Journal of Lie Theory
Volume **34** (2024) 511–530

On The Stability of Tensor Product Representations of Classical Groups

Dibyendu Biswas

Communicated by S. Sahi

Abstract. From an irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ there is a natural way to construct an irreducible representations of $\mathrm{GL}(n+1,\mathbb{C})$ by adding a zero at the end of the highest weight $\underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ with $\lambda_i \geq 0$ of the irreducible representation of $\mathrm{GL}(n,\mathbb{C})$. The paper considers the decomposition of tensor products of irreducible representation of $\mathrm{GL}(n,\mathbb{C})$ and of the corresponding irreducible representations of $\mathrm{GL}(n+1,\mathbb{C})$ and proves a stability result about such tensor products. We go on to discuss similar questions for classical groups.

Mathematics Subject Classification: Primary 22E46, 20G05; secondary 05E10.

Key Words: Classical groups, tensor product, Pieri's rule, Littlewood-Richardson rule, Weyl character formula.

1. Introduction

An important aspect of classical groups is that they lie in nested families:

$$\mathrm{GL}(n,\mathbb{C}) \subseteq \mathrm{GL}(n+1,\mathbb{C}) \subseteq \mathrm{GL}(n+2,\mathbb{C}) \subseteq \cdots$$
$$\mathrm{Sp}(2n,\mathbb{C}) \subseteq \mathrm{Sp}(2n+2,\mathbb{C}) \subseteq \mathrm{Sp}(2n+4,\mathbb{C}) \subseteq \cdots$$
$$\mathrm{SO}(2n+1,\mathbb{C}) \subseteq \mathrm{SO}(2n+3,\mathbb{C}) \subseteq \mathrm{SO}(2n+5,\mathbb{C}) \subseteq \cdots$$
$$\mathrm{SO}(2n,\mathbb{C}) \subseteq \mathrm{SO}(2n+2,\mathbb{C}) \subseteq \mathrm{SO}(2n+4,\mathbb{C}) \subseteq \cdots .$$

Further, in each case, an n-tuple of integers $\underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ gives rise to an irreducible representation $\Pi_{\underline{\lambda}}$ of highest weight $\underline{\lambda}$ of the corresponding group of rank n. Adding a zero at the end of $\underline{\lambda}$, we thus have a natural map from irreducible representations of $\mathrm{GL}(n,\mathbb{C})$ to irreducible representations of $\mathrm{GL}(n+1,\mathbb{C})$, and similarly, for other classical groups. In fact, let us write any of the above nested sequences of groups as

$$G_n \subseteq G_{n+1} \subseteq G_{n+2} \subseteq \cdots .$$

One can ask how this natural map from irreducible representations of G_n to irreducible representations of G_{n+1} behaves for tensor products. This paper aims to study this question. We find that the decomposition of $\Pi_{\underline{\lambda}} \otimes \Pi_{\underline{\mu}}$ as a sum of irreducible representations of G_r, where $\Pi_{\underline{\lambda}}$ and $\Pi_{\underline{\mu}}$ are irreducible representations of G_n, is independent of r as soon as $r \geq 2n$, which we may then call *stable tensor product* of $\Pi_{\underline{\lambda}}$ and $\Pi_{\underline{\mu}}$, with the understanding that the stable tensor product of

ISSN 0949–5932 / \$2.50

$\Pi_{\underline{\lambda}}$ and $\Pi_{\underline{\mu}}$ is not only a representation of G_n but of all G_r for any $r \geq n$. Furthermore, we found to our surprise that the stable tensor product $\Pi_{\underline{\lambda}} \otimes \Pi_{\underline{\mu}}$ of G_n is independent of the classical group $(\mathrm{Sp}(2n,\mathbb{C}),\ \mathrm{SO}(2n+1,\mathbb{C}),\ \mathrm{SO}(2n,\mathbb{C}))$ chosen; see table for some of these conclusion for one specific example.

It is possible that many of the theorems about the tensor product of irreducible representations of G_n have an analogue for stable tensor product, such as the Littlewood-Richardson rule [2], [8] or saturation conjecture, a theorem due to Knutson-Tao [6] for $\mathrm{GL}(n,\mathbb{C})$, although at this point we are not sure if the statements simplify or become more involved. In Section 3 we begin by discussing the case of $\mathrm{GL}(n,\mathbb{C})$. The main tool here and in fact for all the cases studied in this work is Pieri's rule which describes the tensor product $\Psi_{\underline{\lambda}} \otimes \mathrm{Sym}^k(V)$ where $\Psi_{\underline{\lambda}}$ is the irreducible representation of $\mathrm{GL}(n,\mathbb{C}) = \mathrm{GL}(V)$ with highest weight $\underline{\lambda}$. Pieri's rule also proves as a consequence that every irreducible representation of highest weight of $\underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ of $\mathrm{GL}(V)$ is a sub-representation of $\mathrm{Sym}^{\lambda_1}(V) \otimes \mathrm{Sym}^{\lambda_2}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n}(V)$, where it appears with multiplicity exactly one, and the other constituents have highest weight which is "lower" than the highest weight of this. Pieri's rule together with this consequence allows us in Section 3 to prove our theorems comparing tensor products for $\mathrm{GL}(n,\mathbb{C})$ and for $\mathrm{GL}(n+1,\mathbb{C})$.

In Section 7 we compare the multiplicities in the tensor product of all the classical groups G_{N-1} with the same in G_N if N is the stability level corresponding to that tensor product which we call "just before stability results".

Here is the main theorem of this paper for classical groups proving stability theorem for classical groups of the same kind as asserted in the previous paragraph for the general linear groups. At the same time, we also prove that the multiplicities occurring in the tensor product are independent of which classical group we deal with. (For a sequence of integers $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r : \lambda_i \geq 0\}$, define $l(\underline{\lambda})$, the length of $\underline{\lambda}$, to be the largest integer s such that $\lambda_s \neq 0$.)

Theorem 1.1. *Let G_n be any one of the groups $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, or $\mathrm{SO}(2n,\mathbb{C})$. Let $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$ and $\underline{\mu} = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0\}$ be two sequence of integers, and $\Pi^n_{\underline{\lambda}}$ and $\Pi^n_{\underline{\mu}}$ the corresponding irreducible representations of G_n for $n \geq$ max $\{l(\underline{\lambda}), l(\underline{\mu})\}$. Write the tensor product of $\Pi^n_{\underline{\lambda}}$ and $\Pi^n_{\underline{\mu}}$ as,*

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}} = \sum_{\underline{\nu}} N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)\, \Pi^n_{\underline{\nu}}. \tag{1}$$

Suppose $n_0 = l(\underline{\lambda}) + l(\underline{\mu})$. Then,

1. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) = 0$ *if* $l(\underline{\nu}) \geq n_0 + 1$,
2. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n+1)$ *for* $n \geq \begin{cases} n_0 & \text{if } G_n = \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(2n+1,\mathbb{C}); \\ n_0+1 & \text{if } G_n = \mathrm{SO}(2n,\mathbb{C}), \end{cases}$
3. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)$ *is independent of the group* $G_n = \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(2n+1,\mathbb{C})$ *if* $n \geq n_0$,
4. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)$ *is independent of the group* $G_n = \mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, $\mathrm{SO}(2n,\mathbb{C})$ *if* $n \geq n_0 + 1$.

We now end the introduction by relating our work with other works on this topic. Indeed there are several works around the theme of stability theorems for classical groups, especially GL_n. Among the earliest references we could find is the work of [5], where parts (3) and (4) of Theorem 1.1 are proven. Other parts of the theorem seem new. The papers [10], [3] and [7] prove analogous stability theorems, but their theorems are not as precise as the theorems we prove. Our proofs are totally different from [5], [3], [7], and depend on simple inductive arguments using only the Pieri rule, and also give results just before the stability range which we have not seen discussed before anywhere. For related results, especially for classical groups, see [1].

2. Notations

The paper deals with specific embeddings of general linear groups $\mathrm{GL}(n,\mathbb{C})$ inside $\mathrm{GL}(n+1,\mathbb{C})$, and by iteration, embeddings of $\mathrm{GL}(n,\mathbb{C})$ inside $\mathrm{GL}(n+k,\mathbb{C})$ for all integers $k \geq 0$. Similarly, we need specific embeddings of classical groups in larger classical groups. We fix these embeddings in this section.

Let $V_i = \{e_1, e_2, \ldots, e_i\}$ be a vector space of dimension i for $i \geq 1$ with this specific basis. Thus $V_i \subseteq V_{i+1}$, giving rise to an embedding on $\mathrm{GL}(V_i)$ inside $\mathrm{GL}(V_{i+1})$ in which

$$\mathrm{GL}(V_i) = \{g \in \mathrm{Aut}(V_{i+1}) \text{ preserving } V_i \text{ and with } ge_{i+1} = e_{i+1}\}.$$

When we talk of $\mathrm{GL}(n,\mathbb{C})$, we mean $\mathrm{GL}(V_n)$, embedded in $\mathrm{GL}(n+1,\mathbb{C}) = \mathrm{GL}(V_{n+1})$ as

$$g \in \mathrm{GL}(n,\mathbb{C}) \longrightarrow \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right) \in \mathrm{GL}(n+1,\mathbb{C}).$$

Similarly, we define embeddings of classical groups. First, we do it for symplectic groups. Let $W_{2n} = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a $\mathbb{C}$-vector space of dimension $2n$ with the symplectic bilinear form B

$$\begin{aligned} B(e_i, f_j) &= \delta_{ij} \quad \forall 1 \leq i, j \leq n, \\ B(e_i, e_j) &= B(f_i, f_j) = 0 \quad \forall i, j, 1 \leq i, j \leq n. \end{aligned}$$

Then $\mathrm{Sp}(2n,\mathbb{C}) = \{g \in \mathrm{Aut}(W_{2n}) | B(gv_1, gv_2) = B(v_1, v_2), \forall v_1, v_2 \in W_{2n}\}$.

