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# Finite-Dimensional Construction of Self-Duality and Related Moduli Spaces over a Riemann Surface as Stratified Holomorphic Symplectic Spaces

**Johannes Huebschmann**

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*Dedicated to the memory of Peter Slodowy*

**Abstract.** In terms of appropriate extended moduli spaces, we develop a finite-dimensional construction of the self-duality and related moduli spaces over a Riemann surface as stratified holomorphic symplectic spaces by singular finite-dimensional holomorphic symplectic reduction.

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*Key Words:* Self-duality moduli space, analytic representation variety, holomorphic symplectic reduction, stratified holomorphic symplectic space, stratified hyperkähler space, extended moduli space.

## 1. Introduction

Our aim is to offer a description of the singularities for a class of moduli spaces including those of the self duality equations over a Riemann surface in the realm of what we call *stratified holomorphic symplectic spaces*, a stratified holomorphic symplectic space with a single stratum being a holomorphic symplectic manifold. To this end, we develop a purely finite-dimensional construction of the moduli spaces under discussion.

In [29, Thm. 6.7], N. Hitchin constructed, by infinite-dimensional methods, the moduli space of irreducible solutions to the SO(3) self-duality equations over a closed Riemann surface and showed that this moduli space acquires the structure of a hyperkähler manifold. Hitchin also considered corresponding moduli spaces for the group $GL_2$ and odd degree and showed they form complete moduli spaces [29, Thm. 6.1]. Extensions of the construction of such a self-duality moduli space as a hyperkähler manifold relative to a Lie group more general than SO(3) and $GL_2$ are in the literature; see [30, Section 6] and the literature there as well as what is said below. These moduli spaces have also come to be known as moduli spaces of stable *Yang-Mills-Higgs*-bundles or *Higgs*-bundles. In itself, such a moduli space is an interesting object, not only of Riemannian geometry but also of symplectic geometry; it has no singularities, and the resulting hyperkähler manifold exhibits interesting fundamental algebro-geometric properties as well.

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The entire moduli space of solutions (suitably defined) to the self-duality equations with regard to some fixed Lie group typically has singularities. This is true already for the moduli space of solutions to the SO(3) self-duality equations. We here address the issue of singularities seriously.

A hyperkähler manifold has a family of underlying holomorphic symplectic Kähler structures; given one such holomorphic symplectic Kähler structure, the hyperkähler constraint is a somewhat special one on the Kähler structure: a holomorphic symplectic Kähler structure underlies a hyperkähler structure if and only if the two tensor fields which the real and imaginary parts of the holomorphic symplectic structure define via the Kähler metric are almost complex structures.

The hyperkähler structure on a (non-singular) Yang-Mills-Higgs moduli space arises from the holomorphic structure of the underlying bundle in such a way that the complex structure of the fiber and the topology of the bundle determine one of the underlying holomorphic symplectic structures. We show here that, in the presence of singularities, that holomorphic symplectic structure on the (suitably defined) non-singular stratum of the entire moduli space (equivalently: on the moduli space of stable points) extends to a (suitably defined) stratified holomorphic symplectic structure on the entire space.

We derive this description of the singularities from a purely finite-dimensional construction for these moduli spaces realized, according to the *nonabelian Hodge* correspondence, as spaces of representations in a complex reductive Lie group of the fundamental group of the surface and twisted versions thereof. In particular, Simpson extended Hitchin's original result by showing that polystable Higgs bundles correspond to solutions to the self duality equations [67, 68] (Hitchin-Kobayashi correspondence for Higgs bundles) and Corlette [9] and Donaldson [12] established a correspondence between solutions to the self-duality equations and representations of the fundamental group (Hitchin-Kobayashi correspondence for complex connections).

The papers [19], [32], [40], [41], building on [43] and [76], settle a similar issue: a purely finite-dimensional construction of the moduli spaces of semistable holomorphic vector bundles on a Riemann surface, possibly punctured, and of generalizations thereof as stratified symplectic spaces in the sense of [71], realized, according to the *Hitchin-Kobayashi* correspondence for principal bundles on a Riemann surface, as spaces of twisted representations of the fundamental group in a compact Lie group. The construction proceeds by ordinary symplectic reduction applied to a finite-dimensional *extended moduli space* arising from a product of $2\ell$ copies of the Lie group (the group $\mathrm{U}(n)$ for the case of holomorphic rank $n$ vector bundles) where $\ell$ is the genus of the surface or, in the presence of punctures, from a variant thereof. This structure depends on the Lie group, a choice of an invariant inner product on its Lie algebra, and the topology of a corresponding bundle, but is independent of any complex structure on the underlying surface.

Here we proceed in the same way: According to the nonabelian Hodge correspondence, we construct analytic twisted representation varieties associated with the fundamental group by holomorphic symplectic reduction applied to a finite-dimensional extended moduli space arising from a product of $2\ell$ copies of the corresponding complexified Lie group, a complex reductive Lie group. Thus one can view such a twisted representation variety as a complexification of a twisted representation space of the kind we explored in [32]; since the polar decomposition and the inner product on the Lie algebra determine a diffeomorphism between the total space of the real cotangent bundle and the complexification of a compact Lie group, see Section 5 for details, one can also view such a twisted representation variety as the total space of a real cotangent bundle (beware: in the presence of singularities the interpretation of the term cotangent bundle is not immediate) of a twisted representation space of the kind we examined in [32]. Our main result, Theorem 5.1, says that the complex structure of the Lie group, a chosen invariant inner product on the Lie algebra, and a certain additional ingredient which corresponds to the topology of an associated bundle determine a stratified holomorphic symplectic structure on a twisted representation variety of the kind we study in this paper. Our approach includes in particular a new construction of a *Betti* moduli space (in the terminology of [67, 68]) as an analytic space and puts a stratified holomorphic symplectic structure on such a space. To avoid confusion we note that the terminology in [67, 68] is “character variety” for our “representation variety”.

To carry out the requisite holomorphic symplectic reduction and to extract structural information on the reduced level we extend results due to Mayrand [54]. Mayrand, in turn, builds on [71], [10, Theorem 2.1], an analytic version of the Kempf-Ness theorem due to Heinzer-Loose [26, Introduction §1.3 p. 289, §3.3 Theorem p. 295], and a holomorphic slice theorem [26, §2.7 Theorem p. 292], [70, Theorem 1.12 p. 100]. In the etale world, this kind of slice theorem goes back to [48]. Mayrand works exclusively with hyperkähler manifolds, and in Section 3 we show that his arguments apply to the more general setting in the present paper. Accordingly, Theorems 3.19, 3.20, 3.21, 3.24, 3.27, 3.29, 3.32, 3.33 parallel or extend results in [54]. Mayrand’s crucial technical result [54, Theorem 1.3] is a *holomorphic symplectic slice theorem* for Hamiltonian hyperkähler manifolds—it gives a local normal form for a holomorphic momentum mapping formally exactly of the same kind as the *Guillemin-Sternberg-Marle* local normal form of a real momentum mapping [17, 50, 51]—and Theorem 3.24 extends this observation to a *holomorphic symplectic slice theorem* for Hamiltonian holomorphic symplectic Kähler manifolds. In the algebraic setting, [47, Theorem 3] already establishes such a symplectic slice theorem for an algebraic Hamiltonian action of a reductive group on a non-singular affine symplectic variety.

Narasimhan-Seshadri constructed the moduli spaces of semistable holomorphic vector bundles by geometric invariant theory as normal projective varieties [57], together with a Kähler structure in the coprime case [55], see also [58]; Atiyah-Bott obtained these spaces by infinite dimensional methods and recovered the Kähler structure in the coprime case [4, Section 9 p. 587]. However it may happen that such a moduli space is non-singular as a projective variety but still exhibits singularities as a stratified symplectic space in the sense that it has more than one stratum. This happens, e.g., for the moduli space of semistable rank 2 degree zero holomorphic vector bundles with zero determinant on a Riemann surface of genus 2: This moduli space is a 3-dimensional complex projective space, and the non-stable semistable points constitute a Kummer surface [56]; the stratified symplectic Poisson structure is defined everywhere but the symplectic structures live only on the strata [35]. It is very likely that similar phenomena occur for the self duality moduli spaces as stratified holomorphic symplectic spaces. This is presumably in particular true for example for the self duality moduli space that corresponds to the moduli space of semistable rank 2 degree zero holomorphic vector bundles with zero determinant on a Riemann surface of genus 2. We expect this moduli space to be a complex manifold but to have more than one stratum as a stratified holomorphic symplectic space.

The paper [7] offers a construction of "wild character varieties" as algebraic varieties. In our approach analyticity is essential, and there is little overlap between [7] and the present paper. In particular, the presently available technology does not enable us to construct an algebraic variant of the kind of stratified holomorphic symplectic structure we introduce in this paper. Also, some of the results in [7] must be regarded as incomplete since [7] does not justify the existence of the Poisson brackets in the singular case. The technology we develop in this paper provides the necessary additional argument. We explain this briefly in Appendix I.

At the risk of being repetitive we note that, in what follows, the *singularities of a stratified complex analytic space* are the points in the complement of the top stratum; these are not necessarily the singularities of that complex analytic space [1], that is, a point in the complement of the top stratum is not necessarily a singularity relative to the complex analytic structure.

In terms of the notation $I$ and $J$ in [29], the purely finite-dimensional approach in the present paper covers the singularity structure of the moduli space with regard to the complex structure $J$. The paper [13] offers a description of the singularity structure of the moduli space relative to the complex structure $I$. This structure depends on the complex structure of the underlying surface. The techniques in[13] are infinite-dimensional and similar to technology developed in [31, 33] for the real case.

## 2. Group cohomology construction of a Hamiltonian holomorphic symplectic Kähler structure

To take care over the terminology: A complex *reductive* Lie group is an affine complex algebraic group that is reductive in the sense that every rational representation is completely reducible; also the terminology *linearly reductive* is in the literature. Equivalently, a complex reductive Lie group arises as the complexification of a compact Lie group, and an affine complex complex algebraic group is reductive if and only if the unipotent radical of its connected component of the identity (in the classical topology) is trivial. Thus a complex reductive Lie group is not necessarily connected.

Let $G$ be a complex reductive Lie group and $\cdot$ a non-degenerate $\mathbb{C}$-valued invariant symmetric bilinear form on its Lie algebra $\mathfrak{g}$. The Maurer-Cartan calculus in [19], [32, Section 1], [40], [41], [76] is then available over $\mathbb{C}$ for the group $G$. To recall its ingredients, let $\mathcal{A}$ denote the de Rham forms and, for a differential form $\alpha$ on $G$, let $\alpha_j$ $(j = 1, 2)$ denote the pullback of $\alpha$ by the projection $p_j$ to the $j$'th component:

1. the left invariant Maurer-Cartan form $\omega \in \mathcal{A}(G, \mathfrak{g})$ and the right invariant Maurer-Cartan form $\overline{\omega} \in \mathcal{A}(G, \mathfrak{g})$;
2. the triple product $\tau(x, y, z) = \frac{1}{2}[x, y] \cdot z,\ x, y, z \in \mathfrak{g}$;

3. the Cartan 3-form $\lambda = \frac{1}{12}[\omega,\omega]\cdot\omega$;
4. the 2-form $\Omega = \frac{1}{2}\omega_1 \cdot \overline{\omega}_2 \in \mathcal{A}^2(G\times G)$;
5. the equivariant 1-form $\vartheta\colon \mathfrak{g} \to \mathcal{A}^1(G)$ given by

$$\vartheta(X) = \tfrac{1}{2}X\cdot(\omega+\overline{\omega}),\ X\in\mathfrak{g}. \tag{1}$$

Let $\Sigma$ be a closed (real) surface of genus $\ell \geq 1$, and consider the standard presentation $\mathcal{P}$ of the fundamental group $\pi$ of $\Sigma$, viz.,

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r\rangle, \quad r = \Pi[x_j, y_j]. \tag{2}$$

The relator $r$ induces a complex algebraic map

$$r\colon G^{2\ell} \longrightarrow G. \tag{3}$$

Let $O \subseteq \mathfrak{g}$ be the open $G$-invariant subset of $\mathfrak{g}$ where the exponential mapping from $\mathfrak{g}$ to $G$ is regular; the reader will notice that $O$ contains the center of $\mathfrak{g}$. Define the space $\mathcal{H}(\mathcal{P},G)$ by requiring that

$$\begin{array}{ccc} \mathcal{H}(\mathcal{P},G) & \xrightarrow{r_O} & O \\ {\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\exp} \\ G^{2\ell} & \xrightarrow[r]{} & G \end{array} \tag{4}$$

be a pullback diagram; here we denote by $\eta$ and $r_O$ the induced maps. The space $\mathcal{H}(\mathcal{P},G)$ is a complex manifold and the induced map $\eta$ from $\mathcal{H}(\mathcal{P},G)$ to $G^{2\ell}$ is a holomorphic codimension zero immersion whence $\mathcal{H}(\mathcal{P},G)$ has the same dimension as $G^{2\ell}$.

Let $F$ be the free group on $x_1, y_1, \ldots, x_\ell, y_\ell$. Evaluation yields a bijection

$$\mathrm{Hom}(F,G) \to G^{2\ell}.$$

This induces an injection of $\mathrm{Hom}(\pi,G)$ into $\mathcal{H}(\mathcal{P},G)$ and, in this way, we view $\mathrm{Hom}(\pi,G)$ as a subspace of $\mathcal{H}(\mathcal{P},G)$.

