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# Polynomial Modules over a Class of GIM Lie Algebras

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**Abstract.** We construct and classify all rank one polynomial modules over the GIM Lie algebra $\mathfrak{g}_n$ $(n \geq 3)$ with structural matrix

$$\begin{bmatrix} 2 & -1 & & & 1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 1 & & & -1 & 2 \end{bmatrix}_{n\times n}.$$

Moreover, the simplicity of these modules is studied.

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*Key Words:* GIM Lie algebra, Cartan subalgebra, polynomial module, non-weight module.

## 1. Introduction

In 1986, Slodowy [14] introduced the generalized intersection matrix (GIM for short). A GIM is an $n \times n$ square integer matrix $O = (m_{ij})_{n\times n}$ with the conditions

$$\begin{aligned} &m_{ii} = 2 \text{ for } i \in \{1, \dots, n\}, \\ &m_{ij} < 0 \text{ if and only if } \ m_{ji} < 0, and \\ &m_{ij} > 0 \text{ if and only if } \ m_{ji} > 0 \text{ for } i \neq j. \end{aligned}$$

It is clear that a GIM generalizes the notion of the generalized Cartan matrix. The GIM Lie algebra (see [4, 13]) associated with a GIM is a generalization of the Kac-Moody Lie algebra, which was introduced by Slodowy. Moreover, a GIM Lie algebra is isomorphic to an involutory subalgebra of some Kac-Moody Lie algebra with a symmetric Dynkin diagram (see [2, 14]).

In the past decades, many authors made contributions on the GIM Lie algebras [3, 4, 1, 7, 10, 19] and their $q$-deformations [8, 17, 18]. In [3], Berman, Jurisich, and Tan showed that a new class of Lie algebras by generators and relations which simultaneously generalize the Borcherds Lie algebras and the GIM Lie algebras were subalgebras of Borcherds Lie algebras. [19] considered a class of Lie algebras arising from symmetrizable GIM. In [17], Lusztig symmetries of Quantized GIM Lie algebras were studied. [8] proved that the quantized GIM algebras for simple-laced cases were isomorphic to a subalgebra of a quantum universal enveloping algebra.

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In this paper, we focus on the GIM Lie algebra $\mathfrak{g}_n$ $(n \geq 3)$ with the structural matrix

$$O_n = (m_{ij})_{n\times n} = \begin{bmatrix} 2 & -1 & & & 1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 1 & & & -1 & 2 \end{bmatrix}_{n\times n}. \quad (1)$$

$\mathfrak{g}_n$ is perhaps the simplest oriented GIM Lie algebra since it can be mapped into affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}_{2n}}$ by Berman's homomorphism $\varphi$ in Theorem 1.31 of [2]. In [9], the finite dimensional simple modules over $\mathfrak{g}_n$ were classified.

Let $L$ be a Lie algebra with a Cartan subalgebra $\mathfrak{h}$. An $L$-module $V$ is called a polynomial module if it is a finite rank free $U(\mathfrak{h})$-module. The polynomial modules over Lie algebras constitute an important class of non-weight modules, they were first introduced by Nilsson [11] for the algebra of complex matrices $\mathfrak{sl}_{n+1}(\mathbb{C})$ in 2015. These modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$ were introduced at the same time using different approach in [16]. Since then such modules have been extensively studied for many Lie algebras and Lie superalgebras, see [6], [12], [15] and [21], etc. In this paper, we study the rank one polynomial modules over the GIM Lie algebra $\mathfrak{g}_n$ $(n \geq 3)$.

This paper is organized as follows. In Section 2, we recall some notations and collect some known facts about the GIM Lie algebras and the complex matrices algebra. In Section 3, we first construct two classes of rank one polynomial modules over the GIM Lie algebra $\mathfrak{g}_3$. We further determine the simplicity of these modules. To be precise, one class is generically simple and each reducible module has a unique submodule (Theorem 3.2). Another class is simple (Theorem 3.4). In Subsection 3.2, we give the classification of all rank one polynomial modules over $\mathfrak{g}_3$ in Theorem 3.5. Section 4 is devoted to classifying the rank one polynomial modules over $\mathfrak{g}_n$ $(n \geq 4)$ (Theorem 4.3). We show that those modules are all simple (Theorem 4.2).

## 2. Basics

In this section, we collect some basic definitions and results needed for our study. Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{N}$ the sets of complex numbers, nonzero complex numbers, positive integers, respectively. We use $U(L)$ to denote the universal enveloping algebra of a Lie algebra $L$. For a set $S$, we define the indicator functions

$$\delta_{s\in S} = \begin{cases} 1, & \text{if } s \in S, \\ 0, & \text{if } s \notin S, \end{cases} \qquad \delta_{s\notin S} = \begin{cases} 1, & \text{if } s \notin S, \\ 0, & \text{if } s \in S. \end{cases} \quad (2)$$

Let us first recall some results about the complex matrices algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ $(n \geq 1)$. Denote by $E_{i,j}$ the $(n+1)\times(n+1)$ unit matrix with 1 at the $(i,j)$-entry and zero everywhere else. For $1 \leq i \leq n$, let $\mathbf{e}_i = E_{i,i+1}, \mathbf{f}_i = E_{i+1,i}, \mathbf{h}_i = E_{i,i} - E_{i+1,i+1}$.

Then $\{\mathbf{e}_i, \mathbf{f}_i, \mathbf{h}_i \mid i = 1,\ldots,n\}$ is the set of Chevalley generators of $\mathfrak{sl}_{n+1}(\mathbb{C})$. Let $(H_1,\ldots,H_n)^T = A^{-1}(\mathbf{h}_1,\ldots,\mathbf{h}_n)^T$, i.e., $(\mathbf{h}_1,\ldots,\mathbf{h}_n)^T = A(H_1,\ldots,H_n)^T$, where $T$ means taking the transpose of the matrix, and

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{n\times n}.$$

Then $\mathcal{H} = \mathrm{span}_{\mathbb{C}}\{H_i \mid i = 1, \dots, n\}$ is a Cartan subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$ and

$$[H_i, \mathbf{e}_j] = \delta_{ij}\mathbf{e}_j, \quad [H_i, \mathbf{f}_j] = \delta_{ij}\mathbf{f}_j, \quad 1 \le i, j \le n.$$

Let $\mathcal{Z}$ be a finite-dimensional abelian Lie algebra. For any $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$, $b \in U(\mathcal{Z})$ and $S \subseteq \{1, \dots, n+1\}$, we define the action of $\mathfrak{sl}_{n+1}(\mathbb{C}) \oplus \mathcal{Z}$ on $\mathcal{M}(\mathbf{a}, b, S) := U(\mathcal{H} \oplus \mathcal{Z})$ as follows, for all $g \in U(\mathcal{H} \oplus \mathcal{Z})$,

$$\begin{aligned} &H_i \cdot g = H_i g, \quad z \cdot g = zg, \quad 1 \le i \le n, z \in \mathcal{Z}, \\ &\mathbf{e}_i \cdot g = a_i(\delta_{i\in S} + \delta_{i\notin S}(H_i - H_{i-1} - b - 1))(\delta_{i+1\in S}(H_{i+1} - H_i - b) + \delta_{i+1\notin S})\tau_i(g), \\ &\mathbf{f}_i \cdot g = a_i^{-1}(\delta_{i\in S}(H_i - H_{i-1} - b) + \delta_{i\notin S})(\delta_{i+1\in S} + \delta_{i+1\notin S}(H_{i+1} - H_i - b - 1))\tau_i^{-1}(g), \end{aligned}$$

where $H_0 = H_{n+1} := 0$, $\tau_i$ $(1 \le i \le n)$ is the automorphism of $U(\mathcal{H} \oplus \mathcal{Z})$ which sends $H_i$ to $H_i - 1$ and fixes other $H_j, j \ne i$ and $\mathcal{Z}$.

In [11], Nilsson determined the rank one polynomial modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$. [5] rewrote the results in [11] and discussed the rank one polynomial modules over $\mathfrak{sl}_{n+1}(\mathbb{C}) \oplus \mathcal{Z}$.

**Theorem 2.1.** *Keep notations as above.*

(1) (*see* [5, 11]) *If* $\mathcal{Z} = \mathbb{C}$, *then* $\mathcal{M}(\mathbf{a}, b, S) = U(\mathcal{H})$ *is an* $\mathfrak{sl}_{n+1}(\mathbb{C})$*-module which is free of rank one when restricted to* $U(\mathcal{H})$. *Moreover, any rank one polynomial module over* $\mathfrak{sl}_{n+1}(\mathbb{C})$ *is isomorphic to* $\mathcal{M}(\mathbf{a}, b, S)$ *for some* $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$, $b \in \mathbb{C}$, $S \subseteq \{1, \dots, n+1\}$.

(2) (*see* [5]) $\mathcal{M}(\mathbf{a}, b, S) = U(\mathcal{H} \oplus \mathcal{Z})$ *is an* $(\mathfrak{sl}_{n+1}(\mathbb{C}) \oplus \mathcal{Z})$*-module which is free of rank one when restricted to* $U(\mathcal{H} \oplus \mathcal{Z})$. *Moreover, any rank one polynomial module over* $\mathfrak{sl}_{n+1}(\mathbb{C}) \oplus \mathcal{Z}$ *is isomorphic to* $\mathcal{M}(\mathbf{a}, b, S)$ *for some* $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$, $b \in U(\mathcal{Z})$, $S \subseteq \{1, \dots, n+1\}$.

For the rest of this paper, we assume that $n \ge 3$. The GIM Lie algebra $\mathfrak{g}_n (n \ge 3)$ with the structural matrix $O_n = (m_{ij})_{n\times n}$ given by (1) is a Lie algebra over $\mathbb{C}$ generated by the elements $e_i, f_i, h_i$ $(1 \le i \le n)$ with subject to the generating relations

$$\begin{aligned} &[h_i, h_j] = 0, \quad [h_i, e_j] = m_{ij}e_j, \quad [h_i, f_j] = -m_{ij}f_j, \quad [e_i, f_i] = h_i \text{ for } 1 \le i, j \le n, \\ &[e_i, f_j] = 0, \quad (\mathrm{ad}e_i)^{-m_{ij}+1}e_j = (\mathrm{ad}f_i)^{-m_{ij}+1}f_j = 0 \quad \text{for } m_{ij} \le 0, \\ &[e_i, e_j] = [f_i, f_j] = 0, \quad (\mathrm{ad}e_i)^{m_{ij}+1}f_j = (\mathrm{ad}f_i)^{m_{ij}+1}e_j = 0 \quad \text{for } m_{ij} > 0, i \ne j. \end{aligned}$$

For convenience, we also say that $\mathfrak{g}_n$ is of type $O_n$. It is clear that $\mathfrak{h} := \mathrm{span}_{\mathbb{C}}\{h_i \mid i = 1, \dots, n\}$ is the Cartan subalgebra of $\mathfrak{g}_n$. Let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. Since $\det(O_n) \ne 0$, it is easy to find a basis $\alpha_1, \dots, \alpha_n$ of $\mathfrak{h}^*$ such that $\alpha_i(h_j) = m_{ij}$ for all $1 \le i, j \le n$. Now we take another basis $\varepsilon_1, \dots, \varepsilon_n$ for $\mathfrak{h}$ such that

$$(\varepsilon_1, \dots, \varepsilon_n)^T = O_n^{-1}(h_1, \dots, h_n)^T,$$

where $T$ means taking the transpose of the matrix.

Then the polynomial algebra $\mathbb{C}[\varepsilon_1,\dots,\varepsilon_n]$ is the universal enveloping algebra $U(\mathfrak{h})$ of $\mathfrak{h}$. Moreover, we can rewrite the definition of the GIM Lie algebra $\mathfrak{g}_n$ (cf. [9]) as follows

$$[\varepsilon_i,\varepsilon_j]=0,\quad [\varepsilon_i,e_j]=\delta_{ij}e_j,\quad [\varepsilon_i,f_j]=-\delta_{ij}f_j,\quad 1\le i,j\le n, \tag{3}$$
$$[e_1,f_1]=2\varepsilon_1-\varepsilon_2+\varepsilon_n,\quad [e_n,f_n]=\varepsilon_1-\varepsilon_{n-1}+2\varepsilon_n, \tag{4}$$
$$[e_i,f_i]=-\varepsilon_{i-1}+2\varepsilon_i-\varepsilon_{i+1},\quad 2\le i\le n-1, \tag{5}$$
$$[e_i,f_j]=0,\quad 1\le i\ne j\le n, \{i,j\}\ne\{1,n\}, \tag{6}$$
$$[e_i,e_j]=[f_i,f_j]=0,\quad |i-j|>1, \tag{7}$$
$$[e_i,[e_i,e_j]]=[f_i,[f_i,f_j]]=0,\quad |i-j|=1, \tag{8}$$
$$[e_1,[e_1,f_n]]=[f_1,[f_1,e_n]]=0,\quad [e_n,[e_n,f_1]]=[f_n,[f_n,e_1]]=0. \tag{9}$$

**Definition 2.2.** For a fixed $1\le i\le n$, define an algebra automorphism

$$\sigma_i:\mathcal{U}(\mathfrak{h})\to\mathcal{U}(\mathfrak{h})$$

by $\sigma_i(\varepsilon_k):=\varepsilon_k-\delta_{ik}$ for any $k\in\{1,\dots,n\}$.