There is a natural embeddings of $\mathrm{Sp}(2n,\mathbb{C})$ inside $\mathrm{Sp}(2n+2,\mathbb{C})$ as

$$g \in \mathrm{Sp}(2n,\mathbb{C}) \longrightarrow \left(\begin{array}{c|c} g & O_{n\times 2} \\ \hline O_{2\times n} & I_{2\times 2} \end{array}\right) \in \mathrm{Sp}(2n+2,\mathbb{C}).$$

Next we define $\mathrm{SO}(n,\mathbb{C})$. For this $V_n = \{e_1, e_2, \ldots, e_n\}$, let B be the symmetric bilinear form on V_n by

$$B(e_i, e_j) = \delta_{ij} \quad \forall 1 \leq i, j \leq n.$$

Define

$$\mathrm{SO}(n,\mathbb{C}) = \mathrm{SO}(V_n) = \{g \in \mathrm{Aut}(V_n) | B(gv_1, gv_2) = B(v_1, v_2), \det g = 1\}.$$

Again there is an embedding of $\mathrm{SO}(n,\mathbb{C})$ inside $\mathrm{SO}(n+1,\mathbb{C})$ as

$$g \in \mathrm{SO}(n,\mathbb{C}) \longrightarrow \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right) \in \mathrm{SO}(n+1,\mathbb{C}).$$

By iteration there is also an embedding of $\mathrm{SO}(n,\mathbb{C})$ inside $\mathrm{SO}(n+2,\mathbb{C})$ or more generally $\mathrm{SO}(n+d,\mathbb{C})$ for all $d \geq 0$.

All the representations considered in this paper are finite-dimensional algebraic representations of algebraic groups and are over the field $\mathbb{C}$. Recall that a finite-dimensional irreducible (algebraic) representation of $\mathrm{GL}(n,\mathbb{C})$ is parametrized by an n-tuple $(\lambda_1,\ldots,\lambda_n)$ of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

In this paper, we consider only the polynomial representations of $\mathrm{GL}(n,\mathbb{C})$, thus we will always impose the condition that $\lambda_n \geq 0$. Often, we are reduced to the condition $\lambda_n \geq 0$ because of twisting by the determinant character $\det\colon \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^*$, but in any case, for our work, we prefer to assume throughout the paper $\lambda_n \geq 0$.

Similarly, irreducible finite dimensional representations of the classical groups $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, $\mathrm{SO}(2n,\mathbb{C})$ are parametrized by their highest weights $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with λ_i integers which are ≥ 0 $\forall i$ for $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$, but for $\mathrm{SO}(2n,\mathbb{C})$, it may happen that $\lambda_n \leq 0$; for $\mathrm{SO}(2n,\mathbb{C})$, $\lambda_{n-1} \geq |\lambda_n|$. We will often use the well-known fact about classical groups that any finite-dimensional(algebraic) representation of them is completely reducible. If we were to consider non-algebraic representations, then the following example suggested by the referee is not completely reducible. Let

$$\rho : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{C})$$

defined by

$$\rho(g) = \begin{pmatrix} 1 & \log|\det(g)| \\ 0 & 1 \end{pmatrix}.$$

Note that ρ is an indecomposable two-dimensional representation that is not irreducible. However, in this paper, we will have no occasion to use non-algebraic representations.

3. General linear group

Any sequence of integers $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r\}$ defines an irreducible representation $\Psi^r_{\underline{\lambda}}$ of $\mathrm{GL}(r,\mathbb{C})$ for all $r \geq s = l(\underline{\lambda})$ with the highest weight $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r\}$.

Let $\underline{\lambda}$ and $\underline{\mu}$ be two sequence of integers, and $r \geq 1$, an integer such that $r \geq l(\underline{\lambda})$ and $r \geq l(\underline{\mu})$ then it makes sense to talk about the tensor product : $\Psi^r_{\underline{\lambda}} \otimes \Psi^r_{\underline{\mu}}$ of the representation of $\mathrm{GL}(r,\mathbb{C})$ as r varies.

Our first theorem proves that the representations $\Psi^r_{\underline{\lambda}} \otimes \Psi^r_{\underline{\mu}}$ are "independent" of r if $r \geq l(\underline{\lambda}) + l(\underline{\mu})$. Here is a more precise statement.

Theorem 3.1. *Let $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$, $\underline{\mu} = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0\}$ be two sequence of integers with $r \geq$ max $\{l(\underline{\lambda}), l(\underline{\mu})\}$. Write*

$$\Psi^r_{\underline{\lambda}} \otimes \Psi^r_{\underline{\mu}} = \sum_{\underline{\nu}} C^{\underline{\nu}}_{\underline{\lambda}\,\underline{\mu}}(r)\, \Psi^r_{\underline{\nu}}, \qquad C^{\underline{\nu}}_{\underline{\lambda}\,\underline{\mu}}(r) \in \mathbb{Z}_{\geq 0}.$$

Then if $r \geq l(\underline{\lambda})+l(\underline{\mu})$, *we have* $C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(r) = C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(r+1)$ *for all sequence of integers* $\underline{\nu} = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r\}$.

The proof of this theorem will be a consequence of Pieri's Rule.

Lemma 3.2. (Pieri's Rule) *Let* V *be an* r*-dimensional* $\mathbb{C}$*-vector space, and let* $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$ *be a sequence of integers and* $\Psi^r_{\underline{\lambda}}$ *the irreducible highest weight module of* $\mathrm{GL}(r,\mathbb{C})$. *Then*

$$\Psi^r_{\underline{\lambda}} \otimes \mathrm{Sym}^k(V) = \bigoplus_{\underline{\mu}} \Psi^r_{\underline{\mu}},$$

where $\underline{\mu} = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0\}$ *is a sequence of integers. Each irreducible representation* $\Psi_{\underline{\mu}}$ *appears with multiplicity 1, and exactly those* $\underline{\mu}$ *appear which are obtained for* $\underline{\lambda}$ *by adding* k *boxes to the Young diagram of* $\underline{\lambda}$ *such that no two boxes are added to the same column.*

Remark 3.3. For general linear group, $\mathrm{Sym}^k(V)$ is an irreducible representation of $\mathrm{GL}(V)$, and $\mathrm{Sym}^k(V) = \Psi^r_{(k)}$ where $r = \dim(V)$.

Corollary 3.4. *If* $\Psi^r_{\underline{\lambda}} \otimes \mathrm{Sym}^k(V) = \bigoplus_{\underline{\mu}} \Psi^r_{\underline{\mu}}$, *then* $l(\underline{\mu}) \leq l(\underline{\lambda}) + 1$.

This corollary will not be needed for this section, but we will have occasion to use it later.

Lemma 3.5. *Let* $\Psi_{\underline{\lambda}}$, $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$, *be the irreducible representation of* $\mathrm{GL}(V)$ *with highest weight* $\underline{\lambda}$, *then*

$$\Psi_{\underline{\lambda}} \subseteq \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r}(V).$$

Proof. The proof of this lemma will be by induction on r using Pieri's Rule. Thus we assume that the lemma is true if $l(\underline{\lambda}) \leq r-1$ and then we prove it for $\underline{\lambda}$ with $l(\underline{\lambda}) = r$.

Let $\underline{\lambda}^{'} = \{\lambda_1 \geq \cdots \geq \lambda_{r-1}\}$. By induction hypothesis

$$\Psi_{\underline{\lambda}'} \subseteq \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_{r-1}}(V).$$

Therefore $\Psi_{\underline{\lambda}'} \otimes \mathrm{Sym}^{\lambda_r}(V) \subseteq \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r}(V)$.

By Pieri's Rule $\Psi_{\underline{\lambda}} \subseteq \Psi_{\underline{\lambda}'} \otimes \mathrm{Sym}^{\lambda_r}(V)$, hence

$$\Psi_{\underline{\lambda}} \subseteq \mathrm{Sym}^{\lambda_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r}(V).$$ ■

Proof. (of Theorem 3.1) It suffices to prove that in the decomposition,

$$\Psi^r_{\underline{\lambda}} \otimes \Psi^r_{\underline{\mu}} = \sum C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(r)\, \Psi^r_{\underline{\nu}},$$

if $C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(r) \neq 0$, then $l(\underline{\nu}) \leq l(\underline{\lambda}) + l(\underline{\mu})$. By Lemma 3.5,

$$\Psi^r_{\underline{\mu}} \subseteq \mathrm{Sym}^{\mu_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\mu_r}(V).$$

Therefore $\Psi^r_{\underline{\lambda}} \otimes \Psi^r_{\underline{\mu}} \subseteq \Psi^r_{\underline{\lambda}} \otimes \mathrm{Sym}^{\mu_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\mu_r}(V)$.

