Journal of Lie Theory
Volume **34** (2024) 531–540


# A Remark on Ado's Theorem for Principal Ideal Domains

**Andoni Zozaya**

Communicated by G. I. Olshanski

**Abstract.** Ado's Theorem had been extended to principal ideal domains independently by Churkin and Weigel. They proved that if $R$ is a principal ideal domain of characteristic zero and $\mathfrak{L}$ is a Lie algebra over $R$ which is also a free $R$-module of finite rank, then $\mathfrak{L}$ admits a finite faithful Lie algebra representation over $R$. We present a quantitative proof of this result, providing explicit bounds on the degree of the Lie algebra representations in terms of the rank as a free module. To achieve it, we generalise an established embedding theorem for complex Lie algebras: any Lie algebra as above embeds in a larger Lie algebra that decomposes as the direct sum of its nilpotent radical and another subalgebra.

*Mathematics Subject Classification:* 17B10, 17B30, 17B35.

*Key Words:* Ado's Theorem, Lie algebras, degree of representations.

## 1. Introduction

Ado's Theorem [1] states that every finite dimensional Lie algebra $\mathfrak{L}$ over a field $K$ of characteristic zero admits a finite faithful Lie algebra representation, that is, there exists a finite dimensional $K$-vector space $V$ and a $K$-Lie algebra monomorphism $\Phi\colon \mathfrak{L} \hookrightarrow \mathrm{End}_K(V)$. Obviously, this is equivalent to the existence of a finite matricial representation $\tilde{\Phi}\colon \mathfrak{L} \hookrightarrow \mathrm{M}_n(K)$, and the integer $n = \dim_K V$ is called the degree of the representation.

From most of the proofs of Ado's Theorem it follows that $\deg \Phi$, the degree of the representation $\Phi$, is bounded in terms of $\dim_K \mathfrak{L}$, the $K$-vector space dimension of $\mathfrak{L}$. That is to say, if we define the degree of a Lie algebra $\mathfrak{L}$ as

$$\deg \mathfrak{L} := \min \{\deg \Phi \mid \Phi \text{ faithful Lie algebra representation of } \mathfrak{L}\}, \tag{1}$$

then the integer $\deg \mathfrak{L}$ is bounded only in terms of $\dim_K \mathfrak{L}$.

Some quantitative bounds are known for $\deg \mathfrak{L}$, specially when $\mathfrak{L}$ is nilpotent. For instance, if $\mathfrak{L}$ is a nilpotent Lie algebra of dimension $d$ and nilpotency class $c$, according to [3, Corollary to Theorem 4], $\deg \mathfrak{L} \leq \frac{d^{c+1}-1}{d-1}$ or according to [6, Corollary 5.1], $\deg \mathfrak{L} \leq \binom{d+c}{c}$. Remarkably, if the nilpotency class is fixed, $\deg \mathfrak{L}$ is polynomial in $\dim_K \mathfrak{L}$.

---

The author acknowledges support by the Basque Government, project IT483-22, and the Spanish Government, project PID2020-117281GB-I00, partly with ERDF funds. This paper forms part of the author's PhD thesis, funded by the Spanish Ministry of Science, Innovation and Universities' grant FPU17/04822.

ISSN 0949–5932 / \$2.50 

In general, for a nilpotent $d$-dimensional $K$-Lie algebra $\mathfrak{L}$, Burde [4] proved that

$$\deg \mathfrak{L} \leq \eta \frac{2^d}{\sqrt{d}}, \quad \text{where } \ \eta \sim 2.763.$$

Iwasawa [8] extended Ado's Theorem to Lie algebras over fields of positive characteristic, and there are further generalizations in the base ring. Following the terminology of [13], for a general ring $R$ we denote by $R$-Lie lattice an $R$-Lie algebra that is a free $R$-module of finite rank as well. Actually, any $R$-Lie algebra that admits a finite matricial representation is indeed an $R$-Lie lattice.

Conversely, suppose that $R$ is a principal ideal domain (PID) of characteristic zero or a general ring of positive characteristic, Churkin [5] and Weigel [13] proved that every $R$-Lie lattice $\mathfrak{L}$ admits a finite faithful $R$-Lie algebra representation $\Phi\colon \mathfrak{L} \hookrightarrow \mathrm{End}_R(V)$, where $V$ is a free $R$-module of finite rank. Like for fields, the degree of the preceding representation $\Phi$ is defined to be $\mathrm{rk}_R V$, the rank of $V$ as a free $R$-module, and the degree of an $R$-Lie lattice is defined exactly as in (1).

Suppose that $R$ is a PID of characteristic zero. Both [5] and [13] follow Jacobson's proof of the Theorem of Ado (see [9, Chapter VI]), which in its turn, is based on a proof due by Harish-Chandra [7], but, unlike for fields, it cannot be directly affirmed that $\deg \mathfrak{L}$ depends uniquely on $\mathrm{rk}_R \mathfrak{L}$. In fact, in [13, Proposition 3.4], the degree-to-be is finite because $R$ is a Noetherian ring, and so a particular ascending chain of ideals must be stationary. However, the length of the chain, which eventually will be the degree of the representation, might not be bounded in terms of $\mathrm{rk}_R \mathfrak{L}$.

In this note, we collect several existing proofs of Ado's Theorem, and by adapting them to PIDs we prove the following:

**Theorem 1.1.** *Let $R$ be a PID of characteristic zero and let $\mathfrak{L}$ be an $R$-Lie lattice of rank $r$. Then,* $\deg \mathfrak{L} \leq r + \eta \frac{2^r}{\sqrt{r}}$, *where* $\eta \sim 2.763$.

