \chi(b) = a^2 \star_4 b = a(a \star_4 b) = aaba^2a^2aba^2 = a^2ba^2ba^2 = b$. Hence, $\chi(a \star_3 b) = \chi(a) \star_4 \chi(b)$. Similarly, we can show that $\chi(x \star_3 y) = \chi(x) \star_4 \chi(y)$, for all $x, y \in A_4$. Therefore, (A_4, \cdot, \star_3) and (A_4, \cdot, \star_4) are isomorphic. ■

Now, we discuss Schur multipliers and Lie exterior squares of some given multiplicative Lie algebras.

Theorem 3.3. *Let K be a group.*

1. *Suppose K is an abelian group with trivial multiplicative Lie algebra structure. Then the Lie exterior square $K \wedge^L K \cong \tilde{M}(K)$ with trivial multiplicative Lie algebra structure. Moreover,*

$$\tilde{M}(K) \cong M(K) \times M(K).$$

2. *Suppose K is a Lie simple group with trivial Schur multiplier. Then*

$$\tilde{M}(K) \cong \text{Hom}([K, K], \mathbb{C}^*).$$

Moreover, the Lie exterior square $K \wedge^L K \cong [K, K] \times \text{Hom}([K, K], \mathbb{C}^)$ with improper multiplicative Lie algebra structure.*

Proof. By sequence (3) and Definition 2.17, we have $K \wedge^L K \cong \tilde{M}(K)$ with trivial multiplicative Lie algebra structure. The rest follows from the Proposition 2.12 and Corollary 2.15. ■

Corollary 3.4. *Let K be a group with trivial Schur multiplier and a multiplicative Lie algebra structure \star such that $[K, K]$ is a non abelian simple group. Then $\tilde{M}(K)$ is trivial and $K \wedge^L K \cong [K, K]$ with the improper multiplicative Lie algebra structure.*

Proof. Since the Schur multiplier of K is trivial, $K \wedge K \cong [K, K]$. By Remark 2.14 $\tilde{M}(K) \cong \text{Hom}\left(\frac{[K, K]}{J}, \mathbb{C}^*\right)$, where J is the subgroup generated by $\{((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y) \mid x, y, z \in K\}$. Also, by Proposition 2.10, the multiplicative Lie algebra structure \star is uniquely determined by a K -equivariant homomorphism $\phi : [K, K] \rightarrow K$. Since $[K, K]$ is simple, either ϕ is trivial or injective.

If ϕ is trivial, then $J = \{1\}$.

Now, suppose that ϕ is injective. Since we have $\phi([x \star y, {}^y z]) = (x \star y) \star^y z$, we obtain $\phi([x \star y, {}^y z][y \star z, {}^z x][z \star x, {}^x y]) = 1$. So $J = \{1\}$. Hence, $\tilde{M}(K)$ is trivial.

By sequence (3) and Definition 2.17, we have $K \wedge^L K \cong [K, K]$ with the improper multiplicative Lie algebra structure. ■

Corollary 3.5. *Let K be a non abelian simple group with trivial Schur multiplier and a multiplicative Lie algebra structure \star . Then $\tilde{M}(K)$ is trivial and $K \wedge^L K \cong K$ with the improper multiplicative Lie algebra structure.*

Proof. Follows from Corollary 3.4. ■

Now, we calculate the Schur multiplier of the Klein four group, the direct product of two cyclic group, the Dihedral group and the Quaternion group with arbitrary multiplicative Lie algebra structures from [11].

Proposition 3.6. *Consider the Klein four group $V_4 = \langle a, b \mid a^2 = 1 = b^2, ab = ba \rangle$ with multiplicative Lie algebra structure \star given by $a \star b = a$. Then*

$$\tilde{M}(V_4) \cong V_4.$$

Proof. By Remark 2.14, we have

$$1 \longrightarrow \text{Hom}\left(\frac{M(V_4)}{J}, \mathbb{C}^*\right) \longrightarrow \tilde{M}(V_4) \xrightarrow{\tilde{p}} M(V_4),$$

where J is the subgroup of $\wedge^2 V_4$ generated by

$$\{((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y)\}.$$

Claim: $J = \{1\}$

For $x = a, y = a, z = ab$, we have

$$((a \star a) \wedge ab)((a \star ab) \wedge a)((ab \star a) \wedge a) = (1 \wedge ab)(a \wedge a)(a \wedge a) = 1.$$