Obviously $\sigma_i\sigma_j=\sigma_j\sigma_i$ for $1\le i,j\le n$. For any $g=g(\varepsilon_1,\dots,\varepsilon_n)\in\mathbb{C}[\varepsilon_1,\dots,\varepsilon_n]$, we have $\sigma_i(g)=g(\varepsilon_1,\dots,\varepsilon_i-1,\dots,\varepsilon_n)$, $\sigma_i^{-1}(g)=g(\varepsilon_1,\dots,\varepsilon_i+1,\dots,\varepsilon_n)$.

Let $\mathcal{N}$ be a rank one polynomial module over the GIM Lie algebra $\mathfrak{g}_n$ $(n\ge 3)$. Then $\mathcal{N}$ is actually a free $U(\mathfrak{h})$-module. In consequence we can identify $\mathcal{N}$ with $U(\mathfrak{h})=\mathbb{C}[\varepsilon_1,\dots,\varepsilon_n]$ as a vector space.

## 3. Rank one polynomial modules over $\mathfrak{g}_3$

In this section, we classify all rank one polynomial modules over the GIM Lie algebra $\mathfrak{g}_3$. We further determine the simplicity of these modules.

For convenience, we rewrite the definition of $\mathfrak{g}_3$. GIM Lie algebra $\mathfrak{g}_3$ with structural matrix $O_3$ is a Lie algebra generated by the elements $e_i,f_i,\varepsilon_i$ $(i=1,2,3)$ with subject to the following generating relations

$$[\varepsilon_i,\varepsilon_j]=0,\quad [\varepsilon_i,e_j]=\delta_{ij}e_j,\quad [\varepsilon_i,f_j]=-\delta_{ij}f_j,\quad i,j=1,2,3, \tag{10}$$
$$[e_1,f_1]=2\varepsilon_1-\varepsilon_2+\varepsilon_3,\ [e_2,f_2]=-\varepsilon_1+2\varepsilon_2-\varepsilon_3,\ [e_3,f_3]=\varepsilon_1-\varepsilon_2+2\varepsilon_3, \tag{11}$$
$$[e_1,f_2]=[e_2,f_1]=[e_2,f_3]=[e_3,f_2]=[e_1,e_3]=[f_1,f_3]=0, \tag{12}$$
$$[e_1,[e_1,e_2]]=[f_1,[f_1,f_2]]=0,\quad [e_2,[e_2,e_1]]=[f_2,[f_2,f_1]]=0, \tag{13}$$
$$[e_2,[e_2,e_3]]=[f_2,[f_2,f_3]]=0,\quad [e_3,[e_3,e_2]]=[f_3,[f_3,f_2]]=0, \tag{14}$$
$$[e_1,[e_1,f_3]]=[f_1,[f_1,e_3]]=0,\quad [e_3,[e_3,f_1]]=[f_3,[f_3,e_1]]=0. \tag{15}$$

In this case, $\mathfrak{h}=\operatorname{span}_{\mathbb{C}}\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ is a Cartan subalgebra of $\mathfrak{g}_3$. Thus we have $\mathcal{U}(\mathfrak{h})=\mathbb{C}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$.

### 3.1. $\mathfrak{g}_3$-module structures on $U(\mathfrak{h})$

The main goal of this subsection is to construct two classes of rank one polynomial modules over the GIM Lie algebra $\mathfrak{g}_3$ and determine the simplicity of these modules. Fix any $\mathbf{a}=(a_1,a_2,a_3)\in(\mathbb{C}^*)^3$, $b\in\mathbb{C}$, $S\subseteq\{1,2,3,4\}$. Let $\mathcal{N}_1(\mathbf{a},b,S):=\mathbb{C}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$. For any $g\in\mathbb{C}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$, we define the action of $\mathfrak{g}_3$ on $\mathcal{N}_1(\mathbf{a},b,S)$ as follows

$$\begin{cases} \varepsilon_i \cdot g = \varepsilon_i g, \quad i = 1,2,3, \\ e_1 \cdot g = a_1(\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}\sigma_1(g), \\ f_1 \cdot g = a_1^{-1}(\varepsilon_1 + b + 1)^{\delta_{1\in S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\notin S}}\sigma_1^{-1}(g), \\ e_2 \cdot g = a_2(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\notin S}}(-\varepsilon_2 + b + 1)^{\delta_{3\in S}}\sigma_2(g), \\ f_2 \cdot g = a_2^{-1}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}(-\varepsilon_2 + b)^{\delta_{3\notin S}}\sigma_2^{-1}(g), \\ e_3 \cdot g = a_3(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b - 1)^{\delta_{2\in S}}(-\varepsilon_3 - b)^{\delta_{4\in S}}\sigma_3(g), \\ f_3 \cdot g = a_3^{-1}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b)^{\delta_{2\notin S}}(-\varepsilon_3 - b - 1)^{\delta_{4\notin S}}\sigma_3^{-1}(g). \end{cases} \tag{16}$$

where $\sigma_i$ $(i = 1,2,3)$ is defined by Definition 2.2.

Now we give one of the main results of this subsection.

**Proposition 3.1.** *Fix* $\mathbf{a} = (a_1, a_2, a_3) \in (\mathbb{C}^*)^3$, $b \in \mathbb{C}$, *and* $S \subseteq \{1,2,3,4\}$, *then* $\mathcal{N}_1(\mathbf{a}, b, S)$ *is a rank one polynomial module over* $\mathfrak{g}_3$.

**Proof.** According to the definition of (16), we must prove that the actions of $e_i, f_i$ (i=1,2,3) on $\mathcal{N}_1(\mathbf{a}, b, S)$ satisfy the relations of $\mathfrak{g}_3$. According to (16) we have for any $g = g(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathcal{N}_1(\mathbf{a}, b, S)$

$$\begin{aligned} &\varepsilon_1 \cdot e_1 \cdot g - e_1 \cdot \varepsilon_1 \cdot g \\ &\quad = a_1\varepsilon_1 \cdot ((\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}\sigma_1(g)) - e_1 \cdot (\varepsilon_1 g) \\ &\quad = a_1\varepsilon_1(\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}\sigma_1(g) \\ &\qquad - a_1(\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}(\varepsilon_1 - 1)\sigma_1(g) \\ &\quad = a_1(\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}\sigma_1(g), \end{aligned}$$

Thus $[\varepsilon_1, e_1] \cdot g = e_1 \cdot g$ holds. The other relations in (10) can be proved similarly. From the following equalities

$$\begin{aligned} &e_1 \cdot f_1 \cdot g - f_1 \cdot e_1 \cdot g \\ &= (\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}(\varepsilon_1 + b)^{\delta_{1\in S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\notin S}}g \\ &\quad - (\varepsilon_1 + b + 1)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\in S}}(\varepsilon_1 + b + 1)^{\delta_{1\in S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\notin S}}g \\ &= (\varepsilon_1 + b)(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)g - (\varepsilon_1 + b + 1)(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)g \\ &= (2\varepsilon_1 - \varepsilon_2 + \varepsilon_3)g = [e_1, f_1] \cdot g, \\ &e_2 \cdot f_2 \cdot g - f_2 \cdot e_2 \cdot g \\ &= (-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\notin S}}(-\varepsilon_2 + b + 1)^{\delta_{3\in S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\in S}}(-\varepsilon_2 + b + 1)^{\delta_{3\notin S}}g \\ &\quad - (-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\notin S}}(-\varepsilon_2 + b)^{\delta_{3\in S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}(-\varepsilon_2 + b)^{\delta_{3\notin S}}g \\ &= (-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)(-\varepsilon_2 + b + 1)g - (-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)(-\varepsilon_2 + b)g \\ &= (-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3)g = [e_2, f_2] \cdot g, \\ &e_3 \cdot f_3 \cdot g - f_3 \cdot e_3 \cdot g \\ &= (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b - 1)^{\delta_{2\in S}}(-\varepsilon_3 - b)^{\delta_{4\in S}}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b - 1)^{\delta_{2\notin S}}(-\varepsilon_3 - b)^{\delta_{4\notin S}}g \\ &\quad - (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b)^{\delta_{2\in S}}(-\varepsilon_3 - b - 1)^{\delta_{4\in S}}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b)^{\delta_{2\notin S}}(-\varepsilon_3 - b - 1)^{\delta_{4\notin S}}g \\ &= (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b - 1)(-\varepsilon_3 - b)g - (\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b)(-\varepsilon_3 - b - 1)g \\ &= (\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3)g = [e_3, f_3] \cdot g, \end{aligned}$$

we see that the relations in (11) are satisfied. For relations in (12), we only check $[e_1, f_2] \cdot g = 0$. The proof for the others is similar.

Explicitly,

$$\begin{aligned} & e_1\cdot f_2\cdot g-f_2\cdot e_1\cdot g \\ & =a_1a_2^{-1}(\varepsilon_1+b)^{\delta_{1\notin S}}\big((-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+2)\big)^{\delta_{2\in S}}(-\varepsilon_2+b)^{\delta_{3\notin S}} \\ & \quad \sigma_1(\sigma_2^{-1}(g))-a_1a_2^{-1}(\varepsilon_1+b)^{\delta_{1\notin S}}\big((-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+2)(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)\big)^{\delta_{2\in S}} \\ & \quad (-\varepsilon_2+b)^{\delta_{3\notin S}}\sigma_2^{-1}(\sigma_1(g))=0, \end{aligned}$$

where the last equality is obtained by $\sigma_1\sigma_2=\sigma_2\sigma_1$. This yields $[e_1,f_2]\cdot g=0$. Since

$$\begin{aligned} (e_1e_2-e_2e_1)\cdot g &= e_1\cdot e_2\cdot g-e_2\cdot e_1\cdot g \\ &= a_1a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)\sigma_1(\sigma_2(g)) \\ &\quad -a_1a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b)\sigma_2(\sigma_1(g)) \\ &= a_1a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\sigma_1\sigma_2)(g), \end{aligned}$$

we obtain

$$\begin{aligned} & (e_1^2e_2-e_1e_2e_1)\cdot g=e_1\cdot(e_1\cdot e_2\cdot g-e_2\cdot e_1\cdot g) \\ & =a_1a_2e_1\cdot\big((\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\sigma_1\sigma_2)(g)\big) \\ & =a_1^2a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)^{\delta_{2\in S}}\sigma_1\big((\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\sigma_1\sigma_2)(g)\big) \\ & =a_1^2a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)^{\delta_{2\in S}}(\varepsilon_1+b-1)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\sigma_1^2\sigma_2)(g), \end{aligned}$$

and

$$\begin{aligned} & (e_1e_2e_1-e_2e_1^2)\cdot g=e_1\cdot e_2\cdot(e_1\cdot g)-e_2\cdot e_1\cdot(e_1\cdot g) \\ & =a_1a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\sigma_1\sigma_2)(e_1\cdot g) \\ & =a_1^2a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\sigma_1\sigma_2)\big((\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)^{\delta_{2\in S}}\sigma_1(g)\big) \\ & =a_1^2a_2(\varepsilon_1+b)^{\delta_{1\notin S}}(-\varepsilon_2+b+1)^{\delta_{3\in S}}(\varepsilon_1+b-1)^{\delta_{1\notin S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)^{\delta_{2\in S}}(\sigma_1^2\sigma_2)(g), \end{aligned}$$

where the last equality is implied by the definition of $\sigma_i$ and $\sigma_1\sigma_2=\sigma_2\sigma_1$. Using these relations, we obtain

$$(e_1^2e_2-2e_1e_2e_1+e_2e_1^2)\cdot g=(e_1^2e_2-e_1e_2e_1)\cdot g-(e_1e_2e_1-e_2e_1^2)\cdot g=0.$$