Let $c \in C_2(F)$ be a 2-chain whose image in $C_2(\pi)$ is closed and represents a generator of $\mathrm{H}_2(\pi)\cong\mathbb{Z}$. Our approach is independent of a choice of complex structure on $\Sigma$ and hence we need not worry about the choice of an orientation. Let

$$\omega_{c,\mathcal{P}} = \eta^*(\omega_c) - r_O^*(\beta) \tag{5}$$

be the closed $G$-invariant 2-form on $\mathcal{H}(\mathcal{P},G)$ in [32, Thm. 1], let $\psi\colon \mathfrak{g}\to\mathfrak{g}^*$ denote the adjoint of the 2-form $\cdot$ on $\mathfrak{g}$, and recall that the composite

$$\mu_{c,\mathcal{P}}\colon \mathcal{H}(\mathcal{P},G) \xrightarrow{r_O} O \subseteq \mathfrak{g} \xrightarrow{\psi} \mathfrak{g}^* \tag{6}$$

is an equivariantly closed extension of $\omega_{c,\mathcal{P}}$ [32, Thm. 2] (written there as $\mu$). As for how $\psi$ arises in this context, see also the remark at the end of Section 1 of [32]. By construction, under the present circumstances, $\omega_{c,\mathcal{P}}$ and $\mu_{c,\mathcal{P}}$ are holomorphic. Let $\mathcal{M}(\mathcal{P},G)$ be the subspace of $\mathcal{H}(\mathcal{P},G)$ where the 2-form $\omega_{c,\mathcal{P}}$ is non-degenerate; this is an open $G$-invariant subset containing the pre-image $r^{-1}(\mathfrak{z})$. Abusing the notation slightly, denote the restriction of $\mu_{c,\mathcal{P}}$ to $\mathcal{M}(\mathcal{P},G)$ by $\mu_{c,\mathcal{P}}\colon\mathcal{M}(\mathcal{P},G)\to\mathfrak{g}^*$ as well. Then the triple

$$(\mathcal{M}(\mathcal{P},G), \omega_{c,\mathcal{P}}, \mu_{c,\mathcal{P}}) \tag{7}$$

is a complex $G$-Hamiltonian manifold.

Applying the procedure of symplectic reduction naively to

$$(\mathcal{M}(\mathcal{P},G),\omega_{c,\mathcal{P}},\mu_{(\mathcal{P},G)})$$

poses problems since we need a "good" analytic (or Hilbert) $G$-quotient [24] of analytic sets of the kind $\mu^{-1}_{(\mathcal{P},G)}(q)$ for points $q$ in the dual $\mathfrak{z}^*$ of the center $\mathfrak{z}$ of $\mathfrak{g}$, this dual $\mathfrak{z}^*$ being well-defined since $\mathfrak{z}$ is a direct summand of $\mathfrak{g}$. In the next section we show how results in [26] and [54] enable us to overcome these difficulties.

**Remark 2.1.** An extended moduli space arises as a special case of a general construction which renders lattice gauge theory rigorous [34]. ■

## 3. Reduction of Hamiltonian holomorphic symplectic Kähler manifolds

For a smooth symplectic manifold with a Hamiltonian action of a compact Lie group, Sjamaar-Lerman proved that the reduced space acquires a stratified symplectic structure [71]. Their arguments rely on the *Guillemin-Sternberg-Marle* local normal form of the momentum mapping [17, 50, 51]. Sjamaar-Lerman [71] noted that this normal form implies that, locally, such a reduced space is isomorphic to one arising from linear symplectic reduction and thereby extended the *Darboux* theorem to such reduced spaces. Also, from the local model, they deduced that the orbit type decomposition is a *Whitney stratification*.

In [54], Mayrand addresses these issues in the holomorphic setting. He settles them merely for Hamiltonian hyperkähler manifolds but his arguments, suitably extended, work for Hamiltonian holomorphic symplectic Kähler manifolds, and this extension clarifies the nature of the arguments, simplifies the exposition and, as we show in this paper, opens a wealth of attractive examples. Here we extend this approach to Hamiltonian holomorphic symplectic Kähler manifolds, taylored to our purposes.

### 3.1. Decomposed and stratified spaces

A *decomposed* space is a space $X$ together with a family of pairwise disjoint subspaces that are smooth manifolds, the *pieces* of the decomposition, such that $X$ is the union of the pieces. For a decomposed space $X$, we use the notation $C^\infty(X)$ for an algebra of real-valued continuous functions on $X$, a *smooth structure* [64], which, on each piece of the decomposition, are ordinary smooth functions; we then denote by $C^\infty(X,\mathbb{C})$ the obvious extension of $C^\infty(X)$ to an algebra of complex-valued continuous functions on $X$. There is no claim to the effect that the restriction $C^\infty(X)\to C^\infty(S)$ to a stratum $S$ be onto; in the situations under discussion below, the image of the restriction will contain the compactly supported functions on that stratum, and this suffices for characterizing the various geometric structures under discussion; thus, there is no need to "sheafify" the smooth structures. Below we use the term 'stratification' and 'stratified' space but, deliberately, we do not make this precise. In particular, 'stratified' space could simply mean 'decomposed' space, and the definitions still make sense. Mather's definition [52] provides a good understanding of the idea of a statification; see also [60] and the literature there. For intelligibility we recall that a stratification (in the sense of Mather) of a space $X$ is a map $\mathcal{S}$ which assigns to each point $x$ of $X$ the set germ $\mathcal{S}_x$ of a locally closed subset of $X$ such that the following holds: *For each $x\in X$ there is an open* *neighborhood $U$ of $x$ and a decomposition $\mathcal{Z}_U$ of $U$ such that, for $y \in U$, the set germ $\mathcal{S}_y$ coincides with the set germ of the unique piece $R_y \in \mathcal{Z}_U$ which contains $y$ as an element.*

Recall that a *stratified Kähler space* [36, 37, 38] consists of a complex analytic space $X$, together with

(i) a complex analytic stratification (a not necessarily proper refinement of the standard complex analytic stratification, cf. [1]), and with

(ii) a real stratified symplectic structure $(C^\infty X, \{\,\cdot\,,\,\cdot\,\})$ [71] which is compatible with the complex analytic structure.

The two structures being *compatible* means the following:

(i) For each point $q$ of $X$ and each holomorphic function $f$ defined on an open neighborhood $U$ of $q$, there is an open neighborhood $V$ of $q$ with $V \subset U$ such that, on $V$, $f$ is the restriction of a function in $C^\infty(X, \mathbb{C})$;

(ii) on each stratum, the symplectic structure determined by the symplectic Poisson structure (on that stratum) combines with the complex analytic structure to a Kähler structure.

We extend this terminology to the hyperkähler setting as follows; we recall that the three Kähler forms $\omega_{\mathsf{I}}$, $\omega_{\mathsf{J}}$, $\omega_{\mathsf{K}}$ of a hyperkähler structure $(g, \mathsf{I}, \mathsf{J}, \mathsf{K})$ on a smooth manifold $M$ encapsulate the entire hyperkähler structure: with the notation $\omega_{\mathsf{I}}^\sharp \colon \mathrm{T}M \to \mathrm{T}^*M$ etc. for the adjoint of $\omega_{\mathsf{I}}$ etc., necessarily $\omega_{\mathsf{J}}^{-1} \circ \omega_{\mathsf{I}} = \mathsf{K}$ etc.

**Definition 3.1.** (1) A *stratified hyperkähler space* consists of a stratified space $X$, a smooth structure $C^\infty(X)$ on $X$, and three Poisson structures $\{\,\cdot\,,\,\cdot\,\}_1$, $\{\,\cdot\,,\,\cdot\,\}_2$, $\{\,\cdot\,,\,\cdot\,\}_3$ on $C^\infty(X)$ so that, on each stratum, for $j = 1, 2, 3$, the bracket $\{\,\cdot\,,\,\cdot\,\}_j$ is the Poisson structure associated with a symplectic structure $\omega_j$ and that $\omega_1$, $\omega_2$, $\omega_3$ constitute a hyperkähler structure on that stratum.

(2) A *stratified holomorphic symplectic space* consists of a complex analytic space $(X, \mathcal{O}_X)$ together with a complex analytic stratification and a holomorphic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_X$ on the sheaf $\mathcal{O}_X$ of germs of holomorphic functions on $X$ which, on each stratum, restricts to the holomorphic Poisson structure associated with a holomorphic symplectic structure on that stratum.

(3) A *stratified holomorphic symplectic Kähler space* consists of a stratified Kähler space $(X, C^\infty(X), \mathcal{O}_X, \{\,\cdot\,,\,\cdot\,\}_{\mathbb{R}})$, together with a holomorphic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_{\mathbb{C}}$ on the sheaf $\mathcal{O}_X$ of germs of holomorphic functions on $X$ which, on each stratum, restricts to the holomorphic Poisson structure associated with a holomorphic symplectic structure on that stratum.

(4) A *weak stratified hyperkähler space* is a stratified holomorphic symplectic Kähler space $(X, C^\infty(X), \mathcal{O}_X, \{\,\cdot\,,\,\cdot\,\}_{\mathbb{R}}, \{\,\cdot\,,\,\cdot\,\}_{\mathbb{C}})$ such that, on each stratum, the pieces of structure combine to an ordinary hyperkähler structure.

(5) A *stratified hyperkähler space* is a stratified space together with (i) three complex analytic structures $\mathcal{O}_{\mathsf{I}}$, $\mathcal{O}_{\mathsf{J}}$, $\mathcal{O}_{\mathsf{K}}$ which are compatible with the stratification and (ii) three pairwise compatible real Poisson structures $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}$ such that

$$(\mathcal{O}_{\mathsf{I}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}), (\mathcal{O}_{\mathsf{J}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}), (\mathcal{O}_{\mathsf{K}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}})$$

are holomorphic Poisson structures which are compatible with the stratification and, on each stratum, restrict to an ordinary hyperkähler structure. ■

A stratified hyperkähler structure generates a sphere of complex analytic and compatible real Poisson structures.

**Remark 3.2.** Mayrand gives the definition of a stratified hyperkähler space as [53, Definition 3.1.9]. ■

### 3.2. Quotients

Let $G$ be a topological group. For a $G$-space $X$, a *$G$-subset* is a subset of $X$ that is closed under the $G$-action. For $G$-space $Y$, we say a $G$-invariant map $\pi\colon Y \to Y_0$ to a space $Y_0$ (with trivial $G$-action) is a *$G$-reduction* if

($G$-red 1) every fiber $\pi^{-1}(y_0) \subseteq Y$, as $y_0$ ranges over $Y_0$, contains exactly one closed $G$-orbit.

Let $Y$ be a $G$-space. As in [49, §1.1 p. 173], consider the following property, see also (∗) [61, §7.2 p. 420]:

(A) Each $G$-orbit in $Y$ contains in its closure a unique closed $G$-orbit.

Let $Y$ be a $G$-space enjoying property (A). Extending a construction in [49, §1.1 p. 173], see also [61, §7.2 p. 421], define the *quotient $Y//G$ of $Y$ by $G$* to be the space whose points are the closed $G$-orbits in $Y$, the *$G$-quotient map* $\pi_Y\colon Y \to Y//G$ to be the map which assigns to a point of $Y$ the unique closed orbit in the closure of its $G$-orbit, and endow $Y//G$ with the quotient topology. Then $\pi_Y\colon Y \to Y//G$ is a $G$-reduction.

### 3.3. Holomorphic symplectic reduction

Let $(M, \omega_{\mathbb{C}})$ be a holomorphic symplectic manifold, let $G$ be a complex reductive Lie group, and suppose $G$ acts holomorphically on $M$ preserving $\omega_{\mathbb{C}}$. We say $(M, \omega_{\mathbb{C}})$ is a *$G$-holomorphic symplectic manifold.* Write the Lie algebra of $G$ as $\mathfrak{g}$ and suppose that, furthermore, the $G$-action is Hamiltonian, with holomorphic momentum mapping $\mu_{\mathbb{C}}\colon M \to \mathfrak{g}^*$. We refer to $(M, \omega_{\mathbb{C}}, \mu_{\mathbb{C}})$ as a *$G$-Hamiltonian holomorphic symplectic manifold.*

Our goal is to build the analogue of the stratified symplectic structure on the reduced space for the real case recalled at the beginning of this section. The present aim is to show that the analytic variant of Kempf-Ness theory in [23] yields the requisite complex analytic quotient of the zero locus $\mu_{\mathbb{C}}^{-1}(0)$ as a complex analytic space. To this end suppose that $M$ possesses, independently of $\omega_{\mathbb{C}}$, an ordinary real Kähler form $\omega_{\mathbb{R}}$ invariant under a maximal compact subgroup $K$ of $G$, and suppose the $K$-action on $M$ is Hamiltonian with momentum mapping $\mu_{\mathbb{R}}\colon M \to \mathfrak{k}^*$. Consider the subspace

$$M^{\mu_{\mathbb{R}}-\mathrm{ss}} = \{q \in M; \overline{Gq} \cap \mu_{\mathbb{R}}^{-1}(0) \neq \emptyset\} \tag{8}$$

of *momentum semistable points of $M$* with respect to $\mu_{\mathbb{R}}$, cf. [23, Section 0] for the terminology; these are the *analytically semistable points* in the sense of [70, Definition 2.2 p. 109]. The following summarizes various results in the literature.