For $x = a, y = b, z = ab$, we have

$$((a \star b) \wedge ab)((b \star ab) \wedge a)((ab \star a) \wedge b) = (a \wedge ab)(a \wedge a)(a \wedge b) = (a \wedge b)(a \wedge b) = 1.$$

Similarly, for any combination of x, y and z , we can see that

$$((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y) = 1.$$

Hence $J = \{1\}$.

Claim: \tilde{p} is surjective.

By identities (2) and (3) of Definition 2.11, it is easy to see that

$$h(a, ab) = h(a, b) = h(ab, b) \quad \text{and} \quad h(b, ab) = h(ab, a) = h(b, a).$$

By using identity (4), we have

$$(i) \quad h(a, b)h(a, b)f(a, a) = 1$$

Now, since $h(ab, ab) = 1$, we have

$$(ii) \quad h(b, a)h(a, b)f(a, a) = 1.$$

On comparing (i) and (ii), we have $h(a, b) = h(b, a)$. Therefore,

$$h(a, ab) = h(ab, b) = h(b, ab) = h(ab, a) = h(b, a) = h(a, b) = \sqrt{f(a, a)^{-1}}$$

Thus for any element $f \in M(V_4)$, we have a map $h : V_4 \times V_4 \rightarrow \mathbb{C}^*$ defined by $h(a, b) = \sqrt{f(a, a)^{-1}}$. So, the homomorphism \tilde{p} is surjective. Hence, we have a short exact sequence

$$1 \longrightarrow Hom(\wedge^2 V_4, \mathbb{C}^*) \longrightarrow \tilde{M}(V_4) \xrightarrow{\tilde{p}} M(V_4) \longrightarrow 1$$

Now, the map $t : M(V_4) \rightarrow \tilde{M}(V_4)$ defined by $t(\bar{f}) = (f, \bar{h})$ gives a splitting. Hence $\tilde{M}(V_4) \cong M(V_4) \times Hom(M(V_4), \mathbb{C}^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong V_4$. ■

Remark 3.7. Consider the Klein four group $V_4 = \langle a, b \mid a^2 = 1 = b^2, ab = ba \rangle$ with trivial multiplicative Lie algebra structure. Then from the short exact sequence (3), we have

$$V_4 \wedge^L V_4 \cong \tilde{M}(V_4) \cong V_4$$

with trivial multiplicative Lie algebra structure.

Proposition 3.8. Consider the Klein four group $V_4 = \langle a, b \mid a^2 = 1 = b^2, ab = ba \rangle$ with multiplicative Lie algebra structure \star given by $a \star b = a$. Then

$$V_4 \wedge^L V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

with trivial multiplicative Lie algebra structure.

Proof. Since $\tilde{M}(V_4) \cong V_4$ and $(V_4 \star V_4)[V_4, V_4] \cong \mathbb{Z}_2$, from the sequence (3), we have

$$1 \rightarrow V_4 \rightarrow V_4 \wedge^L V_4 \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

Since $\tilde{M}(V_4) \cong V_4$ and $(V_4 \star V_4)[V_4, V_4] \cong \mathbb{Z}_2$, we have $|V_4 \wedge^L V_4| = 8$. From (5) and (8) of Definition 2.17, we have $(a \wedge b)(u \wedge v) = (u \wedge v)(a \wedge b)$, $[a, b]_0(u \wedge v) = (u \wedge v)[a, b]_0$, and $[a, b]_0[u, v]_0 = [u, v]_0[a, b]_0$. So, $V_4 \wedge^L V_4$ is an abelian group of order 8 which contains V_4 . This implies $V_4 \wedge^L V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $V_4 \wedge^L V_4 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. But from the Definition 2.17, order of every non trivial element in $V_4 \wedge^L V_4$ is 2. Hence, $V_4 \wedge^L V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Moreover, from (5) and (9) of Definition 2.17, we have

$$[a, b]_0 \tilde{\star}[u, v]_0 = [a, b]_0 \tilde{\star}(u \wedge v) = (a \wedge b) \tilde{\star}(u \wedge v) = 1.$$