Therefore by Pieri's Rule, if

$$\Psi^r_{\underline{\nu}} \subseteq \Psi^r_{\underline{\lambda}} \otimes \mathrm{Sym}^{\mu_1}(V) \otimes \cdots \otimes \mathrm{Sym}^{\mu_r}(V),$$

$l(\underline{\nu}) \leq l(\underline{\lambda}) + l(\underline{\mu})$. By Theorem 4.1 proved in the next sections, it follows that

$$C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(r) = C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(r+1)\,. \qquad \blacksquare$$

Remark 3.6. The very last step in the proof of Theorem 3.1 can be done by an inductive argument as we will do for the classical groups. But we have given another proof of the last step which may be of independent interest.

4. Relating tensor products for GL(n) and GL(n+1)

For $\underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ with corresponding irreducible highest weight representation $\Psi_{\underline{\lambda}}$ of $\mathrm{GL}(n,\mathbb{C})$, it character, the Schur function

$$S_{\underline{\lambda}} \in \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n},$$

is the character of the representation $\Psi_{\underline{\lambda}}$ at the diagonal matrix

$$\begin{pmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{pmatrix}.$$

Under the natural homomorphism of algebras,

$$\begin{aligned} p_n : \mathbb{Z}[X_1, \ldots, X_{n+1}]^{\mathfrak{S}_{n+1}} &\longrightarrow \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}, \\ X_i &\longmapsto X_i, \quad i \leq n, \\ X_{n+1} &\longmapsto 0, \end{aligned}$$

it follows from the Weyl character formula or more directly from the corresponding determinantal formula that

$$p_n(S_{\underline{\lambda}}) = \begin{cases} 0, & \text{if } \underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_{n+1} \geq 0), \lambda_{n+1} \neq 0, \\ S_{\underline{\lambda}}, & \text{if } \underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_{n+1} \geq 0), \lambda_{n+1} = 0. \end{cases}$$

In particular, if ω_i, $1 \leq i \leq (n+1)$ are the fundamental representations of $\mathrm{GL}(n+1,\mathbb{C})$ as well as $\mathrm{GL}(n,\mathbb{C})$ (in which case we naturally take $1 \leq i \leq n$),

$$p_n(\omega_i) = \begin{cases} 0, & \text{if } i = n+1, \\ \omega_i, & \text{if } i \leq n. \end{cases}$$

The following form of the next theorem is similar to, but stronger than, [10, Proposition 1.5(4)] which uses the Littlewood-Richardson rule for a proof. We thank Prof. Schwarz for pointing this out. Although this theorem captures "stability" well, it does not subsume our Theorem 3.1, neither of the Theorems 3.1 and Theorem 4.1 is a consequence of the other.

Theorem 4.1. *Let* $G_n = \mathrm{GL}(n,\mathbb{C})$. *Let* $\underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ *and* $\underline{\mu} = (\mu_1 \geq \ldots \geq \mu_n \geq 0)$ *be two sequence of integers with* $l(\underline{\lambda}) \leq n$, $l(\underline{\mu}) \leq n$.
Let $\Psi^n_{\underline{\lambda}}$, $\Psi^n_{\underline{\mu}}$, *and* $\Psi^{n+1}_{\underline{\lambda}}$, $\Psi^{n+1}_{\underline{\mu}}$ *be the corresponding irreducible highest weight representations of* G_n, *and* G_{n+1} *respectively. Assume that*

$$\Psi^n_{\underline{\lambda}} \otimes \Psi^n_{\underline{\mu}} = \sum C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) \Psi^n_{\underline{\nu}}, \tag{2}$$

$$\Psi^{n+1}_{\underline{\lambda}} \otimes \Psi^{n+1}_{\underline{\mu}} = \sum C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n+1) \Psi^{n+1}_{\underline{\nu}} \tag{3}$$

are the decomposition of the tensor products of $\Psi^n_{\underline{\lambda}}$ *and* $\Psi^n_{\underline{\mu}}$, *and* $\Psi^{n+1}_{\underline{\lambda}}$ *and* $\Psi^{n+1}_{\underline{\mu}}$ *as a direct sum of irreducible representation of* G_n, *and* G_{n+1} *respectively. Suppose* $l(\underline{\nu}) \leq n$ *then,*

$$C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) = C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n+1).$$

Proof. As observed before the statements of the theorem, we have a natural homomorphism of algebras $p_n : \mathbb{Z}[X_1, \ldots, X_{n+1}]^{\mathfrak{S}_{n+1}} \longrightarrow \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ such that

1. $p_n(\Psi^{n+1}_\lambda) = \Psi^n_\lambda$, whenever $l(\lambda) \leq n$,
2. $p_n(\Psi^{n+1}_\lambda) = 0$, if $l(\lambda) > n$.

Applying the homomorphism p_n to equation (3), we get equation (2), proving that if $l(\underline{\nu}) \leq n$, then,

$$C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) = C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n+1). \qquad \blacksquare$$

Example 4.2. We give decomposition of $\pi_{\underline{\lambda}} \otimes \pi_{\underline{\mu}}$ for $\underline{\lambda} = (2,1,1,0,0,\ldots)$ and $\underline{\mu} = (1,1,0,0,0,\ldots)$ on appropriate general linear groups. All these calculations were done on *Lie Software*.

Groups	$\Psi_{\underline{\lambda}} \otimes \Psi_{\underline{\mu}}$ as sum of irreducible representations, where $\underline{\lambda} = (2,1,1,0,0,\ldots)$ and $\underline{\mu} = (1,1,0,0,0,\ldots)$
GL(3)	(2,2,2) + (3,2,1)
GL(4)	(2,2,2,0)+ (3,2,1,0) + (2,2,1,1) + (3,1,1,1)
GL(5)	(2,2,2,0,0) + (3,2,1,0,0)+ (2,2,1,1,0) +(3,1,1,1,0) + (2,1,1,1,1)
GL(6)	(2,2,2,0,0,0) + (3,2,1,0,0,0)+ (2,2,1,1,0,0) +(3,1,1,1,0,0) + (2,1,1,1,1,0)

Table 1: Decomposition for $\mathrm{GL}(n)$

Remark 4.3. In Table 1 for $\mathrm{GL}(n)$, the tensor product stabilizes at GL(5) level, with $5 = l(\underline{\lambda}) + l(\underline{\mu})$, i.e., the result for the tensor product for GL(6) is obtained by just adding a 0 at the end of the corresponding result for GL(5) which verifies the Theorem 4.1.

5. Pieri's formula for classical groups and consequences

For V a finite dimensional vector space over $\mathbb{C}$, let $q : V \to \mathbb{C}$ be a non-degenerate quadratic form on V, considered as an element of $\mathrm{Sym}^2(V^*)$, giving rise to the contraction map

$$\mathrm{Sym}^2(V^*) \times \mathrm{Sym}^k(V) \to \mathrm{Sym}^{k-2}(V).$$

Let $\Pi_{(k)}$ be the kernel of the contraction map from $\mathrm{Sym}^k V$ to $\mathrm{Sym}^{k-2} V$. Then $\Pi_{(k)}$ is the irreducible representation of $\mathrm{SO}(V, \mathbb{C})$ with highest weight ke_1 and

$$\mathrm{Sym}^k V = \Pi_{(k)} \oplus \Pi_{(k-2)} \oplus \cdots \oplus \Pi_{(k-2p)},$$

where p is the largest integer $\leq k/2$ (c.f. [2, 19.5]).

For a sequence of integers $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$, let $l(\lambda)$ be the length of $\underline{\lambda}$, the largest integer s such that $\lambda_s \neq 0$. Let $|\underline{\lambda}|$ be the sum of nonzero parts of a weight $\underline{\lambda}$. For any two sequence of integers $\underline{\lambda}$ and $\underline{\xi}$ such that the Young diagram of $\underline{\lambda}$ contains the Young diagram of $\underline{\xi}$, define

$$|\underline{\lambda}/\underline{\xi}| = |\underline{\lambda}| - |\underline{\xi}|.$$

We say that $\underline{\lambda}/\underline{\xi}$ is a horizontal strips if $\lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \cdots \geq \lambda_l \geq \xi_l \geq \cdots$.

We will use Pieri's rule from [9]. Pieri's rule in [9] for $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n)$ has slightly different formulation but we will look only at those irreducible representations whose highest weights have the last coordinate equal to zero for $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ or the last two coordinates are zero for $\mathrm{SO}(2n)$. In this case, Pieri's rule from [9] simplifies to the following theorem.