In particular, we recover for PIDs the best bound yet known over fields of characteristic zero. More concretely, in Sections 3 and 4 we reproduce quantitative results about the representability of nilpotent and splittable $R$-Lie lattices, and in Section 5 (see Theorem 5.3) we prove the following:

**Theorem 1.2.** *Let $R$ be a PID of characteristic zero and let $\mathfrak{L}$ be an $R$-Lie lattice. There exists an $R$-Lie lattice of the form $\bar{\mathfrak{L}} = R_n(\bar{\mathfrak{L}}) \rtimes \mathfrak{S}$ extending $\mathfrak{L}$, where $R_n(\bar{\mathfrak{L}})$ is the nilpotent radical of $\bar{\mathfrak{L}}$.*

This result is based on the analogue for complex Lie algebras proved by Neretin [11], and previously discussed in [10, 12]. Lastly, Theorem 1.1 is proved in Section 6 using Theorem 1.2 and the arguments of the previous sections.

Finally, we must note that for rings of positive characteristic, the generalisation is proved reproducing word-by-word the original demonstration of Iwasawa, and therefore we obtain the same bound we had for these fields, namely

$$\deg \mathfrak{L} \leq n^{\mathrm{rk}^3 \mathfrak{L}},$$

where $n = \mathrm{char}\, R$ (compare with [2, Section 6.24]).

**Notation.** Hereinafter $R$ will always be a PID of characteristic zero, and we will use $K$ for fields. For an $R$-Lie lattice $\mathfrak{L}$, $R_n(\mathfrak{L})$ and $R_s(\mathfrak{L})$ refer to the nilpotent and solvable radicals of $\mathfrak{L}$. We denote by $\dim_K$ the $K$-vector space dimension, by $\mathrm{rk}_R$ (rk when $R$ is clear from the context) the rank of a free $R$-module, by $\langle X\rangle_R$ the $R$-module generated by a set $X$, and $\mathfrak{I} \leq \mathfrak{L}$ and $\mathfrak{I} \trianglelefteq \mathfrak{L}$ represent respectively that $\mathfrak{I}$ is a subalgebra and an ideal of $\mathfrak{L}$. We will use the abbreviation $[\mathfrak{I}_1, \dots, \mathfrak{I}_n] = [[\mathfrak{I}_1, \dots, \mathfrak{I}_{n-1}], \mathfrak{I}_n]$ for iterated Lie brackets, and throughout the manuscript "$:=$" is used to mean *defined to be* in contrast with "$=$", which is used to denote *equal to*.

Finally, we recall that an $R$-submodule $\mathfrak{I} \leq \mathfrak{L}$ is isolated if whenever $rx \in \mathfrak{I}$ for some $r \in R$ and $x \in \mathfrak{L}$, then $x \in \mathfrak{I}$, that is, the quotient $R$-module $\frac{\mathfrak{L}}{\mathfrak{I}}$ is torsion-free, and thus free.

## 2. Preliminaries: adjoint and regular representations

There are two natural Lie algebra representations in any $R$-Lie lattice $\mathfrak{L}$. On the one hand, by virtue of Jacobi's identity the adjoint representation $\mathrm{Ad}\colon \mathfrak{L} \to \mathrm{End}_R(\mathfrak{L})$, $x \mapsto \mathrm{ad}_x$, where $\mathrm{ad}_x\colon \mathfrak{L} \to \mathfrak{L}$, $y \mapsto [x, y]$, is a finite Lie algebra representation. However, this representation is not faithful as its kernel is the centre of $\mathfrak{L}$, namely

$$Z(\mathfrak{L}) = \{x \in \mathfrak{L} \mid [x, y] = 0 \ \forall y \ \in \mathfrak{L}\}.$$

In particular, when $\mathfrak{L}$ is a semisimple $R$-Lie lattice, i.e. $\mathfrak{L}$ has no abelian ideal, then $\deg \mathfrak{L} \leq \mathrm{rk}_R \mathfrak{L}$.

Typically, Ado's Theorem is proved by firstly constructing a *finite* representation $\Phi\colon \mathfrak{L} \to \mathrm{End}_R(W)$ that is faithful in $Z(\mathfrak{L})$, and then taking the finite faithful representation $\mathrm{Ad} \oplus \Phi$.

On the other hand, $\mathfrak{L}$ acts on its universal enveloping algebra $\mathcal{U}_R(\mathfrak{L})$. Indeed, the tensor algebra of $\mathfrak{L}$ is

$$\mathbf{T}_R(\mathfrak{L}) = R \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \cdots \oplus \mathfrak{L}_i \oplus \dots,$$

where $\mathfrak{L}_i := \mathfrak{L} \otimes \overset{(i)}{\dots} \otimes \mathfrak{L}$ is the tensor $R$-module and the multiplication in $\mathbf{T}_R(\mathfrak{L})$ is defined extending by linearity the rule

$$(x_1 \otimes \cdots \otimes x_i) \otimes (y_1 \otimes \cdots \otimes y_j) = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j.$$

Then the universal enveloping algebra of $\mathfrak{L}$ is

$$\mathcal{U}_R(\mathfrak{L}) := \frac{\mathbf{T}_R(\mathfrak{L})}{\mathfrak{R}},$$

where $\mathfrak{R}$ is the ideal generated by the elements

$$[x, y] - (x \otimes y - y \otimes x), \ \forall x, y \in \mathfrak{L}. \tag{2}$$

The image of the tensor $x_{i_1} \otimes \cdots \otimes x_{i_t}$ in $\mathcal{U}_R(\mathfrak{L})$ will be simply denoted by the monomial $x_{i_1} \dots x_{i_t}$.