Hence $V_4 \wedge^L V_4$ is trivial multiplicative Lie algebra. ■

Proposition 3.9. Consider the abelian group $K = \mathbb{Z}_m \times \mathbb{Z}_n$ of order mn with a multiplicative Lie algebra structure \star . Then

$$\tilde{M}(K) = \mathbb{Z}_d \times \mathbb{Z}_d, \quad \text{where } d = \gcd(m, n).$$

Proof. Suppose $K = \langle a, b \mid a^m = b^n = 1, ab = ba \rangle$ and $a \star b = a^i b^j$, for some suitable $1 \leq i \leq m, 1 \leq j \leq n$ (for details see [Theorem 2.5 (2), [11]]).

By Remark 2.14, we have

$$1 \longrightarrow \text{Hom}\left(\frac{M(\mathbb{Z}_m \times \mathbb{Z}_n)}{J}, \mathbb{C}^*\right) \longrightarrow \tilde{M}(\mathbb{Z}_m \times \mathbb{Z}_n) \xrightarrow{\tilde{p}} M(\mathbb{Z}_m \times \mathbb{Z}_n),$$

where J is the subgroup of $\wedge^2(\mathbb{Z}_m \times \mathbb{Z}_n)$ generated by

$$\{(x \star y) \wedge^y z, (y \star z) \wedge^z x, (z \star x) \wedge^x y\}.$$

Claim: $J = \{1\}$.

For $x = a, y = b, z = ab$, we have

$$\begin{aligned} ((a \star b) \wedge ab)((b \star ab) \wedge a)((ab \star a) \wedge b) &= (a^i b^j \wedge ab)(a^{-i} b^{-j} \wedge a)(a^{-i} b^{-j} \wedge b) \\ &= (b^j \wedge a)(a^i \wedge b)(b^{-j} \wedge a)(a^{-i} \wedge b) = 1. \end{aligned}$$

For $x = a, y = a, z = ab$, we have

$$((a \star a) \wedge ab)((a \star ab) \wedge a)((ab \star a) \wedge a) = (1 \wedge ab)(a^i b^j \wedge a)(a^{-i} b^{-j} \wedge a) = 1.$$

Similarly, for all other combinations of x, y and z , we can see that

$$((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y) = 1.$$

Hence $J = \{1\}$.

Claim: \tilde{p} is surjective.

By identities (1), (2) and (3) of Definition 2.11, it is easy to see that

1. $h(x, 1) = h(1, x) = h(x, x) = 1$ for all $x \in K$.
This implies that $h(a^i, a^j) = 1 = h(b^i, b^j)$,
2. $h(a^p, a^q b^r) = h(a^p, b^r) f(a^{-q}, a^q)^{-1} f(a^q, a^p \star b^r) f(a^q(a^p \star b^r), a^{-q})$,
3. $h(a^q b^r, a^p) = h(b^r, a^p) f(a^{-q}, a^q)^{-1} f(a^q, b^r \star a^p) f(a^q(b^r \star a^p), a^{-q})$,
4. $h(b^p, a^q b^r) = h(b^p, a^q) f(a^{-q}, a^q)^{-1} f(a^q, a^{-q})$,
5. $h(a^q b^r, b^p) = h(a^q, b^p) f(a^{-q}, a^q)^{-1} f(a^q, a^{-q})$,
6. $h(a^p b^q, a^r b^s) = h(a^p b^q, a^r) h(a^p b^q, b^s) f(a^{-r}, a^r)^{-1} f(a^r, (a^p b^q) \star b^s) f(a^r((a^p b^q) \star b^s), a^{-r}) f((a^p b^q) \star a^r, (a^p b^q) \star b^s)$.

Also, we can see that $h(a^p, b^r)$ can be written in terms of $h(a, b)^{pr}$.