Theorem 5.1. *Let G_n be any of the classical groups $\mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{SO}(2n+1, \mathbb{C})$, $\mathrm{SO}(2n, \mathbb{C})$. Let $\Pi_{\underline{\lambda}}$ be the irreducible highest weight representation of any of the groups G_n with highest weight $\underline{\lambda} = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$.*

Assume that if $G_n = \mathrm{Sp}(2n, \mathbb{C})$ or $\mathrm{SO}(2n+1, \mathbb{C})$, we have $\lambda_n = 0$, whereas if $G_n = \mathrm{SO}(2n, \mathbb{C})$, we have $\lambda_{n-1} = \lambda_n = 0$; then

$$\Pi_{\underline{\lambda}} \otimes \Pi_{(k)} = \bigoplus_{\underline{\mu}} N^{\underline{\mu}}_{\underline{\lambda},k} \Pi_{\underline{\mu}},$$

with $N^{\underline{\mu}}_{\underline{\lambda},k}$ given by:

$$N^{\underline{\mu}}_{\underline{\lambda},k} = \#\left\{\underline{\xi} : \text{ sequence of integers } \underline{\xi} \text{ satisfying the following two conditions}\right\},$$

1. *$\underline{\lambda}/\underline{\xi}$ and $\underline{\mu}/\underline{\xi}$ are both horizontal strips,*
2. *$|\underline{\lambda}/\underline{\xi}| + |\underline{\mu}/\underline{\xi}| = k$.*

Proof. The symplectic Pieri's rule (Lemma 7.1) is the same as what is asserted above. For odd orthogonal groups the Pieri's rule (Lemma 7.2) allows the condition $|\underline{\lambda}/\underline{\xi}| + |\underline{\mu}/\underline{\xi}| = k$ and $k-1$. But the condition $|\underline{\lambda}/\underline{\xi}| + |\underline{\mu}/\underline{\xi}| = k-1$ does not occur as $l(\underline{\lambda}) \neq n$. Hence the Pieri's rule (Lemma 7.2) simplifies to the formula asserted above. Similarly for the even orthogonal groups, in Lemma 7.3, the condition $\lambda_{n-1} = \lambda_n = 0$ implies $\xi_{n-1} = \xi_n = 0$, hence the Pieri's rule (Lemma 7.3) reduces to the assertion in our theorem. ∎

Corollary 5.2. *Let G_n be any of the classical groups* $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, $\mathrm{SO}(2n,\mathbb{C})$. *Let $\Pi_{\underline{\lambda}}$ be an irreducible highest weight representation of G_n with highest weight $\underline{\lambda}=(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. Assume that if $G_n = \mathrm{Sp}(2n,\mathbb{C})$ or $\mathrm{SO}(2n+1,\mathbb{C})$, $\lambda_n = 0$ and if $G_n = \mathrm{SO}(2n,\mathbb{C})$, $\lambda_{n-1}=\lambda_n=0$. Then,*

$$\Pi_{\underline{\lambda}} \subseteq \Pi_{(\lambda_1)} \otimes \Pi_{(\lambda_2)} \otimes \cdots \otimes \Pi_{(\lambda_n)}.$$

Proof. Assuming the conditions in the theorem, we have $\underline{\lambda}=(\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ where $r=n-1$ for $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$, and $r=n-2$ for $\mathrm{SO}(2n,\mathbb{C})$. Note that $\Pi_{(0)}=\mathbb{C}$, so it is enough to prove that

$$\Pi_{\underline{\lambda}} \subseteq \Pi_{(\lambda_1)} \otimes \Pi_{(\lambda_2)} \otimes \cdots \otimes \Pi_{(\lambda_r)}.$$

We give proof by induction on r, using Pieri's Theorem 5.1. Thus we assume that the Corollary is true if $l(\underline{\lambda}) \leq r-1$, and then prove it for $\underline{\lambda}$ with $l(\underline{\lambda})=r$.
Let $\underline{\lambda}^{'}=\{\lambda_1 \geq \cdots \geq \lambda_{r-1}\}$. By induction hypothesis,

$$\Pi_{\underline{\lambda}'} \subseteq \Pi_{(\lambda_1)} \otimes \Pi_{(\lambda_2)} \otimes \cdots \otimes \Pi_{(\lambda_{r-1})}.$$

Therefore $$\Pi_{\underline{\lambda}'} \otimes \Pi_{(\lambda_r)} \subseteq \Pi_{(\lambda_1)} \otimes \Pi_{(\lambda_2)} \otimes \cdots \otimes \Pi_{(\lambda_r)}.$$

Now $N^{\underline{\lambda}}_{\underline{\lambda}^{'},\lambda_r}$ is a positive integer as $\underline{\lambda}^{'}$ satisfies both the conditions in Pieri's Theorem 5.1, so

$$\Pi_{\underline{\lambda}} \subseteq \Pi_{\underline{\lambda}'} \otimes \Pi_{(\lambda_r)},$$

hence $$\Pi_{\underline{\lambda}} \subseteq \Pi_{(\lambda_1)} \otimes \Pi_{(\lambda_2)} \otimes \cdots \otimes \Pi_{(\lambda_r)}.$$ ■

Corollary 5.3. *Let G_n be any of the classical groups* $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, $\mathrm{SO}(2n,\mathbb{C})$. *Let $\Pi_{\underline{\lambda}}$ be an irreducible highest weight representation of G_n with highest weight $\underline{\lambda}=(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. Assume that if $G_n = \mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$, $\lambda_n = 0$ and if $G_n = \mathrm{SO}(2n,\mathbb{C})$, $\lambda_{n-1}=\lambda_n=0$. If*

$$\Pi_{\underline{\lambda}} \otimes \Pi_{(k)} = \bigoplus_{\underline{\mu}} N^{\underline{\mu}}_{\underline{\lambda},k}(n)\, \Pi_{\underline{\mu}},$$

with $N^{\underline{\mu}}_{\underline{\lambda},k}(n) > 0$, then $l(\underline{\mu}) \leq l(\underline{\lambda})+1$.

Proof. Let $l(\underline{\lambda})=l$. So by assumption $l \leq n-1$ for $\mathrm{Sp}(2n,\mathbb{C})$ and $\mathrm{SO}(2n+1,\mathbb{C})$ and $l \leq n-2$ for $\mathrm{SO}(2n,\mathbb{C})$. By Pieri's Theorem 5.1, $\underline{\lambda}/\underline{\xi}$ and $\underline{\mu}/\underline{\xi}$ are horizontal strips,

$$\lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \cdots \geq \lambda_l \geq \xi_l \geq \lambda_{l+1} \geq \xi_{l+1} \geq \cdots,$$

$$\mu_1 \geq \xi_1 \geq \mu_2 \geq \xi_2 \geq \cdots \geq \mu_{l+1} \geq \xi_{l+1} \geq \mu_{l+2} \geq \cdots.$$

So $\lambda_{l+1}=0$ implies $\xi_{l+1}=0$ which further gives $\mu_{l+2}=0$. Hence $l(\underline{\mu}) \leq l(\underline{\lambda})+1$. ■

For our purposes, these corollaries are enough but these corollaries are true without putting $\lambda_n=0$ or $\lambda_{n-1}=\lambda_n=0$. More precise versions of the corollaries will need more precise versions of Pieri's rule which are slightly different for different classical groups.

Corollary 5.4. *Let G_n be any of the classical groups* $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, $\mathrm{SO}(2n,\mathbb{C})$. *Let $\Pi_{\underline{\mu}}$ be an irreducible highest weight representation with highest weight $\underline{\mu} = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n \geq 0)$. Suppose,*

$$r = \begin{cases} n, & \textit{if } \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(2n+1,\mathbb{C}), \\ n-1, & \textit{if } \mathrm{SO}(2n,\mathbb{C}). \end{cases}$$

Assume $l(\underline{\mu}) = r$ and define $\underline{\mu}^{'} = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{r-1})$, then we have the following isomorphism of G_n-modules,

$$\Pi_{\underline{\mu}'} \otimes \Pi_{\mu_r} \cong \Pi_{\underline{\mu}} \bigoplus_{\substack{\nu_r < \mu_r \\ l(\underline{\nu}) \leq r}} N^{\underline{\nu}}_{\underline{\mu}^{'}, \mu_r} \Pi_{\underline{\nu}}.$$

Proof. We will appeal to Theorem 5.1 to prove this corollary.
Let $\underline{\nu} = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n \geq 0)$. We will prove that $\nu_r = \mu_r$ implies $\underline{\nu} = \underline{\mu}$. So it is enough to prove

$$N^{\underline{\nu}}_{\underline{\mu}^{'}, \mu_r} = \begin{cases} 1, & \text{when } \underline{\nu} = \underline{\mu}, \\ 0, & \text{when } \nu_r > \mu_r, \end{cases}$$

where $N^{\underline{\nu}}_{\underline{\mu}^{'}, \mu_r}$ is the cardinality of the set of $\underline{\xi}$ satisfying the conditions that $\underline{\mu}^{'}/\underline{\xi}$ and $\underline{\nu}/\underline{\xi}$ are both horizontal strips and $|\underline{\mu}^{'}/\underline{\xi}| + |\underline{\nu}/\underline{\xi}| = \mu_r$. Horizontal strip condition of $\underline{\mu}^{'}/\underline{\xi}$ implies $\xi_r = 0$ and we get the equation

$$\sum_{i=1}^{r-1} (\mu_i - \xi_i) + \sum_{i=1}^{r-1} (\nu_i - \xi_i) = \mu_r - \nu_r \tag{4}$$

from the condition $|\underline{\mu}^{'}/\underline{\xi}| + |\underline{\nu}/\underline{\xi}| = \mu_r$.
If $\mu_r = \nu_r$ then RHS of the equation (4) becomes zero, and since $\underline{\mu}^{'}/\underline{\xi}$ and $\underline{\nu}/\underline{\xi}$ are horizontal strip, we have $\mu_i \geq \xi_i$ and $\nu_i \geq \xi_i$, hence $\xi_i = \mu_i$ and $\nu_i = \xi_i$ for $1 \leq i \leq r-1$. So we get $\nu_i = \mu_i$ for $1 \leq i \leq r-1$ and $\underline{\xi} = \underline{\mu}^{'}$. The former implies $\underline{\nu} = \underline{\mu}$ and the later tells

$$N^{\underline{\nu}}_{\underline{\mu}^{'}, \mu_r} = 1 \text{ for } \underline{\nu} = \underline{\mu}.$$

If $\nu_r > \mu_r$, then RHS of the equation (4) becomes negative which is not possible as LHS is always positive as $\underline{\mu}^{'}/\underline{\xi}$ and $\underline{\nu}/\underline{\xi}$ are horizontal strips. So no such $\underline{\xi}$ exists for this case and hence $N^{\underline{\nu}}_{\underline{\mu}^{'}, \mu_r} = 0$. ■

6. Classical groups

For the convenience of the reader, we state Theorem 1.1 from the introduction again.