This yields that $[e_1,[e_1,e_2]]\cdot g=0$. It follows from (16) and Definition 2.2 that

$$\begin{aligned} & (f_1e_3-e_3f_1)\cdot g=f_1\cdot e_3\cdot g-e_3\cdot f_1\cdot g \\ & =a_1^{-1}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b)^{\delta_{2\notin S}}(\varepsilon_1-\varepsilon_2+\varepsilon_3-b)^{\delta_{2\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}} \\ & \quad \sigma_1^{-1}(\sigma_3(g))-a_1^{-1}a_3(\varepsilon_1-\varepsilon_2+\varepsilon_3-b-1)^{\delta_{2\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\varepsilon_1+b+1)^{\delta_{1\in S}} \\ & \quad (-\varepsilon_1+\varepsilon_2-\varepsilon_3+b-1)^{\delta_{2\notin S}}\sigma_3(\sigma_1^{-1}(g)) \\ & =a_1^{-1}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\sigma_1^{-1}\sigma_3)(g). \end{aligned}$$

This implies that

$$\begin{aligned} & (f_1^2e_3-f_1e_3f_1)\cdot g=f_1\cdot(f_1\cdot e_3-f_1\cdot e_3\cdot f_1)\cdot g \\ & =a_1^{-1}a_3f_1\cdot\big((\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\sigma_1^{-1}\sigma_3)(g)\big) \\ & =a_1^{-2}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b)^{\delta_{2\notin S}} \\ & \quad \sigma_1^{-1}\big((\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\sigma_1^{-1}\sigma_3)(g)\big) \\ & =a_1^{-2}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b)^{\delta_{2\notin S}}(\varepsilon_1+b+2)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\sigma_1^{-2}\sigma_3)(g), \end{aligned}$$

and

$$\begin{aligned}
&(f_1e_3f_1 - e_3f_1^2)\cdot g = f_1\cdot e_3\cdot(f_1\cdot g) - e_3\cdot f_1\cdot(f_1\cdot g)\\
&= a_1^{-1}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\sigma_1^{-1}\sigma_3)(f_1\cdot g)\\
&= a_1^{-2}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}\\
&\quad(\sigma_1^{-1}\sigma_3)((\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b)^{\delta_{2\notin S}}\sigma_1^{-1}(g))\\
&= a_1^{-2}a_3(\varepsilon_1+b+1)^{\delta_{1\in S}}(-\varepsilon_3-b)^{\delta_{4\in S}}(\varepsilon_1+b+2)^{\delta_{1\in S}}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b)^{\delta_{2\notin S}}(\sigma_1^{-2}\sigma_3)(g).
\end{aligned}$$

Thus $(f_1^2e_3 - 2f_1e_3f_1 + e_3f_1^2)\cdot g = (f_1^2e_3 - f_1e_3f_1)\cdot g - (f_1e_3f_1 - e_3f_1^2)\cdot g = 0$. Therefore $[f_1,[f_1,e_3]]\cdot g = 0$. The proof for the other relations in (13)–(15) can be verified similarly, we omit the details. Then we complete the proof. ∎

Now we exploit the module structure of $\mathcal{N}_1(\mathbf{a},b,S)$ and provide a sufficient and necessary condition for simplicity of $\mathcal{N}_1(\mathbf{a},b,S)$.

**Theorem 3.2.** *Let $\mathbf{a}\in(\mathbb{C}^*)^3$, $b\in\mathbb{C}$, and $S\subseteq\{1,2,3,4\}$. Then the polynomial module $\mathcal{N}_1(\mathbf{a},b,S)$ over $\mathfrak{g}_3$ is simple except the case that $\delta_{1\in S}=\delta_{2\in S}=\delta_{3\in S}=\delta_{4\notin S}$ and $-4b-3$ is a positive integer. Moreover, $\mathcal{N}_1(\mathbf{a},b,S)$ has a unique nonzero proper submodule if it is not simple.*

**Proof.** Suppose that $M$ is a nonzero submodule of $\mathcal{N}_1(\mathbf{a},b,S)$. Let $g(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ be a nonzero polynomial in $M$. We consider four cases separately.

**Case 1.** $\delta_{1\in S}=\delta_{2\notin S}$.

From the relations (16), we see that either

$$(a_1^{-1}e_1)^k\cdot g(\varepsilon_1,\varepsilon_2,\varepsilon_3) = g(\varepsilon_1-k,\varepsilon_2,\varepsilon_3)\in M,\quad k\in\mathbb{N},$$

or

$$(a_1f_1)^k\cdot g(\varepsilon_1,\varepsilon_2,\varepsilon_3) = g(\varepsilon_1+k,\varepsilon_2,\varepsilon_3)\in M,\quad k\in\mathbb{N}.$$

Then we may obtain a nonzero polynomial $g_1(\varepsilon_2,\varepsilon_3)$ that lies in $M$. In fact, suppose for example that

$$g(\varepsilon_1-k,\varepsilon_2,\varepsilon_3)=\sum_{i=0}^{r}(\varepsilon_1-k)^i g'_i(\varepsilon_2,\varepsilon_3)=\sum_{i=0}^{r}k^i g'_i(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in M,$$

where $g'_r(\varepsilon_2,\varepsilon_3)\neq 0$ and $r$ is minimal. Then taking $k=1,\dots,r$, we deduce that $g_1(\varepsilon_2,\varepsilon_3):=g'_r(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(-1)^r g'_r(\varepsilon_2,\varepsilon_3)\in M$ as required, here $g_1(\varepsilon_2,\varepsilon_3)$ is the leading coefficient of the polynomial $g(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ in the variable $\varepsilon_1$.

If $\delta_{2\in S}=\delta_{3\notin S}$, we may obtain a nonzero polynomial $g_2(\varepsilon_3)\in M$ from $g_1(\varepsilon_2,\varepsilon_3)\in M$ in a similar way.

If $2,3\in S$, from the equality

$$a_2f_2\cdot g_1(\varepsilon_2,\varepsilon_3)=(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)g_1(\varepsilon_2+1,\varepsilon_3)\in M,$$

we obtain that the coefficient $-g_1(\varepsilon_2+1,\varepsilon_3)$ of $\varepsilon_1$ belongs to $M$. Thus we can deduce that there exists a nonzero polynomial $g_2(\varepsilon_3)\in M$.

If $2,3\notin S$, we can obtain a nonzero polynomial $g_2(\varepsilon_3)\in M$ in a similar way.

Next if $\delta_{2\in S}=\delta_{4\in S}$, we can obtain $1\in M$ by the actions of $(a_3^{-1}e_3)^k$ or of $(a_3f_3)^k$ on $g_2(\varepsilon_3)$ with enough many $k\in\mathbb{N}$. If $\delta_{2\in S}=\delta_{4\notin S}$, using a similar method to the case $2,3\in S$ (or $2,3\notin S$), we can prove $1\in M$. Therefore $\mathcal{N}_1(\mathbf{a},b,S)$ is simple in this case.

**Case 2.** $\delta_{2\in S}=\delta_{3\notin S}$ or $\delta_{2\in S}=\delta_{4\in S}$.
The proof is similar to Case 1.

**Case 3.** $\delta_{1\in S}=\delta_{2\in S}=\delta_{3\in S}=\delta_{4\notin S}=1$.

In this case, we have

$$\begin{aligned} e_1\cdot g&=a_1(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)\sigma_1(g), & f_1\cdot g&={a_1}^{-1}(\varepsilon_1+b+1)\sigma_1^{-1}(g),\\ e_2\cdot g&=a_2(-\varepsilon_2+b+1)\sigma_2(g), & f_2\cdot g&={a_2}^{-1}(-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)\,\sigma_2^{-1}(g),\\ e_3\cdot g&=a_3(\varepsilon_1-\varepsilon_2+\varepsilon_3-b-1)\sigma_3(g), & f_3\cdot g&={a_3}^{-1}(-\varepsilon_3-b-1)\sigma_3^{-1}(g). \end{aligned}$$

For $k_1,k_2,k_3\in\mathbb{N}$, let

$$H_{k_1,k_2,k_3}=\Big(\prod_{r=1}^{k_1}(\varepsilon_1+b+r)\Big)\Big(\prod_{s=1}^{k_2}(-\varepsilon_2+b+s)\Big)\Big(\prod_{t=1}^{k_3}(\varepsilon_3+b+t)\Big).$$

Then

$$\{H_{k_1,k_2,k_3}\mid k_1,k_2,k_3\in\mathbb{N}\}$$

is a basis of $U(\mathfrak{h})$. Now suppose that $H_{k_1,k_2,k_3}$ be a basis element occurring with nonzero coefficient in $g(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ expressed in this basis. Then we have

$$\begin{aligned} a_1f_1\cdot H_{k_1,k_2,k_3}&=(\varepsilon_1+b+k_1+1)H_{k_1,k_2,k_3},\\ a_2^{-1}e_2\cdot H_{k_1,k_2,k_3}&=(-\varepsilon_2+b+k_2+1)H_{k_1,k_2,k_3},\\ a_3f_3\cdot H_{k_1,k_2,k_3}&=(\varepsilon_3+b+k_3+1)H_{k_1,k_2,k_3}. \end{aligned}$$

Hence

$$\begin{aligned} (a_1f_1-\varepsilon_1-b)\cdot H_{k_1,k_2,k_3}&=(k_1+1)H_{k_1,k_2,k_3},\\ (a_2^{-1}e_2+\varepsilon_2-b)\cdot H_{k_1,k_2,k_3}&=(k_2+1)H_{k_1,k_2,k_3},\\ (a_3f_3-\varepsilon_3-b)\cdot H_{k_1,k_2,k_3}&=(k_3+1)H_{k_1,k_2,k_3}. \end{aligned}$$

Then from any element $u=\sum_{k_1,k_2,k_3}a_{k_1,k_2,k_3}H_{k_1,k_2,k_3}$, we can obtain $H_{k_1,k_2,k_3}$ for any nonzero coefficient $a_{k_1,k_2,k_3}$. Moreover, it follows that any simple submodule can be generated by a single homogeneous element $H_{k_1,k_2,k_3}$. Furthermore, for general $k_1>0,k_2,k_3\in\mathbb{N}$, we have

$$\begin{aligned} a_1^{-1}e_1\cdot H_{k_1,k_2,k_3}&=-(\varepsilon_1+b)((\varepsilon_1+b+k_1)+(-\varepsilon_2+b+k_2+1)\\ &\quad+(\varepsilon_3+b+k_3+1))H_{k_1-1,k_2,k_3}+(4b+k_1+k_2+k_3+3)(\varepsilon_1+b)H_{k_1-1,k_2,k_3}\\ &=-(\varepsilon_1+b)(H_{k_1,k_2,k_3}+H_{k_1-1,k_2+1,k_3}+H_{k_1-1,k_2,k_3+1})\\ &\quad+(4b+k_1+k_2+k_3+3)H_{k_1,k_2,k_3}-k_1(4b+k_1+k_2+k_3+3)H_{k_1-1,k_2,k_3}, \end{aligned}$$

which contains a lower degree term $-k_1(4b+k_1+k_2+k_3+3)H_{k_1-1,k_2,k_3}$ for all $k_1>0$ and $k_1+k_2+k_3\neq-4b-3$. Together with similar computations of $a_2f_2\cdot H_{k_1,k_2,k_3}, a_3^{-1}e_3\cdot H_{k_1,k_2,k_3}$, we know that $H_{k_1-1,k_2,k_3}, H_{k_1,k_2-1,k_3}, H_{k_1,k_2,k_3-1}\in M$ if $k_1>0$ and $k_1+k_2+k_3\neq-4b-3$. It then follows by induction that $1=H_{0,0,0}\in M$.

Thus $\mathcal{N}_1(\mathbf{a},b,S)$ is simple except that $-4b-3$ is a positive integer.

Now assume that $-4b-3$ is a positive integer. Let $-4b=n\in\mathbb{N}$ and $n>3$. Define

$$M_1=\operatorname{span}_{\mathbb{C}}\{H_{k_1,k_2,k_3}\mid k_1+k_2+k_3\geq n-3\}.$$

It is clear that $f_1\cdot M_1, e_2\cdot M_1, f_3\cdot M_1\subseteq M_1$. Moreover,

$$e_1\cdot H_{k_1,k_2,k_3}, f_2\cdot H_{k_1,k_2,k_3}, e_3\cdot H_{k_1,k_2,k_3}\in M_1$$

for $k_1+k_2+k_3>n-3$.