**Proposition 3.3.** *Suppose the subspace $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ of momentum semistable points in $M$ is non-empty.*

(1) *The subspace $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ is $G$-invariant and open in $M$, indeed, the smallest $G$-invariant open subspace of $M$ containing $\mu_{\mathbb{R}}^{-1}(0)$.*

(2) *The zero locus $\mu_{\mathbb{R}}^{-1}(0)$ is a* Kempf-Ness set (fiber critical set)*, that is,*

(KN 1) *for $x \in M^{\mu_{\mathbb{R}}-\mathrm{ss}}$, the orbit $Gx$ is closed in $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ if and only if*

$$Gx \cap \mu_{\mathbb{R}}^{-1}(0) \neq \emptyset;$$

(KN 2) *for $x \in \mu_{\mathbb{R}}^{-1}(0)$, the $K$-orbit $Kx$ coincides with $Gx \cap \mu_{\mathbb{R}}^{-1}(0)$.*

(3) *The $G$-manifold $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ admits a $G$-reduction $\pi\colon M^{\mu_{\mathbb{R}}-\mathrm{ss}} \to M^{\mu_{\mathbb{R}}-\mathrm{ss}}//G$ in such a way that the inclusion $\mu_{\mathbb{R}}^{-1}(0) \subseteq M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ induces a homeomorphism*

$$M//_{\mu_{\mathbb{R}}}K = \mu_{\mathbb{R}}^{-1}(0)/K \to M^{\mu_{\mathbb{R}}-\mathrm{ss}}//G. \tag{9}$$

*In particular, the quotient space $\mu_{\mathbb{R}}^{-1}(0)/K \cong M^{\mu_{\mathbb{R}}-\mathrm{ss}}//G$ is a Hausdorff space.*

(4) *The subspace $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ is dense in $M$.*

Claims (1)–(3) are due to [26] (§ 1.3 Thm. p. 289, § 3.3 Thm. p. 295). Under the additional assumption that the gradient flow of the negative of the norm square of $\mu_{\mathbb{R}}$ be globally defined they are in [70, Proposition 2.4 p. 110, Theorem 2.5 p. 112]; this assumption holds, e.g., when $\mu_{\mathbb{R}}$ is proper. Under even more restrictive hypotheses these observations are due to [44]. Claim (4) is [23, Lemma in Section 9 p. 83].

The reasoning in [26] for the openness of $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ is somewhat cryptic. This openness is certainly well understood among the experts. However, the non-expert will have difficulties extracting a proof from the literature. We therefore take the liberty of sketching a proof, concocted with the help of P. Heinzer. A proof substantially different from that we are about to reproduce is in [28].

Consider a Kaehler manifold $(X, \omega)$ with a holomorphic $G$-action whose restriction to a maximal compact subgroup $K$ preserves $\omega$. Recall a $K$-invariant function $\varphi\colon X \to \mathbb{R}$ is a *(Kaehler) potential* when

$$\omega = -\tfrac{1}{2} dd^c\phi = i\partial\overline{\partial}\phi,\ d^c = i(\partial - \overline{\partial}); \tag{10}$$

then $\varphi$ is necessarily strictly plurisubharmonic and, with the notation $\xi_X$ for the vector field on $X$ which a member $\xi$ of the Lie algebra $\mathfrak{k}$ of $K$ induces, the identity

$$\xi \circ \mu = \tfrac{1}{2}(d^c\phi)(\xi_X) = \tfrac{1}{2}(d\phi)(J\xi_X),\ \xi \in \mathfrak{k}, \tag{11}$$

characterizes a $K$-momentum mapping $\mu\colon X \to \mathfrak{k}^*$ which renders the $K$-action on $X$ Hamiltonian with respect to $\omega$. For a Hausdorff $G$-quotient $\pi\colon X \to Q$ (provided it exists) a *relative exhaustion* [24, § 3.1 p. 330, § 3.3 p. 336] is a smooth $K$-invariant function $\psi\colon X \to \mathbb{R}$ that is bounded from below and has the property that

$$\psi \times \pi\colon X \to \mathbb{R} \times Q \tag{12}$$

is proper. When $X$ is Stein, a Hausdorff quotient exists as a Stein space [72], [24, Proposition 3.1.2 p. 328]. Let $N$ be a Stein manifold with a holomorphic $G$-action and a strictly plurisubharmonic relative exhaustion function $\varphi\colon N \to \mathbb{R}$ invariant under a maximal compact subgroup $K$ of $G$. We then say $(N, G, K, \varphi)$ is a *relative exhaustion Stein $G$-manifold.* Recall a real function $f$ on a (reasonable topological) space $D$ is an *exhaustion function* if $\{z; f(z) < r\} \subseteq D$ is relatively compact in $D$ for any real $r$. Here is [25, Lemma 1 p. 131] in another guise:

**Proposition 3.4.** *For a relative exhaustion Stein $G$-manifold $(N, G, K, \psi)$, the momentum semistable subspace relative to the momentum mapping $\mu\colon N \to \mathfrak{k}^*$ which $\psi$ induces via* (11) *coincides with $N$.*

**Proof.** Since the restriction of $\psi$ to a fiber is proper and bounded, it is an exhaustion on that fiber and, in view of (11), the restriction of $\psi$ to a closed orbit has a critical point, necessarily an absolute minimum. Hence, with respect to the Stein quotient map $\pi\colon N \to Q$,

$$\mu^{-1}(0) = \{p; \psi|_{\pi^{-1}\pi(p)} \text{ attains its minimum at } p\}. \tag{13}$$

This observation implies that the restriction $\pi|_{\mu^{-1}(0)}$ of $\pi$ to $\mu^{-1}(0)$ is surjective and induces a continuous bijective map $\mu^{-1}(0)/K \to Q$. See, e.g., the proof of [24, Section 3 Proposition 3.1.5 p. 329]. Since $\psi$ is bounded from below and $\psi \times \pi$ proper, the map $\mu^{-1}(0)/K \to Q$ is a homeomorphism [25, Lemma 1 p. 131], [24, Section 3 Proposition 3.1.7 p. 331]. ■

**Proof of openness of $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ in $M$.** Let $q$ be a point of the zero locus $\mu_{\mathbb{R}}^{-1}(0)$. The proof of the slice theorem [26, §2.7 Theorem p. 292] yields an open slice neighborhood $N$ of $q$ in $M$ that underlies a relative exhaustion Stein $G$-manifold $(N, G, K, \psi)$ in such a way that $\psi$ determines the restrictions to $N$ of $\omega_{\mathbb{R}}$ and $\mu_{\mathbb{R}}$. The construction of $\psi$ builds on a similar construction in [25] and in particular relies on [25, Lemma 2]. ■

With these preparations out of the way, let

$$\mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-\mathrm{ss}} = \mu_{\mathbb{C}}^{-1}(0) \cap M^{\mu_{\mathbb{R}}-\mathrm{ss}}, \quad \text{and} \quad \mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})\colon M \longrightarrow \mathfrak{k}^* \times \mathfrak{g}^*.$$

Then

$$M_0 := \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-\mathrm{ss}}//G \tag{14}$$

is the analytic quotient of $\mu_{\mathbb{C}}^{-1}(0)$ we seek, and the inclusion $\mu_{\mathbb{R}}^{-1}(0) \subseteq M^{\mu_{\mathbb{R}}-\mathrm{ss}}$ induces a homeomorphism

$$\mu^{-1}(0)/K \longrightarrow M_0 = \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-\mathrm{ss}}//G. \tag{15}$$

The orbit space $\mu^{-1}(0)/K$ is a topological model for the analytic quotient $M_0$ of $\mu_{\mathbb{C}}^{-1}(0)$.

### 3.4. Holomorphic local model

**3.4.1.** $\mathrm{T}^*G$**.** Endow $\mathrm{T}^*G$ with the algebraic cotangent bundle symplectic structure and identify $\mathrm{T}^*G$ biholomorphically with $G \times \mathfrak{g}^*$ via left translation. Accordingly the action of $G \times G$ on $\mathrm{T}^*G$ which left and right translation on $G$ induces takes the form

$$G \times G \times G \times \mathfrak{g}^* \longrightarrow G \times \mathfrak{g}^*,\ (x, y, u, \xi) \longmapsto (xuy^{-1}, \mathrm{Ad}_y^*\xi), \tag{16}$$

and the association

$$G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^*,\ (x, \xi) \longmapsto (\mathrm{Ad}_x^*\xi, -\xi). \tag{17}$$

characterizes the algebraic $G \times G$-momentum mapping that turns $\mathrm{T}^*G \cong G \times \mathfrak{g}^*$ into a $G \times G$-Hamiltonian complex algebraic manifold relative to the algebraic cotangent bundle symplectic structure.

**3.4.2. Geometry of the local model.** Let $H$ be a reductive subgroup of $G$ and $V$ a complex symplectic representation of $H$. Write the complex symplectic form on $V$ as $\omega_V$. The familiar algebraic momentum mapping

$$\Phi_V : V \longrightarrow \mathfrak{h}^*, \ \Phi_V(v)(x) = \tfrac{1}{2}\omega_V(xv, v), \tag{18}$$

turns $V$ into a complex algebraic Hamiltonian $H$-space. Relative to the embedding of $H$ into $G \times G$ via the second copy of $G$, take the product momentum mapping

$$\lambda : G \times \mathfrak{g}^* \times V \longrightarrow \mathfrak{h}^*, \ \lambda(x, \xi, v) = \Phi_V(v) - \xi|_{\mathfrak{h}}. \tag{19}$$

Zero is a regular value of $\lambda$, the reduced space $E = (\mathrm{T}^*G \times V)//_\lambda H$ is a complex algebraic manifold, acquires an algebraic symplectic structure which we write as $\omega_E$ and, furthermore, via the first copy of $G$, an algebraic Hamiltonian $G$-action, with algebraic momentum mapping coming from (17).

Take a $K$-invariant hermitian inner product on $\mathfrak{g}$ and let $\mathfrak{m}$ be the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$. This identifies $\mathfrak{h}^*$ with the annihilator $\mathfrak{m}^o$ of $\mathfrak{m}$ in $\mathfrak{g}^*$ and $\mathfrak{m}^*$ with the annihilator $\mathfrak{h}^o$ of $\mathfrak{h}$ in $\mathfrak{g}^*$, and we thereby view $\Phi_V$ as taking values in $\mathfrak{g}^*$. Then $E$ appears as the total space of the algebraic vector bundle $E = G \times_H (\mathfrak{h}^o \times V) \to G/H$, and the algebraic momentum mapping reads

$$\kappa : G \times_H (\mathfrak{h}^o \times V) \longrightarrow \mathfrak{g}^*, \ [x, \xi, v] \longmapsto \mathrm{Ad}^*_x(\xi + \Phi_V(v)). \tag{20}$$

Furthermore, the zero section embedding $G/H \to E$ is isotropic relative to $\omega_E$.
The canonical injection $V \to G \times_H (\mathfrak{h}^o \times V)$ induces, via the commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & G \times_H (\mathfrak{h}^o \times V) \\ {\scriptstyle \Phi_V}\downarrow & & \downarrow{\scriptstyle \kappa} \\ \mathfrak{m}^o & \longrightarrow & \mathfrak{g}^*, \end{array} \tag{21}$$

an isomorphism

$$\Phi_V^{-1}(0)//H \longrightarrow \kappa^{-1}(0)//G \tag{22}$$

of algebraic GIT-quotients.

**3.4.3. Topology of the local model in terms of Kempf-Ness theory.** Write $V_0 = \Phi_V^{-1}(0)//H$ and let $\pi : \Phi_V^{-1}(0) \to V_0$ denote the quotient map. Let $\sigma_V$ be a (real) Kähler form on $V$ invariant under $L = H \cap K$, let $\mathfrak{l}$ denote the Lie algebra of $L$, let

$$\mu_{\sigma_V} : V \longrightarrow \mathfrak{l}^*, \ x \circ \mu_{\sigma_V}(v) = \tfrac{1}{2}\sigma_V(xv, v), \ v \in V, \ \mathfrak{l} \ni x : \mathfrak{l}^* \to \mathbb{R}, \tag{23}$$

be the associated momentum mapping having the value zero at the origin, and consider

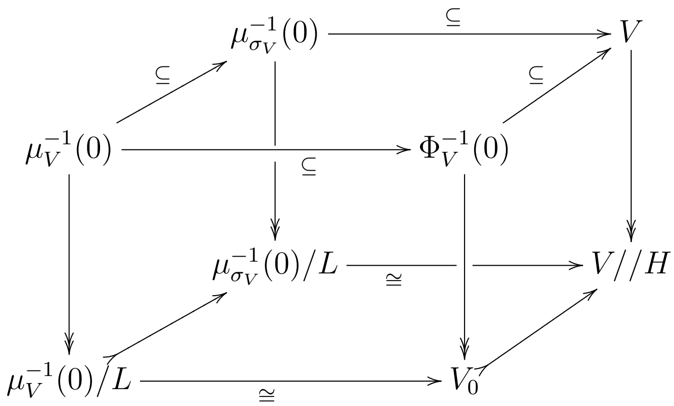
$$\mu_V = (\mu_{\sigma_V}, \Phi_V) : V \longrightarrow \mathfrak{l}^* \times \mathfrak{h}^*. \tag{24}$$

The injection $\mu_V^{-1}(0) \subseteq \Phi_V^{-1}(0)$ induces a homeomorphism

$$\mu_V^{-1}(0)/L \to V_0 = \Phi_V^{-1}(0)//H.$$

The injection $\mu_{\sigma_V}^{-1}(0) \subseteq V$ induces, likewise, a homeomorphism $\mu_{\sigma_V}^{-1}(0)/L \to V//H$.

