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Irreducibility of Wave-Front Sets for Depth Zero Cuspidal Representations

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Abstract. We show that recent results imply a positive answer to the question of Moeglin-Waldspurger on wave-front sets in the case of depth zero cuspidal representations. Namely, we deduce that for large enough residue characteristic, the Zariski closure of the wave-front set of any depth zero irreducible cuspidal representation of any reductive group over a non-Archimedean local field is an irreducible variety. In more details, we use results of Barbasch and Moy, DeBacker, and Okaka to reduce the statement to an analogous statement for finite groups of Lie type, which was proven by Lusztig, Achar and Aubert, and Taylor.

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1. Introduction

In this paper we prove the following theorem.

Theorem A. *For any* $n \in \mathbb{N}$ *there exists* $T \in \mathbb{N}$ *such that for any*

- *prime* $p > T$,
- *local field* F *of residue characteristic* p *such that* $val_F(p) < n$,
- *reductive group* $\mathbf{G}$ *defined over* F *such that* $\dim \mathbf{G} < n$,
- *cuspidal irreducible representation* π *of depth zero of* $\mathbf{G}(F)$,

the Zariski closure of the wave-front set $\mathrm{WF}(\pi)$ *in* $\mathfrak{g}^*$ *is irreducible, where* $\mathfrak{g}$ *denotes the Lie algebra of* $\mathbf{G}$.

In §2 we formulate a more explicit version of this theorem.

1.1. Idea of the proof

The natural analogue of Theorem A for finite groups of Lie type was proven in [1, 9, 15]. Barbasch and Moy [2] provide a method to describe the wave-front set of a depth zero representation of $\mathbf{G}(F)$ in terms of certain representations of certain finite groups of Lie type. In general, the description is quite complicated, but for cuspidal representations we make this description very explicit (see Corollary 5.4 below). This explicit description together with the result of [1, 9, 15] implies Theorem A.

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1.2. Related results

The irreducibility of the wave-front set of irreducible representations of finite groups of Lie type was conjectured (in a different language) in [7]. This conjecture was proven in [1, 9, 15].

For irreducible representations of p-adic groups the irreducibility of the wave-front set was suggested in [12] and proven for some cases, including all irreducible representations of GL_n, see [12, Chapter II]. In [17, 18], the irreducibility of the wave-front set was proven for many cases including anti-tempered and tempered unipotent representations of groups in the inner class of the split form of $\mathrm{SO}(2n+1)$. Recently, it was proven for irreducible Iwahori-spherical depth zero representations with a *real infinitesimal character*, see [3, Theorem 1.3.1].

Very recently, examples of irreducible representations of p-adic groups with reducible wave-front sets were given in [16].
Theorem A is independently proven in [4, §2.5].

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2. Formulation of the main result

Throughout the paper we fix a reductive algebraic group $\mathbf{G}$ defined over a local non-Archimedean field F.

2.1. Notation We denote:

- k – the residue field of F,
- $\mathfrak{g}$ – the Lie algebra of $\mathbf{G}$,
- $\mathbf{G}'$ – the derived group of $\mathbf{G}$,
- $\mathfrak{g}'$ – the Lie algebra of $\mathbf{G}'$,
- $G = \mathbf{G}(F)$,
- $BT(G)$ – the Bruhat-Tits building of $\mathbf{G}$,
- $\mathrm{irr}(G)$ – the set of (isomorphism classes of) irreducible smooth representations of G.

For any $x \in BT(G)$ and $r \in \mathbb{R}$ we further denote:

- $G_{x,r}$ and G_{x,r^+} – the Moy-Prasad subgroups (see [10, §2.6] where they are denoted by $\mathcal{P}_{x,r}, \mathcal{P}_{x,r^+}$).
- $M_x := G_{x,0}/G_{x,0^+}$.
- $\mathbf{M}_x$ – the natural reductive group defined over k s.t. $M_x \cong \mathbf{M}_x(k)$ (see [10, §3.2])
- Q_x – the normalizer of $G_{x,0}$ inside G.
- $G_r := \bigcup_{x \in BT(G)} G_{x,r}$
- $G_{r^+} := \bigcup_{x \in BT(G)} G_{x,r^+}$
- $\mathfrak{g}_{x,r}, \mathfrak{g}_{x,r^+}, \mathfrak{m}_x, \mathfrak{g}_r, \mathfrak{g}_{r^+}$ – the Lie algebra versions of the above (see [10, §3]).