Now, since $h(ab, ab) = 1$. It can be seen that $h(b, a)$ can be written in terms of $h(a, b)$. By identity (4) of Definition 2.11, we have

$$h(a^i b^j, b) h(a^{m-i} b^{n-j}, b) f(a^i b^j \star b, a^{-i} b^{-j} \star b) = 1.$$

This implies that

$$h(a^i, b) h(a^{m-i}, b) f(a^{-i}, a^i)^{-1} f(a^i, a^{-i}) f(a^i, a^{m-1})^{-1} f(a^{m-i}, a^i) f(a^i b^j \star b, a^{-i} b^{-j} \star b) = 1.$$

So, $h(a, b)$ can be written completely in terms of values of f . Thus for any $f \in M(\mathbb{Z}_m \times \mathbb{Z}_n)$, we have a map $h : (\mathbb{Z}_m \times \mathbb{Z}_n) \times (\mathbb{Z}_m \times \mathbb{Z}_n) \rightarrow \mathbb{C}^*$ defined in terms of f . So, the homomorphism \tilde{p} is surjective.

Hence, we have a short exact sequence

$$1 \longrightarrow \text{Hom}(\wedge^2(\mathbb{Z}_m \times \mathbb{Z}_n), \mathbb{C}^*) \longrightarrow \tilde{M}(\mathbb{Z}_m \times \mathbb{Z}_n) \xrightarrow{\tilde{p}} M(\mathbb{Z}_m \times \mathbb{Z}_n) \longrightarrow 1.$$

Now, the map $t : M(\mathbb{Z}_m \times \mathbb{Z}_n) \rightarrow \tilde{M}(\mathbb{Z}_m \times \mathbb{Z}_n)$ defined by $t(\bar{f}) = (f, \bar{h})$ gives a splitting. Hence $\tilde{M}(\mathbb{Z}_m \times \mathbb{Z}_n) \cong M(\mathbb{Z}_m \times \mathbb{Z}_n) \times \text{Hom}(M(\mathbb{Z}_m \times \mathbb{Z}_n), \mathbb{C}^*) \cong \mathbb{Z}_d \times \mathbb{Z}_d$. ■

Remark 3.10. Proposition 3.6 is the particular case of Proposition 3.9. But here, we like to give the proof of Proposition 3.6 so that one can easily understand the proof of the general case.

Proposition 3.11. Consider the dihedral group $D_n = \langle a, b \mid a^2 = b^n = 1, aba = b^{-1} \rangle$ with a multiplicative Lie algebra structure \star .

1. If n is odd, then $\tilde{M}(D_n) \cong \mathbb{Z}_n$.
2. If n is even, then $\tilde{M}(D_n) \cong \mathbb{Z}_2 \times \mathbb{Z}_n$.

Proof. By Theorem 2.5 (3) of [11], it is enough to assume that the multiplicative Lie algebra structure is given by $a \star b = b^i$, for any fix $1 \leq i \leq n$. By Remark 2.14, we have

$$1 \longrightarrow \text{Hom}\left(\frac{\wedge^2(D_n)}{J}, \mathbb{C}^*\right) \longrightarrow \tilde{M}(D_n) \xrightarrow{\tilde{p}} M(D_n),$$

where J is the subgroup of $\wedge^2 D_n$ generated by

$$\{((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y)\}.$$

Claim: $J = \{1\}$

For $x = a, y = b^p, z = ab^q$, we have

$$((b^p \star a) \wedge^a ab^q)((a \star ab^q) \wedge^{ab^q} b^p)((ab^q \star b^p) \wedge^{b^p} a) = (b^{-pi} \wedge ab^{-q})(b^{pi} \wedge ab^{-2p}) = 1.$$

For $x = y = a, z = b^p$, we have

$$((a \star a) \wedge^a b^p)((a \star b^p) \wedge^{b^p} a)((b^p \star a) \wedge^a a) = (b^{pi} \wedge ab^{-2p})(b^{-pi} \wedge a) = 1.$$

Similarly, for any combination of x, y and z , we can see that

$$((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y) = 1.$$

Hence $J = \{1\}$.

Case 1: n is odd

In this case, $M(D_n) = \{1\}$. So $\tilde{M}(D_n) \cong \text{Hom}(\wedge^2(D_n), \mathbb{C}^*) \cong \mathbb{Z}_n$.