The left-hand side characterizes the topology and the right-hand side the complex algebraic structure of $V//H$, and the diagram




(25)

is commutative. By construction, the domain of each inclusion written as $\subseteq$ and of each injection written as $\rightarrowtail$ carries the induced topology, the range of each surjection written as $\twoheadrightarrow$ carries the quotient topology, and the arrows labeled $\cong$ are homeomorphisms. Indeed, as for the topologies of the spaces in the upper square the claim is immediate, and the homeomorphisms result form GIT. Since the group $L$ is compact, it is immediate that $\mu_V^{-1}(0)/L$ carries the topology induced from $\mu_{\sigma_V}^{-1}(0)/L$. Hence $V_0$ carries the topology induced from $V//H$.

**Proposition 3.5.** *A subset $U$ of $V_0 = \Phi_V^{-1}(0)//H$ is open if and only if, relative to the quotient map $\pi\colon \Phi_V^{-1}(0) \to V_0 = \Phi_V^{-1}(0)//H$, there is an $H$-saturated open subset $W$ of $V$ such that $\pi^{-1}(U) = \Phi_V^{-1}(0) \cap W$.*

**Proof.** A subset $U$ of $V_0 = \Phi_V^{-1}(0)//H$ is open if and only if there is an open subset $W'$ of $V//H$ such that $U = V_0 \cap W'$. The pre-image of $W'$ in $V$ is $H$-saturated. This implies the claim. ■

**3.4.4. Variation of the Hamiltonian structure of the local model.** Let $\eta_{\mathbb{C}}$ be a $G$-Hamiltonian holomorphic symplectic structure on $E = G \times_H (\mathfrak{h}^o \oplus V)$ with momentum mapping $\mu_{\mathbb{C}}\colon E \to \mathfrak{g}^*$ and suppose that the zero section embedding $G/H \to E$ is isotropic relative to $\eta_{\mathbb{C}}$. Proposition 2 in [47, §3.2 p. 222], taken up in the proof of [54, Theorem 1.3], says the following.

**Proposition 3.6.** *There is a $G$-equivariant biholomorphism $\chi\colon E \to E$ such that $\chi^*(\eta_{\mathbb{C}})$ and the canonical algebraic symplectic structure $\omega_E$ on $E$ recalled in Subsection* 3.4.2 *coincide on the image $Z$ of the zero section embedding $G/H \to E$.* ■

The holomorphic extension [47, §3.3 p. 223], [54, Section 3] of the Darboux-Weinnstein theorem [75, Theorem 4.1, Corollary 4.3], reproduced in [18, Theorem 22.1], [5, Theorem 6], [59, 7.3.1 Theorem], implies the following.

**Proposition 3.7.** *Suppose that the restrictions of $\eta_{\mathbb{C}}$ and $\omega_E$ to the image $Z$ of the zero section embedding $G/H \to E$ coincide. Then there are open $G$-invariant neighborhoods $U_0$ and $U_1$ of $Z$ in $E$ and a $G$-equivariant biholomorphism $\vartheta\colon U_0 \to U_1$ such that $\vartheta^*(\eta_{\mathbb{C}}) = \omega_E$ and $\vartheta|_Z = \mathrm{Id}_Z$.* ■

**3.4.5. Affine complex structure on $V_0$.** Consider the commutative diagram

$$\begin{array}{ccc} \Phi_V^{-1}(0) & \xrightarrow{\subseteq} & V \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi} \\ V_0 & \xrightarrow[\subseteq]{} & V//H. \end{array} \tag{26}$$

On the one hand, the space $V_0$ acquires the structure of an affine variety as a subvariety of $V//H$ and, on the other hand, it acquires such a structure as an affine GIT-quotient of the complex algebraic set $\Phi_V^{-1}(0)$. At the risk of making a mountain out of a molehill, we now show the two structures coincide; this prepares for the analytic case in Subsection 3.4.6 below:

The affine coordinate ring $\mathbb{C}[V//H]$ of the algebraic GIT-quotient $V//H$ is the ring $\mathbb{C}[V]^H$ of $H$-invariants in the affine coordinate ring $\mathbb{C}[V]$ of $V$ and, when we view $V_0$ as an affine subvariety of $V//H$, that is, as the affine subvariety $\Pi(\Phi_V^{-1}(0))$ of $V//H$, the affine coordinate ring $\mathbb{C}[\Pi(\Phi_V^{-1}(0))]$ thereof arises as the quotient of $\mathbb{C}[V]^H$ modulo the vanishing ideal of $V_0$ in $\mathbb{C}[V]^H$. Thus a (continuous) complex-valued function $f$ on $V_0$ belongs to this coordinate ring of $V_0$ if and only if there exists an $H$-invariant function $\widehat{f}$ in the affine coordinate ring $\mathbb{C}[V]$ of $V$ that renders a diagram of the kind

$$\begin{array}{ccc} \Phi_V^{-1}(0) & \xrightarrow{\subseteq} & V \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\widehat{f}} \\ V_0 & \xrightarrow[f]{} & \mathbb{C} \end{array} \tag{27}$$

commutative. On the other hand, the affine coordinate ring $\mathbb{C}[\Phi_V^{-1}(0)//H]$ of the algebraic GIT-quotient $\Phi_V^{-1}(0)//H$ is the ring $\mathbb{C}[\Phi_V^{-1}(0)]^H$ of $H$-invariants in the affine coordinate ring $\mathbb{C}[\Phi_V^{-1}(0)]$ of the complex algebraic set $\Phi_V^{-1}(0)$. Thus a (continuous) complex-valued function $f$ on $V_0$ belongs to $\mathbb{C}[\Phi_V^{-1}(0)//H]$ if and only if there exists a function $f^\sharp$ in the affine coordinate ring $\mathbb{C}[V]$ of $V$ that renders a diagram of the kind (27) commutative, with $f^\sharp$ substituted for $\widehat{f}$. While the composite $f\circ\pi$ is $H$-invariant, there is, at first, no reason for $f^\sharp$ to be $H$-invariant. Rendering $f^\sharp$ invariant under the maximal compact subgroup $L$ of $H$ yields an $H$-invariant extension, however: the function $\widehat{f}$ which the identity

$$\widehat{f}(v) = \int_L f^\sharp(xv)dx \tag{28}$$

characterizes is $L$-invariant and hence $H$-invariant. Consequently the canonical injection $\mathbb{C}[\Pi(\Phi_V^{-1}(0))] \rightarrowtail \mathbb{C}[\Phi_V^{-1}(0)//H]$ is an isomorphism whence the canonical map from $\Phi_V^{-1}(0)//H$ to $\Pi(\Phi_V^{-1}(0))$ is an isomorphism of affine varieties. Accordingly,

$$V_0 = \mathrm{Hom}_{Alg}(\mathbb{C}[\Phi_V^{-1}(0)]^H, \mathbb{C}) = \mathrm{Spec}(\mathbb{C}[\Phi_V^{-1}(0)]^H). \tag{29}$$

Since $\mathbb{C}[V]^H = \mathbb{C}[V]^L$ and $\mathbb{C}[\Phi_V^{-1}(0)]^H = \mathbb{C}[\Phi_V^{-1}(0)]^L$, we can also think of the affine $H$-quotients $\pi$ and $\Pi$ as affine $L$-quotients, that is, $V_0 = \Phi_V^{-1}(0)//L$ and $V//H = V//L$.

**3.4.6. Complex analytic structure on $V_0$.** The (affine space associated to the) vector space $V$ carries its standard complex analytic structure, and restriction of holomorphic functions endows the complex algebraic set $\Phi_V^{-1}(0)$ with a complex analytic structure. Since $V$ is a complex $H$-representation, setting $V//H = V//L$ makes sense, that is, the quotient is independent of the choice of maximal compact subgroup [27, §1 p. 235], and the sheaf of germs of holomorphic functions on $V//H = V//L$ arising from the assignment to an open set $U$ in $V//L$ of the $L$-invariant holomorphic functions on the $H$-invariant open set $\Pi^{-1}(U)$ of $V$ turns the quotient $V//H = V//L$ into a complex analytic space. Setting $\Phi_V^{-1}(0)//H = \Phi_V^{-1}(0)//L$ makes, likewise, sense, and the sheaf $\mathcal{O}_{V_0}$ of germs of holomorphic functions on $V_0$, viewed as the quotient $\Phi_V^{-1}(0)//L$, arises as follows: Let $U$ be an open set in $V_0$; then $\pi^{-1}(U)$ is open in $\Phi_V^{-1}(0)$, that is, for some open set $U'$ in $V$, the subset $\pi^{-1}(U)$ coincides with $\Phi_V^{-1}(0) \cap U'$; a complex-valued function $f$ on $U$ is holomorphic, i.e., belongs to $\mathcal{O}_{V_0}(U)$, if and only if there exists a holomorphic function $\widehat{f}$ on $U'$ that renders a diagram of the kind (27), with $U'$ and $U$ substituted for $V$ and $V_0$, respectively, commutative. On the other hand, the sheaf $\mathcal{O}_{(V//L)|V_0}$ of germs of holomorphic functions on $V_0$ as a subvariety of $V//L$ arises from restriction.

Results in [22], see also [21], imply the following.

**Proposition 3.8.** *The canonical morphism $\mathcal{O}_{(V//L)|V_0} \to \mathcal{O}_{V_0}$ of sheaves is an isomorphism, that is, the complex analytic structure on $V_0$ as a quotient of $\Phi_V^{-1}(0)$ and that as a subvariety of $V//L$ coincide.*

Proposition 3.8 is of crucial importance for the construction of the Poisson structure in Subsection 3.4.8 below. It is an immediate consequence of the following elementary:

**Proposition 3.9.** *For $U$ open in $V_0$ and a holomorphic function $f$ on $U$, there is an open $L$-invariant set $U'$ in $V$ containing $\pi^{-1}(U)$ and an $L$-invariant holomorphic function $\widehat{f}$ on $U'$ that render a diagram of the kind* (27)*, with $U'$ and $U$ substituted for $V$ and $V_0$, respectively, commutative.*

**Proof.** For a closed subset $A$ of $V$, the set $LA$ is as well closed. Hence, for an open neighborhood $W$ of $\Phi_V^{-1}(0)$ in $V$, the set $W' = \cap_{x\in L} xW$ is open, necessarily $L$-invariant, and plainly an open neighborhood of $\Phi_V^{-1}(0)$ in $V$. (I am indebted to P. Heinzner for having communicated this argument to me.) Consequently, for $U$ open in $V_0$ and a holomorphic function $f$ on $U$, there is an open $L$-invariant set $U'$ in $V$ containing $\pi^{-1}(U)$ and a holomorphic function $f^\sharp$ on $U'$ that make, with $f^\sharp$ substituted for $\widehat{f}$, a diagram of the kind (27) commutative, and rendering $f^\sharp$ invariant under $L$, cf. (28), yields an $L$-invariant function $\widehat{f}$ on $U'$ rendering a diagram of the kind (27) commutative. ∎

**Corollary 3.10.** *For an open set $U$ of $V_0$, a function $f\colon U \to \mathbb{C}$ is holomorphic if and only if there is an open $H$-invariant subset $U'$ of $V$ containing $\pi^{-1}(U)$ and an $H$-invariant function $\widehat{f}$ on $U'$ rendering a diagram of the kind* (27)*, with $U'$ and $U$ substituted for $V$ and $V_0$, respectively, commutative.* ∎

**Remark 3.11.** The Stein space geometric invariant theory built in [22] establishes the not a priori clear fact that $V_0$ and $V//L$, endowed with the structure sheaves explained above, are the categorical quotients in the analytic setting, with analytic quotient maps $\Pi$ and $\pi$ [22, Lemma §6.4 p. 657, §6.4 Theorem p. 658, §6.5 p. 659]. ■

**Remark 3.12.** For a closed analytic subspace $Y$ of a Stein space $X$, the restriction morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ between the sheaves of germs of analytic functions is an epimorphism but, in view of Cartan's Theorem B [8], this can fail for non Stein domains. The construction of the complex analytic structure (structure sheaf) on $V_0$ proceeds by restriction of analytic functions and is independent of any assumption of extendibility of analytic functions defined on a complex analytic subspace.

