Strong Integrality of Inversion Subgroups of Kac-Moody Groups

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Abstract. Let A be a symmetrizable generalized Cartan matrix with corresponding Kac-Moody algebra $\mathfrak{g}$ over $\mathbb{Q}$. Let $V = V^\lambda$ be an integrable highest weight $\mathfrak{g}$-module with dominant regular integral weight λ and representation $\rho : \mathfrak{g} \to \operatorname{End}(V)$, and let $V_\mathbb{Z} = V_\mathbb{Z}^\lambda$ be a $\mathbb{Z}$-form of V. Let $G_V(\mathbb{Q})$ be the associated minimal Kac-Moody group generated by the automorphisms $\exp(t\rho(e_i))$ and $\exp(t\rho(f_i))$ of V, where e_i and f_i are the Chevalley-Serre generators and $t \in \mathbb{Q}$. Let $G(\mathbb{Z})$ be the group generated by $\exp(t\rho(e_i))$ and $\exp(t\rho(f_i))$ for $t \in \mathbb{Z}$. Let $\Gamma(\mathbb{Z})$ be the Chevalley subgroup of $G_V(\mathbb{Q})$, that is, the subgroup that stabilizes the lattice $V_\mathbb{Z}$ in V. For a subgroup M of $G_V(\mathbb{Q})$, we say that M is integral if $M \cap G(\mathbb{Z}) = M \cap \Gamma(\mathbb{Z})$ and that M is strongly integral if there exists $v \in V_\mathbb{Z}$ such that $g \cdot v \in V_\mathbb{Z}$ implies $g \in G(\mathbb{Z})$ for all $g \in M$. We prove strong integrality of inversion subgroups $U_{(w)}$ of $G_V(\mathbb{Q})$ for w in the Weyl group, where $U_{(w)}$ is the group generated by positive real root groups that are flipped to negative root groups by w^{-1}. We use this to prove strong integrality of subgroups of the unipotent subgroup U of $G_V(\mathbb{Q})$ that are generated by commuting real root groups. When A has rank 2, this gives strong integrality of subgroups U_1 and U_2 where $U = U_1 * U_2$ and each U_i is generated by 'half' the positive real roots.

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1. Introduction

This paper concerns the generalization to infinite dimensional symmetrizable Kac-Moody algebras of the construction of Chevalley groups associated to finite dimensional semisimple Lie algebras. Let A be a symmetrizable generalized Cartan matrix. Let $\mathfrak{g}$ be a Kac-Moody algebra over $\mathbb{Q}$ associated with A. We denote simple roots by $\alpha_1, \ldots, \alpha_\ell$, the set of roots by Δ, the set of positive (respectively negative) roots by $\Delta_\pm$, and the set of real roots by Δ^{re}. If A is not of finite type, then $\mathfrak{g}$ is infinite dimensional. Let $V = V^\lambda$ be an integrable highest-weight $\mathfrak{g}$-module with dominant regular integral highest weight λ, highest weight vector v_λ, and defining homomorphism $\rho : \mathfrak{g} \to \operatorname{End}(V)$. Let $\chi_{\pm\alpha_i}(t) \in \operatorname{GL}(V)$ denote $\exp(t\rho(e_i))$ (respectively $\exp(t\rho(f_i))$), for each $i \in \{1, \ldots, \ell\}$, where $t \in \mathbb{Q}$ and e_i (respectively f_i) is the Chevalley-Serre generator corresponding to the simple root $\pm\alpha_i$.

A (split) *minimal representation-theoretic Kac-Moody group* over $\mathbb{Q}$ is the group

$$G_V(\mathbb{Q}) = \langle \chi_{\alpha_i}(t), \ \chi_{-\alpha_i}(t) \mid i \in \{1, \ldots, \ell\}, \ t \in \mathbb{Q} \rangle.$$

This is a representation theoretic construction of the value of the constructive Tits functor $\widetilde{\mathfrak{G}}$ [19] over $\mathbb{Q}$ (see for example [2]). For $s \in \mathbb{Q}^\times$, let

$$\widetilde{w}_{\alpha_i}(s) = \chi_{\alpha_i}(s) \chi_{-\alpha}(-s^{-1}) \chi_{\alpha_i}(s), \quad h_{\alpha_i}(s) \ = \ \widetilde{w}_{\alpha_i}(s) \widetilde{w}_{\alpha_i}(1)^{-1},$$

and
$$H = \langle h_{\alpha_i}(s) \mid s \in \mathbb{Q}^\times \rangle \cong (\mathbb{Q}^\times)^\ell.$$

Let $\mathcal{U}_\mathbb{Z} \subseteq \mathcal{U}$ denote the $\mathbb{Z}$-form of the universal enveloping algebra $\mathcal{U}$ of $\mathfrak{g}$ ([19]; see also Section 3 below). As in [2], we set $V_\mathbb{Z} = \mathcal{U}_\mathbb{Z} \cdot v_\lambda$. The representation V depends on the choice of a dominant regular integral weight λ. For $\lambda_1 \neq \lambda_2$ the groups $G_{V^{\lambda_1}}(\mathbb{Q})$ and $G_{V^{\lambda_2}}(\mathbb{Q})$ are isomorphic [10]. Furthermore the construction of $V_\mathbb{Z}$ depends on the choice of a highest weight vector v_λ. We fix suitable λ and v_λ throughout. Let

$$G(\mathbb{Z}) = \langle \chi_{\alpha_i}(t), \ \chi_{-\alpha_i}(t) \mid i \in \{1, \ldots, \ell\}, \ t \in \mathbb{Z} \rangle.$$

We define the *Chevalley subgroup* of $G_V(\mathbb{Q})$ to be the subgroup of $G_V(\mathbb{Q})$ that stabilizes the integral lattice $V_\mathbb{Z}$:

$$\Gamma(\mathbb{Z}) = \{g \in G_V(\mathbb{Q}) \mid g \cdot V_\mathbb{Z} = V_\mathbb{Z}\}.$$

For fixed V and $V_\mathbb{Z}$, we say that $G_V(\mathbb{Q})$ is *integral* if $G(\mathbb{Z}) = \Gamma(\mathbb{Z})$. It is easy to show that $G(\mathbb{Z}) \subseteq \Gamma(\mathbb{Z})$. Thus it remains to determine if $\Gamma(\mathbb{Z}) \subseteq G(\mathbb{Z})$.

Our motivation for this work is that over a general commutative ring R, there is still no widely agreed upon definition of a Kac-Moody group $G(R)$, except in the affine case ([1], [7]). The groundwork for the notion of a Kac-Moody group over an arbitrary commutative ring was provided by Tits [19] who defined a functor $\widetilde{\mathfrak{G}}$ from commutative rings to groups. However, Tits' functor is not uniquely determined over $\mathbb{Z}$. The groups $\Gamma(\mathbb{Z}) = \text{Stab}_{G_V(\mathbb{Q})}(V_\mathbb{Z})$ and $G(\mathbb{Z})$ considered here are both possible candidates for the notion of a Kac-Moody group over $\mathbb{Z}$. Thus an essential question is to determine if these groups coincide.

The answer to this question has not yet been established for any family of Kac-Moody groups, except in the finite dimensional case. Here we consider integrality of certain subgroups of $G_V(\mathbb{Q})$. We call a subgroup M of $G_V(\mathbb{Q})$ *integral* if $M \cap G(\mathbb{Z}) = M \cap \Gamma(\mathbb{Z})$. This is equivalent to showing that for all $g \in M$, $g \cdot V_\mathbb{Z} = V_\mathbb{Z}$ implies that $g \in G(\mathbb{Z})$. Note that $M_1 \subseteq M_2$ and M_2 integral implies that M_1 is integral. However M_1 and M_2 integral *does not* imply that the subgroup $\langle M_1, M_2 \rangle$ generated by M_1 and M_2 is integral.