For $k_1+k_2+k_3=n-3$, we have

$$\begin{aligned} a_1^{-1}e_1\cdot H_{0,k_2,k_3} &= (-\varepsilon_1+\varepsilon_2-\varepsilon_3+b+1)H_{0,k_2,k_3}\in M_1,\\ a_1^{-1}e_1\cdot H_{k_1,k_2,k_3} &= -(\varepsilon_1+b)((\varepsilon_1+b+k_1)+(-\varepsilon_2+b+k_2+1)\\ &\quad+(\varepsilon_3+b+k_3+1))H_{k_1-1,k_2,k_3}\\ &= -(\varepsilon_1+b)(H_{k_1,k_2,k_3}+H_{k_1-1,k_2+1,k_3}+H_{k_1-1,k_2,k_3+1})\in M_1. \end{aligned}$$

Similarly, we find that $a_2f_2\cdot H_{k_1,k_2,k_3}, a_3^{-1}e_1\cdot H_{k_1,k_2,k_3}\in M_1$. Thus $M_1$ is a submodule of $\mathcal{N}_1(\mathbf{a},b,S)$ and $M_1$ has to be simple.

If $\mathcal{N}_1(\mathbf{a},b,S)$ has another submodule $M_2\neq M_1$: Similar to above we can assume that $H_{k_1,k_2,k_3}\in M_2$ for $k_1+k_2+k_3<n-3$. By using

$$\begin{aligned} a_1^{-1}e_1\cdot H_{k_1,k_2,k_3} = & -(\varepsilon_1+b)(H_{k_1,k_2,k_3}+H_{k_1-1,k_2+1,k_3}+H_{k_1-1,k_2,k_3+1})\\ & +(k_1+k_2+k_3-n+3)H_{k_1,k_2,k_3}\\ & -k_1(k_1+k_2+k_3-n+3)H_{k_1-1,k_2,k_3}, \end{aligned}$$

and similar computations of $a_2f_2\cdot H_{k_1,k_2,k_3}, a_3^{-1}e_3\cdot H_{k_1,k_2,k_3}$, we know that $1\in M_2$. Thus $M_1$ is the unique proper submodule.

**Case 4.** $\delta_{1\in S}=\delta_{2\in S}=\delta_{3\in S}=\delta_{4\notin S}=0$. The proof is similar to Case 3. The proof is complete. ■

We now construct another module over $\mathfrak{g}_3$. Fix $\mathbf{a}=(a_1,a_2,a_3)\in(\mathbb{C}^*)^3$, $S\subseteq\{1,2,3\}$ and set $\mathcal{N}_2(\mathbf{a},S):=\mathbb{C}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$. For any $g\in\mathbb{C}[\varepsilon_1,\varepsilon_2,\varepsilon_3]$, we define the action of $\mathfrak{g}_3$ on $\mathcal{N}_2(\mathbf{a},S)$ as follows

$$\begin{cases} \varepsilon_i\cdot g=\varepsilon_i g, \quad i=1,2,3,\\ e_1\cdot g=a_1(\varepsilon_1+\varepsilon_3-\frac{1}{2})^{\delta_{1\notin S}}(-\varepsilon_1+\varepsilon_2+\frac{1}{2})^{\delta_{2\in S}}\sigma_1(g),\\ f_1\cdot g=a_1^{-1}(\varepsilon_1+\varepsilon_3+\frac{1}{2})^{\delta_{1\in S}}(-\varepsilon_1+\varepsilon_2-\frac{1}{2})^{\delta_{2\notin S}}\sigma_1^{-1}(g),\\ e_2\cdot g=a_2(-\varepsilon_1+\varepsilon_2-\frac{1}{2})^{\delta_{2\notin S}}(-\varepsilon_2+\varepsilon_3+\frac{1}{2})^{\delta_{3\in S}}\sigma_2(g),\\ f_2\cdot g=a_2^{-1}(-\varepsilon_1+\varepsilon_2+\frac{1}{2})^{\delta_{2\in S}}(-\varepsilon_2+\varepsilon_3-\frac{1}{2})^{\delta_{3\notin S}}\sigma_2^{-1}(g),\\ e_3\cdot g=a_3(-\varepsilon_2+\varepsilon_3-\frac{1}{2})^{\delta_{3\notin S}}(-\varepsilon_1-\varepsilon_3+\frac{1}{2})^{\delta_{1\notin S}}\sigma_3(g),\\ f_3\cdot g=a_3^{-1}(-\varepsilon_2+\varepsilon_3+\frac{1}{2})^{\delta_{3\in S}}(-\varepsilon_1-\varepsilon_3-\frac{1}{2})^{\delta_{1\in S}}\sigma_3^{-1}(g), \end{cases} \tag{17}$$

where $\sigma_i$ $(i=1,2,3)$ is defined by Definition 2.2.

**Proposition 3.3.** *Fix $\mathbf{a}\in(\mathbb{C}^*)^3$ and $S\subseteq\{1,2,3\}$, then $\mathcal{N}_2(\mathbf{a},S)$ is a rank one polynomial module over $\mathfrak{g}_3$.*

**Proof.** The proof is similar to Proposition 3.1. We omit the details. ■

**Theorem 3.4.** *For all $\mathbf{a}=(a_1,a_2,a_3)\in(\mathbb{C}^*)^3$ and $S\subseteq\{1,2,3\}$, the polynomial module $\mathcal{N}_2(\mathbf{a},S)$ over $\mathfrak{g}_3$ is simple.*

**Proof.** Let $M$ be a nonzero submodule of $\mathcal{N}_2(\mathbf{a},b,S)$. Assume $0\neq g(\varepsilon_1,\varepsilon_2,\varepsilon_3)\in M$.

**Case 1.** $S=\emptyset$. In this case, from

$$(a_3f_3)^k\cdot g(\varepsilon_1,\varepsilon_2,\varepsilon_3)=g(\varepsilon_1,\varepsilon_2,\varepsilon_3+k)\in M,\quad k\in\mathbb{N},$$

we may obtain a nonzero polynomial $g_1(\varepsilon_1,\varepsilon_2)\in M$.

Moreover, by

$$a_1^{-1}e_1 \cdot g_1(\varepsilon_1, \varepsilon_2) = (\varepsilon_1 + \varepsilon_3 - \frac{1}{2})g_1(\varepsilon_1 - 1, \varepsilon_2) \in M,$$
$$a_1^{-1}a_3 f_3 \cdot e_1 \cdot g_1(\varepsilon_1, \varepsilon_2) = (\varepsilon_1 + \varepsilon_3 + 1 - \frac{1}{2})g_1(\varepsilon_1 - 1, \varepsilon_2) \in M,$$

we can obtain $g_1(\varepsilon_1 - k, \varepsilon_2) \in M$ for all $k \in \mathbb{N}$. Thus a nonzero polynomial $g_2(\varepsilon_2)$ in $M$ can be obtained. Similarly, from the facts that

$$a_2 f_2 \cdot g_2(\varepsilon_2) = (-\varepsilon_2 + \varepsilon_3 - \frac{1}{2})g_2(\varepsilon_2 + 1) \in M,$$
$$a_2 a_3 f_3 \cdot f_2 \cdot g_2(\varepsilon_2) = (-\varepsilon_2 + \varepsilon_3 + 1 - \frac{1}{2})g_2(\varepsilon_2 + 1) \in M,$$

we deduce that $1 \in M$. Thus $\mathcal{N}_2(\mathbf{a}, S)$ is simple.

**Case 2.** $S = \{1, 2, 3\}$. The proof is similar to Case 1.

**Case 3.** $|S| = 1$. In this case, we only prove for $S = \{1\}$. The proof for the other two cases is similar.

By assumption, we have

$$(a_1^{-1}e_1)^k \cdot g(\varepsilon_1, \varepsilon_2, \varepsilon_3) = g(\varepsilon_1 - k, \varepsilon_2, \varepsilon_3) \in M, \quad k \in \mathbb{N},$$

then we obtain a nonzero polynomial $g_1(\varepsilon_2, \varepsilon_3)$ that lies in $M$. Since

$$a_2^{-1}e_2 \cdot g_1(\varepsilon_2, \varepsilon_3) = (-\varepsilon_1 + \varepsilon_2 - \frac{1}{2})g_1(\varepsilon_2 - 1, \varepsilon_3) \in M,$$
$$a_1^{-1}a_2^{-1}e_1 \cdot e_2 \cdot g_1(\varepsilon_2, \varepsilon_3) = (-\varepsilon_1 + \varepsilon_2 + 1 - \frac{1}{2})g_1(\varepsilon_2 - 1, \varepsilon_3) \in M,$$

we see that there exists a nonzero polynomial $g_2(\varepsilon_3) \in M$. Then from the equalities

$$a_3 f_3 \cdot g_2(\varepsilon_3) = (-\varepsilon_1 - \varepsilon_3 - \frac{1}{2})g_2(\varepsilon_3 + 1) \in M,$$
$$a_1^{-1}a_3 e_1 \cdot f_3 \cdot g_2(\varepsilon_3) = (-\varepsilon_1 - \varepsilon_2 + 1 - \frac{1}{2})g_2(\varepsilon_3 + 1) \in M,$$

we obtain that $1 \in M$. It follows that $\mathcal{N}_2(\mathbf{a}, S)$ is a simple module.

**Case 4.** $|S| = 2$. The proof is similar to Case 3. Therefore, $\mathcal{N}_2(\mathbf{a}, S)$ is simple in all these cases. The proof is complete. ∎

### 3.2. Classification of rank one polynomial modules over $\mathfrak{g}_3$

In this subsection, we give the complete classification of rank one polynomial modules over $\mathfrak{g}_3$.

**Theorem 3.5.** *Let $\mathcal{N}$ be a rank one polynomial module over $\mathfrak{g}_3$. Then one of the following holds.*

(i) *$\mathcal{N}$ is isomorphic to $\mathcal{N}_1(\mathbf{a}, b, S)$ defined by (16) for some $\mathbf{a} = (a_1, a_2, a_3) \in (\mathbb{C}^*)^3$, $b \in \mathbb{C}$, $S \subseteq \{1, 2, 3, 4\}$;*

(ii) *$\mathcal{N}$ is isomorphic to $\mathcal{N}_2(\mathbf{a}, S)$ defined by (17) for some $\mathbf{a} = (a_1, a_2, a_3) \in (\mathbb{C}^*)^3$, $S \subseteq \{1, 2, 3\}$.*

**Proof.** We divide the proof into five steps.

**Step 1.** Identify $\mathfrak{sl}_3(\mathbb{C})$ as a Lie subalgebra of $\mathfrak{g}_3$ in the following way:

$$\mathbf{e}_i = e_i, \quad \mathbf{f}_i = f_i, \quad \mathbf{h}_i = h_i, \quad i = 1, 2.$$

Then
$$H_1 = \frac{2}{3}(2\varepsilon_1 - \varepsilon_2 + \varepsilon_3) + \frac{1}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) = \varepsilon_1 + \frac{1}{3}\varepsilon_3,$$
$$H_2 = \frac{1}{3}(2\varepsilon_1 - \varepsilon_2 + \varepsilon_3) + \frac{2}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) = \varepsilon_2 - \frac{1}{3}\varepsilon_3,$$

and $\mathcal{N}$ is a module over $\mathfrak{sl}_3(\mathbb{C}) + \mathbb{C}\varepsilon_3$. These together with Theorem 2.1 give

$$\begin{aligned}
e_1 \cdot g &= a_1(H_1 - b_1 - 1)^{\delta_{1\notin S_1}}(H_2 - H_1 - b_1)^{\delta_{2\in S_1}}\sigma_1(g)\\
&= a_1(\varepsilon_1 + \frac{1}{3}\varepsilon_3 - b_1 - 1)^{\delta_{1\notin S_1}}(\varepsilon_2 - \varepsilon_1 - \frac{2}{3}\varepsilon_3 - b_1)^{\delta_{2\in S_1}}\sigma_1(g),\\
f_1 \cdot g &= a_1^{-1}(H_1 - b_1)^{\delta_{1\in S_1}}(H_2 - H_1 - b_1 - 1)^{\delta_{2\notin S_1}}\sigma_1^{-1}(g)\\
&= a_1^{-1}(\varepsilon_1 + \frac{1}{3}\varepsilon_3 - b_1)^{\delta_{1\in S_1}}(\varepsilon_2 - \varepsilon_1 - \frac{2}{3}\varepsilon_3 - b_1 - 1)^{\delta_{2\notin S_1}}\sigma_1^{-1}(g),\\
e_2 \cdot g &= a_2(H_2 - H_1 - b_1 - 1)^{\delta_{2\notin S_1}}(-\varepsilon_2 + \frac{1}{3}\varepsilon_3 - b_1)^{\delta_{3\in S_1}}\sigma_2(g)\\
&= a_2(\varepsilon_2 - \varepsilon_1 - \frac{2}{3}\varepsilon_3 - b_1 - 1)^{\delta_{2\notin S_1}}(-H_2 - b_1)^{\delta_{3\in S_1}}\sigma_2(g),\\
f_2 \cdot g &= a_2^{-1}(H_2 - H_2 - b_1)^{\delta_{2\in S_1}}(-H_2 - b_1 - 1)^{\delta_{3\notin S_1}}\sigma_2^{-1}(g)\\
&= a_2^{-1}(\varepsilon_2 - \varepsilon_1 - \frac{2}{3}\varepsilon_3 - b_1)^{\delta_{2\in S_1}}(-\varepsilon_2 + \frac{1}{3}\varepsilon_3 - b_1 - 1)^{\delta_{3\notin S_1}}\sigma_2^{-1}(g),
\end{aligned}$$

for some $S_1 \subseteq \{1,2,3\}$, $a_1, a_2 \in \mathbb{C}^*$, and $b_1 \in \mathbb{C}[\varepsilon_3]$.