Definition 2.1. We say that the pair $(F, \mathbf{G})$ is *acceptable*, if the following conditions are satisfied:

(1) The pair $(F, \mathbf{G})$ satisfies the Hales-Moy-Prasad conjecture for depth 0 representations, *i.e.* for any depth 0 representation $\rho \in \mathrm{irr}(G)$, the Harish-Chandra-Howe character expansion for ρ is valid on the set G_{0+} of topologically unipotent elements in G.

(2) The series defining the exponential map $\mathfrak{g}' \to \mathbf{G}$ given by the adjoint representation converge on $\mathfrak{g}_{0+} \cap \mathfrak{g}'$.

(3) For any $x \in BT(G)$, we have $\mathrm{char}\, k > 3(h_x - 1)$, where h_x is the Coxeter number of $\mathbf{M}_x$. Note that in particular this implies that p is a good prime for $\mathbf{M}_x$.

Proposition 2.2. *For any $n \in \mathbb{N}$ there exists $T \in \mathbb{N}$ such that for any*

- *prime $p > T$*
- *local field F of residue characteristic p such that $val_F(p) < n$*
- *reductive group $\mathbf{G}$ defined over F such that $\dim \mathbf{G} < n$,*

the pair $(F, \mathbf{G})$ is acceptable.

Proof. In order to satisfy condition (3) we take $T > 3n$. In order to satisfy condition (2) we take $T > n^2$. It suffices by [2, Lemma 3.2]. Finally, one can choose T such that (1) is satisfied by [5, §2.2, §3.4, Theorem 3.5.2] and condition (2). ∎

Definition 2.3. For $\pi \in \mathrm{irr}(G)$ denote by $\mathrm{WF}(\pi)$ the wave-front set of the character of π over the point $1 \in G$. It also equals the union of the closures of all the orbits $O \subset \mathfrak{g}^*(F)$ that appear with non-zero coefficients in the Harish-Chandra-Howe expansion.

The following is a more explicit version of Theorem A.

Theorem 2.4. *Assume that $(F, \mathbf{G})$ is an acceptable pair. Let π be a cuspidal irreducible representation of depth zero of $\mathbf{G}(F)$. Then the Zariski closure of the wave-front set $\mathrm{WF}(\pi)$ in $\mathfrak{g}^*$ is irreducible.*

From now till the end of the paper we will assume that $(F, \mathbf{G})$ is an acceptable pair. The rest of the paper is dedicated to the proof of this theorem.

3. Wave-front sets and generalized Gelfand-Graev models for finite groups of Lie type

Let k be a finite field, $\mathbf{M}$ be a reductive group defined over k, and $M := \mathbf{M}(k)$ be its group of k-points. We assume that $\mathrm{char}\, k > 3(h-1)$, where h is the Coxeter number of $\mathbf{M}$. In particular, this implies that $\mathrm{char}\, k$ is a good prime for $\mathbf{M}$. Let $\mathfrak{m}$ denote the Lie algebra of $\mathbf{M}$.

Definition 3.1.
- For every nilpotent element $N \in \mathfrak{m}(k)$, let Γ_N denote the *generalized Gelfand-Graev model* attached to N, as in [2, §2.2]. Since the isomorphism class of Γ_N only depends on the orbit of N under the adjoint action of M, we will also use the notation Γ_O for every nilpotent orbit $O \subset \mathfrak{m}(k)$.
- Let σ be a finite-dimensional representation of M. Let $GG(\sigma)$ denote the union of all nilpotent M-orbits $O \subset \mathfrak{m}(k)$ satisfying $\langle \tau, \Gamma_O \rangle \neq 0$, where $\langle \tau, \Gamma_O \rangle$ denotes the intertwining number.
- Let $\mathrm{WF}(\sigma)$ denote the Zariski closure of $\mathbf{M} \cdot GG(\sigma)$ in $\mathfrak{m}$.