In the finite dimensional case, the property of integrality for semisimple algebraic groups $G_V(\mathbb{Q})$, with root system Δ and Lie algebra $\mathfrak{g}$, was established in [4]. Chevalley constructed (what was his notion of) an affine group scheme associated to $G_V(\mathbb{Q})$ and $V_\mathbb{Z}$ and showed that its coordinate algebra is generated over $\mathbb{Z}$. This was a starting point for Grothendieck and Demazure who organized the SGA3 seminar ([6]) where they developed the theory of split reductive group schemes over arbitrary base schemes. Chevalley's notion of an affine group scheme $G_\mathbb{Z}$ over $\mathbb{Z}$ was replaced by the modern definition due to Demazure and Grothendieck.

Let L be a lattice satisfying $Q \subseteq L \subseteq P$, where Q is the root lattice and P is the weight lattice of $\mathfrak{g}$. Chevalley showed that (Δ, L) defines an affine group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ such that

$$\Gamma(\mathbb{Z}) = G_{\mathbb{Z}}(\mathbb{Z}) = \mathrm{Hom}(\mathrm{Spec}(\mathbb{Z}), G_{\mathbb{Z}})$$

is its set of integral points ([4]). The group $G_{\mathbb{Z}}(\mathbb{Q}) \cong G_V(\mathbb{Q})$ is now known as a Chevalley group. If $L = Q$, then $G_{\mathbb{Z}}(\mathbb{Q})$ is called *adjoint* and if $L = P$, then $G_{\mathbb{Z}}(\mathbb{Q})$ is called *simply connected*. It follows that the group $\Gamma(\mathbb{Z})$ is generated by the automorphisms

$$\{\chi_{\alpha_i}(t), \chi_{-\alpha_i}(t) \mid i \in \{1, \ldots, \ell\},\ t \in \mathbb{Z}\},$$

so if $G_{\mathbb{Z}}(\mathbb{Q})$ is simply connected, $G_{\mathbb{Z}}(\mathbb{Z})$ coincides with $\Gamma(\mathbb{Z})$ and hence is integral.

To our knowledge, there is no proof in the literature that $G_{\mathbb{Z}}(\mathbb{Z}) = \Gamma(\mathbb{Z})$ for semisimple linear algebraic groups $G_V(\mathbb{Q})$ that doesn't use the language of group schemes. In certain cases, it is possible to verify integrality using properties of the linear algebraic group and underlying representation. For example, it is easy to see that $\mathrm{SL}_n(\mathbb{Z})$ is the stabilizer of the standard lattice $\mathbb{Z}^n$ in $\mathbb{Q}^n$. See also [16], where Soulé showed that the group $\mathbb{E}_7(\mathbb{Z})$ coincides with $\mathbb{E}_7(\mathbb{R}) \cap \mathrm{Sp}_{56}(\mathbb{Z})$ where $\mathbb{E}_7(\mathbb{R})$ is the non-compact real form and $\mathrm{Sp}_{56}(\mathbb{Z})$ is the stabilizer of the standard lattice (and the canonical symplectic form) in the fundamental representation of dimension 56 of the Lie algebra $\mathfrak{e}_7$.

The methods used to prove integrality in the finite dimensional case do not extend to Kac-Moody groups $G_V(\mathbb{Q})$. Here we address the question of integrality of subgroups of $G_V(\mathbb{Q})$ using only the definition of $G_V(\mathbb{Q})$ acting on the integrable highest weight module V. We introduce the following slight modification of the notion of integrality.

Definition 1.1. Let M be a subgroup of the Kac-Moody group $G_V(\mathbb{Q})$ and let $v_0 \in V_{\mathbb{Z}}$. We say that M is *strongly integral with respect to* $v_0 \in V_{\mathbb{Z}}$ if, for all $g \in M$, $g \cdot v_0 \in V_{\mathbb{Z}}$ implies that $g \in G(\mathbb{Z})$.

This is stronger than integrality since it only requires us to test the action on a single vector v_0, rather than on the entire set $V_{\mathbb{Z}}$. It is easy to see that strong integrality implies integrality (Lemma 4.1).

The (positive) unipotent subgroup of $G_V(\mathbb{Q})$ is

$$U = \langle \chi_{\alpha_i}(t) \mid i \in \{1, \ldots, \ell\},\ t \in \mathbb{Q} \rangle.$$

The integral unipotent subgroup of $G_V(\mathbb{Q})$ is

$$U(\mathbb{Z}) = \langle \chi_{\alpha_i}(t) \mid i \in \{1, \ldots, \ell\},\ t \in \mathbb{Z} \rangle.$$

Unipotent subgroups play an important role in the study of the structure and representation theory of Kac-Moody groups. They are constituents of group decompositions such as Iwasawa and Birkhoff decompositions, which provide important tools for studying Kac-Moody groups and their applications.

In this paper, as a first step towards proving integrality of U, we prove strong integrality of inversion subgroups of U defined as follows: For w in the Weyl group W, the *inversion subgroup* is

$$U_{(w)} := \langle U_{\beta} \mid \beta \in \Phi_{(w)} \rangle,$$

where $\Phi_{(w)} = \{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_-\}$ and $U_\beta = \{\chi_\beta(t) \mid t \in \mathbb{Q}\}$ for real root group generators $\chi_\beta(t)$ (see Subsection 3.2). In consequence $U_{(w)}$ is integral if $U_{(w)} \cap G(\mathbb{Z}) = U_{(w)} \cap \Gamma(\mathbb{Z})$. We note that the left hand side of this equality equals $U_{(w)} \cap U(\mathbb{Z})$. We show (Corollary 5.4) that

$$U_{(w)} \cap U(\mathbb{Z}) = U_{(w)}(\mathbb{Z}) := \langle U_\alpha(\mathbb{Z}) \mid \alpha \in \Phi_{(w)} \rangle, \quad \text{where } U_\alpha(\mathbb{Z}) := \{\chi_\beta(t) \mid t \in \mathbb{Z}\}.$$

Our main result is the following (Section 6).

Theorem 1.2. (Strong integrality of inversion subgroups) *Let U be the positive unipotent subgroup of $G_V(\mathbb{Q})$. For $w \in W$, $U_{(w)}$ is strongly integral with respect to $\widetilde{w} \cdot v_\lambda$.*

Corollary 1.3. (Integrality of inversion subgroups) *Let U be the positive unipotent subgroup of $G_V(\mathbb{Q})$. For any $w \in W$, $U_{(w)}$ is integral. That is for all $g \in U_{(w)}$, $g \cdot V_\mathbb{Z} \subseteq V_\mathbb{Z}$ implies that $g \in U_{(w)}(\mathbb{Z})$.*

If the matrix A is of finite type (see Subsection 2.1), then the group

$$G_V(\mathbb{Q}) = \langle \chi_{\alpha_i}(t), \ \chi_{-\alpha_i}(t) \mid i \in \{1, \ldots, \ell\}, \ t \in \mathbb{Q} \rangle$$

is a semisimple linear algebraic group. Furthermore, if the set of weights of $V = V^\lambda$ contains all the fundamental weights, then $G_V(\mathbb{Q})$ coincides with the simply connected Chevalley group $G_\mathbb{Z}(\mathbb{Q})$ associated with A. Our methods give a proof of integrality of unipotent subgroups of simply connected Chevalley groups $G_\mathbb{Z}(\mathbb{Q}) \cong G_V(\mathbb{Q})$, using only the action of $G_V(\mathbb{Q})$ on V. As a corollary of Theorem 1.2, we have the following (Subsection 7.1).

Corollary 1.4. *Suppose that the matrix A has finite type. Suppose also that the set of weights of V contains all the fundamental weights. Then the positive unipotent subgroup U of $G_V(\mathbb{Q})$ is integral.*

Theorem 1.2 also yields integrality of other subgroups of U. In Subsection 7.2 we prove:

Theorem 7.2. *If M is a subgroup of U generated by commuting real root subgroups, then M is integral.*

In Section 7.3, we consider the case where G has rank 2. In this case $U = U_1 * U_2$ where each U_i is generated by 'half' the positive real roots (see [3]).