Since $R$ is a PID and $\mathfrak{L}$ is finitely generated, we know that the natural inclusion $\iota\colon \mathfrak{L} \cong \mathfrak{L}_1 \hookrightarrow \mathcal{U}_R(\mathfrak{L})$ is a monomorphism (see [13, Theorem 3.2]), and, therefore, we can assume that $\mathfrak{L} \subseteq U_R(\mathfrak{L})$. The universal enveloping algebra is characterised by the Poincaré-Birkhoff-Witt Theorem:

**Theorem 2.1.** (cf. [13, Theorem 3.2]) *Let $\mathfrak{L}$ be an $R$-Lie lattice with basis $\{x_1, \dots, x_r\}$. Then $\mathcal{U}_R(\mathfrak{L})$ is a free $R$-module with basis*

$$\{x_1^{\alpha_1} \dots x_r^{\alpha_r} \mid \alpha_i \in \mathbb{N}_0\}, \tag{3}$$

*where $x_1^0 \dots x_r^0 = 1$ is the identity of $R$.*

The idea is that given two monomials, their product can expressed as a suitable linear combination of elements of the form (3) by successively applying the identity $x_j x_i = x_i x_j - [x_i, x_j]$ to reorder the indeterminates.

As already mentioned, $\mathfrak{L}$ acts on $\mathcal{U}_R(\mathfrak{L})$ by multiplication and this gives rise to the (left) regular representation $\mathcal{R}\colon \mathfrak{L} \hookrightarrow \mathrm{End}_R(\mathcal{U}_R(\mathfrak{L}))$, where $\mathcal{R}(x)$ is the left multiplication map $\ell_x\colon \mathcal{U}_R(\mathfrak{L}) \to \mathcal{U}_R(\mathfrak{L}),\ y \mapsto xy$.

By virtue of (2), $\mathcal{R}$ is a Lie algebra representation, and it is faithful as

$$\ell_x(1) = x \neq y = \ell_y(1) \text{ for all } x, y \in \mathfrak{L}.$$

Although $\mathcal{R}$ is not finite, by virtue of the universal property of the enveloping algebra (see [13, Proposition 3.1]), every finite Lie algebra representation of $\mathfrak{L}$ factors through $\mathcal{U}_R(\mathfrak{L})$. Hence, $\mathfrak{L}$ admits a finite faithful Lie algebra representation if and only if there exists an isolated ideal $\mathfrak{X} \trianglelefteq \mathcal{U}_R(\mathfrak{L})$ such that $\mathfrak{X} \cap \mathfrak{L} = \{0\}$. Indeed, it is enough to consider the induced action on the free $R$-module $\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{X}}$.

## 3. Nilpotent Lie lattices

For nilpotent $R$-Lie lattices the construction of Birkhoff [3] is still valid over PIDs. Suppose that $\mathfrak{L}$ is a nilpotent $R$-Lie lattice of nilpotency class $c$, and let $K$ be the fraction field of $R$. Since the Lie bracket is bilinear, $\mathfrak{L}_K = \mathfrak{L} \otimes_R K$ is also a nilpotent Lie algebra of nilpotency class $c$.

For each $i \in \{1, \dots, c\}$, define the isolated ideal $\mathfrak{L}_i = [\mathfrak{L}_K, \overset{(i)}{\dots}, \mathfrak{L}_K] \cap \mathfrak{L} \trianglelefteq \mathfrak{L}$, and choose a basis $\{x_1, \dots, x_r\}$ for $\mathfrak{L}$ as free $R$-module in such way that the first elements $x_1, \dots, x_{r_1}$ are an $R$-basis for $\mathfrak{L}_c$, the first elements $x_1, \dots, x_{r_2}$ ($r_2 > r_1$) are an $R$-basis for $\mathfrak{L}_{c-1}$ and so forth. By Theorem 2.1, the elements of $\mathcal{U}_R(\mathfrak{L})$ are of the form $\sum_{\alpha \in \mathbb{N}_0^{(r)}} c_\alpha \mathbf{x}^\alpha$, where $\mathbf{x}^\alpha$ stands for $x_1^{\alpha_1} \dots x_r^{\alpha_r}$. Accordingly, define a weight function $\omega\colon \mathcal{U}_R(\mathfrak{L}) \to \mathbb{N}_0 \cup \{\infty\}$ in the following way:

$$\omega(x_i) = \max\{m \mid x_i \in \mathfrak{L}_m\}, \qquad \omega(\mathbf{x}^\alpha) = \sum_{i=1}^{r} \alpha_i \omega(x_i),$$

$$\omega\left(\sum_\alpha c_\alpha \mathbf{x}^\alpha\right) = \min\{\omega(\mathbf{x}^\alpha) \mid c_\alpha \neq 0\}, \qquad \omega(0) = \infty.$$

Observe that $\omega([x_i, x_j]) \geq \omega(x_i) + \omega(x_j)$ for all $i, j \in \{1, \dots, r\}$, and so

$$\omega(uv) \geq \omega(u) + \omega(v)\ \forall u, v \in \mathcal{U}_R(\mathfrak{L}). \tag{4}$$

For each $m \in \mathbb{N}_0$, consider the isolated $R$-modules

$$\mathfrak{U}^m(\mathfrak{L}) := \{u \in \mathcal{U}_R(\mathfrak{L}) \mid \omega(u) > m\}$$

or simply $\mathfrak{U}^m$ when the lattice is clear from the context.

By (4), $\mathfrak{U}^m(\mathfrak{L})$ is an ideal and thus for every $x \in \mathfrak{L}$ we have that $\ell_x(\mathfrak{U}^m) \subseteq \mathfrak{U}^m$, so for any $m \in \mathbb{N}$ the regular representation induces the finite representation

$$\mathcal{R}_m \colon \mathfrak{L} \to \mathrm{End}_R\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^m(\mathfrak{L})}\right),\ x \mapsto \ell_x,$$

whose kernel is $\mathfrak{L} \cap \mathfrak{U}^m(\mathfrak{L})$ – with an abuse of notation, when $f \in \mathrm{End}_R(\mathcal{U}_R(\mathfrak{L}))$ satisfies $f(\mathfrak{X}) \subseteq \mathfrak{X}$ for some ideal $\mathfrak{X} \trianglelefteq \mathcal{U}_R(\mathfrak{L})$, we will keep $f$ to denote the endomorphism of $\mathrm{End}_R\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{X}}\right)$ defined as $x + \mathfrak{X} \mapsto f(x) + \mathfrak{X}$ –.