Case 2: n is even

First of all, we show that \tilde{p} is surjective. Using (1), (2) and (3) of Definition 2.11, we have the following:

1. $h(x, 1) = h(1, x) = h(x, x) = 1$ for all x in D_n ,
2. $h(a, ab^j) = h(a, a)h(a, b^j)f(a, a)^{-1}f(a, b^{ji})f(ab^{ji}, a)f(1, {}^a b^{ji})$
 $= h(a, b^j)f(a, a)^{-1}f(a, b^{ji})f(ab^{ji}, a),$
3. $h(ab^j, a) = h(b^j, a)h(a, a)f(a, a)^{-1}f(a, b^{-ji})f(ab^{-ji}, a)f({}^a b^{-ji}, 1)$
 $= h(b^j, a)f(a, a)^{-1}f(a, b^{-ji})f(ab^{-ji}, a),$
4. $h(b^j, ab^k) = h(b^j, a),$
5. $h(ab^k, b^j) = h(a, b^j),$

6. $h(ab^j, ab^k)$
 $= h(a, b^k)h(b^j, a)f(a, a)^{-2}f(a, b^{-ji})f(ab^{-ji}, a)f(a, b^{ki})f(ab^{ki}, a)f(b^{ji}, b^{-ki})$.
7. Since $h(ab^j, ab^j) = 1$,
 $h(a, b^j)h(b^j, a)f(a, a)^{-2}f(a, b^{-ji})f(ab^{-ji}, a)f(a, b^{ji})f(ab^{ji}, a)f(b^{ji}, b^{-ji}) = 1$,
8. $h(a, b^2) = h(a, b)^2f(b^{-1}, b)^{-1}f(b, b^i)f(bb^i, b^{-1})f(b^i, b^i)$. Similarly, $h(a, b^j)$ can be written in terms of $h(a, b)^j$.

From the above identities, it is clear that for any given f , image of any elements of $K \times K$ under the map h is determined by the image of (a, b) under the map h .

Now, using (5) of Definition 2.11, we have

$$(i) \quad h(a, b^{-1}) = h({}^a a, {}^a b) = h(a, b)f(a, a \star b)f(a, a)^{-1}f(a(a \star b), a) \\ = h(a, b)f(a, b^i)f(a, a)^{-1}f(ab^i, a)h(a, b^{-1}) = h(a, b)z,$$

where $z = f(a, b^i)f(a, a)^{-1}f(ab^i, a)$. Since $1 = h(a, 1) = h(a, bb^{-1})$ we obtain $h(a, b)h(a, b^{-1})f(b^{-1}, b)^{-1}f(b, b^{-i})f(b^{-i+1}, b^{-1})f(b^i, b^{-i}) = 1$. By using (i), we have $h(a, b)h(a, b^{-1})w = h(a, b)h(a, b)zw$, where

$$w = f(b^{-1}, b)^{-1}f(b, b^{-i})f(b^{-i+1}, b^{-1})f(b^i, b^{-i}).$$

So, $h(a, b) = \sqrt{(zw)^{-1}}$. Thus for any $f \in M(D_n)$, we have a map $h : D_n \times D_n \rightarrow \mathbb{C}^*$ defined by $h(a, b) = \sqrt{(zw)^{-1}}$. So, the homomorphism \tilde{p} is surjective. Hence, we have a short exact sequence

$$1 \longrightarrow \text{Hom}(\wedge^2 D_n, \mathbb{C}^*) \longrightarrow \tilde{M}(D_n) \xrightarrow{\tilde{p}} M(D_n) \longrightarrow 1$$

Now, the map $t : M(D_n) \rightarrow \tilde{M}(D_n)$ defined by $t(\bar{f}) = (\bar{f}, h)$ gives a splitting.