An analytic map defined on a Stein space is *Stein* when the pre-image of a Stein open is Stein. In the analytic setting, the maps $\Pi$ and $\pi$ in (26) are Stein. Every Stein subvariety admits a Stein neighborhood [69]. This fact allows for a characterization of the analytic structure on $V_0$ entirely in terms of Stein open sets, and this characterization is enough to arrive at the holomorphic Poisson structure in Proposition 3.15 below. ■

**3.4.7. Algebraic Poisson structure.** The complex symplectic form $\omega_V$ on $V$ determines an $H$-invariant algebraic Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathbb{C}[V]$ and hence an algebraic Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathbb{C}[V]^H = \mathbb{C}[V//H]$. Recall that the identity

$$(\delta_V(y))(v) = (\Phi_V(v))(y),\ y \in \mathfrak{h},\ v \in V, \tag{30}$$

characterizes the comomentum $\delta_V \colon \mathfrak{h} \to \mathbb{C}[V]$ which the momentum mapping (18) induces, and let $I_{\Phi_V}$ be the ideal in $\mathbb{C}[V]$ which $\delta_V(\mathfrak{h}) \subseteq \mathbb{C}[V]$ generates. By construction, the vanishing ideal $I(\Phi_V^{-1}(0))$ of the algebraic set $\Phi_V^{-1}(0)$ is the radical $\sqrt{I_{\Phi_V}}$ of the ideal $I_{\Phi_V}$ in $\mathbb{C}[V]$.

As before, let $L$ be a maximal compact subgroup of $H$. Since $L$ is compact, taking $L$-invariants, we obtain an injection $(I(\Phi_V^{-1}(0)))^L \to \mathbb{C}[V]^L$. Recall

$$\mathbb{C}[V_0] = \mathbb{C}[\Phi_V^{-1}(0)]^H = \mathbb{C}[\Phi_V^{-1}(0)]^L.$$

Inspection of the commutative diagram

$$\begin{array}{ccccc}
(I(\Phi_V^{-1}(0)))^L & \rightarrowtail & \mathbb{C}[V]^L & \twoheadrightarrow & \mathbb{C}[V_0] \\
\downarrow & & \downarrow & & \downarrow \\
I(\Phi_V^{-1}(0)) & \rightarrowtail & \mathbb{C}[V] & \twoheadrightarrow & \mathbb{C}[\Phi_V^{-1}(0)]
\end{array}$$

shows that the canonical epimorphism $\mathbb{C}[V]^L \to \mathbb{C}[V_0]$ induces an isomorphism

$$\mathbb{C}[V]^L/(I(\Phi_V^{-1}(0)))^L \longrightarrow \mathbb{C}[V_0] = \mathbb{C}[\Phi_V^{-1}(0)]^L. \tag{31}$$

**Theorem 3.13.** *The ideal $(I(\Phi_V^{-1}(0)))^L = (I(\Phi_V^{-1}(0)))^H$ in $\mathbb{C}[V]^L = \mathbb{C}[V]^H$ is a Poisson ideal. Consequently the algebraic Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on the algebra $\mathbb{C}[V]^H = \mathbb{C}[V//H]$ induces an algebraic Poisson bracket on $\mathbb{C}[V_0]$.*

**Proof.** The comomentum $\delta_V\colon \mathfrak{h} \to \mathbb{C}[V]$ is a morphism of Lie algebras. Let $u \in \mathbb{C}[V]$ and $v = \sum \alpha_j y_j \in I_{\Phi_V}$, with $\alpha_j \in \mathbb{C}[V]$ and $y_j \in \delta_V(\mathfrak{h})$. Then

$$\{u,v\} = \sum \{u,\alpha_j\}y_j + \sum \alpha_j\{u,y_j\}.$$

When $u \in \mathbb{C}[V]^L$, the bracket $\{u,y_j\}$ vanishes for every $j$ whence

$$\{u,v\} = \sum \{u,\alpha_j\}y_j \in I_{\Phi_V}.$$

Hence the ideal $I^L_{\Phi_V}$ of $L$-invariant functions in the ideal $I_{\Phi_V}$ in $\mathbb{C}[V]$ which the image $\delta_V(\mathfrak{h}) \subseteq \mathbb{C}[V]$ generates is a Poisson ideal in $\mathbb{C}[V]^L$. Then the radical $\sqrt{I^L_{\Phi_V}}$ of the ideal $I^L_{\Phi_V}$ in $\mathbb{C}[V]^L$ is a Poisson ideal as well. Indeed, if an ideal $I$ in a commutative algebra $A$ over the rationals is closed under a derivation $D$ of $A$, so is its radical $\sqrt{I}$ [42, Lemma I.1.8 p. 12]. For if $a \in \sqrt{I} \subseteq A$, the power $a^n$, for some $n > 0$, belongs to $I$, and an inductive argument shows $(Da)^{2n-1} \in I$ [42, Lemma I.1.7 p. 12]. Hence, for $u \in \mathbb{C}[V]^L$ and $v \in \sqrt{I^L_{\Phi_V}}$, the value $\{u,v\}$ belongs to $\sqrt{I^L_{\Phi_V}}$ as well. (I am indebted to O. Gabber for having pointed me to [42] and for having clarified the reasoning.)

The canonical map
$$\sqrt{I^L_{\Phi_V}} \longrightarrow \sqrt{I_{\Phi_V}}^{\,L} = (I(\Phi_V^{-1}(0)))^L \tag{32}$$

identifies the radical $\sqrt{I^L_{\Phi_V}}$ with the vanishing ideal $(I(\Phi_V^{-1}(0)))^L$. Indeed, consider a member $y$ of $\sqrt{I_{\Phi_V}}^{\,L}$. Thus, $y^\ell \in I_{\Phi_V}$, for some $\ell > 0$, and $y$ is invariant under $L$. But then $y^\ell \in I^L_{\Phi_V}$, whence $y \in \sqrt{I^L_{\Phi_V}}$. Consequently

$$\mathbb{C}[\Phi_V^{-1}(0)]^L \cong \mathbb{C}[V]^L/\sqrt{I^L_{\Phi_V}} \tag{33}$$

whence the affine coordinate ring $\mathbb{C}[V_0] = \mathbb{C}[\Phi_V^{-1}(0)]^H = \mathbb{C}[\Phi_V^{-1}(0)]^L$ inherits a Poisson structure. ∎

**Remark 3.14.** O. Gabber kindly communicated the following direct reasoning for the claims in [42, Lemmata I.1.7, I.1.8 p. 12] to me: *For a commutative algebra $A$ over a field of characteristic zero endowed with a derivation $D$, the radical $\sqrt{I}$ of an ideal $I$ closed under $D$ is closed under $D$ as well.* Indeed, let $a \in A$ and suppose the power $a^n$, for some $n > 0$, belongs to $I$. Then $D^n a^n \in I$. On the other hand

$$D^n a^n = n!(Da)^n + ab$$

for some $b \in A$. Hence $(Da)^n \in \sqrt{I}$ whence $Da \in \sqrt{I}$. ∎

**3.4.8. Holomorphic Poisson structure.** Maintain the choice of a maximal compact subgroup $L$ of $H$. We remind the reader that under the present circumstances the $L$-invariants (in $\mathcal{O}_V(V)$ etc.) coincide with the $H$-invariants.

**Proposition 3.15.** *The symplectic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_V$ on $\mathcal{O}_V$ induces a Poisson structure $\{\,\cdot\,,\,\cdot\,\}_{V_0}$ on $\mathcal{O}_{V_0}$.*

**Proof.** Let $U$ be an open set in $V_0$ and let $f$ and $h$ be holomorphic functions on $U$. By Proposition 3.9, there is an open $L$-invariant subset $U'$ of $V$ containing $\pi^{-1}(U)$ together with $L$-invariant holomorphic functions $\widehat{f}$ and $\widehat{h}$ on $U'$ that both render a diagram of the kind (27) commutative. Since the symplectic form $\omega_V$ on $V$ is $H$-invariant and hence $L$-invariant, the symplectic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_V$ on $\mathcal{O}_V(U')$ induces a Poisson structure on the subalgebra $\mathcal{O}_V(U')^L$ of $L$-invariants. Thus we must show that restricting $\{\widehat{f},\widehat{h}\}$ to $\pi^{-1}(U)$ unambiguously yields a Poisson bracket $\{f,h\}_{V_0}$ for $f$ and $h$. This fact is a consequence of Lemma 3.16 below. ∎

**Lemma 3.16.** *For an open $L$-invariant subset $U$ of $V$ that contains $\Phi_V^{-1}(0)$, relative to the symplectic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_V$ on the subalgebra $\mathcal{O}_V(U)^L$ of $L$-invariants which the $H$-and hence $L$-invariant symplectic form $\omega_V$ on $V$ induces, the ideal of $L$-invariant functions in $\mathcal{O}_V(U)$ that vanish on $\Phi_V^{-1}(0)$ is a Poisson ideal in $\mathcal{O}_V(U)^L$.*

**Proof.** The argument for [3, Theorem 1 p. 35] shows that the ideal of $L$-invariant functions in $\mathcal{O}_V(U)$ that vanish on $\Phi_V^{-1}(0)$ is a Poisson ideal in $\mathcal{O}_V(U)^L$. Since this is, perhaps, not entirely obvious, we reproduce the details in the present holomorphic setting: Let $f$ and $h$ be $L$-invariant holomorphic functions on $U$ with $h|_{\Phi_V^{-1}(0)}=0$ and let $q$ be a point of $\Phi_V^{-1}(0)$. Thus $\Phi_V(q)=0$. Write the holomorphic Hamiltonian vector field of $f$ as $X_f$ and, for $Y\in\mathfrak{l}=\mathrm{Lie}(L)$, let $Y_V$ denote the linear holomorphic (algebraic) vector field on $V$ which $Y$ induces. We must show that

$$\{f,h\}_V(q)=-(X_fh)(q)=0. \tag{34}$$

Since $f$ is $L$-invariant, necessarily

$$\{Y\circ\Phi_V,f\}_V=-Y_Vf=0,\quad \text{for } Y\in\mathfrak{l}.$$

Consequently, for $Y\in\mathfrak{l}$, the algebraic function $Y\circ\Phi_V$ is constant along the holomorphic integral curves of $X_f$. Hence the holomorphic integral curve $z\mapsto\varphi_z^f(q)$ of $X_f$ having $\varphi_0^f(q)=q$ ($z$ in a neighborhood of $0\in\mathbb{C}$) lies in $\Phi_V^{-1}\Phi_V(q)=\Phi_V^{-1}(0)$. Differentiating the function $z\mapsto h(\varphi_z^f(q))$ with respect to the variable $z$ and evaluating at $z=0$ we find (34). ∎

**Remark 3.17.** For $U=V$, Lemma 3.16 says that, in the algebra of $L$-invariant entire holomorphic functions on $V$, the ideal of $L$-invariant entire holomorphic functions that vanish on $\Phi_V^{-1}(0)$ is a Poisson ideal. This is formally the statement of Theorem 3.13, with entire holomorphic functions substituted for algebraic functions. The statement of Lemma 3.16 is, essentially, [3, Theorem 1, p. 35], however. ∎

**Remark 3.18.** [54, Proposition 2.12(ii)] says the following: "For a complex reductive group, let $\pi\colon X\to X//G$ be an analytic Hilbert quotient. If $Y\subseteq X$ is a $G$-invariant closed complex analytic subspace, then $Y//G:=\pi(Y)$ is a closed complex analytic subspace of $X//G$ and the restriction $Y\to Y//G$ is an analytic Hilbert quotient." By means of this proposition, [54, Lemma 4.13] establishes, with a reasoning formally involving the statement of Corollary 3.10 above, the existence of a holomorphic Poisson structure similar to that spelled out in Proposition 3.15. For the proof of [54, Proposition 2.12(ii)], Mayrand refers to [27, §1(ii)]. The material in

Subsection 3.4.6 above establishes the requisite analytical details in an elementary manner. This observation concerns also the reduced Poisson structure in [13]. ■

The following is an immediate consequence of Proposition 3.15.

**Theorem 3.19.** *Let $(V, \omega_{\mathbb{C}})$ be a complex symplectic representation of a complex reductive Lie group $H$, let $\sigma_V$ be a (real) Kähler form on $V$ invariant under a maximal compact subgroup $L$ of $H$, let $\mu_{\sigma_V}\colon V \to \mathfrak{l}^*$ and $\Phi_V\colon V \to \mathfrak{h}^*$ denote the associated momentum mappings, and let*

$$V_0 = (\mu_{\sigma_V}^{-1}(0) \cap \Phi_V^{-1}(0))/L \cong \Phi_V^{-1}(0)//H,$$

*endowed with the reduced complex analytic structure $\mathcal{O}_{V_0}$ discussed in* § 3.4.6*. The holomorphic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_V$ on $\mathcal{O}_V$ induces a Poisson structure $\{\,\cdot\,,\,\cdot\,\}_{V_0}$ on $\mathcal{O}_{V_0}$.* ■

**3.4.9. Hyperkähler case.** Let $V \cong \mathbb{H}^n$ $(n \geq 1)$ be a quaternionic vector space. Let $\mathsf{I}$, $\mathsf{J}$, $\mathsf{K}$ denote three complex structures that behave like quaternions (generate the quaternion group of order eight) and generate the quaternionic structure, and let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a real hyperkähler metric (i.e., $\langle\,\cdot\,,\,\cdot\,\rangle$ renders $\mathsf{I}$, $\mathsf{J}$, $\mathsf{K}$ skew). This turns $V$ into a hyperkähler manifold with Kähler forms $\omega_{\mathsf{I}}(\cdot,\cdot) = \langle \mathsf{I}\cdot,\cdot\rangle$, $\omega_{\mathsf{J}}(\cdot,\cdot) = \langle \mathsf{J}\cdot,\cdot\rangle$, $\omega_{\mathsf{K}}(\cdot,\cdot) = \langle \mathsf{K}\cdot,\cdot\rangle$. Define

$$\Phi_{\mathsf{I}}\colon V \to \mathfrak{l}^*, \quad x \circ \Phi_{\mathsf{I}}(v) = \tfrac{1}{2}\omega_{\mathsf{I}}(xv, v), \tag{35}$$

$$\Phi_{\mathsf{J}}\colon V \to \mathfrak{l}^*, \quad x \circ \Phi_{\mathsf{J}}(v) = \tfrac{1}{2}\omega_{\mathsf{J}}(xv, v), \tag{36}$$

$$\Phi_{\mathsf{K}}\colon V \to \mathfrak{l}^*, \quad x \circ \Phi_{\mathsf{K}}(v) = \tfrac{1}{2}\omega_{\mathsf{K}}(xv, v). \tag{37}$$