Since $\mathfrak{L} \cap \mathfrak{U}^c(\mathfrak{L}) = \{0\}$, $\mathcal{R}_c$ is a finite faithful representation and its degree is

$$|\{\mathbf{x}^\alpha \mid \omega(\mathbf{x}^\alpha) \leq c\}|,$$

as these monomials are a basis for $\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^c}$. Finally, this number was bounded by Burde (see [4, Lemma 5(3) and Proposition 6]):

$$\deg \mathcal{R}_c = \mathrm{rk}_R\left(\frac{\mathcal{U}_R(\mathfrak{L})}{\mathfrak{U}^c(\mathfrak{L})}\right) = |\{\mathbf{x}^\alpha \mid \omega(\mathbf{x}^\alpha) \leq c\}| \leq \eta \frac{2^r}{\sqrt{r}}, \tag{5}$$

where $\eta = \sqrt{\frac{2}{\pi}} \prod_{l=1}^{\infty} \frac{2^l}{2^l - 1} \sim 2.763$.

## 4. Splittable Lie lattices

We say that an $R$-Lie lattice $\mathfrak{L}$ is splittable if the short exact sequence

$$0 \to R_n(\mathfrak{L}) \ \to\ \frac{\mathfrak{L}}{R_n(\mathfrak{L})} \ \to\ 0$$

splits in the category of $R$-Lie algebras, that is, if there exists an $R$-Lie subalgebra $\mathfrak{S} \leq \mathfrak{L}$ such that $\mathfrak{L} = R_n(\mathfrak{L}) \rtimes \mathfrak{S}$.

In the splittable case we can blend the preceding regular representation for $R_n(\mathfrak{L})$ and representations induced from derivations, namely endomorphisms $D \in \mathrm{End}_R(\mathfrak{L})$ that satisfy Leibniz identity, i.e.

$$D([x, y]) = [x, D(y)] + [D(x), y]\ \forall x, y \in \mathfrak{L}.$$

The collection of all derivations of $\mathfrak{L}$ is denoted by $\mathrm{Der}_R(\mathfrak{L})$. For example, by virtue of Jacobi's identity, $\mathrm{ad}_x$ is a derivation for every $x \in \mathfrak{L}$. Starting from $D \in \mathrm{Der}_R(\mathfrak{L})$ we can define a derivation $D^*$ of $\mathcal{U}_R(\mathfrak{L})$ by imposing Leibniz identity, that is, by taking the linear extension of the rule

$$D^*(x_{i_1} \dots x_{i_t}) = \sum_j x_{i_1} \dots x_{i_{j-1}} D(x_{i_j}) x_{i_{j+1}} \dots x_{i_t},$$

together with $D^*(1) = 0$ as it must happen for every derivation of an algebra with identity. In keeping with the notation of the previous section, we have:

**Lemma 4.1.** *Let $\mathfrak{L}$ be a nilpotent $R$-Lie lattice and $D \in \mathrm{Der}_R(\mathfrak{L})$ a derivation. Then $D^*(\mathfrak{U}^m(\mathfrak{L})) \subseteq \mathfrak{U}^m(\mathfrak{L})$ for every $m \in \mathbb{N}$.*

**Proof.** Let $c$ be the nilpotency class of $\mathfrak{L}$, and let $\{x_1, \ldots, x_r\}$ be the basis of $\mathfrak{L}$ with respect to which the weight function $\omega$ has been defined. Since $D$ is a derivation, $D(\mathfrak{L}_i) \subseteq \mathfrak{L}_i$ for all $i \in \{1, \ldots, c\}$, so $\omega(D(x_i)) \geq \omega(x_i)$ for all $i \in \{1, \ldots, r\}$. Hence, if $x_{i_1} \ldots x_{i_t} \in \mathfrak{U}^m(\mathfrak{L})$, by (4),

$$\omega\left(D^*(x_{i_1} \ldots x_{i_t})\right) \geq \min_{j=1,\ldots,t} \left\{\omega(x_{i_1} \ldots D(x_{i_j}) \ldots x_{i_t})\right\} \geq \omega(x_{i_1} \ldots x_{i_t}) > m. \qquad \blacksquare$$

**Proposition 4.2.** [Zassenhaus extension, cf. [9, Chapter VI.2, Theorem 1]] *Let $\mathfrak{L}$ be a splittable $R$-Lie lattice and let $c$ be the nilpotency class of $R_n(\mathfrak{L})$. Then, there exists a finite $R$-Lie algebra representation*

$$\Phi\colon \mathfrak{L} \to \mathrm{End}_R\left(\frac{\mathcal{U}_R(R_n(\mathfrak{L}))}{\mathfrak{U}^c(R_n(\mathfrak{L}))}\right)$$

*that is injective in $R_n(\mathfrak{L})$ and such that*

$$\deg \Phi \leq \eta \frac{2^{\operatorname{rk} R_n(\mathfrak{L})}}{\sqrt{\operatorname{rk} R_n(\mathfrak{L})}}, \quad \textit{where} \quad \eta \sim 2.763. \tag{6}$$