Hence $\tilde{M}(D_n) \cong M(D_n) \times \text{Hom}(\wedge^2 D_n, \mathbb{C}^*) \cong \mathbb{Z}_2 \times \mathbb{Z}_n$. ■

Proposition 3.12. *Consider the dihedral group $D_n = \langle a, b \mid a^2 = b^n = 1, aba = b^{-1} \rangle$ (where n is odd) with a multiplicative Lie algebra structure $a \star b = b^i$ for any fix $1 \leq i \leq n$. Then $D_n \wedge^L D_n \cong \mathbb{Z}_n \times \mathbb{Z}_n$ is trivial multiplicative Lie algebra.*

Proof. Since $M(D_n) \cong \{1\}$ and $\tilde{M}(D_n) \cong \mathbb{Z}_n$, $D_n \wedge D_n \cong \mathbb{Z}_n$ and by the sequence (3), we have $1 \rightarrow \mathbb{Z}_n \rightarrow D_n \wedge^L D_n \rightarrow D_n \wedge D_n \rightarrow 1$. So, $|D_n \wedge^L D_n| = n^2$.

From (5) and (8) of Definition 2.17, we get $(a \wedge b)(u \wedge v) = (u \wedge v)(a \wedge b)$, $[a, b]_0(u \wedge v) = (u \wedge v)[a, b]_0$, and $[a, b]_0[u, v]_0 = [u, v]_0[a, b]_0$. So, $D_n \wedge^L D_n$ is an abelian group of order n^2 , which contains \mathbb{Z}_n . This implies $D_n \wedge^L D_n \cong \mathbb{Z}_n \times \mathbb{Z}_n$.

Moreover, from (5) and (9) of Definition 2.17, we have

$$[a, b]_0 \tilde{\star}[u, v]_0 = [a, b]_0 \tilde{\star}(u \wedge v) = (a \wedge b) \tilde{\star}(u \wedge v) = 1.$$

Hence, $D_n \wedge^L D_n$ is a trivial multiplicative Lie algebra. ■

Proposition 3.13. *Consider the quaternion group $Q_n = \langle a, b \mid a^2 = b^n, aba^{-1} = b^{-1} \rangle$ of order $4n$ with multiplicative Lie algebra structure $a \star b = b^i$, where $1 \leq i \leq 2n-1$. Then $\tilde{M}(Q_n) \cong \mathbb{Z}_{\text{gcd}(n, i)}$.*

Proof. We know that $M(Q_n) = 1$. From the short exact sequence

$$1 \longrightarrow M(Q_n) \longrightarrow Q_n \wedge Q_n \longrightarrow [Q_n, Q_n] \longrightarrow 1,$$

we have $Q_n \wedge Q_n \cong [Q_n, Q_n] \cong \langle b^2 \rangle \cong \mathbb{Z}_n$.

By Remark 2.14, we have

$$1 \longrightarrow \text{Hom}\left(\frac{\wedge^2(Q_n)}{J}, \mathbb{C}^*\right) \longrightarrow \tilde{M}(Q_n) \xrightarrow{\tilde{p}} M(Q_n),$$

where J is the subgroup of $\wedge^2 Q_n$ generated by

$$\{(x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y)\}.$$

Since, $M(Q_n) = 1$, we have $\tilde{M}(Q_n) \cong \text{Hom}\left(\frac{\wedge^2(Q_n)}{J}, \mathbb{C}^*\right) \cong \text{Hom}\left(\frac{\mathbb{Z}_n}{J}, \mathbb{C}^*\right)$.

Claim: $J = \mathbb{Z}_m$, where $m = \frac{n}{\text{gcd}(n, i)}$.

Case 1: For $x = a, y = b^p, z = ab^q$, we have

$$[a \star b^p, {}^{b^p} ab^q][b^p \star ab^q, {}^{ab^q} a][ab^q \star a, {}^a b^p] = [b^{ip}, {}^{b^p} ab^q][b^{-ip}, {}^{ab^q} a][a b^{-iq}, {}^a b^p] = b^{2ip} \cdot 1 \cdot 1 = b^{2ip}.$$

Case 2: For $x = y = a, z = b^p$, we have

$$[a \star a, {}^a b^p][a \star b^p, {}^{b^p} a][b^p \star a, {}^a a] = [1, {}^a b^p][b^{ip}, {}^{b^p} a][b^{-ip}, {}^a a] = 1 \cdot b^{2i} \cdot b^{-2i} = 1.$$