**Theorem 3.20.** *Let $L$ be a compact Lie group acting linearly on $V$ and preserving the linear hyperkähler structure $\langle\,\cdot\,,\,\cdot\,\rangle$, $\mathsf{I}$, $\mathsf{J}$, $\mathsf{K}$. The three complex structures $\mathsf{I}$, $\mathsf{J}$, $\mathsf{K}$ determine, on the hyperkähler quotient $V_0 = \Phi^{-1}(0)/L$ relative to the hyperkähler momentum mapping*

$$\Phi\colon V \longrightarrow \mathfrak{l}^* \otimes \mathbb{R}^3,\ (x_1, x_2, x_3) \circ \Phi(v) = \tfrac{1}{2}(\omega_{\mathsf{I}}(x_1 v, v), \omega_{\mathsf{J}}(x_2 v, v), \omega_{\mathsf{K}}(x_3 v, v)), \tag{38}$$

*three respective complex analytic structures $\mathcal{O}_{\mathsf{I}}$, $\mathcal{O}_{\mathsf{J}}$, $\mathcal{O}_{\mathsf{K}}$, and the corresponding Kähler forms on $V$ determine three pairwise compatible real Poisson structures $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}$ on, respectively $\mathcal{O}_{\mathsf{I}}$, $\mathcal{O}_{\mathsf{J}}$, $\mathcal{O}_{\mathsf{K}}$, such that*

$$(\mathcal{O}_{\mathsf{I}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}),\ (\mathcal{O}_{\mathsf{J}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}),\ (\mathcal{O}_{\mathsf{K}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}}) \tag{39}$$

*are holomorphic Poisson structures on $V_0$. These generate a sphere of holomorphic Poisson structures on $V_0$.*

**Proof.** Relative to $(\mathsf{I}, \omega_{\mathsf{I}}, \omega_{\mathsf{J}} + i\omega_{\mathsf{K}}, \Phi_{\mathsf{J}} + i\Phi_{\mathsf{K}})$, the affine space which underlies the vector space $V$ is an $L^{\mathbb{C}}$-Hamiltonian holomorphic symplectic Kähler manifold. Thm. 3.19 yields the holomorphic Poisson structure $(\mathcal{O}_{\mathsf{I}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}})$. Now, repeat the argument with $(\mathsf{J}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}} + i\omega_{\mathsf{I}}, \Phi_{\mathsf{K}} + i\Phi_{\mathsf{I}})$ and $(\mathsf{K}, \omega_{\mathsf{K}}, \omega_{\mathsf{I}} + i\omega_{\mathsf{J}}, \Phi_{\mathsf{I}} + i\Phi_{\mathsf{J}})$. ■

**3.4.10. Stratification.** Return to the linear $H$-Hamiltonian holomorphic symplectic Kähler manifold

$$(V, \sigma_V, \omega_{\mathbb{C}}, \mu_{\sigma_V}, \Phi_V)$$

studied earlier in this section. The reasoning in [54, §4.7] establishes the following.

**Theorem 3.21.** *Let $(V,\omega_{\mathbb{C}})$ be a complex symplectic representation of a complex reductive Lie group $H$, let $\sigma_V$ be a (real) Kähler form on $V$ invariant under a maximal compact subgroup $L$ of $H$, and let $\mu_{\sigma_V}\colon V \to \mathfrak{l}^*$ and $\Phi_V\colon V \to \mathfrak{h}^*$ denote the associated momentum mappings. The orbit type decomposition of the quotient $V_0 = (\mu_{\sigma_V}^{-1}(0)\cap\Phi_V^{-1}(0))/L \cong \Phi_V^{-1}(0)//H$ is a complex Whitney stratification. Hence the complex analytic structure $\mathcal{O}_{V_0}$ on $V_0$ which the complex structure of $V$ determines and the holomorphic Poisson structure $\{\,\cdot\,,\,\cdot\,\}_{V_0}$ on $\mathcal{O}_{V_0}$ which the complex symplectic structure $\omega_{\mathbb{C}}$ on $V$ induces turns $(V_0,\mathcal{O}_{V_0},\{\,\cdot\,,\,\cdot\,\}_{V_0})$ into a stratified holomorphic symplectic space.* ■

(N.B. In the statement of the theorem, there is a single complex structure on $V$ under discussion.)

**Theorem 3.22.** *Under the circumstances of Theorem* 3.20*, the three holomorphic Poisson structures* (39) *on the hyperkähler quotient $V_0 = \Phi^{-1}(0)/L$ are compatible with the orbit type stratification of $V_0$ and thereby yield a stratified hyperkähler structure. Moreover, the orbit type stratification of $V_0$ is a complex Whitney stratification relative to each of $\mathcal{O}_{\mathsf{I}}$, $\mathcal{O}_{\mathsf{J}}$, $\mathcal{O}_{\mathsf{K}}$.* ■

**Remark 3.23.** In the real setting, in [71, 1.11 Example], Sjamaar-Lerman recall that [3, Theorem 1 p. 35] yields the real stratified symplectic Poisson structure. In [71, 3.1 Proposition], they establish the existence of this Poisson structure by a pointwise reasoning involving the stratification. In [54, Subsection 4.8], Mayrand extends the pointwise reasoning for [71, 3.1 Proposition] in terms of the corresponding stratification to the complex analytic case in the realm of hyperkähler manifolds to construct a Poisson bracket of the kind $\{\,\cdot\,,\,\cdot\,\}_{V_0}$ in Theorem 3.19. The proof of Theorem 3.19 is independent of the stratification. ■

### 3.5. Local structure of the analytic quotient of the Hamiltonian holomorphic symplectic Kähler manifold at the start

Return to the circumstances of Subsection 3.3. Let $p$ be a point of $\mu^{-1}(0) \subseteq M$. By an observation in [26, §2.2], the stabilizer $H_p$ of $p$ is reductive and hence the complexification of a compact group. The $G$-action on $M$ turns the complex symplectic vector space $(\mathrm{T}_pM,\omega_{\mathbb{C}})$ into a symplectic $H_p$-representation, the tangent space $\mathfrak{g}p = \mathrm{T}_p(G\cdot p) \subseteq \mathrm{T}_pM$ to the $G$-orbit at $p$ is a subrepresentation, and so is the skew-orthogonal complement $\mathfrak{g}p^{\omega_{\mathbb{C}}} \subseteq \mathrm{T}_pM$ of $\mathfrak{g}p$. In view of the momentum property, $\mathfrak{g}p^{\omega_{\mathbb{C}}} = \ker(d\mu_{\mathbb{C}})$, and $\mathfrak{g}p \subseteq \mathfrak{g}p^{\omega_{\mathbb{C}}}$ as the annihilator of the restriction of $\omega_{\mathbb{C}}$ to $\mathfrak{g}p^{\omega_{\mathbb{C}}}$. Relative to the hermitian form associated with $\omega_{\mathbb{R}}$, let $V_p$ denote the orthogonal complement of $\mathfrak{g}p$ in $\mathfrak{g}p^{\omega_{\mathbb{C}}}$, so that

$$\mathfrak{g}p^{\omega_{\mathbb{C}}} = \mathfrak{g}p \oplus V_p \tag{40}$$

is a decomposition of $H_p$-representations. Analogously to terminology in [71, Section 2], say $V_p$ is an *infinitesimal holomorphic symplectic slice at $p$ for the $G$-action on $M$*. In the terminology of [59, 7.2.1 Definition p. 276], the complex vector space $V_p$ is a *symplectic normal space* at $p$. The holomorphic symplectic structure $\omega_{\mathbb{C}}$ on $M$ induces a complex symplectic form $\omega_p$ on $V_p$, and the stabilizer $H_p \subseteq G$ of the point $p$ of $M$ acts linearly and symplectically on $V_p$.

Write $E_p = G \times_{H_p} (\mathfrak{h}_p^o \times V_p)$. Combining the holomorphic slice theorem [26, §2.7 Theorem p. 292], [70, Theorem 1.12 p. 100] with a holomorphic version of the Darboux-Weinstein theorem [75, Theorem 4.1, Corollary 4.3], reproduced in [18, Theorem 22.1], [5, Theorem 6], [59, 7.3.1 Theorem], and with some extra work, in [54], Mayrand manages to establish the holomorphic local normal form of the momentum mapping or, equivalently, the holomorphic symplectic slice theorem, in the realm of hyperkähler manifolds. We extend this result as follows.

**Theorem 3.24.** (Holomorphic symplectic slice theorem) *Let $G$ be a complex reductive Lie group, $K$ a maximal compact subgroup, and let $(M, \omega_{\mathbb{C}}, \mu_{\mathbb{C}})$ be a $G$-Hamiltonian holomorphic symplectic manifold which carries a $K$-Hamiltonian Kähler structure $(M, \omega_{\mathbb{R}}, \mu_{\mathbb{R}})$ as well. For an arbitrary point $p$ in $\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)$, by construction necessarily in $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$, cf.* (8)*, there is a $G$-saturated neighborhood $U_p$ of $p$ in $M^{\mu_{\mathbb{R}}-\mathrm{ss}}$, a $G$-saturated neighborhood $U_p'$ in $G \times_{H_p} (\mathfrak{h}_p^o \times V_p)$ of the image ($\cong G/H_p$) of the zero section of the vector bundle $E_p = G \times_{H_p} (\mathfrak{h}_p^o \times V_p) \to G/H_p$, and an isomorphism*

$$(U_p', \omega_{E_p}, \kappa_p) \longrightarrow (U_p, \omega_{\mathbb{C}}, \mu_{\mathbb{C}}) \tag{41}$$

*of $G$-Hamiltonian complex manifolds mapping the point $[e, 0, 0]$ of $G \times_{H_p} (\mathfrak{h}_p^o \times V_p)$ (which the point $(e, 0, 0)$ of $G \times \mathfrak{h}_p^o \times V_p$ represents) to $p$.*

Since Mayrand merely proceeds in the hyperkähler setting, we explain the salient steps of the proof. The strategy of the proof is classical, see [71, 2.5. Proposition], [47, Theorem 3], [59, 7.4.1 Theorem p. 282], [59, 7.5.5 Theorem p. 285].

**Proof.** Let $p$ be a point in $\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)$, and recall

$$\mathfrak{g}p = \mathrm{T}_p(G \cdot p) \cong \mathfrak{g}/\mathfrak{h}_p \cong \mathfrak{m}_p,$$

cf. § 3.4.2 for the notation. The symplectic polar $V_p^{\omega_{\mathbb{C}}} \subseteq \mathrm{T}_pM$ of $V_p$ is an $H_p$-representation and, for some Lagrangian complement $W_p$ of $\mathfrak{g}p$ in $V^{\omega_{\mathbb{C}}}$, necessarily an $H$-representation,

$$V_p^{\omega_{\mathbb{C}}} = W_p \oplus \mathfrak{g}p \tag{42}$$

as $H_p$-representations in such a way that the map

$$\vartheta \colon W_p \longrightarrow (\mathfrak{g}p)^*, \ (\vartheta(Y))(X) = \omega_{\mathbb{C}}(X, Y), \ X \in \mathfrak{g}p, \ Y \in W_p \tag{43}$$

is an $H_p$-linear isomorphism. Thus the resulting decomposition

$$\mathrm{T}_pM \cong \mathfrak{m}_p \oplus \mathfrak{m}_p^* \oplus V_p \cong \mathfrak{m}_p \oplus \mathfrak{h}_p^o \oplus V_p \tag{44}$$

is a complex *Witt-Artin* decomposition (relative to the symplectic structure and momentum mapping), cf., e.g., [59, 7.1.1 Theorem] for the real case; with regard to the tautological symplectic structure on $\mathfrak{m}_p \oplus \mathfrak{m}_p^*$, the decomposition (44) is one of complex symplectic $H_p$-representations.

The $G$-orbit $G \cdot p \subseteq M$ of $p$ in $M$ is a complex submanifold of $M$ and, in view of the holomorphic slice theorem [26, §2.7 Theorem p. 292], there is a locally closed $H_p$-invariant complex submanifold $S_p$ such that the canonical $G$-equivariant holomorphic map

$$G \times_{H_p} S_p \longrightarrow GS_p \tag{45}$$

is a $G$-equivariant biholomorphism onto the open $G$-invariant $G$-saturated neighborhood $GS_p$ of the $G$-orbit $G \cdot p$ of $p$ in $M$.