Theorem 3.2. ([15, Theorem 14.10] *If* char k *is a good prime for* $\mathbf{M}$ *then for every irreducible representation* σ *of* M*, the algebraic variety* $\mathrm{WF}(\sigma)$ *is irreducible.*

4. The results of Barbasch-Moy

In this section we describe the results of [2], as refined in [6, 13], on the relation between wave-front sets of depth zero representations of G and of representations of the finite groups M_x for various $x \in BT(G)$.

The results in [2] require certain assumptions, described in [2, 4.4]. Assumptions (2) and (3) in [2, 4.4] coincide with assumptions (2) and (3) in Definition 2.1. Assumption (1) in [2, 4.4] can be replaced by assumption (1) of Definition 2.1. Indeed, this assumption is only used in [2] in order to deduce the statement of assumption (1) of Definition 2.1. Therefore all the results of [2] are valid for the acceptable pair $(F, \mathbf{G})$.

Definition 4.1. Let

(i) $\mathcal{N}_o(G)$ denote the set of nilpotent G-orbits in $\mathfrak{g}(F)$.

(ii) $\mathcal{I}_o(G) = \{(x, O) \,|x \in BT(G), \, O$ is a nilpotent M_x-orbit in $\mathfrak{m}_x\}$.

(iii) F^{un} be the unramified closure of F.

(iv) We define a pre-order on $\mathcal{N}_o(G)$ in the following way:

$$\Omega \geq \Omega' \quad \text{iff} \quad \overline{\mathbf{G}(F^{un}) \cdot \Omega} \supset \Omega',$$

where $\overline{\mathbf{G}(F^{un}) \cdot \Omega}$ denotes the closure in the local topology on $\mathfrak{g}(F^{un})$.

(v) We define a pre-order on $\mathcal{I}_o(G)$ in the following way:

$$(x', O') \leq (x, O) \text{ iff } x = x' \text{ and } \overline{\mathbf{M}_x \cdot O}^{Zar} \supset O',$$

where $\overline{\mathbf{M}_x \cdot O}^{Zar}$ denotes the Zariski closure.

Theorem 4.2. [6, Lemma 5.3.3.] *For any* $(x, O) \in \mathcal{I}_o(G)$ *there exists a unique* $\Omega \in \mathcal{N}_o(G)$ *with the property that there exists an* $\mathfrak{sl}_2$*-triple* $e, h, f \in \mathfrak{g}_{x,0}$ *satisfying:*

- $e \in \Omega$
- *the projections* $\bar{e}, \bar{h}, \bar{f}$ *form an* $\mathfrak{sl}_2$*-triple in* $\mathfrak{m}_x(k) = \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}$
- $\bar{e} \in O$.

Definition 4.3. We will denote: $\mathcal{L}(x,O) := \Omega$.

Theorem 4.4. [2, 6] *The map $\mathcal{L} : \mathcal{I}_o(G) \to \mathcal{N}_o(G)$ is:*

(i) *surjective, cf.* [6, Theorem 5.6.1],

(ii) *pre-order preserving, cf.* [2, Proposition 3.16].

Definition 4.5. For $x \in BT(G)$ and an M_x-stable subset $\Xi \subset \mathfrak{m}_x$ denote

$$\mathcal{L}_x(\Xi) := \bigcup_{O \in \Xi/M_x} \mathcal{L}(x,O).$$

Definition 4.6. For $\pi \in \operatorname{irr}(G)$ and $x \in BT(G)$, define $\pi_x := \pi^{G_{x,0+}}$ considered as a representation of $M_x = G_{x,0}/G_{x,0^+}$.

Theorem 4.7. ([13, Corollary 1.2], cf. [2, §5]) *Let π be a representation of G of depth zero. Then we have*

$$\overline{\mathrm{WF}(\pi)}^{Zar} = \overline{\bigcup_{x \in BT(G)} \mathcal{L}_x(\mathrm{WF}(\pi_x)(k))}^{Zar},$$

where by $\overline{\mathrm{WF}(\pi)}^{Zar}$ we mean the closure in the Zariski topology.