Theorem 7.4. *Let $G_V(\mathbb{Q})$ be of rank 2 with positive unipotent subgroup $U = U_1 * U_2$ where U_i are abelian if A is symmetric, and nilpotent of class 2 if A is not symmetric. Then the groups U_i are integral.*

Though our methods do not currently extend to a proof of integrality for U, we conjecture[1] that integrality holds for U. It would then be straightforward to show that $G(\mathbb{Z}) = \Gamma(\mathbb{Z})$ in the symmetrizable Kac-Moody case.

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[1] An earlier preprint by a subset of the authors (arXiv:1803.11204v2 [math.RT] Proposition 6.4) contains an error.

2.   Kac-Moody Algebras

In this section, we review some facts about Kac-Moody algebras and their representations.

2.1. Generalized Cartan matrices

Let $I = \{1, 2, \ldots, \ell\}$ and let $A = (a_{ij})_{i,j \in I}$ be a *generalized Cartan matrix*. That is,

$$a_{ij} \in \mathbb{Z}, \quad a_{ii} = 2, \quad a_{ij} \leq 0 \ \text{ for } i \neq j, \quad a_{ij} = 0 \iff a_{ji} = 0,$$

for all $i, j \in I$. The matrix A is *symmetrizable* if there exist positive rational numbers $d_1, \ldots, d_\ell$, such that the matrix $\mathrm{diag}(d_1, \ldots, d_\ell)A$ is symmetric. We say that A is of *finite type* if A is positive definite, and that A is of *affine type* if A is positive semidefinite but not positive definite. If A is not of finite or affine type, we say that A has *indefinite type*. In particular, A is of *hyperbolic type* if A is neither of finite nor affine type, but every proper, indecomposable submatrix is either of finite or of affine type.

We assume throughout that A is a symmetrizable generalized Cartan matrix.

2.2. Generators and relations for $\mathfrak{g}$

Let $\mathfrak{h}$ be a $\mathbb{Q}$-vector space of dimension $2\ell - \mathrm{rank}(A)$, and let $\langle \circ, \circ \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{Q}$ denote the natural nondegenerate bilinear pairing between $\mathfrak{h}$ and its dual. Fix *simple roots* $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathfrak{h}^*$ and *simple coroots* $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subseteq \mathfrak{h}$ such that Π and $\Pi^\vee$ are linearly independent, and $\langle \alpha_j, \alpha_i^\vee \rangle = \alpha_j(\alpha_i^\vee) = a_{ij}$.

As in [11], [12], and [8, Theorem 9.11], the associated *Kac-Moody algebra* $\mathfrak{g}$ is the Lie algebra over $\mathbb{Q}$ with generating set $\mathfrak{h} \cup \{e_i, f_i \mid i \in I\}$ and defining relations:

$$[h, h'] = 0;$$
$$[h, e_i] = \langle \alpha_i, h \rangle e_i; \qquad\qquad [h, f_i] = -\langle \alpha_i, h \rangle f_i;$$
$$[e_i, f_i] = \alpha_i^\vee; \qquad\qquad [e_i, f_j] = 0;$$
$$(\mathrm{ad}\, e_i)^{-a_{ij}+1}(e_j) = 0; \qquad\qquad (\mathrm{ad}\, f_i)^{-a_{ij}+1}(f_j) = 0;$$

for $h, h' \in \mathfrak{h}$, and $i, j \in I$ with $i \neq j$.

2.3. Roots and the Weyl group

The *roots* of $\mathfrak{g}$ are the nonzero $\alpha \in \mathfrak{h}^*$ for which the corresponding root space

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$$

is nontrivial. The simple roots α_i have root spaces $\mathfrak{g}_{\alpha_i} = \mathbb{Q}e_i$. Every root α can be written in the form $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ where the k_i are integers, with either all $k_i \geq 0$, in which case α is called *positive*, or all $k_i \leq 0$, in which case α is called *negative*. We denote the set of roots by Δ and the set of positive (respectively negative) roots by Δ_+ (respectively Δ_-). The *root lattice* is $Q := \mathbb{Z}\Pi \subseteq \mathfrak{h}^*$ and the *coroot lattice* is $Q^\vee := \mathbb{Z}\Pi^\vee \subseteq \mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ has a triangular decomposition ([8, Theorem 1.2])

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \text{where } \mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha.$$

Since A is symmetrizable, $\mathfrak{g}$ admits a nondegenerate symmetric invariant bilinear form $(\circ, \circ)$ satisfying

$$\left(\alpha_i^\vee, \alpha_j^\vee\right) = d_i a_{ij}, \quad (e_i, f_j) = \delta_{ij}, \quad (\mathfrak{h}, \mathfrak{n}_+) = 0, \quad \text{and} \quad (\mathfrak{h}, \mathfrak{n}_-) = 0.$$

We define the *simple reflection* $w_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$w_i(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i.$$

The group $W \subseteq \mathrm{GL}(\mathfrak{h}^*)$ generated by the simple reflections is called the *Weyl group*. The induced bilinear form on $\mathfrak{h}^*$ is W-invariant. The group W comes equipped with a length function $\ell \colon W \longrightarrow \mathbb{Z}_{\geq 0}$ defined as $\ell(w) = k$, where k is the smallest number such that w is the product of k simple root reflections. A word $w = w_{i_1} w_{i_2} \cdots w_{i_k}$ of minimal length is called a *reduced word.*

A root $\alpha \in \Delta$ is called *real* if there exists $w \in W$ such that $w\alpha$ is a simple root. A root α which is not real is called *imaginary.* We denote by Δ^{re} the set of real roots and Δ^{im} the set of imaginary roots. We have $\Delta^{\mathrm{re}} = W\Pi$. Similarly define sets $\Delta_\pm^{\mathrm{re}} := \Delta^{\mathrm{re}} \cap \Delta_\pm$ and $\Delta_\pm^{\mathrm{im}} := \Delta^{\mathrm{im}} \cap \Delta_\pm$. For $\alpha \in \Delta^{\mathrm{re}}$, the reflection in α is defined by

$$w_\alpha(v) = v - \langle v, \alpha^\vee \rangle \alpha,$$

where $\alpha^\vee$ is the coroot of α. Since $w\alpha = \alpha_i$ for some $w \in W$ and $i \in I$, we have $w_\alpha = w w_i w^{-1} \in W$.

2.4. Integrable highest weight modules

The *weight lattice* $P \subseteq \mathfrak{h}^*$ is the dual lattice of $Q^\vee$, that is,

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \ i \in I\}.$$

An element $\lambda \in P$ is *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in I$, and is *regular* if $\langle \lambda, \alpha_i^\vee \rangle \neq 0$ for all $i \in I$. We denote the set of dominant elements of P by P_+ and write $Q_+ = \sum_{i=1}^\ell \mathbb{Z}_{\geq 0} \alpha_i$. The *dominance order* on $\mathfrak{h}^*$ is defined by $\lambda \leq \mu$ if $\mu - \lambda \in Q_+$.

Let V be a $\mathfrak{g}$-module with defining homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$. Then the *weight space* of $\mu \in \mathfrak{h}^*$ is

$$V_\mu := \{v \in V \mid \rho(h)(v) = \mu(h)v \text{ for all } h \in \mathfrak{h}\},$$

and the set of *weights* of V is $\mathrm{wts}(V) = \{\mu \in \mathfrak{h}^* \mid V_\mu \neq 0\}$.

Recall also that $x \in \mathfrak{g}$ acts *locally nilpotently* on V if, for each $v \in V$, there is a natural number $n = n(v)$ such that $\rho(x)^n(v) = 0$. Then V is called an *integrable* module if it is a direct sum of its weight spaces, that is $V = \bigoplus_{\mu \in \mathrm{wts}(V)} V_\mu$, and each generator e_i or f_i acts locally nilpotently on V.