To establish the claim, it suffices to argue in terms of $G \times_{H_p} S_p$, that is, near the point $p$, we can take $M$ to be $G \times_{H_p} S_p$. By construction, the injection $S_p \subseteq M$ induces, via the decomposition (44), an isomorphism $\mathrm{T}_p S_p \to \mathfrak{h}^o \oplus V_p$ and, in view of the decomposition (44), the complex vector space $W_p = \mathrm{T}_p M / T_p(G\cdot p) \cong \mathfrak{h}^o_p \oplus V_p$ serves as an ordinary infinitesimal holomorphic (beware: not symplectic) slice at $p$ for the $G$-action on $M$. Hence parametrizing $S_p$ holomorphically by its tangent space $\mathfrak{h}^o_p \oplus V_p$ at $p$ yields a local biholomorphism between $E_p = G \times_{H_p} (\mathfrak{h}^o_p \oplus V_p)$ and $G \times_{H_p} S_p$ near the point $p$, that is, there is an open $G$-invariant neighborhood $U$ of $p$ in $M^{\mu_\mathbb{R}-ss}$, an open $G$-invariant neighborhood $U'$ in $G \times_{H_p} (\mathfrak{h}^o_p \oplus V_p)$ of the image $Z_p \cong G/H_p$ of the zero section of the vector bundle $G \times_{H_p} (\mathfrak{h}^o_p \oplus V_p) \to G/H_p$, and a $G$-equivariant biholomorphism $\Psi\colon U' \to U$ mapping the point $[1,0,0]$ of $E_p = G \times_{H_p} (\mathfrak{h}^o_p \oplus V_p)$ which the point $(1,0,0)$ of $G \times (\mathfrak{h}^o_p \oplus V_p)$ represents to $p$. By [54, Proposition 3.8], every $G$-invariant neighborhood of $p$ contains a $G$-saturated neighborhood of $p$. Hence we may take $U$ and $U'$ to be $G$-saturated in $M^{\mu_\mathbb{R}-ss}$.

Now, the complex algebraic manifold $E_p = G \times_{H_p} (\mathfrak{h}^o_p \oplus V_p)$ carries the algebraic $G$-invariant symplectic structure $\omega_{E_p}$ and the $G$-invariant holomorphic symplectic structure $\eta_\mathbb{C} = \Psi^*(\omega_\mathbb{C})$, and the zero section $G/H_p \to E_p$ is an isotropic embedding for both. While the restrictions of $\omega_{E_p}$ and $\eta_\mathbb{C}$ to $G/H_p$ need not coincide, Propositions 3.6 and 3.7 imply that there are open $G$-invariant neighborhoods $U_0$ and $U_1$ of the image $Z$ of the zero section in $E$ and a $G$-equivariant biholomorphism $\rho\colon U_0 \to U_1$ such that $\rho^*(\eta_\mathbb{C}) = \omega_{E_p}$, and we may take $U_0$ and $U_1$ to be $G$-saturated. Shrinking the open neighborhoods if need be and combining $\rho$ and $\Psi$ yields the isomorphism (41). Compatibility with the momentum mappings is a consequence of the fact that a momentum mapping is unique up to a constant value in the center. ■

**Remark 3.25.** For an algebraic Hamiltonian action of a reductive group $G$ on a non-singular affine symplectic variety, [47, Theorem 3] – Losev calls it a "symplectic slice theorem"– establishes an analytical equivalence at an arbitrary point having closed orbit between a saturated neighborhood of that point and a saturated neighborhood of the corresponding point of a model space of the kind $G \times_H (\mathfrak{h}^o \oplus V)$. The reader will notice there is no auxiliary Kähler form of the kind $\omega_\mathbb{R}$ (cf. Subsection 3.3 above) or $\sigma_V$ (cf. §3.3.3 above) present in [47, Theorem 3].

### 3.6. Globalization

To spell out global versions of Theorems 3.19, 3.20, 3.21 and 3.22, as before, let $G$ be a complex reductive Lie group and $K$ a maximal compact subgroup.

**Remark 3.26.** It is important to note that, in Theorems 3.27, 3.32, and 3.33 below the complex analytic and the holomorphic Poisson structures are independent of the stratifications (orbit type decompositions) and, in fact, can be understood independently of the corresponding orbit type decomposition; in each case, the orbit type decomposition being a stratification is an additional piece of structure. ■

**Theorem 3.27.** *Let $(M, \omega_\mathbb{C}, \mu_\mathbb{C})$ be a $G$-Hamiltonian holomorphic symplectic manifold endowed with, furthermore, a $K$-hamiltonion Kähler structure $(M, \omega_\mathbb{R}, \mu_\mathbb{R})$. Then the complex structure of $M$ determines, on the reduced space*

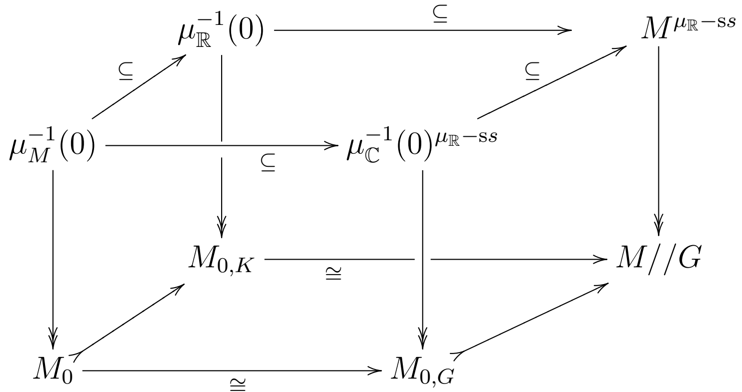
$$M_0 = (\mu_\mathbb{R}^{-1}(0) \cap \mu_\mathbb{C}^{-1}(0))/K \cong \mu_\mathbb{C}^{-1}(0)^{\mu_\mathbb{R}-\mathrm{ss}}//G, \tag{46}$$

*cf.* (14), *a complex analytic structure* $\mathcal{O}_{M_0}$, *and the holomorphic symplectic form* $\omega_{\mathbb{C}}$ *induces a holomorphic Poisson bracket* $\{\,\cdot\,,\,\cdot\,\}_{M_0}$ *on* $\mathcal{O}_{M_0}$. *Moreover, the orbit type decomposition of* $M_0$ *is a complex Whitney stratification and, relative to that stratification,* $(M_0, \mathcal{O}_{M_0}, \{\,\cdot\,,\,\cdot\,\}_{M_0})$ *is a stratified holomorphic symplectic space.*

In terms of the notation

$$\mu_M^{-1}(0) = \mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0), M_{0,K} = \mu_{\mathbb{R}}^{-1}(0)/K, M_{0,G} = \mu_{\mathbb{C}}^{-1}(0)^{\mu_{\mathbb{R}}-ss}//G,$$

diagram (25) globalizes to the commutative diagram:




(47)

**Proof.** This is a straightforward consequence of Theorem 3.24. Indeed, the constructions and arguments given there globalize in an obvious fashion. That the stratification is a complex Whitney stratification is due to Mayrand [54, Theorem 1.4]. His reasoning for the hyperkähler case extends to the present more general case. We leave the details to the reader. ■

**Remark 3.28.** In Theorem 3.27, there is no piece of structure on $M_0$ which the real Kähler form $\omega_{\mathbb{R}}$ induces. In the presence of more structure on $M$, Theorems 3.32 and 3.33 show in particular that $\omega_{\mathbb{R}}$ then induces a stratified Kähler structure on $M_0$. ■

For the application in Section 5, Theorem 3.27 suffices. However, the following results are worth spelling out: Let $(M, \mathsf{I}, \mathsf{J}, \mathsf{K}, \omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}, \mu_{\mathsf{I}}, \mu_{\mathsf{J}}, \mu_{\mathsf{K}})$ be a $K$-tri-Hamiltonian hyperkähler manifold, write $\mu^{-1}(0) = \mu_{\mathsf{I}}^{-1}(0) \cap \mu_{\mathsf{J}}^{-1}(0) \cap \mu_{\mathsf{K}}^{-1}(0)$, and consider the hyperkähler quotient $M_0 = \mu^{-1}(0)/K$. Let $C^\infty(M_0)$ denote the image, under restriction, of $C^\infty(M)^K$ in the algebra of continuous functions on $M_0$.

**Theorem 3.29.** *The three Kähler forms* $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$ *induce three Poisson structures* $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I},0}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J},0}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K},0}$ *on* $C^\infty(M_0)$ *that constitute a stratified hyperkähler structure.*

**Remark 3.30.** In Theorem 3.29, the term "stratified" refers to a notion of stratification in the sense of [14], weaker than that of a Whitney stratification. ■

**Proof.** By [10, Theorem 2.1], on each piece of the orbit type decomposition, the hyperkähler structure on $M$ induces a hyperkähler structure. Accordingly, on a stratum, the three Kähler forms $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$ on $M$ induce the respective Kähler forms $\omega_{\mathsf{I},0}, \omega_{\mathsf{J},0}, \omega_{\mathsf{K},0}$; let $\mathsf{I}_0$, $\mathsf{J}_0$, $\mathsf{K}_0$ denote the corresponding complex structures on that stratum.

To construct the Poisson structures, we adapt the pointwise reasoning in [71, 3.1 Proposition], cf. Remark 3.23, to the present situation as follows:

Let $f, h$ be in $C^\infty(M_0)$ and let $q$ be a point of $M_0$. The point $q$ lies in a unique orbit type piece $S_q$ of the orbit type decomposition of $M_0$, a hyperkähler manifold, so take

$$\{f,h\}_{\mathsf{I},0}(q) = \{f,h\}_{\mathsf{I},0,S_q}(q) \ (\omega_{\mathsf{I},0}\text{-symplectic Poisson bracket in } C^\infty(S_q)). \qquad (48)$$

It then remains to prove that $\{f,h\}_{\mathsf{I},0}$ is a member of $C^\infty(M_0)$.

By construction, there are $K$-invariant smooth functions $\widehat{f}$ and $\widehat{h}$ on $M$ rendering the diagrams

$$\begin{array}{ccc} \mu^{-1}(0) & \longrightarrow & M \\ \downarrow & & \downarrow \widehat{f} \\ M_0 & \xrightarrow[f]{} & \mathbb{R} \end{array} \qquad \begin{array}{ccc} \mu^{-1}(0) & \longrightarrow & M \\ \downarrow & & \downarrow \widehat{h} \\ M_0 & \xrightarrow[h]{} & \mathbb{R} \end{array}$$

commutative. The ordinary $\omega_{\mathsf{I}}$-symplectic Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}$ on $C^\infty(M)$ is $K$-invariant. Hence the smooth function $\{\widehat{f},\widehat{h}\}_{\mathsf{I}}$ on $M$ is $K$-invariant. This function renders the diagram

$$\begin{array}{ccc} \mu^{-1}(0) & \longrightarrow & M \\ \downarrow & & \downarrow \{\widehat{f},\widehat{h}\}_{\mathsf{I}} \\ M_0 & \xrightarrow[\{f,h\}_{\mathsf{I},0}]{} & \mathbb{R} \end{array}$$

commutative. Repeating the argument with regard to $\mathsf{J}$ and $\mathsf{K}$ yields the Poisson brackets $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J},0}$ and $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K},0}$, respectively.

By [54, Theorem 1.2], the orbit type decomposition is a stratification in the sense of [14]. ■

**Remark 3.31.** The reasoning in the above proof shows that the ideal of $K$-invariant functions in $C^\infty(M)^K$ which vanish on $\mu^{-1}(0)$ is Poisson ideal in $C^\infty(M)^K$ relative to each of the Poisson structures $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}$ on $C^\infty(M)$ associated with, respectively, $\omega_{\mathsf{I}}, \omega_{\mathsf{J}}, \omega_{\mathsf{K}}$. It would be interesting to establish this fact by extending the argument for [3, Theorem 1 p. 35], cf. Remark 3.23. ■

Combining Theorems 3.24, 3.27 and 3.29 leads to the following.

**Theorem 3.32.** *Suppose that the $K$-action integrates to a holomorphic $G$-action relative to $\mathsf{I}$. Then the symplectic structure $\omega_{\mathsf{I}}$ induces a stratified Kähler structure $(C^\infty(M_0), \mathcal{O}_{M_0}, \{\,\cdot\,,\,\cdot\,\}_{\mathbb{R}})$ on $M_0$, cf. Proposition* 3.15 *and Remark* 3.23. *Furthermore, this stratified Kähler structure combines with the stratified holomorphic symplectic structure $(\mathcal{O}_{M_0}, \{\,\cdot\,,\,\cdot\,\}_{M_0})$ which $\mathsf{I}$ and $\omega_{\mathsf{J}} + i\omega_{\mathsf{K}}$ together with $\omega_{\mathsf{I}}$, in view of Theorem* 3.27*, determine, to a weak stratified hyperkähler structure on $M_0$ relative to the orbit type stratification of $M_0$. Moreover, this stratification is a complex Whitney stratification relative to $\mathcal{O}_{\mathsf{I}}$. Finally, on the local model in Theorem* 3.24*, more precisely, on the left-hand side $(U'_p, \omega_{E_p}, \kappa_p)$ of* (41)*, the other pieces of structure $\mathsf{J}$, $\mathsf{K}$ and $\omega_{\mathsf{I}}$ on $M$ induce not necessarily linear complex structures and a Kähler form that turn the local model into a $K$-tri-Hamiltonian hyperkähler manifold.* ■

Repeating the reasoning for Theorem 3.32 with regard to the complex structures $\mathsf{J}$ and $\mathsf{K}$ leads to the following, which is [53, Corollary 3.1.10].