5. Proof of Theorem A

We will give an explicit version of the results of [2] for cuspidal representations of depth 0. For this we first recall a construction from [11] that exhausts all depth zero irreducible cuspidal representations of G:

Theorem 5.1. [11, Propositions 6.6 and 6.8] *Let $x \in BT(G)$ s.t. $G_{x,0}$ is a maximal parahoric subgroup of G, and let $\tau_0 \in \operatorname{irr}(Q_x/G_{x,0^+})$ such that $\tau_0|_{M_x}$ is a cuspidal representation. Let τ be the lift of τ_0 to Q_x. Then $\pi := \operatorname{ind}_{Q_x}^G \tau$ is a depth zero cuspidal irreducible representation of G. Moreover, any depth zero cuspidal irreducible representation of G can be obtained in this way.*

Proposition 5.2. *Let $x \in BT(G), \tau_0 \in \operatorname{irr}(Q_x/G_{x,0^+})$, its lift $\tau \in \operatorname{irr}(Q_x)$ and $\pi = \operatorname{ind}_{Q_x}^G \tau \in \operatorname{irr}(G)$ be as in Theorem 5.1, and let $y \in BT(G)$. Then*

(i) *If $\pi_y \neq 0$ then there exists $g \in G$ such that $gG_{x,0}g^{-1} = G_{y,0}$.*

(ii) $\pi_x \simeq (\tau_0)|_{M_x}$.

For the proof we will need the following lemma.

Lemma 5.1. *Let $x, y \in BT(G)$, and let F_x and F_y denote the minimal faces that include them. Assume that F_x is a face of minimal dimension. If $F_x \neq F_y$ then the image of $G_{x,0} \cap G_{y,0^+}$ in $G_{x,0}/G_{x,0^+}$ includes the unipotent radical of a proper parabolic subgroup of $\mathbf{M}_x(k)$.*

This Lemma is well known, however, we could not find an exact reference. Thus, for completeness, we deduce it here from what we found in the literature.

Proof. By passing to an unramified extension of F we can and will assume that $\mathbf{G}$ is quasi-split. Passing to a cover of $\mathbf{G}$ we may further assume that $\mathbf{G}$ is a product of a simply connected group and a torus. Since the torus component is inessential for our claim, we may ignore it and assume that $\mathbf{G}$ is simply connected. We recall that in this case the parahoric subgroup $G_{x,0}$ is just the stabilizer of x.

By [14, Corollary 3.24] there is a parabolic $\mathbf{P} \subset \mathbf{M}_x$ such that $G_{x,0} \cap G_{y,0} = \mathbf{P}(k)$ and $G_{x,0} \cap G_{y,0+} = \mathbf{U}(k)$, where $\mathbf{U} \subset \mathbf{P}$ is the unipotent radical.[1] It is left to show that $\mathbf{P}$ is a proper parabolic.

Choose an apartment $A \subset BT(G)$ that contains the points x and y. Connect these points by a segment $I \subset A$. Note that the intersection $G_{x,0} \cap G_{y,0}$ fixes I. So for any $z \in I$ we have $G_{x,0} \cap G_{z,0} \supset G_{x,0} \cap G_{y,0}$. Therefore without loss of generality we may and will assume that $x \in \bar{F}_y$. The statement in this case follows from [14, Proposition 3.22]. ∎

Proof of Proposition 5.2 We have the following isomorphisms of vector spaces.

$$\begin{aligned}\pi_y &\simeq \bigoplus_{[g]\in Q_x\backslash G/G_{y,0+}} (\mathrm{Ind}_{G_{y,0+}\cap g^{-1}Q_x g}^{G_{y,0+}}(\tau|_{gG_{y,0+}g^{-1}\cap Q_x})^g)^{G_{y,0+}}\\ &\simeq \bigoplus_{[g]\in Q_x\backslash G/G_{y,0+}} (\mathrm{Ind}_{G_{y,0+}\cap Q_{g^{-1}x}}^{G_{y,0+}}(\tau|_{G_{gy,0+}\cap Q_x})^g)^{G_{y,0+}}\\ &\simeq \bigoplus_{[g]\in Q_x\backslash G/G_{y,0+}} ((\tau|_{G_{gy,0+}\cap Q_x})^g)^{G_{y,0+}\cap Q_{g^{-1}x}} \simeq \bigoplus_{[g]\in Q_x\backslash G/G_{y,0+}} \tau^{G_{gy,0+}\cap Q_x}\end{aligned}$$