Case 3: For $x = y = a, z = ab^p$, we have

$$[a \star a, {}^a ab^p][a \star ab^p, {}^{ab^p} a][ab^p \star a, {}^a a] = [1, {}^a ab^p][a b^{ip}, {}^{ab^p} a][a b^{-ip}, {}^a a] = 1 \cdot b^{-2ip} \cdot b^{2ip} = 1.$$

Similarly, for all other combinations of x, y and z , we can see that

$$((x \star y) \wedge^y z)((y \star z) \wedge^z x)((z \star x) \wedge^x y) = 1 \text{ or } b^{2ip}.$$

So, $J = \langle (b^2)^i \rangle \cong \mathbb{Z}_{\frac{n}{\text{gcd}(n, i)}}$. Hence, $\frac{\wedge^2(Q_n)}{J} \cong \mathbb{Z}_{\text{gcd}(n, i)}$. Therefore we finally obtain $\tilde{M}(Q_n) \cong \mathbb{Z}_{\text{gcd}(n, i)}$. ■

Acknowledgements. The first named author sincerely thanks IIIT Allahabad and University Grant Commission (UGC), Government of India, New Delhi, for research fellowship. The second named author sincerely thanks IIIT Allahabad and Ministry of Education, Government of India, for providing institute fellowship.

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Table for the Schur multiplier of some multiplicative Lie algebras:

Group K	$\wedge^2 K$	$M(K)$	Multiplicative Lie algebra structure	$\tilde{M}(K)$
\mathbb{Z}_n	trivial	trivial	Lie simple	trivial
V_4	\mathbb{Z}_2	\mathbb{Z}_2	arbitrary	V_4
$SL(2, 3)$	Q_2	trivial	Lie simple	V_4
$Q_n = \langle a, b \mid a^2 = b^n, aba^{-1} = b^{-1} \rangle$	\mathbb{Z}_n	trivial	$a \star b = b^i$	$\mathbb{Z}_{\gcd(n,i)}$
$D_n = \langle a, b \mid a^2 = b^n = 1, aba = b^{-1} \rangle$	\mathbb{Z}_n	trivial if n is odd and \mathbb{Z}_2 if n is even	arbitrary	\mathbb{Z}_n if n is odd and $\mathbb{Z}_n \times \mathbb{Z}_2$ if n is even.
$\mathbb{Z}_m \times \mathbb{Z}_n = \langle a, b \mid a^m = b^n = 1, ab = ba \rangle$	\mathbb{Z}_d , where $d = (m, n)$	\mathbb{Z}_d , where $d = (m, n)$	arbitrary	$\mathbb{Z}_d \times \mathbb{Z}_d$
$Q_2 \times \mathbb{Z}_p$	\mathbb{Z}_2	trivial	Lie simple	\mathbb{Z}_2
$\langle a, b \mid a^8 = b^p = 1, a^{-1}ba = b^{-1} \rangle$	\mathbb{Z}_p	trivial	Lie simple	\mathbb{Z}_p
$\langle a, b \mid a^8 = b^p = 1, a^{-1}ba = b^\alpha \rangle$, where α is a primitive root of $\alpha^4 \equiv 1 \pmod{p}$, $4 \mid (p-1)$	\mathbb{Z}_p	trivial	Lie simple	\mathbb{Z}_p
$\langle a, b \mid a^8 = b^p = 1, a^{-1}ba = b^\alpha \rangle$, where α is a primitive root of $\alpha^8 \equiv 1 \pmod{p}$, $8 \mid (p-1)$	\mathbb{Z}_p	trivial	Lie simple	\mathbb{Z}_p
$\langle a, b, c, d \mid a^4 = b^2 = c^2 = d^p = 1, ab = ba, ac = ca, bc = cb, d^{-1}ad = b, bd^{-1}bd = c, d^{-1}cd = ab \rangle$	\mathbb{Z}_2^3	trivial	Lie simple	\mathbb{Z}_2^3

Table 1: $\tilde{M}(K)$ for some groups with arbitrary multiplicative Lie algebra structure.

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Received January 24, 2023

and in final form November 25, 2023