**Step 2.** Identify $\mathfrak{sl}_3(\mathbb{C})$ as a Lie subalgebra in the following way:

$$\mathbf{e}_i = e_{i+1}, \quad \mathbf{f}_i = f_{i+1}, \quad \mathbf{h}_i = h_{i+1}, \quad i = 1, 2.$$

Then
$$H_1 = \frac{2}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) + \frac{1}{3}(\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3) = \varepsilon_2 - \frac{1}{3}\varepsilon_1,$$
$$H_2 = \frac{1}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) + \frac{2}{3}(\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3) = \varepsilon_3 + \frac{1}{3}\varepsilon_1,$$

and $\mathcal{N}$ is a module over $\mathfrak{sl}_3(\mathbb{C}) + \mathbb{C}\varepsilon_1$. By Theorem 2.1, we have

$$\begin{aligned}
e_2 \cdot g &= a_2'(\varepsilon_2 - \frac{1}{3}\varepsilon_1 - b_2 - 1)^{\delta_{2\notin S_2}}(\frac{2}{3}\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b_2)^{\delta_{3\in S_2}}\sigma_2(g),\\
f_2 \cdot g &= (a_2')^{-1}(\varepsilon_2 - \frac{1}{3}\varepsilon_1 - b_2)^{\delta_{2\in S_2}}(\frac{2}{3}\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b_2 - 1)^{\delta_{3\notin S_2}}\sigma_2^{-1}(g),\\
e_3 \cdot g &= a_3(\frac{2}{3}\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b_2 - 1)^{\delta_{3\notin S_2}}(-\frac{1}{3}\varepsilon_1 - \varepsilon_3 - b_2)^{\delta_{4\in S_2}}\sigma_3(g),\\
f_3 \cdot g &= a_3^{-1}(\frac{2}{3}\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b_2)^{\delta_{3\in S}}(-\frac{1}{3}\varepsilon_1 - \varepsilon_3 - b_2 - 1)^{\delta_{4\notin S_2}}\sigma_3^{-1}(g),
\end{aligned}$$

for some $S_2 \subseteq \{2,3,4\}$, $a_2', a_3 \in \mathbb{C}^*$, and $b_2 \in \mathbb{C}[\varepsilon_1]$.

**Step 3.** We proceed with the sets $S_1$ and $S_2$ obtained in Steps 1 and 2. Comparing the action of $e_2, f_2$ on $g$, we have either

$$a_2 = a_2',\ 2 \in S_1 \Leftrightarrow 2 \in S_2,$$
$$\varepsilon_2 - \varepsilon_1 - \frac{2}{3}\varepsilon_3 - b_1 - 1 = \varepsilon_2 - \frac{1}{3}\varepsilon_1 - b_2 - 1,$$

or
$$a_2 = -a_2',\ 2 \in S_1 \Leftrightarrow 3 \notin S_2,$$
$$\varepsilon_2 - \varepsilon_1 - \frac{2}{3}\varepsilon_3 - b_1 - 1 = -\frac{2}{3}\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b_2.$$

In each case, $S_1$ and $S_2$ can be identified by a single set $S \subseteq \{1,2,3,4\}$.

**Step 4.** If $a_2 = a_2'$, $2 \in S_1 \Leftrightarrow 2 \in S_2$, we have

$$b_1 = -\frac{2}{3}\varepsilon_3 + b, \quad b_2 = \frac{2}{3}\varepsilon_1 + b$$

for some $b \in \mathbb{C}$. Moreover, we may write

$$\begin{aligned}
e_1 \cdot g &= a_1(\varepsilon_1 + \varepsilon_3 - b - 1)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - b)^{\delta_{2\in S}}\sigma_1(g),\\
f_1 \cdot g &= a_1^{-1}(\varepsilon_1 + \varepsilon_3 - b)^{\delta_{1\in S}}(-\varepsilon_1 + \varepsilon_2 - b - 1)^{\delta_{2\notin S}}\sigma_1^{-1}(g),\\
e_2 \cdot g &= a_2(-\varepsilon_1 + \varepsilon_2 - b - 1)^{\delta_{2\notin S}}(-\varepsilon_2 + \varepsilon_3 - b)^{\delta_{3\in S}}\sigma_2(g),\\
f_2 \cdot g &= a_2^{-1}(-\varepsilon_1 + \varepsilon_2 - b)^{\delta_{2\in S}}(-\varepsilon_2 + \varepsilon_3 - b - 1)^{\delta_{3\notin S}}\sigma_2^{-1}(g),\\
e_3 \cdot g &= a_3(-\varepsilon_2 + \varepsilon_3 - b - 1)^{\delta_{3\notin S}}(-\varepsilon_1 - \varepsilon_3 - b)^{\delta_{4\in S}}\sigma_3(g),\\
f_3 \cdot g &= a_3^{-1}(-\varepsilon_2 + \varepsilon_3 - b)^{\delta_{3\in S}}(-\varepsilon_1 - \varepsilon_3 - b - 1)^{\delta_{4\notin S}}\sigma_3^{-1}(g).
\end{aligned}$$

However, the relations $[e_1, e_3] = [f_1, f_3] = 0$ force $1 \in S \Leftrightarrow 4 \notin S$ and $b = -\frac{1}{2}$. Thus we have

$$\begin{aligned}
e_1 \cdot g &= a_1(\varepsilon_1 + \varepsilon_3 - \frac{1}{2})^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 + \frac{1}{2})^{\delta_{2\in S}}\sigma_1(g),\\
f_1 \cdot g &= a_1^{-1}(\varepsilon_1 + \varepsilon_3 + \frac{1}{2})^{\delta_{1\in S}}(-\varepsilon_1 + \varepsilon_2 - \frac{1}{2})^{\delta_{2\notin S}}\sigma_1^{-1}(g),\\
e_2 \cdot g &= a_2(-\varepsilon_1 + \varepsilon_2 - \frac{1}{2})^{\delta_{2\notin S}}(-\varepsilon_2 + \varepsilon_3 + \frac{1}{2})^{\delta_{3\in S}}\sigma_2(g),\\
f_2 \cdot g &= a_2^{-1}(-\varepsilon_1 + \varepsilon_2 + \frac{1}{2})^{\delta_{2\in S}}(-\varepsilon_2 + \varepsilon_3 - \frac{1}{2})^{\delta_{3\notin S}}\sigma_2^{-1}(g),\\
e_3 \cdot g &= a_3(-\varepsilon_2 + \varepsilon_3 - \frac{1}{2})^{\delta_{3\notin S}}(-\varepsilon_1 - \varepsilon_3 + \frac{1}{2})^{\delta_{1\notin S}}\sigma_3(g),\\
f_3 \cdot g &= a_3^{-1}(-\varepsilon_2 + \varepsilon_3 + \frac{1}{2})^{\delta_{3\in S}}(-\varepsilon_1 - \varepsilon_3 - \frac{1}{2})^{\delta_{1\in S}}\sigma_3^{-1}(g).
\end{aligned}$$

So $\mathcal{N}$ is isomorphic to $\mathcal{N}_2(\mathbf{a}, S)$ for some $\mathbf{a} \in (\mathbb{C}^*)^3$, $S \subseteq \{1, 2, 3\}$.

**Step 5.** If $a_2 = -a_2'$, $2 \in S_1 \Leftrightarrow 3 \notin S_2$, we have

$$b_1 = \frac{1}{3}\varepsilon_3 - b, \quad b_2 = -\frac{1}{3}\varepsilon_1 + b - 1$$

for some $b \in \mathbb{C}$. Moreover, we may write

$$\begin{aligned}
e_1 \cdot g &= a_1(\varepsilon_1 + b)^{\delta_{1\notin S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}\sigma_1(g),\\
f_1 \cdot g &= a_1{}^{-1}(\varepsilon_1 + b + 1)^{\delta_{1\in S}}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\notin S}}\sigma_1^{-1}(g),\\
e_2 \cdot g &= a_2(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b)^{\delta_{2\notin S}}(-\varepsilon_2 + b + 1)^{\delta_{3\in S}}\sigma_2(g),\\
f_2 \cdot g &= a_2{}^{-1}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + b + 1)^{\delta_{2\in S}}(-\varepsilon_2 + b)^{\delta_{3\notin S}}\sigma_2^{-1}(g),\\
e_3 \cdot g &= a_3(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b - 1)^{\delta_{2\in S}}(-\varepsilon_3 - b)^{\delta_{4\in S}}\sigma_3(g),\\
f_3 \cdot g &= a_3{}^{-1}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - b)^{\delta_{2\notin S}}(-\varepsilon_3 - b - 1)^{\delta_{4\notin S}}\sigma_3^{-1}(g).
\end{aligned}$$

We conclude that $\mathcal{N}$ is isomorphic to $\mathcal{N}_1(\mathbf{a}, b, S)$ for some $\mathbf{a} \in (\mathbb{C}^*)^3$, $b \in \mathbb{C}$ and $S \subseteq \{1, 2, 3, 4\}$. ∎

## 4. Rank one polynomial modules over $\mathfrak{g}_n$ $(n \geq 4)$

In this section, we consider the general case. We construct and classify all rank one polynomial modules over the GIM Lie algebra $\mathfrak{g}_n$ $(n \geq 4)$. We always assume that $n \geq 4$ in the following.

### 4.1. $\mathfrak{g}_n$-module structures on $U(\mathfrak{h})$

In this subsection, we construct a class of polynomial modules over the GIM Lie algebra $\mathfrak{g}_n$, which are actually free of rank one when restricted to the Cartan subalgebra $\mathfrak{h} = \mathrm{span}_{\mathbb{C}}\{\varepsilon_i \mid i = 1, \ldots, n\}$.

Fix $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$, $b \in \mathbb{C}$, $S \subseteq \{1, \ldots, n\}$, $g \in \mathbb{C}[\varepsilon_1, \ldots, \varepsilon_n]$, we define the action of $\mathfrak{g}_n$ on $\mathcal{N}(\mathbf{a}, b, S) := \mathbb{C}[\varepsilon_1, \ldots, \varepsilon_n]$ as follows:

$$\begin{cases} \varepsilon_i \cdot g = \varepsilon_i g, \quad 1 \le i \le n, \\ e_1 \cdot g = a_1(\varepsilon_1 + \varepsilon_n - \frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 + \frac{1}{2})^{\delta_{2\in S}}\sigma_1(g), \\ f_1 \cdot g = a_1^{-1}(\varepsilon_1 + \varepsilon_n + \frac{1}{2})^{\delta_{1\in S}}(\varepsilon_2 - \varepsilon_1 - \frac{1}{2})^{\delta_{2\notin S}}\sigma_1^{-1}(g), \\ e_i \cdot g = a_i(\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}\sigma_i(g), \qquad 2 \le i \le n-1, \\ f_i \cdot g = a_i^{-1}(\varepsilon_i - \varepsilon_{i-1} + \frac{1}{2})^{\delta_{i\in S}}(\varepsilon_{i+1} - \varepsilon_i - \frac{1}{2})^{\delta_{i+1\notin S}}\sigma_i^{-1}(g), \quad 2 \le i \le n-1, \\ e_n \cdot g = a_n(\varepsilon_n - \varepsilon_{n-1} - \frac{1}{2})^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n + \frac{1}{2})^{\delta_{1\notin S}}\sigma_n(g), \\ f_n \cdot g = a_n^{-1}(\varepsilon_n - \varepsilon_{n-1} + \frac{1}{2})^{\delta_{n\in S}}(-\varepsilon_1 - \varepsilon_n - \frac{1}{2})^{\delta_{1\in S}}\sigma_n^{-1}(g), \end{cases} \tag{18}$$

where $\sigma_i$ $(1 \le i \le n)$ is defined by Definition 2.2.