We call $\lambda \in \mathrm{wts}(V)$ the *highest weight* of V if it is the largest element of $\mathrm{wts}(V)$ in the dominance order. If V has a highest weight we call it a *highest weight module.* If μ is a weight of a highest weight module V, then $\mu = \lambda - \beta$ for some $\beta \in Q_+$. Among all modules with highest weight $\lambda \in \mathfrak{h}^*$, there is a unique irreducible $\mathfrak{g}$-module (Proposition 9.3 of [8]), which we denote by V^λ. The module V^λ is integrable if and only if $\lambda \in P_+$, that is, λ is a dominant weight (Lemma 10.1, [8]). In this case, we get $\mathrm{wts}(V^\lambda) \subseteq P$.

3.   Representation-theoretic Kac-Moody groups

Let $\mathfrak{g}$ be a Kac-Moody algebra over $\mathbb{Q}$ with symmetrizable generalized Cartan matrix A. Let $V = V^\lambda$ be an integrable highest weight $\mathfrak{g}$-module over $\mathbb{Q}$ for a fixed dominant, regular weight λ and let v_λ be a fixed highest weight vector. (that is, a nonzero element of the weight space V_λ). For $x \in \mathfrak{g}$ and $v \in V$, we denote the action of x on v by $x \cdot v := \rho(x)(v)$.

The Chevalley involution ω is an automorphism of $\mathfrak{g}$ with $\omega^2 = 1$, $\omega(h) = -h$ for all $h \in \mathfrak{h}$, and $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Delta$.

3.1. $\mathbb{Z}$-forms

Let $\mathcal{U} = \mathcal{U}_\mathbb{Q}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let

- $\mathcal{U}_\mathbb{Z}^+$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{e_i{}^m}{m!}$ for $i \in I$ and $m \geq 0$;

- $\mathcal{U}_\mathbb{Z}^-$ be the $\mathbb{Z}$-subalgebra generated by $\dfrac{f_i{}^m}{m!}$ for $i \in I$ and $m \geq 0$; and

- $\mathcal{U}_\mathbb{Z}^0$ be the $\mathbb{Z}$-subalgebra generated by

$$\binom{h}{m} := \frac{h(h-1)\dots(h-m+1)}{m!},$$

   for $h \in Q^\vee$ and $m \geq 0$.

As in [2], [18], and [19], the $\mathbb{Z}$-*form of* $\mathcal{U}$ is the $\mathbb{Z}$-subalgebra $\mathcal{U}_\mathbb{Z}$ generated by $\mathcal{U}_\mathbb{Z}^+$, $\mathcal{U}_\mathbb{Z}^-$, and $\mathcal{U}_\mathbb{Z}^0$. By [19], we have $\mathcal{U}_\mathbb{Z}^- \otimes \mathcal{U}_\mathbb{Z}^0 \otimes \mathcal{U}_\mathbb{Z}^+ \cong \mathcal{U}_\mathbb{Z}$.

We define $\mathfrak{g}_\mathbb{Z} = \mathfrak{g} \cap \mathcal{U}_\mathbb{Z}$ and $\mathfrak{h}_\mathbb{Z} = \mathfrak{h} \cap \mathcal{U}_\mathbb{Z}$.

Note that $\mathfrak{g} = \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ and $\mathfrak{h} = \mathfrak{h}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$.

We have $\mathcal{U}_\mathbb{Z}^+ \cdot v_\lambda = \mathbb{Z}v_\lambda$ since every x_α with $\alpha \in \Delta_+^{\mathrm{re}}$ annihilates v_λ. Also $\mathcal{U}_\mathbb{Z}^0 \cdot v_\lambda = \mathbb{Z}v_\lambda$ since $\mathcal{U}_\mathbb{Z}^0$ acts on v_λ as scalar multiplication by an integer. Therefore $\mathcal{U}_\mathbb{Z} \cdot v_\lambda = \mathcal{U}_\mathbb{Z}^- \cdot v_\lambda$. We define the $\mathbb{Z}$-*form of* V to be

$$V_\mathbb{Z} := \mathcal{U}_\mathbb{Z} \cdot v_\lambda = \mathcal{U}_\mathbb{Z}^- \cdot v_\lambda$$

and note that $V = V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$. For $\mu \in \mathrm{wts}(V)$, set $V_{\mu,\mathbb{Z}} = V_\mu \cap V_\mathbb{Z}$.

3.2. The groups $G_V(\mathbb{Q})$ and $G(\mathbb{Z})$

For $i \in I$ and $t \in \mathbb{Q}$ set

$$\chi_{\alpha_i}(t) = \exp(t\rho(e_i)) = \sum_{m=0}^\infty \frac{t^m \rho(e_i)^m}{m!}, \quad \chi_{-\alpha_i}(t) = \exp(t\rho(f_i)) = \sum_{m=0}^\infty \frac{t^m \rho(f_i)^m}{m!},$$

where ρ is the defining homomorphism for $V = V^\lambda$. Since V is integrable, the $\chi_{\pm\alpha_i}(t)$ are well-defined elements of $\mathrm{GL}(V)$. We let $G_V(\mathbb{Q}) \leq \mathrm{GL}(V)$ be the group generated by $\chi_{\pm\alpha_i}(t)$ for $i \in I$ and $t \in \mathbb{Q}$. We refer to $G_V(\mathbb{Q})$ as a *minimal representation-theoretic Kac-Moody group*. Similarly we define $G(\mathbb{Z})$ to be the group generated by $\chi_{\pm\alpha_i}(t)$ for $i \in I$ and $t \in \mathbb{Z}$. We denote the action of $g \in G_V(\mathbb{Q})$ on $v \in V$ by $g \cdot v$.

For each $i \in I$ and $s \in \mathbb{Q}^\times$, we recall that

$$\widetilde{w}_i(s) := \chi_{\alpha_i}(s)\chi_{-\alpha_i}(-s^{-1})\chi_{\alpha_i}(s) \quad \text{and} \quad \widetilde{w}_i := \widetilde{w}_i(1).$$

Let $h_{\alpha_i}(s) = \widetilde{w}_i(s)\widetilde{w}_i(1)^{-1}$ for $s \in \mathbb{Q}^\times$ and $i \in I$.

Let $H \subseteq G_V(\mathbb{Q})$ be the subgroup generated by the $h_{\alpha_i}(s)$. Let N denote the subgroup of $G_V(\mathbb{Q})$ generated by H and the $\widetilde{w}_i$ for $i \in I$. By Lemma 4 in Section 5.4 of [19], there exists a unique endomorphism $N \longrightarrow W$ with kernel H. For $w \in W$, take a reduced word $w = w_{i_1} w_{i_2} \cdots w_{i_k}$ and define

$$\widetilde{w} := \widetilde{w}_{i_1} \widetilde{w}_{i_2} \cdots \widetilde{w}_{i_k}.$$

Now $\widetilde{w}_i \in G(\mathbb{Z}) \subseteq \Gamma(\mathbb{Z})$ and so $\widetilde{w} \cdot V_\mathbb{Z} = V_\mathbb{Z}$ for all $w \in W$.

Note that the adjoint representation makes $\mathfrak{g}$ into a G-module and we also get $\mathrm{Ad}(\widetilde{w})(\mathfrak{g}_\mathbb{Z}) = \mathfrak{g}_\mathbb{Z}$ as in [13]. Let $\alpha \in \Delta_+^{\mathrm{re}}$, then $\alpha = w\alpha_i$ for some $w \in W$ and $i \in I$. We define the *(real) root vectors* ([13], Section 4.1) by

$$x_\alpha := \mathrm{Ad}(\widetilde{w})(e_i) \in \mathfrak{g}_\alpha \cap \mathfrak{g}_\mathbb{Z} \quad \text{and} \quad x_{-\alpha} := \mathrm{Ad}(\widetilde{w})(f_i) \in \mathfrak{g}_{-\alpha} \cap \mathfrak{g}_\mathbb{Z},$$

so that $x_{-\alpha} = -\omega(x_\alpha)$ and $[x_\alpha, x_{-\alpha}] = \alpha^\vee$. In particular, $x_{\alpha_i} = e_i$ and $x_{-\alpha_i} = f_i$. The set $\{x_\alpha, x_{-\alpha}, \alpha^\vee\}$ forms a basis for a subalgebra isomorphic to $\mathfrak{sl}_2$.