**Proof.** Denote $R_n(\mathfrak{L})$ by $\mathfrak{N}$, then $\mathfrak{L} = \mathfrak{N} \rtimes \mathfrak{S}$ for some $R$-Lie subalgebra $\mathfrak{S} \leq \mathfrak{L}$. By Lemma 4.1, $\mathrm{ad}_x^*\left(\mathfrak{U}^c(\mathfrak{N})\right) \subseteq \mathfrak{U}^c(\mathfrak{N})$ for all $x \in \mathfrak{L}$, so we can define the map

$$\Phi\colon \mathfrak{L} = \mathfrak{N} \oplus \mathfrak{S} \to \mathrm{End}_R\left(\frac{\mathcal{U}_R(\mathfrak{N})}{\mathfrak{U}^c(\mathfrak{N})}\right), \ n + s \mapsto \ell_n + \mathrm{ad}_s^*.$$

In order to show that it is an $R$-Lie algebra homomorphism, it suffices to confirm that

$$\Phi\left([s, n]\right) = [\Phi(s), \Phi(n)] = [\mathrm{ad}_s^*, \ell_n]$$

for all $n \in \mathfrak{N}$ and $s \in \mathfrak{S}$. Indeed, for any $n \in \mathfrak{N}$ and $D \in \mathrm{Der}_R\left(\mathcal{U}_R(\mathfrak{N})\right)$ :

$$[D, \ell_n](u) = D \circ \ell_n(u) - \ell_n \circ D(u) = D(n)u = \ell_{D(n)}(u) \ \forall u \in \mathfrak{N},$$

and, since $\mathfrak{N}$ is an ideal, $[s, n] \in \mathfrak{N}$, so

$$\Phi\left([s, n]\right) = \ell_{[s,n]} = \ell_{\mathrm{ad}_s^*(n)} = [\mathrm{ad}_s^*, \ell_n] = [\Phi(s), \Phi(n)].$$

Consequently, $\Phi$ is a finite $R$-Lie algebra representation. In addition, $\Phi|_{\mathfrak{N}}$ is nothing but the faithful representation $\mathcal{R}_c$ of $\mathfrak{N}$. Finally, (6) follows from (5). $\blacksquare$

## 5. Embedding theorem

In [9, Chapter IV.2], and the succeeding works following it, Levi's Theorem is crucial; namely, if $K$ is a field of characteristic zero there exists a semisimple Lie algebra $\mathfrak{S} \leq \mathfrak{L}$ such that $\mathfrak{L} = R_s(\mathfrak{L}) \rtimes \mathfrak{S}$ (see [9, Chapter III.9]). However, this results is not longer true for PIDs. For instance, the $\mathbb{Z}$-Lie algebra $\mathfrak{sl}_2(2\mathbb{Z}) \oplus \mathfrak{t}_2(2\mathbb{Z})$ – the direct sum of $2 \times 2$ matrices of trace zero and $2 \times 2$ upper triangular matrices with coefficients in $2\mathbb{Z}$ – does not admit such a decomposition (see [5, Example in p.838]). Nevertheless, every Lie lattice embeds in a splittable (in the sense of Section 4) $R$-Lie lattice. In effect, over algebraically closed fields this result was first proved for solvable Lie algebras by Mal'cev [10] and Reed [12], and using similar ideas Neretin [11] proved the following (albeit [11] is about complex Lie algebras, the proof is still valid, mutatis mutandi, for any field of characteristic zero):

**Theorem 5.1.** (cf. [11, Lemma 1]) *Let $K$ be a field of characteristic zero and $\mathfrak{L}$ a finite dimensional $K$-Lie algebra. There exists a splittable $K$-Lie algebra $\bar{\mathfrak{L}} = R_n(\bar{\mathfrak{L}}) \rtimes \mathfrak{S}$, where $\mathfrak{S}$ is reductive, extending $\mathfrak{L}$.*

The above theorem is proved by successively applying elementary expansions. Indeed, suppose that we have a $K$-Lie algebra $\mathfrak{K} = \mathfrak{M} \rtimes \mathfrak{S}$ extending $\mathfrak{L}$ such that $\mathfrak{M}$ is a solvable ideal containing $R_n(\mathfrak{K})$ and $\mathfrak{S}$ is a reductive – direct sum of a semisimple and an abelian algebra – subalgebra that acts fully irreducibly on $\mathfrak{M}$. We shall construct another Lie algebra $\mathfrak{K}'$ extending $\mathfrak{K}$ that satisfies those same conditions.

By [9, Chapter III.7, Theorem 13], $[\mathfrak{M}, \mathfrak{K}] \leq R_n(\mathfrak{K})$. Thus, unless $\mathfrak{M}$ is nilpotent, there exists an ideal $\mathfrak{I} \trianglelefteq \mathfrak{M}$ of codimension one containing $R_n(\mathfrak{K})$. Since the action of $\mathfrak{S}$ is fully irreducible, there exists an element $y \in \mathfrak{M} \setminus R_n(\mathfrak{K})$ such that $\mathfrak{M} = \mathfrak{I} \oplus Ky$ as $\mathfrak{S}$-modules, in particular, $[y, \mathfrak{S}] \subseteq R_n(\mathfrak{K}) \cap Ky = \{0\}$. Moreover, according to the Jordan-Chevalley decomposition (see [9, Chapter III.11, Theorem 16]), the derivation $\mathrm{ad}_y \in \mathrm{Der}_R(\mathfrak{K})$ decomposes as $d_{s,y} + d_{n,y}$ where $d_{s,y}$ and $d_{n,y}$ are respectively a semisimple and a nilpotent $K$-linear endomorphism.