**Proposition 4.1.** *Fix $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$, $b \in \mathbb{C}$, and $S \subseteq \{1, \ldots, n\}$. Then $\mathcal{N}(\mathbf{a}, b, S)$ is a rank one polynomial modules over $\mathfrak{g}_n$.*

**Proof.** Using (18), we should verify that the actions of all $\varepsilon_i, e_i, f_i$ $(1 \le i \le n)$ on $\mathcal{N}(\mathbf{a}, b, S)$ satisfy the relations of $\mathfrak{g}_n$. Let $g \in \mathcal{N}(\mathbf{a}, b, S)$.

For any $1 \le i \le n, 2 \le j \le n-1$, using (18) and $\sigma_j(\varepsilon_i g) = (\varepsilon_i - \delta_{ij})\sigma_j(g)$, we have

$$\begin{aligned} &\varepsilon_i \cdot e_j \cdot g - e_j \cdot \varepsilon_i \cdot g \\ &= a_j \varepsilon_i \cdot ((\varepsilon_j - \varepsilon_{j-1} - \frac{1}{2})^{\delta_{j\notin S}}(\varepsilon_{j+1} - \varepsilon_j + \frac{1}{2})^{\delta_{j+1\in S}}\sigma_j(g)) - e_j \cdot (\varepsilon_i g) \\ &= a_j \varepsilon_i(\varepsilon_j - \varepsilon_{j-1} - \frac{1}{2})^{\delta_{j\notin S}}(\varepsilon_{j+1} - \varepsilon_j + \frac{1}{2})^{\delta_{j+1\in S}}\sigma_j(g) - a_j(\varepsilon_j \\ &\quad - \varepsilon_{j-1} - \frac{1}{2})^{\delta_{j\notin S}}(\varepsilon_{j+1} - \varepsilon_j + \frac{1}{2})^{\delta_{j+1\in S}}\sigma_j(\varepsilon_i g) \\ &= a_j(\varepsilon_j - \varepsilon_{j-1} - \frac{1}{2})^{\delta_{j\notin S}}(\varepsilon_{j+1} - \varepsilon_j + \frac{1}{2})^{\delta_{j+1\in S}}\sigma_j(g), \end{aligned}$$

which implies $[\varepsilon_i, e_j] \cdot g = \delta_{ij} e_j \cdot g$. Similarly we can prove $[\varepsilon_i, e_1] \cdot g = \delta_{i1} e_1 \cdot g$ and $[\varepsilon_i, e_n] \cdot g = \delta_{in} e_n \cdot g$ for $1 \le i \le n$. Therefore $[\varepsilon_i, e_j] \cdot g = \delta_{ij} e_j \cdot g$ for $1 \le i, j \le n$.

According to (18), we have

$$\begin{aligned} &e_1 \cdot f_1 \cdot g - f_1 \cdot e_1 \cdot g \\ &= e_1 \cdot (\varepsilon_1 + \varepsilon_n + \frac{1}{2})^{\delta_{1\in S}}(\varepsilon_2 - \varepsilon_1 - \frac{1}{2})^{\delta_{2\notin S}}\sigma_1^{-1}(g) \\ &\quad - f_1 \cdot (\varepsilon_1 + \varepsilon_n - \frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 + \frac{1}{2})^{\delta_{2\in S}}\sigma_1(g) \\ &= (\varepsilon_1 + \varepsilon_n - \frac{1}{2})^{\delta_{1\in S}}(\varepsilon_2 - \varepsilon_1 + \frac{1}{2})^{\delta_{2\notin S}}(\varepsilon_1 + \varepsilon_n - \frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 + \frac{1}{2})^{\delta_{2\in S}} g \\ &\quad - (\varepsilon_1 + \varepsilon_n + \frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 - \frac{1}{2})^{\delta_{2\in S}}(\varepsilon_1 + \varepsilon_n + \frac{1}{2})^{\delta_{1\in S}}(\varepsilon_2 - \varepsilon_1 - \frac{1}{2})^{\delta_{2\notin S}} g \\ &= (\varepsilon_1 + \varepsilon_n - \frac{1}{2})(\varepsilon_2 - \varepsilon_1 + \frac{1}{2})g - (\varepsilon_1 + \varepsilon_n + \frac{1}{2})(\varepsilon_2 - \varepsilon_1 - \frac{1}{2})g \\ &= (2\varepsilon_1 - \varepsilon_2 + \varepsilon_n)g, \end{aligned}$$

and

$$
\begin{aligned}
& e_n \cdot f_n \cdot g - f_n \cdot e_n \cdot g \\
& = e_n \cdot (\varepsilon_n - \varepsilon_{n-1} + \frac{1}{2})^{\delta_{n\in S}}(-\varepsilon_1 - \varepsilon_n - \frac{1}{2})^{\delta_{1\in S}}\sigma_n^{-1}(g) \\
& \quad - f_n \cdot (\varepsilon_n - \varepsilon_{n-1} - \frac{1}{2})^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n + \frac{1}{2})^{\delta_{1\notin S}}\sigma_n(g) \\
& = (\varepsilon_n - \varepsilon_{n-1} - \frac{1}{2})^{\delta_{n\in S}}(-\varepsilon_1 - \varepsilon_n + \frac{1}{2})^{\delta_{1\in S}}(\varepsilon_n - \varepsilon_{n-1} - \frac{1}{2})^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n + \frac{1}{2})^{\delta_{1\notin S}}g \\
& \quad - (\varepsilon_n - \varepsilon_{n-1} + \frac{1}{2})^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n - \frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_n - \varepsilon_{n-1} + \frac{1}{2})^{\delta_{n\in S}}(-\varepsilon_1 - \varepsilon_n - \frac{1}{2})^{\delta_{1\in S}}g \\
& = (\varepsilon_n - \varepsilon_{n-1} - \frac{1}{2})(-\varepsilon_1 - \varepsilon_n + \frac{1}{2})g - (\varepsilon_n - \varepsilon_{n-1} + \frac{1}{2})(-\varepsilon_1 - \varepsilon_n - \frac{1}{2})g \\
& = (\varepsilon_1 - \varepsilon_{n-1} + 2\varepsilon_n)g.
\end{aligned}
$$

Thus relations (4) hold. For any $2 \leq i \leq n-1$, since

$$
\begin{aligned}
& e_i \cdot f_i \cdot g - f_i \cdot e_i \cdot g \\
& = (\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}\sigma_i((\varepsilon_i - \varepsilon_{i-1} + \frac{1}{2})^{\delta_{i\in S}}(\varepsilon_{i+1} - \varepsilon_i - \frac{1}{2})^{\delta_{i+1\notin S}}\sigma_i^{-1}(g)) \\
& \quad - (\varepsilon_i - \varepsilon_{i-1} + \frac{1}{2})^{\delta_{i\in S}}(\varepsilon_{i+1} - \varepsilon_i - \frac{1}{2})^{\delta_{i+1\notin S}} \\
& \sigma_i^{-1}((\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}\sigma_i(g)) \\
& = (\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}(\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\in S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\notin S}}g \\
& \quad - (\varepsilon_i - \varepsilon_{i-1} + \frac{1}{2})^{\delta_{i\in S}}(\varepsilon_{i+1} - \varepsilon_i - \frac{1}{2})^{\delta_{i+1\notin S}}(\varepsilon_i - \varepsilon_{i-1} + \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i - \frac{1}{2})^{\delta_{i+1\in S}}g \\
& = (\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2}) - (\varepsilon_i - \varepsilon_{i-1} + \frac{1}{2})(\varepsilon_{i+1} - \varepsilon_i - \frac{1}{2}) \\
& = (-\varepsilon_{i-1} + 2\varepsilon_i - \varepsilon_{i+1})g,
\end{aligned}
$$

the relations in (5) are verified.

Using $\sigma_i\sigma_j = \sigma_j\sigma_i$, $\sigma_i(\varepsilon_j) = \varepsilon_j$ for $|i-j| > 1$, and some simple computations, we can get $[e_i, e_j] \cdot g = [f_i, f_j] \cdot g = 0$ for $|i-j| > 1$.

For the relations in (6), we only check $[e_i, f_j] \cdot g = 0$ for $2 \leq i \neq j \leq n-1$. The other cases can be proved similarly. The case $|i-j| > 1$ is easy to prove. If $|i-j| = 1$, then

$$
\begin{aligned}
& e_i \cdot f_j \cdot g - f_j \cdot e_i \cdot g \\
& = a_i a_j^{-1}(\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}\sigma_i((\varepsilon_j - \varepsilon_{j-1} + \frac{1}{2})^{\delta_{j\in S}} \\
& \quad (\varepsilon_{j+1} - \varepsilon_j - \frac{1}{2})^{\delta_{j+1\notin S}}\sigma_j^{-1}(g)) - a_i a_j^{-1}(\varepsilon_j - \varepsilon_{j-1} + \frac{1}{2})^{\delta_{j\in S}}(\varepsilon_{j+1} - \varepsilon_j - \frac{1}{2})^{\delta_{j+1\notin S}} \\
& \quad \sigma_j^{-1}((\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}\sigma_i(g)) \\
& = a_i a_j^{-1}((\varepsilon_i - \varepsilon_{i-1} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \frac{1}{2})^{\delta_{i+1\in S}}(\varepsilon_j - \varepsilon_{j-1} + \delta_{i,j-1} + \frac{1}{2})^{\delta_{j\in S}} \\
& \quad (\varepsilon_{j+1} - \varepsilon_j - \delta_{i,j+1} - \frac{1}{2})^{\delta_{j+1\notin S}} - (\varepsilon_j - \varepsilon_{j-1} + \frac{1}{2})^{\delta_{j\in S}}(\varepsilon_{j+1} - \varepsilon_j - \frac{1}{2})^{\delta_{j+1\notin S}} \\
& \quad (\varepsilon_i - \varepsilon_{i-1} - \delta_{i-1,j} - \frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i + \delta_{i+1,j} + \frac{1}{2})^{\delta_{i+1\in S}})(\sigma_i\sigma_j^{-1})(g).
\end{aligned}
$$

Taking $j=i+1$, we obtain

$$\begin{aligned}&[e_i,f_{i+1}]\cdot g=e_i\cdot f_{i+1}\cdot g-f_{i+1}\cdot e_i\cdot g\\&=a_ia_{i+1}^{-1}((\varepsilon_i-\varepsilon_{i-1}-\frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1}-\varepsilon_i+\frac{1}{2})^{\delta_{i+1\in S}}(\varepsilon_{i+1}-\varepsilon_i+\frac{3}{2})^{\delta_{i+1\in S}}\\&\quad\cdot(\varepsilon_{i+2}-\varepsilon_{i+1}-\frac{1}{2})^{\delta_{i+2\notin S}}-(\varepsilon_{i+1}-\varepsilon_i+\frac{1}{2})^{\delta_{i+1\in S}}(\varepsilon_{i+2}-\varepsilon_{i+1}-\frac{1}{2})^{\delta_{i+2\notin S}}\\&\quad\cdot(\varepsilon_i-\varepsilon_{i-1}-\frac{1}{2})^{\delta_{i\notin S}}(\varepsilon_{i+1}-\varepsilon_i+\frac{3}{2})^{\delta_{i+1\in S}})(\sigma_i\sigma_{i+1}^{-1})(g)=0.\end{aligned}$$

Similarly, we have $[e_i,f_{i-1}]\cdot g=0$.