Lemma 3.1.   *Let $w \in W$, $\mu \in \mathrm{wts}(V)$, $\alpha \in \Delta_+^{\mathrm{re}}$, $m \in \mathbb{N}$.*

(a) $\widetilde{w} \cdot V_{\mu,\mathbb{Z}} = V_{w\mu,\mathbb{Z}}$.

(b) $V_{w\lambda,\mathbb{Z}} = \widetilde{w} \cdot V_{\lambda,\mathbb{Z}} = \mathbb{Z}v_{w\lambda}$ *where* $v_{w\lambda} = \widetilde{w} \cdot v_\lambda$.

(c) $\dfrac{x_{\pm\alpha}^m}{m!} \in \mathcal{U}_\mathbb{Z}$ *and* $\dfrac{x_{\pm\alpha}^m}{m!} \cdot V_{\mu,\mathbb{Z}} \subseteq V_{\mu\pm m\alpha,\mathbb{Z}}$.

Proof.   Lemma 3.8 (a) in [8] implies $\widetilde{w}_i \cdot V_\mu = V_{w_i\mu}$ for $i \in I$, and so $\widetilde{w} \cdot V_\mu = V_{w\mu}$ by induction. Combining this with $\widetilde{w} \cdot V_\mathbb{Z} = V_\mathbb{Z}$ and $V = \bigoplus_{\mu \in \mathrm{wts}(V)} V_\mu$ we get (a). Parts (b) and (c) are now straightforward.   ∎

For $t \in \mathbb{Q}$, set $\chi_\alpha(t) := \exp(t\rho(x_\alpha)) = \sum_{m=0}^\infty \dfrac{t^m \rho(x_\alpha)^m}{m!} \in \mathrm{GL}(V)$.

We have $\chi_\alpha(t) = \widetilde{w}\chi_{\alpha_i}(t)\widetilde{w}^{-1} \in G_V(\mathbb{Q})$. We define the *(real) root group* and its $\mathbb{Z}$-subgroup as

$$U_\alpha = \{\chi_\alpha(t) \mid t \in \mathbb{Q}\} \quad \text{and} \quad U_\alpha(\mathbb{Z}) = \{\chi_\alpha(t) \mid t \in \mathbb{Z}\}.$$

Given a set of positive real roots $\Omega \subseteq \Delta_+^{\mathrm{re}}$, we define

$$U_\Omega = \langle U_\alpha \mid \alpha \in \Omega \rangle \quad \text{and} \quad U_\Omega(\mathbb{Z}) = \langle U_\alpha(\mathbb{Z}) \mid \alpha \in \Omega \rangle.$$

In particular, the *unipotent subgroup* is $U := U_{\Delta_+^{\mathrm{re}}}$ and its $\mathbb{Z}$-subgroup is therefore $U(\mathbb{Z}) := U_{\Delta_+^{\mathrm{re}}}(\mathbb{Z})$. It is clear that $U_\Omega(\mathbb{Z}) \subseteq U_\Omega \cap G(\mathbb{Z})$. The reverse inclusion is conjectured to be true for reasonable choices of Ω but will not be needed for the results in this paper.

We end this section with the following:

Lemma 3.2.   *For all $u \in U$, $u \cdot v_\lambda = v_\lambda$.*

Proof.   It suffices to prove this statement for a generator $\chi_\alpha(t)$ of $U_\alpha \subseteq U$. Now

$$\chi_\alpha(t) \cdot v_\lambda = v_\lambda + tx_\alpha \cdot v_\lambda + t^2 \frac{x_\alpha^2}{2!} \cdot v_\lambda + \cdots + t^m \frac{x_\alpha^m}{m!} \cdot v_\lambda,$$

for some m since x_α acts nilpotently. But $x_\alpha^i \cdot v_\lambda \in V_{\lambda + i\alpha} = 0$, since $\lambda + i\alpha$ is not a weight of V for $i > 0$. Thus, $\chi_\alpha(t) \cdot v_\lambda = v_\lambda$.   ∎

4.   Strong integrality of real root groups

Throughout this section, fix $\alpha \in \Delta^{\mathrm{re}}$. The main result of this section is to prove strong integrality of the real root group

$$U_\alpha = \{\chi_\alpha(t) \mid t \in \mathbb{Q}\}.$$

This provides the base case for Theorem 1.2. Integrality of U_α will follow from the following lemma.

Lemma 4.1.   *Strong integrality implies integrality.*

Proof. Let $M \subseteq G$ be strongly integral with respect to $v_0 \in V_\mathbb{Z}$. Let $g \in M \cap \Gamma(\mathbb{Z})$. Then $g \cdot v \in V_\mathbb{Z}$ for all $v \in V_\mathbb{Z}$. In particular $g \cdot v_0 \in V_\mathbb{Z}$ and so $g \in G(\mathbb{Z})$ by strong integrality.   ∎

Define
$$x_\alpha^{(m)} := \frac{x_\alpha^m}{m!}.$$
We will need the following lemma.

Lemma 4.2.   *Let μ be a weight such that $\mu + \alpha$ is not a weight of V. Further assume that $v_\mu \in V_\mu$ and $n := \langle \mu, \alpha^\vee \rangle > 0$. Then*

$$x_\alpha^{(n)} x_{-\alpha}^{(n)} \cdot v_\mu = v_\mu.$$

Proof.   First we prove that

$$[x_\alpha, x_{-\alpha}^k] = kx_{-\alpha}^{k-1} \left(\alpha^\vee - (k-1) \right)$$

by induction on $k \in \mathbb{N}$:

$$
\begin{aligned}
[x_\alpha^k, x_{-\alpha}] &= x_\alpha x_{-\alpha}^k - x_{-\alpha}^k x_\alpha \\
&= x_\alpha^k x_{-\alpha} - x_{-\alpha}^{k-1} x_\alpha x_{-\alpha} + x_{-\alpha}^{k-1} x_\alpha x_{-\alpha} - x_{-\alpha}^k x_\alpha \\
&= [x_\alpha, x_{-\alpha}^{k-1}] x_{-\alpha} + x_{-\alpha}^{k-1} [x_\alpha, x_{-\alpha}] \\
&= (k-1) x_{-\alpha}^{k-2} \left(\alpha^\vee - k + 2 \right) x_{-\alpha} + x_{-\alpha}^{k-1} \alpha^\vee \\
&= (k-1) x_{-\alpha}^{k-2} (\alpha^\vee x_{-\alpha}) - (k-1)(k-2) x_{-\alpha}^{k-1} + x_{-\alpha}^{k-1} m \alpha^\vee \\
&= (k-1) x_{-\alpha}^{k-2} (x_{-\alpha} \alpha^\vee - 2x_{-\alpha}) - (k-1)(k-2) x_{-\alpha}^{k-1} + x_{-\alpha}^{k-1} \alpha^\vee \\
&= kx_{-\alpha}^{k-1} \alpha^\vee - k(k-1) x_{-\alpha}^{k-1}.
\end{aligned}
$$

Note that $x_\alpha \cdot v_\mu \in V_{\mu+\alpha} = 0$, so $x_\alpha \cdot v_\mu = 0$. Also $\alpha^\vee \cdot v_\mu = \langle \mu, \alpha^\vee \rangle v_\mu = n v_\mu$. Next we prove

$$x_\alpha^m x_{-\alpha}^m \cdot v_\mu = m! \prod_{j=0}^{m-1} (n-j) \cdot v_\mu$$

by induction on m:

$$\begin{aligned}
x_\alpha^m x_{-\alpha}^m \cdot v_\mu &= x_\alpha^{m-1} \left([x_\alpha, x_{-\alpha}^m] + x_{-\alpha}^m x_\alpha \right) \cdot v_\mu \\
&= x_\alpha^{m-1} [x_\alpha, x_{-\alpha}^m] \cdot v_\mu \\
&= x_\alpha^{m-1} x_{-\alpha}^{m-1} m \left(\alpha^\vee - (m-1) \right) \cdot v_\mu \\
&= m \left(n - (m-1) \right) x_\alpha^{m-1} x_{-\alpha}^{m-1} \cdot v_\mu \\
&= m! \prod_{j=0}^{m-1} (n-j) \cdot v_\mu.
\end{aligned}$$