Theorem 6.1. *Let*

$$\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\} \quad \textit{and} \quad \underline{\mu} = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0\}$$

be two sequence of integers, and $\Pi^n_{\underline{\lambda}}$ and $\Pi^n_{\underline{\mu}}$ the corresponding irreducible representations of $G_n =$ $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, or $\mathrm{SO}(2n,\mathbb{C})$ for any $n \geq$ max $\{l(\underline{\lambda}), l(\underline{\mu})\}$. Write the tensor product of $\Pi^n_{\underline{\lambda}}$ and $\Pi^n_{\underline{\mu}}$ as,

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}} = \sum_{\underline{\nu}} N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)\, \Pi^n_{\underline{\nu}}. \tag{5}$$

Suppose $n_0 = l(\underline{\lambda}) + l(\underline{\mu})$. *Then,*

1. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) = 0 \quad \text{if} \quad l(\underline{\nu}) \geq n_0 + 1$,
2. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n+1)$ *for* $n \geq \begin{cases} n_0, & \text{if } G_n = \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(2n+1,\mathbb{C}); \\ n_0 + 1, & \text{if } G_n = \mathrm{SO}(2n,\mathbb{C}), \end{cases}$
3. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)$ *is independent of the group* $G_n = \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(2n+1,\mathbb{C})$ *if* $n \geq n_0$,
4. $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)$ *is independent of the group* G_n *if* $n \geq n_0 + 1$.

Proof. The proof of this theorem will be by an inductive process which will prove all four conclusions (1), (2), (3) and (4) at the same time. The induction involved will be using an ordering on the sequence of integers $\underline{\mu} = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0\}$ in which we declare that $\underline{\mu} > \underline{\tilde{\mu}} = (\tilde{\mu}_1 \geq \cdots \geq \tilde{\mu}_r \geq 0)$ if

1. either $l(\underline{\mu}) > l(\underline{\tilde{\mu}})$, or
2. If $l(\underline{\mu}) = l(\underline{\tilde{\mu}}) = r$, then $\mu_r > \tilde{\mu}_r$.

With this ordering on the sequence of integers $\underline{\mu} = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0)$, Corollary 5.4 asserts that

$$\Pi^n_{\underline{\mu}'} \otimes \Pi^n_{\mu_r} = \Pi^n_{\underline{\mu}} \bigoplus_{\substack{\nu_r < \mu_r \\ l(\underline{\nu}) \leq r}} N^{\underline{\nu}}_{\underline{\mu}',\mu_r}(n)\ \Pi^n_{\underline{\nu}}.$$

Now we give the inductive argument of the proof of this theorem for $\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}}$ assuming that the theorem is true for $\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\tilde{\mu}}}$ whenever $\underline{\tilde{\mu}} < \underline{\mu}$.

Since we are assuming that our theorem is true for $\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\tilde{\mu}}}$ for $\underline{\tilde{\mu}} < \underline{\mu}$, in particular, it is true for $\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}'}$, and hence also for $\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}'} \otimes \Pi_{\mu_r}$ by Pieri's Theorem 5.1. Hence our theorem is also true for

$$\Pi^n_{\underline{\lambda}} \otimes \left(\Pi^n_{\underline{\mu}} \bigoplus_{\substack{\nu_r < \mu_r \\ l(\underline{\nu}) \leq r}} N^{\underline{\nu}}_{\underline{\mu}',\mu_r}(n) \Pi^n_{\underline{\nu}} \right).$$

Again by inductive hypothesis, the theorem is true for all $\underline{\nu}$ with $\underline{\nu} < \underline{\mu}$ appearing in the above sum. Therefore as a consequence, our theorem is true for the remaining term in the above sum, i.e. $\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}}$. ■

7. Just before stability level

We will follow the notation of Section 5. The theorem 7.4 is a further improvement of the theorem 6.1 in the sense that now one can talk about the multiplicity of the highest weight irreducible representation appearing in the tensor product of all classical groups G_{N-1}, if N is the stability level corresponding to this tensor product. We also discuss about the parity of $|\underline{\nu}|$ corresponding to $\underline{\nu}$ appearing in the tensor product for some ranges.

Let us recall Pieri's rule for the symplectic groups and the odd orthogonal groups respectively.

Lemma 7.1. (Pieri's Rule [9]) *Let* $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$ *be a sequence of integers and* $\Pi^n_{\underline{\lambda}}$ *be the corresponding irreducible highest weight module of* $\mathrm{Sp}(2n, \mathbb{C})$*. Let* $k \geq 0$ *be a positive integer and* $\Pi^n_{(k)}$ *denotes the irreducible highest weight module of* $\mathrm{Sp}(2n, \mathbb{C})$ *with highest weight* $(k, 0, 0, \ldots, 0)$*. If*

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{(k)} = \bigoplus_{\underline{\mu}} N^{\underline{\mu}}_{\underline{\lambda},k}(n)\, \Pi^n_{\underline{\mu}}, \tag{6}$$

then,

$$N^{\underline{\mu}}_{\underline{\lambda},k}(n) = \#\left\{\underline{\xi} : \underline{\lambda}/\underline{\xi} \text{ and } \underline{\mu}/\underline{\xi} \text{ are both horizontal strips and } |\underline{\lambda}/\underline{\xi}| + |\underline{\mu}/\underline{\xi}| = k\right\}.$$

Lemma 7.2. (Pieri's Rule [9]) *Let* $\underline{\lambda}$ *and* $\underline{\mu}$ *be sequence of integers of length* $\leq n$*,* k *a nonnegative integer and* $\Pi^n_{\underline{\lambda}}$ *be the corresponding irreducible highest weight module of* $\mathrm{SO}(2n+1, \mathbb{C})$*. If*

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{(k)} = \bigoplus_{\underline{\mu}} N^{\underline{\mu}}_{\underline{\lambda},k}(n)\, \Pi^n_{\underline{\mu}}, \tag{7}$$

then,

$$N^{\underline{\mu}}_{\underline{\lambda},k}(n) = \#\left\{\underline{\xi} : \text{sequence of integers } \underline{\xi} \text{ satisfying the following three conditions}\right\}.$$

1. $\underline{\lambda}/\underline{\xi}$ *and* $\underline{\mu}/\underline{\xi}$ *are both horizontal strips,*
2. $|\underline{\lambda}/\underline{\xi}| + |\underline{\mu}/\underline{\xi}| = k$ *or* $k-1$*,*
3. *If* $|\underline{\lambda}/\underline{\xi}| + |\underline{\mu}/\underline{\xi}| = k-1$*, then* $l(\underline{\xi}) = l(\underline{\lambda}) = n$*.*

These two Lemmas are needed for the proof of the 2nd part of the Theorem 7.4.

Lemma 7.3. (Pieri's Rule [9]) *Let* $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}$ *be a sequence of integers,* k *a positive integer and* $\Pi^n_{\underline{\lambda}}$ *be the corresponding irreducible highest weight module of* $\mathrm{SO}(2n, \mathbb{C})$*. Let* $\underline{\mu} = \{\mu_1 \geq \cdots \geq \mu_{n-1} \geq |\mu_n| \geq 0\}$ *be a sequence of integers. If*

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{(k)} = \bigoplus_{\underline{\mu}} N^{\underline{\mu}}_{\underline{\lambda},k}(n)\, \Pi^n_{\underline{\mu}}, \tag{8}$$

then,

$$N^{\underline{\mu}}_{\underline{\lambda},k}(n) = \#\left\{\underline{\xi} : \text{sequence of integers } \underline{\xi} \text{ satisfying the following four conditions}\right\}.$$

1. $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq |\xi_n|$,
2. $\lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \lambda_n \geq \xi_n$ *and*
   $\mu_1 \geq \xi_1 \geq \mu_2 \geq \xi_2 \geq \cdots \geq \xi_{n-1} \geq \mu_n \geq \xi_n$,
3. $\sum_{i=1}^n (\lambda_i - \xi_i) + \sum_{i=1}^n (\mu_i - \xi_i) = k$,
4. $\xi_n \in \{\lambda_n, \mu_n\}$.

Lemma 7.3 is needed to prove 3rd part of the Theorem 7.4.