**Remark 5.2.** Both $d_{s,y}$ and $d_{n,y}$ are in $\mathrm{Der}_K(\mathfrak{K})$. Indeed, when $K$ is algebraically closed it was proved in [12, Proposition 3], as $d_{s,n}(v) = \lambda v$ provided that $v$ belongs to the generalised $\lambda$-eigenspace of $\mathrm{ad}_y$. In general, let us write $S = d_{s,y}$ and $N = d_{n,y}$ and let $\bar{K}$ be the algebraic closure of $K$. Suppose that $\bar{S} + \bar{N}$ is the Jordan-Chevalley decomposition of $\mathrm{ad}_y$ in $\mathfrak{K}_{\bar{K}} = \mathfrak{K} \otimes_K \bar{K}$. Then

$$S \otimes \bar{K} + N \otimes \bar{K} = \mathrm{ad}_y = \bar{S} + \bar{N}$$

are two decompositions of $\mathrm{ad}_y \in \mathrm{End}_{\bar{K}}(\mathfrak{K}_{\bar{K}})$, so by the uniqueness $\bar{S} = S \otimes \bar{K}$ and $\bar{N} = N \otimes \bar{K}$. Finally, since $\bar{S}$ and $\bar{N}$ satisfy Leibniz identity, so do $S$ and $N$. ∎

Thus, we can construct a so-called elementary expansion, namely the $K$-Lie algebra

$$\mathfrak{K}' = \mathfrak{I} \oplus \mathfrak{S} \oplus Kx' \oplus Kz' \qquad ,$$

where $x'$ and $z'$ are formal symbols satisfying

$$[x', u] = d_{n,y}(u), \quad [z', u] = d_{s,y}(u), \quad [x', z'] = 0$$

for every $u \in \mathfrak{I} \oplus \mathfrak{S}$, and where we keep the original Lie bracket for the elements of $\mathfrak{I} \oplus \mathfrak{S}$. Observe that $\mathfrak{K} = \mathfrak{I} \oplus Ky \oplus \mathfrak{S}$ embeds as a Lie algebra in $\mathfrak{K}'$ with respect to $y = x' + z'$.

In addition, $\ker \mathrm{ad}_y \subseteq \ker d_{s,y}$ (see Remark 5.2), so $[z', \mathfrak{S}] = 0$ and $\mathfrak{S}' := \mathfrak{S} \oplus Kz'$ is a reductive Lie algebra. Moreover, $R_n(\mathfrak{K}) \oplus Kx'$ is the nilpotent radical of $\mathfrak{K}'$, $\mathfrak{M}' := \mathfrak{I} \oplus Kx'$ is solvable, and, since $d_{s,y}$ is a semisimple operator, the action of $\mathfrak{S}'$ in $\mathfrak{M}'$ is fully reducible. In particular, $\mathfrak{K}' = \mathfrak{M}' \rtimes \mathfrak{S}'$. In passing, note that

$$\dim_K \mathfrak{M}' = \dim_K \mathfrak{M} \text{ and } \dim_K R_n(\mathfrak{K}') = \dim_K R_n(\mathfrak{K}) + 1. \tag{7}$$

Levi's Theorem gives us the first of the step of the above-described procedure. Indeed, $\mathfrak{L} = R_s(\mathfrak{L}) \rtimes \mathfrak{S}$ for a semisimple subalgebra $\mathfrak{S} \leq \mathfrak{L}$, and by virtue of Weyl's Theorem on complete reducibility (see [9, Chapter III.7, Theorem 8]), the action of $\mathfrak{S}$ on $R_s(\mathfrak{L})$ is fully reducible. Fix bases $\{x_1, ..., x_s\}$ of $R_n(\mathfrak{L})$ and $\{z_1, ..., z_t\}$ of $\mathfrak{S}$. In view of (7), repeating the previous process eventually we obtain a $K$-Lie algebra

$$\bar{\mathfrak{L}} = \bar{\mathfrak{N}} \rtimes \bar{\mathfrak{S}} = \langle x_1, \ldots, x_s, x_1', \ldots, x_r' \rangle_K \rtimes \langle z_1, \ldots, z_t, z_1', \ldots, z_r' \rangle_K, \tag{8}$$

where $\bar{\mathfrak{N}}$ is a nilpotent ideal and $\bar{\mathfrak{S}} \leq \bar{\mathfrak{L}}$ is a reductive subalgebra, and a $K$-basis $\{x_1, \ldots, x_s, y_1, \ldots, y_r, z_1, \ldots, z_t\}$ of $\mathfrak{L}$ such that $R_s(\mathfrak{L}) = \langle x_1, \ldots, x_s, y_1, \ldots, y_r \rangle_K$ and $y_i = x_i' + z_i'$. In particular, $\bar{\mathfrak{L}}$ extends $\mathfrak{L}$, and by construction:

(N1) for all $i, j \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, t\}$ we have $\left[z_i', z_j'\right] = [z_i', z_k] = 0$, and therefore $[x_i', z_k] = [y_i, z_k] \in R_n(\mathfrak{L})$ (see [9, Chapter II.7, Theorem 13]);

(N2) for all $i \in \{1, \ldots, s\}$ and $j, k \in \{1, \ldots, r\}$, by [9, Chapter III.6, Theorem 7], we have $d_{n,y_j}(R_s(\mathfrak{L})) \subseteq R_n(\mathfrak{L})$, so

$$\left[x_j', x_i\right] = d_{n,y_j}(x_i) \in R_n(\mathfrak{L}) \quad \text{and} \quad \left[x_j', y_k\right] = d_{n,y_j}(y_k) \in R_n(\mathfrak{L}).$$

In particular, $R_n(\mathfrak{L}) \trianglelefteq R_n(\bar{\mathfrak{L}})$;

(N3) $\dim_K R_n(\bar{\mathfrak{L}}) = s + r = \dim_K R_s(\mathfrak{L})$.

Furthermore, in view of (N1)–(N2), we have that

$$[x_i', \mathfrak{L}] := \{[x_i', u] \mid u \in \mathfrak{L}\} \subseteq R_n(\mathfrak{L}) = \langle x_1, \ldots, x_s \rangle_K \tag{9}$$

for all $i \in \{1, \ldots, r\}$.