Now take $m = n$ to get $x_\alpha^n x_{-\alpha}^n \cdot v_\mu = n!\, n!\, v_\mu$, so $x_\alpha^{(n)} x_{-\alpha}^{(n)} \cdot v_\mu = v_\mu$. ∎

Lemma 4.3 (Lemma 7.2 of [17]). *Let $\alpha \in \Delta^{\mathrm{re}}$ and $t \in \mathbb{Q}$. If μ is a weight of V and $v \in V_\mu$, then*

$$\chi_\alpha(t) \cdot v = v + \sum_{m \in \mathbb{N}} t^m (x_\alpha^{(m)} \cdot v),$$

where $x_\alpha^{(m)} \cdot v \in V_{\mu+m\alpha}$, and only finitely many of the terms $x_\alpha^{(m)} \cdot v$ are non-zero. If moreover $v \in V_{\mu,\mathbb{Z}}$, then $x_\alpha^{(m)} \cdot v \in V_{\mu+m\alpha,\mathbb{Z}}$.

Proof.   If $v \in V_{\mu,\mathbb{Z}}$, then $x_\alpha^{(m)} \cdot v \in V_{\mu+m\alpha,\mathbb{Z}}$ by Lemma 3.1 (c). ∎

Proposition 4.4.   *Let $\alpha \in \Delta_+^{\mathrm{re}}$. If $\chi_\alpha(t) \widetilde{w}_\alpha \cdot v_\lambda \in V_\mathbb{Z}$, then $\chi_\alpha(t) \in U_\alpha(\mathbb{Z})$.*

Proof.   Suppose $\chi_\alpha(t) \cdot v_0 \in V_\mathbb{Z}$ where $v_0 := \widetilde{w}_\alpha \cdot v_\lambda$. Then v_0 is a lowest weight vector for the $\mathfrak{sl}_2$-module corresponding to α. Let $n = \langle \lambda, \alpha^\vee \rangle$ so that $w_\alpha \lambda = \lambda - n\alpha$. Note that $n > 0$ since λ is regular. Now $V_{\lambda-n\alpha,\mathbb{Z}} = \mathbb{Z} v_0$ by Lemma 3.1 (b). But $x_{-\alpha}^{(n)} \cdot v_\lambda$ is also in $V_{\lambda-n\alpha,\mathbb{Z}}$ by Lemma 3.1 (c), so $x_{-\alpha}^{(n)} \cdot v_\lambda = m v_0$ for some integer m.

We have $\chi_\alpha(t) \cdot v_0 = v_0 + t x_\alpha \cdot v_0 + t^2 x_\alpha^{(2)} \cdot v_0 + \cdots + t^n x_\alpha^{(n)} \cdot v_0.$

Now consider the $V_{\lambda,\mathbb{Z}}$ component:

$$t^n x_\alpha^{(n)} \cdot v_0 = \frac{t^n}{m} x_\alpha^{(n)} x_{-\alpha}^{(n)} \cdot v_\lambda = \frac{t^n}{m} v_\lambda$$

by Lemma 4.2. Hence $\frac{t^n}{m} \in \mathbb{Z}$, so $t^n \in \mathbb{Z}$ and so $t \in \mathbb{Z}$. ∎

Since $U_\alpha(\mathbb{Z}) \subseteq U_\alpha \cap G(\mathbb{Z})$, we get:

Corollary 4.5.   *If $\alpha \in \Delta^{\mathrm{re}}$, then U_α is strongly integral with respect to $\widetilde{w}_\alpha \cdot v_\lambda$. Hence U_α is integral.*

It follows immediately that for $\alpha \in \Delta^{\mathrm{re}}$ and $u \in U(\mathbb{Z})$, if $\chi_\alpha(s) u \in \Gamma(\mathbb{Z})$ then $s \in \mathbb{Z}$.

5. Inversion sets and subgroups

5.1. Inversion sets and orderings

Let $w \in W$. We define the corresponding *inversion set* of roots by

$$\Phi_{(w)} \;=\; \{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_-\} = \Delta_+ \cap w(\Delta_-).$$

We note the following standard properties of $\Phi_{(w)}$, which can be proven by induction on $\ell(w)$.

Lemma 5.1.   *Suppose that $w \in W$ has the reduced expression $w_{i_1} w_{i_2} \cdots w_{i_k}$.*

(a) $\Phi_{(w)} = \{\alpha_{i_1},\ w_{i_1}\alpha_{i_2},\ w_{i_1}w_{i_2}\alpha_{i_3},\ \dots,\ w_{i_1}w_{i_2}\cdots w_{i_{k-2}}w_{i_{k-1}}\alpha_{i_k}\} \subseteq \Delta_+^{\mathrm{re}}.$

(b) $\Phi_{(w)}$ *has cardinality* $k = \ell(w)$.

(c) *If* $w' = w_{i_1}w_{i_2}\cdots w_{i_{k-1}}$, *then* $\Phi_{(w)} = \Phi_{(w')} \sqcup \{w'\alpha_{i_k}\}$.

(d) *If* $w'' = w_{i_2}\cdots w_{i_k}$, *then* $\Phi_{(w)} = \{\alpha_{i_1}\} \sqcup w_{i_1}\Phi_{(w'')}$.

(e) *For* $\mu \in \mathfrak{h}^*$, $w\mu = \mu - \langle\mu,\alpha_{i_1}\rangle\alpha_{i_1} - \langle\mu,\alpha_{i_2}\rangle w_{i_1}\alpha_{i_2} - \cdots - \langle\mu,\alpha_{i_k}\rangle w_{i_1}w_{i_2}\cdots w_{i_{k-1}}\alpha_{i_k}.$

We also note the immediate consequences of Lemma 5.1:

Lemma 5.2.

(a) *For* $\alpha \in \Delta_+^{\mathrm{re}}$, *there exists* $w \in W$ *such that* $\alpha \in \Phi_{(w)}$.

(b) *For a finite subset* $\Omega \subseteq \Delta_+^{\mathrm{re}}$, *there exists a finite set* $T \subseteq W$ *such that* $\Omega \subseteq \bigcup_{w \in T} \Phi_w$.

We equip $\Phi_{(w)}$ with an ordering

$$\alpha_{i_1} \prec w_{i_1}\alpha_{i_2} \prec w_{i_1}w_{i_2}\alpha_{i_3} \prec \cdots \prec w_{i_1}w_{i_2}\cdots w_{i_{k-2}}w_{i_{k-1}}\alpha_k,$$

as described by Papi in [14]. Note that this ordering depends of the choice of reduced word for w.

5.2. Inversion subgroups

For $w \in W$, the *inversion subgroup* is defined as

$$U_{(w)} = U_{\Phi_{(w)}}(\mathbb{Q}) = \langle U_\alpha(\mathbb{Q}) \mid \alpha \in \Phi_{(w)}\rangle.$$

Lemma 5.3.   *Each element* $u_{(w)} \in U_{(w)}$ *can be written uniquely as*

$$u_{(w)} = \prod_{\gamma \in \Phi_w} u_\beta,$$

where $u_\beta \in U_\beta$ *and the product is in the ordering on* $\Phi_{(w)}$.

Corollary 5.4. $U_{(w)}(\mathbb{Z}) = U_{(w)} \cap U(\mathbb{Z})$.

For $w \in W$, set $U^{(w)} = U \cap w^{-1}Uw$. We end this section with the following lemma from Subsection 4.2.6 of [15].

Lemma 5.5. $U = U^{(w)}U_{(w)} = U_{(w)}U^{(w)}$.