Theorem 7.4. *Let G_n be any one of the groups* $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$, *or* $\mathrm{SO}(2n,\mathbb{C})$. *Let* $\underline{\lambda} = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0\}$ *and* $\underline{\mu} = \{\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0\}$ *be two sequence of integers, and* $\Pi^n_{\underline{\lambda}}$ *and* $\Pi^n_{\underline{\mu}}$ *the corresponding irreducible representations of* G_n *for* $n \geq$ *max* $\{l(\underline{\lambda}), l(\underline{\mu})\}$. *Write the tensor product of* $\Pi^n_{\underline{\lambda}}$ *and* $\Pi^n_{\underline{\mu}}$ *as,*

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}} = \sum_{\underline{\nu}} N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n)\, \Pi^n_{\underline{\nu}}. \tag{9}$$

Suppose $n_0 = l(\underline{\lambda}) + l(\underline{\mu})$. *Then,*

1. *Let* $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) \neq 0$. *Then,* $|\underline{\nu}| \equiv |\underline{\lambda}| + |\underline{\mu}| \mod 2$,

$$\text{for } n \geq \begin{cases} \max\{l(\underline{\lambda}), l(\underline{\mu})\}, & \text{if } G_n = \mathrm{Sp}(2n,\mathbb{C}), \mathrm{SO}(2n,\mathbb{C}); \\ n_0, & \text{if } G_n = \mathrm{SO}(2n+1,\mathbb{C}). \end{cases}$$

2. *Let* $\underline{\nu} = \{\nu_1 \geq \ldots \geq \nu_{n_0-1} \geq 0\}$. *Define* $d_{\underline{\nu}} := |\underline{\lambda}| + |\underline{\mu}| - |\underline{\nu}|$ *and define* $\underline{\nu}^{(1)} := (\nu_1, \ldots, \nu_{n_0-1}, 1)$ *if* $\nu_{n_0-1} \geq 1$. *Then,*

$$N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n_0 - 1) = \begin{cases} N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n_0), & \text{if } G_n = \mathrm{Sp}(2n,\mathbb{C}); \\ \left.\begin{matrix} N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n_0) & \text{if } d_{\underline{\nu}} \equiv 0 \mod 2, \\ N^{\underline{\nu}^{(1)}}_{\underline{\lambda}\underline{\mu}}(n_0) & \text{if } d_{\underline{\nu}} \equiv 1 \mod 2, \end{matrix}\right\} & \text{if } G_n = \mathrm{SO}(2n+1,\mathbb{C}). \end{cases}$$

3. *Let* $G_n = \mathrm{SO}(2n,\mathbb{C})$. *Let* $\underline{\nu} = \{\nu_1 \geq \ldots \geq \nu_{n_0} \geq 0\}$ *and define* $\underline{\nu}^{(0)} := (\nu_1, \ldots, \nu_{n_0-1}, -\nu_{n_0})$ *if* $\nu_{n_0} \geq 1$. *Then,*
   (a) $N^{\underline{\nu}^{(0)}}_{\underline{\lambda}\underline{\mu}}(n_0) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n_0)$,
   (b) $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n_0) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n_0 + 1)$.

Proof of (1). The parity relationship $|\underline{\nu}| \equiv |\underline{\lambda}| + |\underline{\mu}| \mod 2$ for $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n)$ follows by looking at the action of the central character of these two groups, which is the action of $-\mathrm{Id} \in \mathrm{SO}(2n)$ or $\mathrm{Sp}(2n)$ on any irreducible representation. By Schur's lemma, $-\mathrm{Id}$ acts by ± 1 on every irreducible representation of $\mathrm{SO}(2n,\mathbb{C})$ or $\mathrm{Sp}(2n,\mathbb{C})$, so also on their tensor product. The conclusion about the parity of $|\underline{\nu}|$ with $N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(n) \neq 0$ follows easily. Part (1) of the theorem for the group $\mathrm{SO}(2n,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$ is now finished.

For the groups $\mathrm{SO}(2n+1)$ we do not have a central character, but surprisingly we have a parity relation for the values of $n \geq n_0$. We prove the parity result for $\mathrm{SO}(2n+1)$ by theorem 5.1 (Pieri's rule) and corollary (5.2).

Let $l(\underline{\mu}) = 1$, therefore let $\underline{\mu} = (k, 0, 0, \ldots)$, for k a positive integer. Then $n_0 = l(\underline{\lambda}) + l(\underline{\mu}) = l(\underline{\lambda}) + 1$. If

$$\Pi^n_{\underline{\nu}} \subseteq \Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}} \quad \text{and} \quad \Pi^n_{\underline{\nu}'} \subseteq \Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}},$$

then by Theorem 5.1, any two highest weights $\underline{\nu}$ and $\underline{\nu}'$ satisfy the equations $|\underline{\lambda}| + |\underline{\nu}| - 2|\underline{\xi}| = k$ and $|\underline{\lambda}| + |\underline{\nu}'| - 2|\underline{\xi}'| = k$ for some $\underline{\xi}$, $\underline{\xi}'$ respectively. So

$$|\underline{\nu}| - |\underline{\nu}'| = 2(|\underline{\xi}| - |\underline{\xi}'|). \tag{10}$$

This means that any $|\underline{\nu}|$ and $|\underline{\nu}^{'}|$ have the same parity. Now

$$\Pi^n_{\underline{\lambda}+\underline{\mu}} \subseteq \Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}}$$

always holds (see [4, Ex. 21.7]), where $\underline{\lambda}+\underline{\mu} = (\lambda_1+\mu_1, \lambda_2+\mu_2, \ldots, \lambda_n+\mu_n)$. So taking $\underline{\nu}^{'} = \underline{\lambda}+\underline{\mu}$ in equation (10), implies $|\underline{\nu}| \equiv |\lambda| + |\underline{\mu}| \mod 2$. So part (1) for $G_n = \mathrm{SO}(2n+1, \mathbb{C})$ is true when $l(\underline{\mu}) = 1$.

For a general $\underline{\mu}$, with $l(\underline{\mu}) \geq 1$ by Corollary (5.2),

$$\Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}} \subseteq \Pi^n_{\underline{\lambda}} \otimes (\Pi^n_{\mu_1} \otimes \cdots \otimes \Pi^n_{\mu_n}). \tag{11}$$

Now, if $\Pi^n_{\underline{\nu}}$ and $\Pi^n_{\underline{\nu}'}$ are subsets of $\Pi^n_{\underline{\lambda}} \otimes (\Pi^n_{\mu_1} \otimes \cdots \otimes \Pi^n_{\mu_n})$ in equation (11), by repeated application of Pieri's rule (Theorem 5.1) $|\underline{\nu}|$ and $|\underline{\nu}^{'}|$ have the same parity. Hence, if $\Pi^n_{\underline{\nu}} \subseteq \Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}}$, $|\underline{\nu}|$ has a fixed parity independent of $\underline{\nu}$. Since $\Pi^n_{\underline{\lambda}+\underline{\mu}} \subseteq \Pi^n_{\underline{\lambda}} \otimes \Pi^n_{\underline{\mu}}$, so

$$|\underline{\nu}| \equiv |\lambda| + |\underline{\mu}| \mod 2 \quad \text{for} \quad n \geq n_0.$$

Part (1) of the theorem for the group $\mathrm{SO}(2n+1, \mathbb{C})$ is now finished.

Proof of (2). For $r \leq n$, let $\underline{\lambda} = \{\lambda_1 \geq \cdots \geq \lambda_r \geq 0\}$ be a sequence of integers. Define

$$C(\underline{\lambda}, n) = \{\underline{\xi} = (\xi_1, \ldots, \xi_n) : \xi_i \geq \xi_{i+1} \geq 0,\ \lambda_1 \geq \xi_1 \geq \cdots \geq \lambda_n \geq \xi_n \geq 0\}. \tag{12}$$

Note that $C(\underline{\lambda}, n) = C(\underline{\lambda}, n+1)$ due to the condition $\lambda_{n+1} = \xi_{n+1} = 0$, as it holds that $l(\underline{\lambda}) \leq r \leq n$. Therefore, $\underline{\xi} \in C(\underline{\lambda}, n) \iff \underline{\xi} \in C(\underline{\lambda}, n+1)$. Now we prove part (2) for the symplectic groups.

Symplectic groups: We prove

$$N^{\underline{\nu}}_{\underline{\lambda}\,\underline{\mu}}(n_0 - 1) = N^{\underline{\nu}}_{\underline{\lambda}\,\underline{\mu}}(n_0) \quad \text{for} \quad l(\underline{\nu}) \leq n_0 - 1, \tag{13}$$

by induction similar to that used in Theorem 6.1. In consequence it is sufficient to prove equation (13) for $\underline{\mu} = (k, 0, \ldots)$, where $k \geq 0$ is a positive integer. Let $\underline{\mu} = (k, 0, \ldots)$, then $n_0 = l(\underline{\lambda}) + l(\underline{\mu}) = m+1$, so we prove

$$N^{\underline{\nu}}_{\underline{\lambda},\,k}(m) = N^{\underline{\nu}}_{\underline{\lambda},\,k}(m+1) \quad \text{for} \quad l(\underline{\nu}) \leq m. \tag{14}$$

By Pieri's rule (Lemma 7.1),

$$N^{\underline{\nu}}_{\underline{\lambda},\,k}(m) = \#\big\{\underline{\xi} = (\xi_1 \geq \cdots \geq \xi_m \geq 0) \text{ satisfies condition (15) below}\big\},$$

$$\begin{cases} \text{(i) } \underline{\xi} \in C(\underline{\lambda}, m), \\ \text{(ii) } \underline{\xi} \in C(\underline{\nu}, m), \\ \text{(iii) } \displaystyle\sum_{i=1}^{m} (\lambda_i + \nu_i - 2\xi_i) = k. \end{cases} \tag{15}$$

Note this,

$$\underline{\xi} \in C(\underline{\lambda}, m) \iff \underline{\xi} \in C(\underline{\lambda}, m+1) \quad \text{as } l(\underline{\lambda}) = m, \tag{16}$$

$$\underline{\xi} \in C(\underline{\nu}, m) \iff \underline{\xi} \in C(\underline{\nu}, m+1) \quad \text{as } l(\underline{\nu}) \leq m. \tag{17}$$

Since $m = l(\underline{\lambda})$, $\underline{\xi} \in C(\underline{\lambda}, m)$ implies $\xi_{m+1} = 0$. So we have

$$\sum_{i=1}^{m}(\lambda_i + \nu_i - 2\xi_i) = k \quad \Longleftrightarrow \quad \sum_{i=1}^{m+1}(\lambda_i + \nu_i - 2\xi_i) = k \tag{18}$$

since $\lambda_{m+1} = \nu_{m+1} = \xi_{m+1} = 0$. Combining the equations (16), (17), (18) we get

$$N^{\underline{\nu}}_{\underline{\lambda}, k}(m) = N^{\underline{\nu}}_{\underline{\lambda}, k}(m+1).$$

So the equation (14) is verified. So part(2) of our theorem for the symplectic group follows.