As a consequence, we can prove the following strengthened version of Theorem 1.2:

**Theorem 5.3.** *Let $R$ be a PID of characteristic zero and let $\mathfrak{L}$ be an $R$-Lie lattice. Then $\mathfrak{L}$ embeds into a splittable $R$-Lie lattice $\bar{\mathfrak{L}}$ such that*

(i) $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ *and*

(ii) $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_s(\mathfrak{L})$.

**Proof.** Let $K$ be the fraction field of $R$ and $\mathfrak{L}_K := \mathfrak{L} \otimes_R K$. According to Theorem 5.1, there exists a finite dimensional splittable $K$-Lie algebra $\mathfrak{L}_K' = R_n(\mathfrak{L}_K') \rtimes \mathfrak{S}_K'$ extending $\mathfrak{L}_K$ and satisfying conditions (N1)–(N3). Denote for simplicity $\mathfrak{N}_K' := R_n(\mathfrak{L}_K')$.

In keeping with the notation of (8), let $\{x_1, \ldots, x_s\}$ be a basis for $R_n(\mathfrak{L})$ as free $R$-module, then $R_n(\mathfrak{L}_K) = \langle x_1, \ldots, x_s \rangle_K$ and there exists a $K$-vector space basis of $\mathfrak{N}_K'$ of the form

$$\{x_1, \ldots, x_s, x_1', \ldots, x_r'\},$$

where $s + r = \dim_K R_s(\mathfrak{L}_K) = \operatorname{rk}_R R_s(\mathfrak{L})$, compare with (N3).

Furthermore, by (N2), there exists $\mu \in R \setminus \{0\}$ such that

$$\mathfrak{N} := \langle x_1, \ldots, x_s, \mu x_1', \ldots, \mu x_r' \rangle_R$$

is a nilpotent $R$-Lie lattice of rank $s + r$, $R_n(\mathfrak{L}) = \langle x_1, \ldots, x_s \rangle_R \trianglelefteq \mathfrak{N}$ and

$$[\mu x_j', \mathfrak{L}] \subseteq \langle x_1, \ldots, x_s \rangle_R,$$

for all $j \in \{1, \ldots, r\}$ (using (9) and that $\mathfrak{L}$ is finitely generated for the last condition). In particular,

$$[\mathfrak{N}, \overset{(i)}{\ldots}, \mathfrak{N}, \mathfrak{L}] \leq [\mathfrak{N}, \overset{(i)}{\ldots}, \mathfrak{N}] \; \forall i \in \mathbb{N}. \tag{10}$$

Let $\bar{\mathfrak{S}}$ be the projection of $\mathfrak{L}$ into $\mathfrak{S}'_K$, that is,

$$\bar{\mathfrak{S}} = \{\sigma \in \mathfrak{S}'_K \mid \exists\, x \in \mathfrak{L}, \exists\, n \in \mathfrak{N}'_K \text{ such that } x = n + \sigma\}.$$

Then $\bar{\mathfrak{S}}$ is an $R$-Lie algebra. Indeed, if $x_1 = n_1 + \sigma_1$ and $x_2 = n_2 + \sigma_2 \in \mathfrak{L}$, where $n_i \in \mathfrak{N}'_K$ and $\sigma_i \in \mathfrak{S}'_K$ $(i \in \{1,2\})$, then

$$[x_1, x_2] = [n_1, x_2] + [\sigma_1, n_2] + [\sigma_1, \sigma_2],$$

$[x_1, x_2] \in \mathfrak{L}$, $[n_1, x_2] + [\sigma_1, n_2] \in \mathfrak{N}'_K$ and $[\sigma_1, \sigma_2] \in \mathfrak{S}'_K$. In addition, since $\mathfrak{L}$ is finitely generated, $\bar{\mathfrak{S}}$ is a free $R$-module of finite rank.

Moreover, since $\mathfrak{L}$ is a finitely generated $R$-module there exists $\lambda \in R \setminus \{0\}$ such that

$$\mathfrak{L} \subseteq \frac{1}{\lambda}\mathfrak{N} \oplus \bar{\mathfrak{S}} \tag{11}$$

and

$$\bar{\mathfrak{S}} \subseteq \mathfrak{L} + \frac{1}{\lambda}\mathfrak{N}. \tag{12}$$

Let $c$ be the nilpotency class of $\mathfrak{N}$, define $\mathfrak{N}_i := [\mathfrak{N}, \overset{(i)}{\ldots}, \mathfrak{N}]$ for each $i \in \{1, \ldots, c\}$, and

$$\bar{\mathfrak{N}} := \sum_{i=1}^{c} \frac{1}{\lambda^i}\, \mathfrak{N}_i \leq \mathfrak{N}'_K,$$

which is a free $R$-module of rank $s + r = \operatorname{rk} R_s(\mathfrak{L})$. On the one hand,

$$\left[\frac{1}{\lambda^i}\mathfrak{N}_i, \frac{1}{\lambda^j}\mathfrak{N}_j\right] = \frac{1}{\lambda^{i+j}}\, [\mathfrak{N}_i, \mathfrak{N}_j] \leq \frac{1}{\lambda^{i+j}}\mathfrak{N}_{i+j},$$

so $\bar{\mathfrak{N}}$ is a nilpotent $R$-Lie lattice. On the other hand, by (10) and (12),

$$\begin{aligned}\left[\frac{1}{\lambda^i}\mathfrak{N}_i, \bar{\mathfrak{S}}\right] &\leq \left[\frac{1}{\lambda^i}\mathfrak{N}_i, \mathfrak{L} + \frac{1}{\lambda}\mathfrak{N}\right] \leq \frac{1}{\lambda^i}\, [\mathfrak{N}_i, \mathfrak{L}] + \frac{1}{\lambda^{i+1}}\, [\mathfrak{N}_i, \mathfrak{N}] \\ &\leq \frac{1}{\lambda^i}\mathfrak{N}_i + \frac{1}{\lambda^{i+1}}\mathfrak{N}_{i+1} \leq \bar{\mathfrak{N}}\end{aligned}$$

for every $i \in \{1, \ldots, c\}$.