6.   Integrality of inversion subgroups

To prove Theorem 1.2, we need a preliminary result, which we discuss next. Define

$$x_{\beta_1,\beta_2,\ldots,\beta_k}^{(i_1,i_2,\ldots,i_k)} := x_{\beta_1}^{(i_1)} x_{\beta_2}^{(i_2)} \cdots x_{\beta_k}^{(i_k)}.$$

The proof of the following is similar to Lemma 4.3:

Lemma 6.1.   *Let $\beta_i \in \Delta_+^{\mathrm{re}}$ and $t_i \in \mathbb{Q}$ for $i = 1,\ldots,k$. If μ is a weight of V and $v \in V_\mu$, then*

$$\prod_{i=1}^k \chi_{\beta_i}(t_i) \cdot v = \sum_{i_1,i_2,\ldots,i_k \in \mathbb{Z}_{\geq 0}} t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k} \left(x_{\beta_1,\beta_2,\ldots,\beta_k}^{(i_1,i_2,\ldots,i_k)} \cdot v \right)$$

where $x_{\beta_1,\beta_2,\ldots,\beta_k}^{(i_1,i_2,\ldots,i_k)} \cdot v \in V_{\mu+i_1\beta_1+i_2\beta_2+\cdots+i_k\beta_k}$ and only finitely of the terms $x_{\beta_1,\beta_2,\ldots,\beta_k}^{(i_1,i_2,\ldots,i_k)} \cdot v$ are non-zero. If moreover $v \in V_{\mu,\mathbb{Z}}$, then $x_{\beta_1,\beta_2,\ldots,\beta_k}^{(i_1,i_2,\ldots,i_k)} \cdot v \in V_{\mu+i_1\beta_1+i_2\beta_2+\cdots+i_k\beta_k,\mathbb{Z}}$.

Proposition 6.2.   *Let $w \in W$ and $u_{(w)} \in U_{(w)}$. If $u_{(w)} \widetilde{w} \cdot v_\lambda \in V_\mathbb{Z}$, then $u_{(w)} \in U_{(w)}(\mathbb{Z})$.*

Theorem 1.2 follows immediately from this result since $U_{(w)}(\mathbb{Z}) \subseteq U_{(w)} \cap G(\mathbb{Z})$.

Proof.   We proceed by induction on $k = \ell(w)$. The base case $\ell(w) = 1$ is just Proposition 4.4 for a simple root α_i. Thus we take $\ell(w) \geq 2$ and assume the inductive hypothesis. Let $\Phi_{(w)} = \{\beta_1,\ldots,\beta_k\}$ in the usual order. So $\beta := \beta_1$ is a simple root and $w = w_\beta w''$ with $\ell(w'') = k - 1$, and $\Phi_{(w'')} = \{w_\beta \beta_2,\ldots,w_\beta \beta_k\}$ by Lemma 5.1(5.1). In particular, $w_\beta \beta_2,\ldots,w_\beta \beta_k$ are positive roots. Write

$$u_{(w)} = \prod_{i=1}^k \chi_{\beta_i}(t_i) = \chi_\beta(t) \prod_{i=2}^k \chi_{\beta_i}(t_i),$$

for $t = t_1, t_2, \ldots, t_k \in \mathbb{Q}$.

Write $\mu = w''\lambda$ and $v_\mu = \widetilde{w}'' \cdot v_\lambda$. Let $n = \langle \mu, \beta^\vee \rangle$, so that $w\lambda = w_\beta \mu = \mu - n\beta$. Let $v_{\mu-n\beta} = \widetilde{w}_\beta \cdot v_\mu = \widetilde{w} \cdot v_\lambda$. Then

$$u_{(w)} \widetilde{w} \cdot v_\lambda = u_{(w)} \cdot v_{\mu-n\beta} = \sum_{i_1,i_2,\ldots,i_k \in \mathbb{Z}_{\geq 0}} t_1^{i_1} t_2^{i_2} \cdots t_k^{i_k} \left(x_{\beta_2,\ldots,\beta_k}^{(i_2,\ldots,i_k)} \cdot v_{\mu-n\beta} \right)$$

$$= t^n x_\beta^{(n)} \cdot v_{\mu-n\beta} + \sum_{\substack{i \in \mathbb{Z}_{\leq n},\, i_2,\ldots,i_k \in \mathbb{Z}_{\geq 0} \\ \text{at least one non-zero}}} t^{n-i} t_2^{i_2} \cdots t_k^{i_k} \left(x_{\beta,\beta_2,\ldots,\beta_k}^{(n-i,i_2,\ldots,i_k)} \cdot v_{\mu-n\beta} \right),$$

where we have substituted $i = n - i_1$ and extracted the term with $i, i_2, \ldots, i_k = 0$ from the sum. Therefore $t^n x_\beta^{(n)} \cdot v_{\mu-n\beta} \in V_\mu$ and $x_{\beta,\beta_2,\ldots,\beta_k}^{(n-i,i_2,\ldots,i_k)} \cdot v_{\mu-n\beta} \in V_\nu$ for $\nu = \mu - i\beta + \sum_{j=2}^k i_j \beta_j$. We have

$$(w'')^{-1}(\nu) = \lambda - (w'')^{-1}(i\beta) + (w'')^{-1}\left(\sum_{j=2}^k i_j \beta_j \right) \geq \lambda - i(w'')^{-1}(\beta),$$

but $(w'')^{-1}(\beta) > 0$, so we must have $i \geq 0$ for ν to be a weight of V. So we can take i to be in the range $0 \leq i \leq n$. If $\mu = \nu$, then $i\beta = \sum_{j=2}^{k} i_j \beta_j$, but $w_\beta (i_j \beta_j) \geq 0$ and $w_\beta(i\beta) \leq 0$, so $i, i_1, \ldots, i_k = 0$. Hence the $V_{\mu,\mathbb{Z}}$ component of $u_{(w)} \widetilde{w} \cdot v_\lambda \in V_\mathbb{Z}$ is just $t^n x_\beta^{(n)} \cdot v_{\mu - n\beta}$.

Now $V_{\mu,\mathbb{Z}} = \mathbb{Z} v_\mu$, so $t^n x_\beta^{(n)} \cdot v_{\mu - n\beta} = m v_\mu$ for some integer m. Also we have $x_{-\beta}^{(n)} \cdot v_\mu \in V_{\mu - n\beta, \mathbb{Z}} = \mathbb{Z} v_{\mu - n\beta}$, so $x_{-\beta}^{(n)} \cdot v_\mu = m' v_{\mu - n\beta}$ for some integer m'. Hence

$$mm' v_{\mu - n\beta} = m x_{-\beta}^{(n)} \cdot v_\mu = t^n x_{-\beta}^{(n)} x_\beta^{(n)} \cdot v_{\mu - n\beta} = t^n v_{\mu - n\beta}$$

by Lemma 4.2 for $\alpha = -\beta$, since $\mu - n\beta - \beta \notin \mathrm{wts}(V)$ as

$$w^{-1}(\mu - n\beta - \beta) = w^{-1}(w\lambda - \beta) = \lambda - w^{-1}\beta > \lambda.$$

Hence $t^n = mm'$ is an integer, so t is an integer.

Now
$$u_{(w)} \widetilde{w} = \chi_\beta(t) \prod_{i=2}^{k} \chi_{\beta_i}(t_i) \, \widetilde{w}_\beta \widetilde{w}'' = \chi_\beta(t) \widetilde{w}_\beta u_{(w'')} \widetilde{w}'',$$

where $u_{(w'')} = \prod_{i=2}^{k} \chi_{w_\beta \beta_i}(t_i) \in U_{(w'')}$. Now $\chi_\beta(t) \widetilde{w}_\beta \in G(\mathbb{Z})$, so

$$u_{(w'')} \widetilde{w}'' \cdot v_\lambda = \left(\chi_\beta(t) \widetilde{w}_\beta \right)^{-1} \cdot \left(u_{(w)} \widetilde{w} \cdot v_\lambda \right) \in V_\mathbb{Z},$$

and so $u_{(w'')} \in U_{(w'')}(\mathbb{Z})$ by induction. Hence $u_{(w)} = \chi_\beta(t) \widetilde{w}_\beta u_{(w'')} \widetilde{w}_\beta^{-1} \in U_{(w)}(\mathbb{Z})$. ∎

7.   Further integrality results

7.1. Integrality of finite dimensional unipotent groups

For this subsection, we assume that A has finite type. In this case, A is a Cartan matrix and $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra over $\mathbb{Q}$. The group construction in Section 3 can be carried out with the same external data: a highest weight representation V with dominant, regular highest weight λ, a $\mathbb{Z}$-form of the universal enveloping algebra (constructed by Cartier and Kostant, see [9]) and a lattice $V_\mathbb{Z}$ in V (called an *admissible lattice* in [4]).