Odd orthogonal group: Similar induction arguments are applicable for the odd orthogonal groups, as we did in equations (13) and (14) for the symplectic groups. So it is sufficient to prove

$$N^{\underline{\nu}}_{\underline{\lambda}, k}(m) = \begin{cases} N^{\underline{\nu}}_{\underline{\lambda}, k}\,(m+1) & \text{if } d_{\underline{\nu}} \equiv 0 \mod 2, \\ N^{\underline{\nu}^{(1)}}_{\underline{\lambda}, k}\,(m+1) & \text{if } d_{\underline{\nu}} \equiv 1 \mod 2, \end{cases} \tag{19}$$

for $l(\underline{\nu}) \leq m$, where $m = l(\underline{\lambda})$. For this $m, k, \underline{\nu}$, we define the set $N_m(\underline{\nu})$ by

$$N_m(\underline{\nu}) = \{\underline{\xi} = (\xi_1 \geq \cdots \geq \xi_m \geq 0) \quad | \quad \underline{\xi} \text{ satisfies condition (20) below}\},$$

$$\begin{cases} \text{(i)} & \underline{\xi} \in C(\underline{\lambda}, m), \\ \text{(ii)} & \underline{\xi} \in C(\underline{\nu}, m), \\ \text{(iii)} & \displaystyle\sum_{i=1}^{m}(\lambda_i + \nu_i - 2\xi_i) = k \text{ or } k-1, \\ \text{(iv)} & \text{if } \displaystyle\sum_{i=1}^{m}(\lambda_i + \nu_i - 2\xi_i) = k-1, \text{ then } l(\underline{\lambda}) = l(\underline{\xi}) = m. \end{cases} \tag{20}$$

By Pieri (Lemma 7.2), $$N^{\underline{\nu}}_{\underline{\lambda}, k}(m) = \# N_m(\underline{\nu}). \tag{21}$$

Since $l(\underline{\lambda}) = m$, both the case k and $k-1$ occur in (20), and we can write the set $N_m(\underline{\nu})$ as a disjoint union of two sets

$$N_m(\underline{\nu}) = N'_m(\underline{\nu}) \sqcup N''_m(\underline{\nu}), \tag{22}$$

where $N'_m(\underline{\nu})$ and $N''_m(\underline{\nu})$ are defined by

$$N'_m(\underline{\nu}) = \{\underline{\xi} = (\xi_1 \geq \cdots \geq \xi_m \geq 0) \quad | \quad \underline{\xi} \text{ satisfies condition (23) below}\},$$

$$\begin{cases} \text{(i)} & \underline{\xi} \in C(\underline{\lambda}, m), \\ \text{(ii)} & \underline{\xi} \in C(\underline{\nu}, m), \\ \text{(iii)} & \displaystyle\sum_{i=1}^{m}(\lambda_i + \nu_i - 2\xi_i) = k. \end{cases} \tag{23}$$

$$N''_m(\underline{\nu}) = \{\underline{\xi} = (\xi_1 \geq \cdots \geq \xi_m \geq 1) \quad | \quad \underline{\xi} \text{ satisfies condition (24) below}\},$$

$$\begin{cases} \text{(i)} & \underline{\xi} \in C(\underline{\lambda}, m), \\ \text{(ii)} & \underline{\xi} \in C(\underline{\nu}, m), \\ \text{(iii)} & \displaystyle\sum_{i=1}^{m} (\lambda_i + \nu_i - 2\xi_i) = k - 1. \end{cases} \tag{24}$$

By Pieri's Theorem 5.1, we have $N'_m(\underline{\nu}) = N_{m+1}(\underline{\nu})$,

by the same argument we used to prove the equation (14) for the symplectic group. Again by Pieri's theorem 5.1,

$$N''_m(\underline{\nu}) = N_{m+1}(\underline{\nu}^{(1)}),$$

the equality is due to both (25) and (26) occurring.

$$\sum_{i=1}^{m} (\lambda_i + \nu_i - 2\xi_i) = k - 1 \quad \Longleftrightarrow \quad \sum_{i=1}^{m+1} (\lambda_i + \nu_i^{(1)} - 2\xi_i) = k, \tag{25}$$

$$\{\underline{\xi} \in C(\underline{\nu}, m), \xi_m \geq 1\} \quad \Longleftrightarrow \quad \{\underline{\xi} \in C(\underline{\nu}^{(1)}, m+1), \xi_{m+1} = 0\}. \tag{26}$$

The equivalence (25) holds because $\sum_{i=1}^{m+1} \nu_i^{(1)} = \sum_{i=1}^{m} \nu_i + 1$ as $\underline{\nu}^{(1)} = (\nu_1, \ldots, \nu_m, 1)$ and $\lambda_{m+1} = \xi_{m+1} = 0$ as $m = l(\underline{\lambda})$. (26) trivially holds. By Pieri's Theorem 5.1,

$$N_{\underline{\lambda}, k}^{\underline{\nu}}(m+1) = \# N_{m+1}(\underline{\nu}) \quad \text{and} \quad N_{\underline{\lambda}, k}^{\underline{\nu}^{(1)}}(m+1) = \# N_{m+1}(\underline{\nu}^{(1)}). \tag{27}$$

Combining the equations (21), (22) and (27) we have

$$N_{\underline{\lambda}, k}^{\underline{\nu}}(m) = N_{\underline{\lambda}, k}^{\underline{\nu}}(m+1) + N_{\underline{\lambda}, k}^{\underline{\nu}^{(1)}}(m+1). \tag{28}$$

Using part (1) of the theorem 7.4,

$$\begin{aligned} N_{\underline{\lambda}, k}^{\underline{\nu}}(m+1) &\neq 0 \qquad \text{if } |\underline{\nu}| \equiv |\underline{\lambda}| + |\underline{\mu}| \mod 2, \\ N_{\underline{\lambda}, k}^{\underline{\nu}^{(1)}}(m+1) &\neq 0 \qquad \text{if } |\underline{\nu}^{(1)}| \equiv |\underline{\lambda}| + |\underline{\mu}| \mod 2; \end{aligned}$$

Hence the product

$$N_{\underline{\lambda}, k}^{\underline{\nu}}(m+1) \cdot N_{\underline{\lambda}, k}^{\underline{\nu}^{(1)}}(m+1) = 0,$$

because $|\underline{\nu}|$ and $|\underline{\nu}^{(1)}|$ do not have the same parity. Therefore, both terms in the equation (28) cannot simultaneously be non-zero, which further implies the equation (19). So our theorem holds for odd orthogonal groups.

Proof of (3) Since $l(\underline{\lambda}), l(\underline{\mu}) < n_0$, it is obvious that the outer automorphism of $\mathrm{SO}(2n_0)$ given by conjugating by an element of $\mathrm{O}(2\mathrm{n}_0)$ not in $\mathrm{SO}(2n_0)$ preserves $\Pi_{\underline{\lambda}}^{n_0}$, $\Pi_{\underline{\mu}}^{n_0}$ hence also the tensor product $\Pi_{\underline{\lambda}}^{n_0} \otimes \Pi_{\underline{\mu}}^{n_0}$. Therefore,

$$N_{\underline{\lambda}\underline{\mu}}^{\underline{\nu}^{(0)}}(n_0) = N_{\underline{\lambda}\underline{\mu}}^{\underline{\nu}}(n_0),$$

and part (a) follows.

In part (b), we use the same induction that used for the other two groups in part (2) of this theorem. So it is sufficient to prove

$$N^{\underline{\nu}}_{\underline{\lambda},k}(m) = N^{\underline{\nu}}_{\underline{\lambda},k}(m+1) \quad \text{for} \quad l(\underline{\nu}) \leq m.$$

Define the two sets $N_m(\underline{\nu})$ and $N_{m+1}(\underline{\nu})$ as follows

$$N_m(\underline{\nu}) = \{\underline{\xi} = (\xi_1 \geq \cdots \geq \xi_{m-1} \geq |\xi_m|) \quad | \quad \underline{\xi} \text{ satisfies condition (29) below}\},$$

$$\begin{cases} \text{(i)} & \lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \cdots \geq \lambda_{m-1} \geq \xi_{m-1} \geq 0 \geq \xi_m, \\ \text{(ii)} & \nu_1 \geq \xi_1 \geq \nu_2 \geq \xi_2 \geq \cdots \geq \nu_{m-1} \geq \xi_{m-1} \geq \nu_m \geq \xi_m, \\ \text{(iii)} & \displaystyle\sum_{i=1}^{m} (\lambda_i + \nu_i - 2\xi_i) = k, \\ \text{(iv)} & \xi_m \in \{0, \nu_m\}. \end{cases} \tag{29}$$

$$N_{m+1}(\underline{\nu}) = \{\underline{\xi} = (\xi_1 \geq \cdots \geq \xi_m \geq |\xi_{m+1}|) \quad | \quad \underline{\xi} \text{ satisfies condition (30) below}\},$$

$$\begin{cases} \text{(i)} & \lambda_1 \geq \xi_1 \geq \lambda_2 \geq \xi_2 \geq \cdots \geq \lambda_{m-1} \geq \xi_{m-1} \geq 0 \geq \xi_m \geq 0 \geq \xi_{m+1}, \\ \text{(ii)} & \nu_1 \geq \xi_1 \geq \nu_2 \geq \xi_2 \geq \cdots \geq \nu_{m-1} \geq \xi_{m-1} \geq \nu_m \geq \xi_m \geq 0 \geq \xi_{m+1}, \\ \text{(iii)} & \displaystyle\sum_{i=1}^{m+1} (\lambda_i + \nu_i - 2\xi_i) = k, \\ \text{(iv)} & \xi_{m+1} = 0. \end{cases} \tag{30}$$