Now we prove $[e_i,[e_i,e_j]]\cdot g=0$ for $|i-j|=1$. We only verify $[e_1,[e_1,e_2]]\cdot g=0$, the other cases can be proved similarly. From (18) and Definition 2.2, we obtain

$$\begin{aligned}&e_1\cdot e_2\cdot g-e_2\cdot e_1\cdot g\\&=a_1a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}\sigma_1((\varepsilon_2-\varepsilon_1-\frac{1}{2})^{\delta_{2\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}\sigma_2(g))\\&\quad-a_1a_2(\varepsilon_2-\varepsilon_1-\frac{1}{2})^{\delta_{2\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}\sigma_2((\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}\sigma_1(g))\\&=a_1a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\sigma_1\sigma_2)(g)\\&\quad-a_1a_2(\varepsilon_2-\varepsilon_1-\frac{1}{2})^{\delta_{2\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1-\frac{1}{2})^{\delta_{2\in S}}(\sigma_1\sigma_2)(g)\\&=a_1a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\sigma_1\sigma_2)(g).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&(e_1^2e_2-e_1e_2e_1)\cdot g=e_1\cdot(e_1\cdot e_2\cdot g-e_2\cdot e_1\cdot g)\\&=a_1^2a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}\sigma_1((\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\sigma_1\sigma_2)(g))\\&=a_1^2a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_1+\varepsilon_n-\frac{3}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\sigma_1^2\sigma_2)(g),\end{aligned}$$

and

$$\begin{aligned}&(e_1e_2e_1-e_2e_1^2)\cdot g=e_1\cdot e_2\cdot(e_1\cdot g)-e_2\cdot e_1\cdot(e_1\cdot g)\\&=a_1^2a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\sigma_1\sigma_2)((\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}\sigma_1(g))\\&=a_1^2a_2(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_1+\varepsilon_n-\frac{3}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_3-\varepsilon_2+\frac{1}{2})^{\delta_{3\in S}}(\sigma_1^2\sigma_2)(g).\end{aligned}$$

Using these relations, we obtain

$$(e_1^2e_2-2e_1e_2e_1+e_2e_1^2)\cdot g=(e_1^2e_2-e_1e_2e_1)\cdot g-(e_1e_2e_1-e_2e_1^2)\cdot g=0.$$

This forces $[e_1,[e_1,e_2]]\cdot g=0$. Since $\sigma_1\sigma_n=\sigma_n\sigma_1$, we have

$$\begin{aligned}&e_1\cdot f_n\cdot g-f_n\cdot e_1\cdot g\\&=a_n^{-1}e_1\cdot((\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(-\varepsilon_1-\varepsilon_n-\frac{1}{2})^{\delta_{1\in S}}\sigma_n^{-1}(g))\\&\quad-a_1f_n\cdot((\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}\sigma_1(g))\\&=a_1a_n^{-1}(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}\\&\quad(-\varepsilon_1-\varepsilon_n+\frac{1}{2})^{\delta_{1\in S}}\sigma_1(\sigma_n^{-1}(g))-a_1a_n^{-1}(\varepsilon_1+\varepsilon_n+\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}\\&\quad(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(-\varepsilon_1-\varepsilon_n-\frac{1}{2})^{\delta_{1\in S}}(\sigma_1\sigma_n^{-1})(g)\\&=a_1a_n^{-1}(-1)^{\delta_1\notin S}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(\sigma_1\sigma_n^{-1})(g).\end{aligned}$$

This implies

$$
\begin{aligned}
&(e_1^2 f_n - e_1 f_n e_1)\cdot g = e_1\cdot(e_1\cdot f_n\cdot g - f_n\cdot e_1\cdot g)\\
&= a_1 a_n^{-1}(-1)^{\delta_{1\notin S}} e_1\cdot((\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(\sigma_1\sigma_n^{-1})(g))\\
&= a_1^2 a_n^{-1}(-1)^{\delta_{1\notin S}}(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_2-\varepsilon_1+\frac{3}{2})^{\delta_{2\in S}}\\
&\quad(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(\sigma_1^2\sigma_n^{-1})(g),
\end{aligned}
$$

and

$$
\begin{aligned}
&(e_1 f_n e_1 - f_n e_1^2)\cdot g = e_1\cdot f_n\cdot(e_1\cdot g) - f_n\cdot e_1\cdot(e_1\cdot g)\\
&= a_1 a_n^{-1}(-1)^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(\sigma_1\sigma_n^{-1})(e_1\cdot g)\\
&= a_1^2 a_n^{-1}(-1)^{\delta_{1\notin S}}(\varepsilon_2-\varepsilon_1+\frac{1}{2})^{\delta_{2\in S}}(\varepsilon_n-\varepsilon_{n-1}+\frac{1}{2})^{\delta_{n\in S}}(\varepsilon_1+\varepsilon_n-\frac{1}{2})^{\delta_{1\notin S}}\\
&\quad(\varepsilon_2-\varepsilon_1+\frac{3}{2})^{\delta_{2\in S}}(\sigma_1^2\sigma_n^{-1})(g).
\end{aligned}
$$

Thus we have

$$(e_1^2 f_n - 2e_1 f_n e_1 + f_n e_1^2)\cdot g = (e_1^2 f_n - e_1 f_n e_1)\cdot g - (e_1 f_n e_1 - f_n e_1^2)\cdot g = 0,$$

which forces $[e_1,[e_1,f_n]]\cdot g = 0$. The other relations of (3)-(9) can be proved similarly. This completes the proof. ■

**Theorem 4.2.** *For any* $\mathbf{a}=(a_1,\dots,a_n)\in(\mathbb{C}^*)^n$, $b\in\mathbb{C}$, *and* $S\subseteq\{1,\dots,n\}$, *the polynomial module* $\mathcal{N}(\mathbf{a},b,S)$ *over* $\mathfrak{g}_n$ *is simple.*

**Proof.** Let $M$ be a nonzero submodule of $\mathcal{N}(\mathbf{a},b,S)$ and $0\neq g=g(\varepsilon_1,\dots,\varepsilon_n)\in M$. We divide the following discussion into four cases.

**Case 1.** $S=\emptyset$. For any $k\in\mathbb{N}$, we have

$$(a_n f_n)^k\cdot g(\varepsilon_1,\dots,\varepsilon_n) = g(\varepsilon_1,\dots,\varepsilon_{n-1},\varepsilon_n+k)\in M.$$

Then we obtain a nonzero polynomial $g_1(\varepsilon_1,\dots,\varepsilon_{n-1})$ that lies in $M$. In fact, $g_1(\varepsilon_1,\dots,\varepsilon_{n-1})$ can be chosen as the leading coefficient of the polynomial $g$ in the variable $\varepsilon_n$. Let $s\le n$ and $g'_s(\varepsilon_1,\dots,\varepsilon_s)\in M$, assume that the leading coefficient $g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1})$ of $g'_s(\varepsilon_1,\dots,\varepsilon_s)$ in the variable $\varepsilon_s$ lies in $M$. From

$$a_{s-1}f_{s-1}\cdot g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1}) = (\varepsilon_s-\varepsilon_{s-1}-\frac{1}{2})g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-2},\varepsilon_{s-1}+1),$$

and using $g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1})\in M$, we can obtain $g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-2},\varepsilon_{s-1}+1)\in M$. Acting repeatedly by $f_{s-1}$ and using $g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1})\in M$, we obtain for all $k\in\mathbb{N}$ that $g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-2},\varepsilon_{s-1}+k)\in M$. Thus we obtain a nonzero polynomial $g'_{s-2}(\varepsilon_1,\dots,\varepsilon_{s-2})\in M$. Note that this induction also holds for $s=2$, we see that $1\in M$. Therefore $\mathcal{N}(\mathbf{a},b,S)$ is simple.

**Case 2.** $S=\{1,\dots,n\}$. The proof is similar to Case 1.

**Case 3.** $|S|\neq 0,n$ and $\delta_{1\in S}=\delta_{n\in S}$.

If $\delta_{1\in S}=\delta_{n\in S}=0$ we can deduce similarly to Case 1 that $g_1(\varepsilon_1,\dots,\varepsilon_{n-1})\in M$.

If $\delta_{1\in S}=\delta_{n\in S}=1$, by

$$(a_n^{-1}e_n)^k\cdot g(\varepsilon_1,\dots,\varepsilon_n)=g(\varepsilon_1,\dots,\varepsilon_{n-1},\varepsilon_n-k)\in M,\quad k\in\mathbb{N},$$

we also obtain $g_1(\varepsilon_1,\dots,\varepsilon_{n-1})\in M$.

Assume that the leading coefficient $g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1})$ of $g'_s(\varepsilon_1,\dots,\varepsilon_s)\in M$ in variable $\varepsilon_s$ belongs to $M$ for some $s\le n$. If $s-1\notin S$, then

$$a_{s-1}f_{s-1}\cdot g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1})=(\varepsilon_s-\varepsilon_{s-1}-\frac{1}{2})^{\delta_{s\notin S}}g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-2},\varepsilon_{s-1}+1)\in M.$$

If $s-1\in S$, then

$$a_{s-1}^{-1}e_{s-1}\cdot g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-1})=(\varepsilon_s-\varepsilon_{s-1}+\frac{1}{2})^{\delta_{s\in S}}g'_{s-1}(\varepsilon_1,\dots,\varepsilon_{s-2},\varepsilon_{s-1}-1)\in M.$$

Similar to Case 1, the induction on $s$ can be applied even for $s=2$. We finally obtain $1\in M$. So $\mathcal{N}(\mathbf{a},b,S)$ is simple.

**Case 4.** $|S|\neq 0,n$ and $\delta_{1\in S}\neq\delta_{n\in S}$. In this case, there exists some $1\le s\le n$ such that $\delta_{s\in S}\neq\delta_{s-1\in S}$. If $s\in S$, then $s-1\notin S$. Thus

$$(a_{s-1}^{-1}e_{s-1})^k\cdot g(\varepsilon_1,\dots,\varepsilon_n)=g(\varepsilon_1,\dots,\varepsilon_{s-1}-k,\dots,\varepsilon_n)\in M.$$

If $s\notin S$, then $s-1\in S$. Thus

$$(a_{s-1}f_{s-1})^k\cdot g(\varepsilon_1,\dots,\varepsilon_n)=g(\varepsilon_1,\dots,\varepsilon_{s-1}+k,\dots\varepsilon_n)\in M.$$

Whatever, we can obtain a nonzero polynomial that lies in $M$, which does not contain variable $\epsilon_{s-1}$. By a similar discussion as in Case 3, we can obtain a nonzero polynomial $g'_s(\varepsilon_s,\dots\varepsilon_n)\in M$. If $s+1\in S$, then

$$a_sf_s\cdot g'_s(\varepsilon_s,\dots,\varepsilon_n)=(\varepsilon_s-\varepsilon_{s-1}+\frac{1}{2})^{\delta_{s\in S}}g'_s(\varepsilon_s+1,\dots,\varepsilon_n)\in M.$$

If $s+1\notin S$, then

$$a_s^{-1}e_s\cdot g'_s(\varepsilon_s,\dots,\varepsilon_n)=(\varepsilon_s-\varepsilon_{s-1}-\frac{1}{2})^{\delta_{s\notin S}}g'_s(\varepsilon_s-1,\dots,\varepsilon_n)\in M.$$

By induction on $s$, we can obtain a nonzero polynomial $g'_n(\varepsilon_n)\in M$.

If $1\in S$, then $n\notin S$. Thus we have

$$a_nf_n\cdot g'_n(\varepsilon_n)=(-\varepsilon_1-\varepsilon_n-\frac{1}{2})g'_n(\varepsilon_n+1)\in M.$$

This together with the above discussion implies that the coefficient of the variable $\varepsilon_1$ belongs to $M$. That is $g'_n(\varepsilon_n+1)\in M$. Acting repeatedly by $f_n$, we obtain $g'_n(\varepsilon_n+k)\in M$ for all $k\in\mathbb{N}$. Then we finally obtain $1\in M$ and $\mathcal{N}(\mathbf{a},b,S)$ is a simple module.

If $1\notin S$, the $n\in S$. Thus we have

$$a_n^{-1}e_n\cdot g'_n(\varepsilon_n)=(-\varepsilon_1-\varepsilon_n+\frac{1}{2})g'_n(\varepsilon_n-1)\in M.$$

Similarly, we have $1\in M$. So $\mathcal{N}(\mathbf{a},b,S)$ is simple. ∎

### 4.2. Classification of rank one polynomial modules over $\mathfrak{g}_n$

We show that the modules constructed in subsection 1 constitute a complete classification of rank one polynomial modules over $\mathfrak{g}_n$. Now we show the following crucial result.

**Theorem 4.3.** *Any rank one polynomial module over $\mathfrak{g}_n$ $(n \geq 4)$ is isomorphic to $\mathcal{N}(\mathbf{a}, b, S)$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$, $b \in \mathbb{C}$, $S \subseteq \{1, \ldots, n\}$.*

**Proof.** Let $\mathcal{N}$ be a rank one polynomial module over $\mathfrak{g}_n$ $(n \geq 4)$. The theorem will be proved by the following steps.