The group $G_V(\mathbb{Q})$ is then a semisimple algebraic group and $G_V(\mathbb{Q}) \cong G_\mathbb{Z}(\mathbb{Q})$, where $G_\mathbb{Z}(\mathbb{Q})$ is Chevalley's group scheme, now known as the Chevalley-Demazure group scheme ([5]).

Our construction of the Kac-Moody group $G_V(\mathbb{Q})$ does not explicitly involve a lattice L between the root lattice and the weight lattice, as in Chevalley's construction, though the assumption that V is a highest weight module rules out the adjoint representation and hence the construction of an adjoint form of $G_V(\mathbb{Q})$. With the assumption that the set of weights of V contains all the fundamental weights, our group $G_V(\mathbb{Q})$ coincides with Chevalley's simply connected group $G_\mathbb{Z}(\mathbb{Q})$.

As a corollary of Theorem 1.2, we have the following.

Corollary 7.1.   *Suppose that the matrix A has finite type. Suppose also that the set of weights of V contains all the fundamental weights. Then the unipotent subgroup U of $G_V(\mathbb{Q})$ is integral.*

Proof.   Since A has finite type, the Weyl group W is finite. Moreover, W contains the longest element, denoted w_0, which flips all positive roots to negative roots. That is $\Phi_{(w_0)} = \Delta^+$, which gives $U_{(w_0)} = U$. Combining this with Theorem 1.2 and Lemma 4.1, it follows that U is integral. ∎

7.2. Integrality of groups generated by commuting real root subgroups

We now prove integrality of subgroups of U generated by commuting real root groups.

Theorem 7.2.   *Let U be the positive unipotent subgroup of $G = G^\lambda(\mathbb{Q})$. If M is a subgroup of U generated by commuting real root subgroups, then M is integral.*

Proof.   Assume, for the sake of contradiction, that

$$u = \chi_{\beta_1}(t_{\beta_1})\chi_{\beta_2}(t_{\beta_2})\cdots\chi_{\beta_N}(t_{\beta_N}) \in M$$

is the shortest word such that $u \cdot V_\mathbb{Z} = V_\mathbb{Z}$ but $u \notin U(\mathbb{Z})$. By Corollary 4.5, we must have $N > 1$. Now $\Omega := \{\beta_1, \beta_2, \ldots, \beta_N\} \subseteq \Delta^{\mathrm{re}}_+$ is a finite set of real roots whose corresponding root subgroups commute with each other. By part (b) of Lemma 5.2, there exists $w \in W$ such that $\Omega \cap \Phi_{(w)} \neq \emptyset$. Using the commuting of the real root groups U_{β_j}, we can rearrange the product above as

$$u = u_{(w)}u^{(w)} = u^{(w)}u_{(w)},$$

where $u_{(w)} = \prod_{\beta \in \Omega \cap \Phi_{(w)}} \chi_\beta(t_\beta)$ and $u^{(w)} = \prod_{\beta \in \Omega \setminus \Phi_{(w)}} \chi_\beta(t_\beta)$.

Now we have $u\widetilde{w} \cdot v_\lambda = u_{(w)}u^{(w)}\widetilde{w} \cdot v_\lambda = u_{(w)}\widetilde{w} \cdot (\widetilde{w}^{-1}u^{(w)}\widetilde{w} \cdot v_\lambda)$.

But $\widetilde{w}^{-1}u^{(w)}\widetilde{w} \in U$, so $\widetilde{w}^{-1}u^{(w)}\widetilde{w}\cdot v_\lambda = v_\lambda$ by Lemma 3.2. So $u_{(w)}\widetilde{w}\cdot v_\lambda = u\widetilde{w}\cdot v_\lambda \in V_\mathbb{Z}$. But $U_{(w)}$ is strongly integral with respect to $\widetilde{w}\cdot v_\lambda$ by Theorem 1.2, so $u_{(w)} \in U(\mathbb{Z})$. Hence ${u_{(w)}}^{-1} \in U(\mathbb{Z})$ and so

$$u^{(w)} \cdot V_\mathbb{Z} = u \cdot ({u_{(w)}}^{-1} \cdot V_\mathbb{Z}) = u \cdot V_\mathbb{Z} = V_\mathbb{Z}.$$

Since $\Omega \cap \Phi_{(w)}$ is nonempty, $u^{(w)} = \prod_{\beta \in \Omega \setminus \Phi_{(w)}} \chi_\beta(t_\beta)$ is a shorter word than u and so our assumption implies $u^{(w)} \in U(\mathbb{Z})$. But now $u = u_{(w)}u^{(w)} \in U(\mathbb{Z})$, which is a contradiction. ∎

7.3. Integrality of subgroups of U in the rank 2 Kac-Moody case

We start with a method for constructing infinite dimensional integral unipotent subgroups.

Proposition 7.3.   *Let $\Omega = \bigcup_{n=1}^\infty \Omega_n \subseteq \Delta^{\mathrm{re}}_+$ where $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ and assume that each U_{Ω_n} is integral. Then $U_\Omega = \langle U_\alpha \mid \alpha \in \Omega \rangle$ is integral.*

Proof.   Take $u \in U_\Omega$ with $u \cdot V_\mathbb{Z} = V_\mathbb{Z}$. Then $u = \prod_{i=1}^N \chi_{\beta_i}(a_i)$ for some $\beta_i \in \Omega$, $a_i \in \mathbb{Q}$. But now the finite set of roots $\{\beta_1, \ldots, \beta_N\}$ must be contained in Ω_n for some n. Hence $u \in U_{\Omega_n}$, which is an integral subgroup. So $u \in G(\mathbb{Z})$. ∎

One way to construct such sets is by taking a sequence of simple reflections $w_{i_1}, w_{i_2}, \ldots$ such that every initial subsequence gives a reduced word $w_{i_1} \cdots w_{i_n} \in W$. Then each group $U_{(w_{i_1} \cdots w_{i_n})}$ is integral by Theorem 1.2. Now take

$$\Omega := \bigcup_{n=1}^{\infty} \Phi_{(w_{i_1} \cdots w_{i_n})} = \{\alpha_{i_1}, \ w_{i_1}\alpha_{i_2}, \ w_{i_1}w_{i_2}\alpha_{i_3}, \ \ldots, \ w_{i_1}w_{i_2} \cdots w_{i_{k-2}}w_{i_{k-1}}\alpha_k, \ \ldots\}.$$

Then the subgroup U_Ω is integral.

Now suppose that $\mathfrak{g}$ has rank 2. A detailed description of the structure of Δ^{re} and U in rank 2 can be found in [3]. We define

$$\Omega_1 := \{\alpha_1, \ w_1\alpha_2, \ w_1w_2\alpha_1, \ \ldots\} \quad \text{and} \quad \Omega_2 := \{\alpha_2, \ w_2\alpha_1, \ w_2w_1\alpha_2, \ \ldots\}.$$

In fact, Ω_1 (respectively Ω_2) is the set of all positive real roots on the lower (respectively upper) branches of the hyperbolas defined in [3].

In this case $U = U_1 * U_2$, where for $i = 1, 2$, the groups $U_i := U_{\Omega_i}$ are abelian if A is symmetric, and nilpotent of class 2 if A is not symmetric (see [3]).

The following theorem follows immediately from Proposition 7.3.

Theorem 7.4. *Suppose that $G_V(\mathbb{Q})$ has rank 2 with $U = U_1 * U_2$, where $U_i = U_{\Omega_i}$. Then the groups U_i are integral.*

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