Hence, $\bar{\mathfrak{L}} := \bar{\mathfrak{N}} \rtimes \bar{\mathfrak{S}}$ is an $R$-Lie lattice that extends $\mathfrak{L}$ (see (11)); by construction $\bar{\mathfrak{L}}$ is splittable, $R_n(\mathfrak{L}) \leq \bar{\mathfrak{N}} = R_n(\bar{\mathfrak{L}})$ and $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} \bar{\mathfrak{N}} = s + r = \operatorname{rk} R_s(\mathfrak{L})$. ■

## 6. Proof of Theorem 1.1

Finally, we gather all the ingredients:

**Proof of Theorem 1.1.** Let $\mathfrak{L}$ be an $R$-Lie lattice of rank $r$. According to Theorem 5.3, there exists a splittable $R$-Lie lattice $\bar{\mathfrak{L}} = R_n(\bar{\mathfrak{L}}) \rtimes \mathfrak{S}$ extending $\mathfrak{L}$ such that $R_n(\mathfrak{L}) \leq R_n(\bar{\mathfrak{L}})$ and $\operatorname{rk} R_n(\bar{\mathfrak{L}}) = \operatorname{rk} R_s(\mathfrak{L})$. By Proposition 4.2, there exists an $R$-Lie algebra representation $\Phi$ of $\bar{\mathfrak{L}}$ which is injective in $R_n(\bar{\mathfrak{L}})$ and whose degree is bounded by $f(\operatorname{rk} R_n(\bar{\mathfrak{L}}))$, with $f\colon \mathbb{N}_{\geq 1} \to \mathbb{R}$, $r \mapsto \eta \frac{2^r}{\sqrt{r}}$.

Therefore, $\tilde{\Phi} := \Phi|_{\mathfrak{L}} \oplus \operatorname{Ad}$ is a faithful $R$-Lie algebra representation of $\mathfrak{L}$. Indeed,

$$\ker \tilde{\Phi} = \ker \Phi|_{\mathfrak{L}} \cap \ker \operatorname{Ad} \subseteq (\mathfrak{L} \setminus R_n(\mathfrak{L})) \cap Z(\mathfrak{L}) = \{0\}.$$

Thus, since $\operatorname{rk} R_n\left(\bar{\mathfrak{L}}\right) = \operatorname{rk} R_s(\mathfrak{L})$ and $f$ is non-decreasing,

$$\deg \mathfrak{L} \leq \deg \tilde{\Phi} = \deg \Phi + \deg \operatorname{Ad} \leq f(\operatorname{rk} R_s(\mathfrak{L})) + r \leq f(r) + r.$$ ■

**Remark 6.1.** For a $K$-Lie algebra $\mathfrak{L}$, with $K$ a field of characteristic zero, Harish-Chandra [7] improved the original result of Ado by constructing a finite faithful representation $\Psi\colon \mathfrak{L} \hookrightarrow \mathrm{End}_K(V)$ with the additional property of been a so-called nil-representation, i.e. $\Psi(x)$ is a nilpotent endomorphism for every $x \in R_n(\mathfrak{L})$. Note that the representation $\tilde{\Phi}$ of the preceding proof is also a nil-representation, as both $\Phi$ and $\mathrm{Ad}$ are so.

## References

[1] I. D. Ado: *The representation of Lie algebras by matrices*, Amer. Math. Soc. Translations 21 (1949); translation from Uspekhi Mat. Nauk (NS) 2/6 (1947) 159–173.

[2] Y. Bahturin: *Identitical Relations in Lie Algebras*, De Gruyter Expositions in Mathematics 68, De Gruyter, Berlin (2021).

[3] G. Birkhoff: *Representatibility of Lie algebras and Lie groups by matrices*, Ann. Math. 38 (1937) 526–532.

[4] D. Burde: *On a refinement of Ado's Theorem*, Arch. Math. 70 (1998) 118–127.

[5] V. A. Churkin: *The matrix representation of Lie algebras over rings*, Math. Notes 10 (1971) 835–839; translation from Mat. Zametki 10 (1971) 671–678.

[6] W. A. de Graaf: *Constructing faithful matrix representation of Lie algebras*, in: Proc. of the 1997 Int. Symp. on Symbolic and Algebraic Computation, ISSAC'97 held at Kihei 1997, W. W. Küchlin (ed.), ACM Press, New York (1997) 54–59.

[7] Harish-Chandra: *On representation of Lie algebras*, Ann. Math. 50 (1949) 900–915.

[8] K. Iwasawa: *On the representation of Lie algebras*, Jap. J. Math. 19 (1948) 405–426.

[9] N. Jacobson: *Lie Algebras*, Dover Publications, New York (1979).

[10] A. Mal'cev: *Solvable Lie algebras*, Izvestia Akad. Nauk SSSR 9 (1945) 329–356.

[11] Yu. Neretin: *A construction of finite-dimensional faithful representation of Lie algebras*, in: Proc. 22nd Winter School on "Geometry and Physics" held at Srní 2002, J. Bures (ed.), Rend. Circ. Mat. Palermo, Suppl. No. 71 (2003) 159–161.

[12] B. E. Reed: *Representations of solvable Lie algebras*, Michigan Math. J. 16 (1969) 227–233.

[13] T. Weigel: *A remark on the Ado-Iwasawa Theorem*, J. Algebra 212 (1999) 613–625.

Andoni Zozaya, Department of Mathematics, University of the Basque Country UPV/EHU, Leioa (Bizkaia), Spain; andoni.zozaya@ehu.eus.

Received July 3, 2023
and in final form March 4, 2024