**Step 1.** Identify $\mathfrak{sl}_n(\mathbb{C})$ as a Lie subalgebra of $\mathfrak{g}_n$ in the following way:

$$\mathbf{e_i} = e_i, \quad \mathbf{f_i} = f_i, \quad \mathbf{h_i} = h_i, \quad i = 1, \ldots, n-1.$$

Then $\quad 2H_1 - H_2 = 2\varepsilon_1 - \varepsilon_2 + \varepsilon_n, \quad H_{n-2} + H_{n-1} = -\varepsilon_{n-2} + 2\varepsilon_{n-1} - \varepsilon_n,$

$$-H_{i-1} + 2H_i - H_{i+1} = -\varepsilon_{i-1} + 2\varepsilon_i - \varepsilon_{i+1}, \quad i = 2, \ldots, n-2,$$

which implies $\quad H_i = \varepsilon_i + (1 - \frac{2i}{n})\varepsilon_n, \quad i = 1, \ldots, n-1.$

Note that $\mathcal{N}$ is a free $\mathcal{U}(\mathfrak{h})$-module over $\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}\varepsilon_n$. From Theorem 2.1 (2), for any $S_1 \subseteq \{1, \ldots, n\}, \mathbf{a} = (a_1, \ldots, a_{n-1}) \in (\mathbb{C}^*)^{n-1}$, $b_1 \in \mathbb{C}[\varepsilon_n]$, we have

$$\begin{aligned}
e_1 \cdot g &= a_1^{-1}(\varepsilon_1 + (1 - \frac{2}{n})\varepsilon_n - b_1 - 1)^{\delta_{1 \notin S_1}}(\varepsilon_2 - \varepsilon_1 - \frac{2}{n}\varepsilon_n - b_1)^{\delta_{2 \in S_1}}\sigma_1(g),\\
f_1 \cdot g &= a_1^{-1}(\varepsilon_1 + (1 - \frac{2}{n})\varepsilon_n - b_1)^{\delta_{1 \in S_1}}(\varepsilon_2 - \varepsilon_1 - \frac{2}{n}\varepsilon_n - b_1 - 1)^{\delta_{2 \notin S_1}}\sigma_1^{-1}(g),\\
e_i \cdot g &= a_i(\varepsilon_i - \varepsilon_{i-1} - \frac{2}{n}\varepsilon_n - b_1 - 1)^{\delta_{i \notin S_1}}(\varepsilon_{i+1} - \varepsilon_i - \frac{2}{n}\varepsilon_n - b_1)^{\delta_{i+1 \in S_1}}\sigma_i(g),\\
f_i \cdot g &= a_i^{-1}(\varepsilon_i - \varepsilon_{i-1} - \frac{2}{n}\varepsilon_n - b_1)^{\delta_{i \in S_1}}(\varepsilon_{i+1} - H_i - \frac{2}{n}\varepsilon_n - b_1 - 1)^{\delta_{i+1 \notin S_1}}\sigma_i^{-1}(g),\\
e_{n-1} \cdot g &= a_{n-1}(\varepsilon_{n-1} - \varepsilon_{n-2} - \frac{2}{n}\varepsilon_n - b_1 - 1)^{\delta_{n-1 \notin S_1}}\\
&\qquad (-\varepsilon_{n-1} + (1 - \frac{2}{n})\varepsilon_n - b_1)^{\delta_{n \in S_1}}\sigma_{n-1}(g),\\
f_{n-1} \cdot g &= a_{n-1}^{-1}(\varepsilon_{n-1} - \varepsilon_{n-2} - \frac{2}{n}\varepsilon_n - b_1)^{\delta_{n-1 \in S_1}}\\
&\qquad (-\varepsilon_{n-1} + (1 - \frac{2}{n})\varepsilon_n - b_1 - 1)^{\delta_{n \notin S_1}}\sigma_{n-1}^{-1}(g).
\end{aligned}$$

**Step 2.** Identify $\mathfrak{sl}_3(\mathbb{C})$ as a Lie subalgebra of $\mathfrak{g}_n$ in the following way:

$$\mathbf{e_i} = e_{n+i-2}, \quad \mathbf{f_i} = f_{n+i-2}, \quad \mathbf{h_i} = h_{n+i-2}, \quad i = 1, 2.$$

Then $\quad 2H_1 - H_2 = -\varepsilon_{n-2} + 2\varepsilon_{n-1} - \varepsilon_n, \quad -H_1 + 2H_2 = \varepsilon_1 - \varepsilon_{n-1} + 2\varepsilon_n,$

which implies $H_{n-1} = \frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_{n-2} + \varepsilon_{n-1}$ and $H_n = \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_{n-2} + \varepsilon_n$.
Since $\mathcal{N}$ is a free $\mathcal{U}(\mathfrak{h})$-module over $\mathfrak{sl}_3(\mathbb{C}) \oplus (\oplus_{i=1}^{n-2}\mathbb{C}H_i)$, according to Theorem 2.1 (2), for any $S_2 \subseteq \{n-1, n, n+1\}, \mathbf{a} = (a'_{n-1}, a_n) \in (\mathbb{C}^*)^2$, $b_2 \in \mathbb{C}[\varepsilon_1, \ldots, \varepsilon_{n-2}]$, we have

$$\begin{aligned}
e_{n-1} \cdot g = a'_{n-1}(&\frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_{n-2} + \varepsilon_{n-1} - b_2 - 1)^{\delta_{n-1 \notin S_2}}\\
&(\frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b_2)^{\delta_{n \in S_2}}\sigma_{n-1}(g),
\end{aligned}$$

$$f_{n-1}\cdot g = {a'}_{n-1}^{-1}(\frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_{n-2} + \varepsilon_{n-1} - b_2)^{\delta_{n-1\in S_2}}$$
$$(\frac{1}{3}H_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b_2 - 1)^{\delta_{n\notin S_2}}\sigma_{n-1}^{-1}(g),$$
$$e_n\cdot g = a_n(\frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b_2 - 1)^{\delta_{n\notin S_2}}$$
$$(-\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_n - b_2)^{\delta_{n+1\in S_2}}\sigma_n(g),$$
$$f_n\cdot g = a_n^{-1}(\frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b_2)^{\delta_{n\in S_2}}$$
$$(-\frac{2}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_n - b_2 - 1)^{\delta_{n+1\notin S_2}}\sigma_n^{-1}(g).$$

**Step 3.** Comparing the action of $e_{n-1}, f_{n-1}$ on $g$, we obtain

$$a_{n-1} = {a'}_{n-1}, \quad n-1\in S_1 \Leftrightarrow n-1\in S_2,$$
$$\frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b_2 = -\varepsilon_{n-1} + (1-\frac{2}{n})\varepsilon_n - b_1,$$

or

$$a_{n-1} = -{a'}_{n-1}, \quad n-1\in S_1 \Leftrightarrow n\notin S_2,$$
$$\frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b_2 = -\varepsilon_{n-1} + \varepsilon_{n-2} + \frac{2}{n}\varepsilon_n + b_1 + 1.$$

**Step 4.** If $a_{n-1} = {a'}_{n-1}, n-1\in S_1 \Leftrightarrow n-1\in S_2$, then

$$b_1 = -\frac{2}{n}\varepsilon_n + b, \quad b_2 = \frac{1}{3}\varepsilon_1 + \frac{1}{3}\varepsilon_{n-2} + b, \quad b\in\mathbb{C}.$$

Thus we have

$$e_1\cdot g = a_1(\varepsilon_1 + \varepsilon_n - b - 1))^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 - b)^{\delta_{2\in S}}\sigma_1(g),$$
$$f_1\cdot g = a_1^{-1}(\varepsilon_1 + \varepsilon_n - b)^{\delta_{1\in S}}(\varepsilon_2 - \varepsilon_1 - b - 1)^{\delta_{2\notin S}}\sigma_1^{-1}(g),$$
$$e_i\cdot g = a_i(\varepsilon_i - \varepsilon_{i-1} - b - 1)^{\delta_{i\notin S}}(\varepsilon_{i+1} - \varepsilon_i - b)^{\delta_{i+1\in S}}\sigma_i(g),$$
$$f_i\cdot g = a_i^{-1}(\varepsilon_i - \varepsilon_{i-1} - b)^{(\delta_{i\in S}}(\varepsilon_{i+1} - \varepsilon_i - b - 1)^{\delta_{i+1\notin S}}\sigma_i^{-1}(g),$$
$$e_n\cdot g = a_n(\varepsilon_n - \varepsilon_{n-1} - b - 1)^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n - b)^{\delta_{n+1\in S}}\sigma_n(g),$$
$$f_n\cdot g = a_n^{-1}(\varepsilon_n - \varepsilon_{n-1} - b)^{\delta_{n\in S}}(-\varepsilon_1 - \varepsilon_n - b - 1)^{\delta_{n+1\notin S}}\sigma_n^{-1}(g),$$

where $i = 2,\ldots,n-1, S\subseteq\{1,\ldots,n+1\}$. Since

$$\begin{aligned}[e_1,e_n]\cdot g &= a_n e_1\cdot(\varepsilon_n - \varepsilon_{n-1} - b - 1)^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n - b)^{\delta_{n+1\in S}}\sigma_n(g)\\
&\quad - a_1 e_n\cdot(\varepsilon_1 + \varepsilon_n - b - 1)^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 - b)^{\delta_{2\in S}}\sigma_1(g)\\
&= a_1 a_n(\varepsilon_n - \varepsilon_{n-1} - b - 1)^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n - b + 1)^{\delta_{n+1\in S}}(\varepsilon_1 + \varepsilon_n - b - 1))^{\delta_{1\notin S}}\\
&\quad (\varepsilon_2 - \varepsilon_1 - b)^{\delta_{2\in S}}\sigma_1(\sigma_n(g)) - (\varepsilon_1 + \varepsilon_n - b - 2)^{\delta_{1\notin S}}(\varepsilon_2 - \varepsilon_1 - b)^{\delta_{2\in S}}\\
&\quad (\varepsilon_n - \varepsilon_{n-1} - b - 1)^{\delta_{n\notin S}}(-\varepsilon_1 - \varepsilon_n - b)^{\delta_{n+1\in S}}(\sigma_n\sigma_1)(g)\\
&= 0,\end{aligned}$$

we have

$$(-\varepsilon_1 - \varepsilon_n - b + 1)^{\delta_{n+1\in S}}(\varepsilon_1 + \varepsilon_n - b - 1)^{\delta_{1\notin S}} = (-\varepsilon_1 - \varepsilon_n - b)^{\delta_{n+1\in S}}(\varepsilon_1 + \varepsilon_n - b - 2)^{\delta_{1\notin S}},$$

which implies that $1\in S \Leftrightarrow n+1\notin S, \quad b = -\frac{1}{2}$.

In this case, the module is isomorphic to $\mathcal{N}(\mathbf{a}, b, S)$ defined by (18).

**Step 5.** If $$a_{n-1} = -a'_{n-1}, \quad n-1 \in S_1 \Leftrightarrow n \notin S_2,$$

then $$b_1 = (1 - \frac{2}{n})\varepsilon_n + b, \quad b_2 = \frac{1}{3}\varepsilon_1 - \frac{2}{3}\varepsilon_{n-2} + b, \quad b \in \mathbb{C}.$$

Using the results obtained in Step 1 and Step 2, there exists $S \subseteq \{1, \dots, n+1\}$ such that

$$\begin{aligned} e_1 \cdot g &= a_1(\varepsilon_1 - b - 1))^{\delta_{1 \notin S}}(\varepsilon_2 - \varepsilon_1 - \varepsilon_n - b)^{\delta_{2 \in S}}\sigma_1(g), \\ e_n \cdot g &= a_n(\varepsilon_{n-2} - \varepsilon_{n-1} + \varepsilon_n - b - 1)^{\delta_{n-1 \in S}}(-\varepsilon_1 + \varepsilon_{n-2} - \varepsilon_n - b - 1)^{\delta_{n+1 \in S}}\sigma_n(g), \end{aligned}$$

From $[e_1, e_n] = 0$, we deduce that

$$\begin{aligned} &(\varepsilon_2 - \varepsilon_1 - \varepsilon_n - b)^{\delta_{2 \in S}}(-\varepsilon_1 + \varepsilon_{n-2} - \varepsilon_n - b)^{\delta_{n+1 \in S}} \\ &= (\varepsilon_2 - \varepsilon_1 - \varepsilon_n - b + 1)^{\delta_{2 \in S}}(-\varepsilon_1 + \varepsilon_{n-2} - \varepsilon_n - b - 1)^{\delta_{n+1 \in S}}, \end{aligned}$$

which is impossible. ■

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