By Lemma 7.3, we have $N^{\underline{\nu}}_{\underline{\lambda},k}(m) = \#N_m(\underline{\nu})$ and $N^{\underline{\nu}}_{\underline{\lambda},k}(m+1) = \#N_{m+1}(\underline{\nu})$.
So now it is enough to show $N_m(\underline{\nu}) = N_{m+1}(\underline{\nu})$.
If $\xi_m = 0$, (29) and (30) are identical. If $\xi_m < 0$, then $\nu_m = \xi_m < 0$ by (iv) of (29), which contradicts our assumption $\nu_m \geq 0$. So $\xi_m = 0$. So

$$N_m(\underline{\nu}) = N_{m+1}(\underline{\nu}).$$

So our theorem follows. ■

8. Examples

In this section, we give the decomposition of $\Pi_{\underline{\lambda}} \otimes \Pi_{\underline{\mu}}$ for $\underline{\lambda} = (2,1,1,0,0,\ldots)$, $\underline{\mu} = (1,1,0,0,0,\ldots)$ on appropriate classical groups. All these calculations were done on *Lie Software* to verify various parts of the Theorem 7.4.

Remark 8.1. The tensor product decompositions for Sp(10) and SO(11) are the same, while the same holds true for Sp(12), SO(13), and SO(12), in Table 2. This verifies part (3) and part (4) of Theorem 1.1, respectively.

Group	$\Pi_{\underline{\lambda}} \otimes \Pi_{\underline{\mu}}$ as sum of irreducible representations, where $\underline{\lambda} = (2,1,1,0,0,\dots)$ and $\underline{\mu} = (1,1,0,0,0,\dots)$
Sp(6)	(1,1,0)+ (2,0,0)+ (2,1,1)+ (2,2,0) +(2,2,2) + (3,1,0) + (3,2,1)
SO(7)	(1,1,0)+ (2,0,0)+ 2(2,1,1)+ (2,2,0) +(2,2,2) + (3,1,0) + (3,2,1)+ (1,1,1)+ (2,1,0) + (2,2,1)+ (3,1,1)
SO(6)	(1,1,0)+ (2,0,0)+ 2(2,1,1)+ (2,2,0) +(2,2,2) + (3,1,0) + (3,2,1)
Sp(8)	(1,1,0,0)+ (2,0,0,0)+ 2(2,1,1,0)+ (2,2,0,0) +(2,2,2,0) + (3,1,0,0) + (3,2,1,0)+ (1,1,1,1)+ (2,2,1,1)+ (3,1,1,1)
SO(9)	Same as Sp(8) plus one more representation (2,1,1,1)
SO(8)	$(1,1,0,0)+(2,0,0,0)+3(2,1,1,0)+(2,2,0,0)+(2,2,2,0)+(3,1,0,0)+(3,2,1,0)+(1,1,1,\pm1)+(2,2,1,\pm1)+(3,1,1,\pm1)$
Sp(10)	(1,1,0,0,0)+ (2,0,0,0,0)+ 2(2,1,1,0,0)+ (2,2,0,0,0) +(2,2,2,0,0) + (3,1,0,0,0) + (3,2,1,0,0)+ (1,1,1,1,0)+ (2,2,1,1,0)+ (3,1,1,1,0)+ (2,1,1,1,1)
SO(11)	SAME AS Sp(10)
SO(10)	Same as Sp(10) and SO(11) except that the last representation (2,1,1,1,1) replaced by the two representations $(2,1,1,1,\pm1)$
Sp(12)	Same as for Sp(10) after putting a zero at the end
SO(13)	SAME AS Sp(12)
SO(12)	SAME AS Sp(12)

Table 2: Calculation of $\Pi_{\underline{\lambda}} \otimes \Pi_{\underline{\mu}}$for different groups, where $\underline{\lambda} = (2,1,1,0,0,\dots)$ and $\underline{\mu} = (1,1,0,0,0,\dots)$.

Remark 8.2. From the Table 2 we make the following observations about the parity relationship.

1. Here $|\underline{\lambda}| + |\underline{\mu}| = 4 + 2 = 6$. $|\underline{\nu}|$ always remains even for all $\underline{\nu}$ appearing in the tensor products for Sp(6), Sp(8), Sp(10) and Sp(12). Similarly, it holds for SO(6), SO(8), SO(10) and SO(12).
2. $|\underline{\nu}|$ is even for all $\underline{\nu}$ appearing in the tensor product for SO(11), but not for SO(9). For example, the weights $(2,2,0,0)$ and $(2,1,1,1)$ appearing in the tensor product for SO(9) have different parity.

These two together verify part (7.4) of Theorem 7.4.

Remark 8.3. The following observations verify Part (7.4) and (7.4) of the Theorem 7.4 for the classical groups when we take $\underline{\lambda} = (2,1,1,0,0,\dots)$ and $\underline{\mu} = (1,1,0,0,0,\dots)$. Note that, for this $\underline{\lambda}$ and $\underline{\mu}$, $n_0 = l(\underline{\lambda}) + l(\underline{\mu}) = 3 + 2 = 5$.

1. Comparing the tensor products of Sp(8) and Sp(10), one can check
$$N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(4) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(5)$$
holds for all $\underline{\nu}$ with $l(\underline{\nu}) \leq 4$, in Table 2.

2. All $|\underline{\nu}|$ are even except for the weight $(2,1,1,1)$ in the tensor product decomposition of SO(9), in Table 2. Comparing the tensor product decomposition of SO(9) and SO(11),
$$N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(4) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(5) \text{ for all } \underline{\nu}, \text{ except } \underline{\nu} = (2,1,1,1),$$
where $l(\underline{\nu}) \leq 4$. For the weight $\underline{\nu} = (2,1,1,1)$, we have
$$N^{(2,1,1,1)}_{\underline{\lambda}\underline{\mu}}(4) = N^{(2,1,1,1,1)}_{\underline{\lambda}\underline{\mu}}(5) = 1.$$

3. In Table 2, comparing the tensor product for SO(10) and SO(12), we get for $\underline{\nu} = (2,1,1,1,-1)$,
$$N^{(2,1,1,1,-1)}_{\underline{\lambda}\underline{\mu}}(5) = N^{(2,1,1,1,1)}_{\underline{\lambda}\underline{\mu}}(5) = 1,$$
and for $\underline{\nu} \neq (2,1,1,1,-1)$,
$$N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(5) = N^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}(6).$$

Acknowledgements. This work is part of the author's Ph.D. thesis at IIT Bombay. The author thanks Prof. Dipendra Prasad for suggesting this question, for numerous helpful discussions, and for spending a lot of time reviewing the paper and correcting mistakes. After the first version of this paper was uploaded to arXiv, I learned in correspondence with Prof. Soichi Okada that parts of Theorem 1.1 are due to [5]. We thank Prof. Vinay Wagh for his help in running the Lie software on the local desktop.

References

[1] M. Colarusso, W. Q. Erickson, J. F. Willenbring: *Contingency tables and the generalized Littlewood-Richardson coefficients*, Proc. Amer. Math. Soc. 150/1 (2022) 79–94.

[2] W. Fulton, J. Harris: *Representation Theory*, Graduate Texts in Mathematics 129, Springer, New York (1991).

[3] R. Howe, E.-C. Tan, J. F. Willenbring: *Stable branching rules for classical symmetric pairs*, Trans. Amer. Math. Soc. 357/4 (2005) 1601–1626.

[4] J. E. Humphreys: *Introduction to Lie Algebras and Representation Theory*, 2n revised printing, Graduate Texts in Mathematics 9, Springer, New York (1978).

[5] K. Koike, I. Terada: *Young-diagrammatic methods for the representation theory of the classical groups of type* B_n, C_n, D_n, J. Algebra 107/2 (1987) 466–511.

[6] A. Knutson, T. Tao: *The honeycomb model of* $\mathrm{GL}_n(\mathbf{C})$ *tensor products. I: Proof of the saturation conjecture*, J. Amer. Math. Soc. 12/4 (1999) 1055–1090.

[7] J.-H. Kwon: *Combinatorial extension of stable branching rules for classical groups*, Trans. Amer. Math. Soc. 370/9 (2018) 6125–6152.

[8] I. G. Macdonald: *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York (2015).

[9] S. Okada: *Pieri rules for classical groups and equinumeration between generalized oscillating tableaux and semistandard tableaux*, Electron. J. Combin. 23/4 (2016), art. no. 4.43, 27 pp.

[10] G. W. Schwarz: *On classical invariant theory and binary cubics*, Ann. Inst. Fourier (Grenoble) 37/3 (1987) 191–216.

Dibyendu Biswas, Indian Institute of Technology Bombay, Mumbai, India; dibubis@gmail.com.

Received November 28, 2023
and in final form January 20, 2024