By Lemma 5.1 and the cuspidality of τ_0 we obtain

$$\pi_y \simeq \bigoplus_{[g]\in Q_x\backslash G/G_{y,0+} \text{ s.t. } F_{gy}=F_x} \tau^{G_{x,0+}} \simeq \bigoplus_{[g]\in Q_x\backslash G/G_{y,0+} \text{ s.t. } F_{gy}=F_x} \tau_0.$$

This proves (i). To prove (ii) we use the following isomorphism of representations of $G_{x,0}$.

$$\begin{aligned}\pi_x &\simeq \bigoplus_{[g]\in Q_x\backslash G/G_{x,0}} (\mathrm{Ind}_{G_{x,0}\cap g^{-1}Q_x g}^{G_{x,0}}(\tau|_{gG_{x,0}g^{-1}\cap Q_x})^g)^{G_{x,0+}}\\ &\cong \bigoplus_{[g]\in Q_x\backslash G/G_{x,0}} (\mathrm{Ind}_{G_{x,0}\cap Q_{g^{-1}x}}^{G_{x,0}}(\tau|_{G_{gx,0}\cap Q_x})^g)^{G_{x,0+}} \qquad (1)\end{aligned}$$

For any $g \in G$ we have a vector space isomorphism:

$$\begin{aligned}(\mathrm{Ind}_{G_{x,0}\cap Q_{g^{-1}x}}^{G_{x,0}}(\tau|_{G_{gx,0}\cap Q_x})^g)^{G_{x,0+}} &\cong \bigoplus_{[h]\in Q_x\backslash Q_x g G_{x,0}/G_{x,0+}} (\mathrm{Ind}_{G_{x,0+}\cap Q_{h^{-1}x}}^{G_{x,0+}}(\tau|_{G_{hx,0+}\cap Q_x})^h)^{G_{x,0+}}\\ \cong \bigoplus_{[h]\in Q_x\backslash Q_x g G_{x,0}/G_{x,0+}} ((\tau|_{G_{hx,0+}\cap Q_x})^h)^{G_{x,0+}\cap Q_{h^{-1}x}} &\cong \bigoplus_{[h]\in Q_x\backslash Q_x g G_{x,0}/G_{x,0+}} \tau^{G_{hx,0+}\cap Q_x}.\end{aligned}$$

[1] [14] considers only the case that G is split, but the proof of this statement (as well as the other statements from [14] that we use) does not use require this assumption.

By Lemma 5.1 and the cuspidality of τ, if the space above does not vanish then for some $h \in Q_x g G_{x,0}$ we have $F_x = F_{hx}$. In other words $Q_x g G_{x,0}$ intersects Q_x, and thus $g \in Q_x$. To sum up, if $(\mathrm{Ind}_{G_{x,0}\cap Q_{g^{-1}x}}^{G_{x,0}}(\tau|_{G_{gx,0}\cap Q_x})^g)^{G_{x,0+}} \neq 0$ then $g \in Q_x$. Using (1) we obtain

$$\pi_x \simeq (\tau|_{G_{x,0}})^{G_{x,0+}} = (\tau_0)|_{M_x} \qquad \blacksquare$$

Remark 5.3. Proposition 5.2 can be deduced from the main result of [8]. We thank the referee for informing us on this reference.

Proposition 5.2 and Theorem 4.7 imply the following corollary.

Corollary 5.4. *Let* $x \in BT(G), \tau_0 \in \mathrm{irr}(Q_x/G_{x,0^+})$, *its lift* $\tau \in \mathrm{irr}(Q_x)$ *and* $\pi = \mathrm{ind}_{Q_x}^G \tau \in \mathrm{irr}(G)$ *be as in Theorem* 5.1. *Then*

$$\overline{\mathrm{WF}(\pi)}^{Zar} = \overline{\mathcal{L}_x(\mathrm{WF}(\tau_0))}^{Zar}.$$

In view of Theorem 5.1, this corollary, together with Theorem 4.4 and Theorem 3.2, imply Theorem 2.4.

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