**Theorem 3.33.** *Suppose that the $K$-action integrates to holomorphic $K^{\mathbb{C}}$-actions relative to each of $\mathsf{I}$, $\mathsf{J}$ and $\mathsf{K}$. Then the hyperkähler structure induces three complex analytic structures $\mathcal{O}_{\mathsf{I}}$, $\mathcal{O}_{\mathsf{J}}$, $\mathcal{O}_{\mathsf{K}}$ and three pairwise compatible real Poisson structures $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}}$, $\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}$ on $M_0$ such that*

$$(\mathcal{O}_{\mathsf{I}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}}),\ (\mathcal{O}_{\mathsf{J}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{K}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}}),\ (\mathcal{O}_{\mathsf{K}}, \{\,\cdot\,,\,\cdot\,\}_{\mathsf{I}} + i\{\,\cdot\,,\,\cdot\,\}_{\mathsf{J}}) \quad (49)$$

*are holomorphic Poisson structures. These generate a sphere of holomorphic Poisson structures on $M_0$. Moreover, the orbit type decomposition of $M_0$ is a complex Whitney stratification relative to each of $\mathcal{O}_{\mathsf{I}}$, $\mathcal{O}_{\mathsf{J}}$, $\mathcal{O}_{\mathsf{K}}$, and the three holomorphic Poisson structures constitute a stratified hyperkähler structure on $M_0$.* ■

**Remark 3.34.** Theorem 3.27 together with Theorem 3.24 extends, in a sense, [54, Theorem 1.4] given there in the realm of hyperkähler manifolds to holomorphic symplectic Kähler manifolds but offers a weaker conclusion, however: The strata in Theorem 3.27 are holomorphic symplectic manifolds whereas those in [54, Theorem 1.4] are hyperkähler. Theorem 3.32 together with Theorem 3.24 essentially recovers [54, Theorem 1.4]. ■

**Remark 3.35.** For $Y \in \mathfrak{k}$, let $Y_M$ denote the induced smooth vector field on $M$. The $K$-action on $M$ is integrable for the complex structure $\mathsf{I}$, i.e., extends to a holomorphic $G$-action on $M$, if and only if, for $Y \in \mathfrak{k}$, the smooth vector field $\mathsf{I}Y_M$ on $M$ is complete. Thus, for $M$ compact, the $K$-action is integrable for any complex structure, cf., e.g., [16, Theorem 4.4], and the three holomorphic Poisson structures in Theorem 3.33 constitute a stratified hyperkähler structure on $M_0$. ■

## 4. Twisted algebraic representation varieties

Retain the notation of Section 2. The $G$-subspace $\mathrm{Hom}(\pi, G)$ of $\mathrm{Hom}(F, G) \cong G^{2\ell}$ is Zariski-closed and hence an affine $G$-variety. By definition, the affine categorical quotient $\mathrm{Hom}(\pi, G)//G$ is the affine variety having $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$ as its coordinate ring, and we take the *algebraic representation variety* $\mathrm{Rep}_{\mathrm{alg}}(\pi, G)$ associated with $\pi$ and $G$ to be this quotient; cf., e.g., [68, Proposition 6.1 p. 11]. By general principles, the projection from $\mathrm{Hom}(\pi, G)$ to $\mathrm{Rep}_{\mathrm{alg}}(\pi, G)$ is a $G$-reduction in the sense of Subsection 3.2, cf., e.g., [73, Section 3]. The closed $G$-orbits are the semisimple representations, the quotient $\mathrm{Rep}_{\mathrm{alg}}(\pi, G)$ parametrizes the closed $G$-orbits, i.e., the semisimple representations, and each $G$-orbit in $\mathrm{Hom}(\pi, G)$ has its *semisimplification* as the unique closed $G$-orbit in its closure [63]. The injection $\mathrm{Hom}^{\mathrm{ssimple}}(\pi, G) \subseteq \mathrm{Hom}(\pi, G)$ induces a homeomorphism

$$\mathrm{Hom}^{\mathrm{ssimple}}(\pi, G)/G \longrightarrow \mathrm{Rep}_{\mathrm{alg}}(\pi, G) \quad (50)$$

from the space of $G$-orbits in the subspace $\mathrm{Hom}^{\mathrm{ssimple}}(\pi, G)$ of semisimple representations in $\mathrm{Hom}(\pi, G)$ onto $\mathrm{Rep}_{\mathrm{alg}}(\pi, G)$. These facts hold for both the Zariski and the classical (metric) topology. In the terminology of [67, 68], $\mathrm{Rep}_{\mathrm{alg}}(\pi, G)$ is the ordinary Betti moduli space; in [68, Section 6 p. 11/12], Simpson proceeds more generally for the fundamental group of a Kähler manifold but this need not concern us here. The *nonabelian Hodge theorem* establishes, among others, for $G = \mathrm{GL}(m, \mathbb{C})$, a homeomorphism between the moduli space of semistable rank $m$ topologically trivial Higgs bundles and $\mathrm{Rep}_{\mathrm{alg}}(\pi, G)$ over the surface $\Sigma$. This goes back to [29] for the case of rank two Higgs bundles and to [67, 68] for the general case (in particular for the fundamental group of an arbitrary Kähler manifold). Suitably rephrased, this correspondence extends to arbitrary complex reductive Lie groups of the kind $G$ under discussion.

A classical topological construction provides the means to recover the case of topologically non-trivial bundles. Atiyah-Bott discuss this in detail for connected $K$ [4, Section 6]; see also [32, Section 5], [11, Section 3]: Let $N$ denote the normal closure of $r$ in $F$. Consider the quotient group $\Gamma = F/[F, N]$. The image $[r] \in \Gamma$ of $r \in F$ generates the central subgroup $\mathbb{Z}\langle [r] \rangle = N/[F, N]$ of $\Gamma$, and the resulting extension

$$0 \longrightarrow \mathbb{Z}\langle [r] \rangle \longrightarrow \Gamma \longrightarrow \pi \longrightarrow 1 \tag{51}$$

is central. Since the transgression homomorphism $\mathrm{H}_2(\pi) \to \mathbb{Z}\langle [r] \rangle$ is an isomorphism, the extension (51) is a maximal stem extension (Schur cover) and since, furthermore, the abelianization of $\pi$ is a free abelian group, that maximal stem extension is unique up to within isomorphism [15, §9.9 Theorem 5 p. 214]. Atiyah and Bott use the terminology "universal central extension" to refer to this situation [4, §6].

Let $X$ be a member of the center $\mathfrak{z}$ of $\mathfrak{k}$ such that $\exp(X)$ lies in the center of $K$. When $K$ is connected, $\exp(X)$ lies in the center of $K$ for any $X \in \mathfrak{z}$. The canonical surjection $F \to \Gamma$ induces an injection $\mathrm{Hom}(\Gamma, G) \subseteq \mathrm{Hom}(F, G)$, and this injection identifies a certain subspace of $\mathrm{Hom}(\Gamma, G)$ with the subspace $r^{-1}(\exp(X))$, if non-empty, of $\mathrm{Hom}(F, G)$. Thus, suppose $r^{-1}(\exp(X))$ non-empty. We then denote that subspace of $\mathrm{Hom}(\Gamma, G)$ by $\mathrm{Hom}_X(\Gamma, G)$. The member $X$ of the center $\mathfrak{z}$ recovers a topological characteristic class of a corresponding bundle. See [4, §6], [11, Proposition 3.1] for details. We take the *twisted algebraic representation variety* $\mathrm{Rep}_{X,\mathrm{alg}}(\Gamma, G)$ *associated to* $X \in \mathfrak{z}$ to be the corresponding affine categorical quotient. The homeomorphism (50) generalizes to a homeomorphism

$$\mathrm{Hom}_X^{\mathrm{ssimple}}(\Gamma, G)//G \longrightarrow \mathrm{Rep}_{X,\mathrm{alg}}(\Gamma, G). \tag{52}$$

The nonabelian Hodge correspondence extends to that case and recovers all topological types of Higgs bundles on $\Sigma$. This is the Higgs bundles version of the corresponding observation in [4, §6]; it is a consequence of results in [66].

## 5. Twisted analytic representation varieties as stratified holomorphic symplectic spaces

Let $K$ be a maximal compact subgroup so that $G$ is the complexification $K^{\mathbb{C}}$ of $K$. Endow the Lie algebra $\mathfrak{k}$ of $K$ with an invariant inner product. Left trivialization, the polar decomposition of $G = K^{\mathbb{C}}$ and the inner product on $\mathfrak{k}$ induce a diffeomorphism

$$\mathrm{T}^*K \xrightarrow{\cong} \mathrm{T}K \longrightarrow K \times \mathfrak{k} \longrightarrow K^{\mathbb{C}} = G \tag{53}$$

compatible with $K$-left and right translation in such a way that the complex structure on $K^{\mathbb{C}}$ and the cotangent bundle symplectic structure on $\mathrm{T}^*K$ combine to a $K$-bi-invariant Kähler structure on $G$. In [46], Kronheimer claims this without proof; a proof is in [20]. Moreover, the cotangent bundle momentum mappings for left and right translation combine, relative to the Kähler form, to a momentum mapping $\mu_{\mathrm{cot}}\colon G \to \mathfrak{k}^*$ for the $K$-action on $G$ by conjugation in $G$, and the inner product on $\mathfrak{k}$ induces a non-degenerate $\mathbb{C}$-valued invariant symmetric bilinear form $\cdot$ on $\mathfrak{g}$. Taking the product structure, we obtain a Kähler form $\omega_{\mathbb{R}}$ on $G^{2\ell}$ and, relative to the diagonal $K$-action on $G^{2\ell}$, the action on each copy of $G$ being by conjugation in $G$, a $K$-momentum mapping $\mu_{\mathbb{R}}\colon G^{2\ell} \longrightarrow \mathfrak{k}^*$. Thus $(G^{2\ell}, \omega_{\mathbb{R}}, \mu_{\mathbb{R}})$ is a $K$-Hamiltonian Kähler manifold.

In diagram (4), substitute, for the open $G$-invariant subset $O$ of $\mathfrak{g}$, an open neighborhood in $\mathfrak{g}$ of $0 \in \mathfrak{g}$ where the exponential map is a biholomorphism onto an open neighborhood of the neutral element $e$ of $G$. Then the restriction $\eta\colon \mathcal{M}(\mathcal{P}, G) \to G^{2\ell}$ is a biholomorphism onto a $G$-invariant open neighborhood of $\mathrm{Hom}(\pi, G) = r^{-1}(e) \subseteq G^{2\ell}$, and the $K$-Hamiltonian Kähler structure on $(G^{2\ell}, \omega_{\mathbb{R}}, \mu_{\mathbb{R}})$ induces a $K$-Hamiltonian Kähler structure $(\omega_{\mathbb{R}}, \mu_{\mathbb{R}})$ on $\mathcal{M}(\mathcal{P}, G)$; here we slightly abuse the notation $\omega_{\mathbb{R}}$ and $\mu_{\mathbb{R}}$. Let $\omega_{\mathbb{C}} = \omega_{c,\mathcal{P}}$, cf. (5), and $\mu_{\mathbb{C}} = \mu_{c,\mathcal{P}}$, cf. (6), and, in terms of the notation and terminology in Subsection 3.3, define the *analytic representation variety* associated with $\pi$ and $G$ to be the holomorphic symplectic quotient

$$\mathrm{Rep}_{\mathrm{an}}(\pi, G) = \mathcal{M}(\mathcal{P}, G)//_{\mu_{\mathbb{C}}} G \cong \mu^{-1}(0)/K. \tag{54}$$

As in the previous section, let $X$ be a member of the center $\mathfrak{z}$ of $\mathfrak{k}$ such that $\mu_{\mathbb{C}}^{-1}(X)$ is non-empty. In diagram (4), substitute, for the open $G$-invariant subset $O$ of $\mathfrak{g}$, an open neighborhood in $\mathfrak{g}$ of $X \in \mathfrak{g}$ where the exponential map is a biholomorphism onto an open neighborhood $\widehat{O}$ of $\exp(X) \in G$. As before, the restriction $\eta\colon \mathcal{M}(\mathcal{P}, G) \to G^{2\ell}$ is a biholomorphism onto a $G$-invariant open neighborhood $\widehat{\mathcal{M}}(\mathcal{P}, G)$ in $G^{2\ell}$ of

$$\mathrm{Hom}_X(\Gamma, G) = r^{-1}(\exp(X)) \subseteq G^{2\ell},$$

and the $K$-Hamiltonian Kähler structure on $(G^{2\ell}, \omega_{\mathbb{R}}, \mu_{\mathbb{R}})$ induces a $K$-Hamiltonian Kähler structure $(\omega_{\mathbb{R}}, \mu_{\mathbb{R}})$ on $\mathcal{M}(\mathcal{P}, G)$; here again we slightly abuse the notation $\omega_{\mathbb{R}}$ and $\mu_{\mathbb{R}}$. As before, let $\omega_{\mathbb{C}} = \omega_{c,\mathcal{P}}$, cf. (5), and $\mu_{\mathbb{C}} = \mu_{c,\mathcal{P}}$, cf. (6), in terms of the notation and terminology in Subsection 3.3, let

$$\mu_{\mathbb{C}}^{-1}(X)^{\mu_{\mathbb{R}}-\mathrm{ss}} = \mu_{\mathbb{C}}^{-1}(X) \cap \mathcal{M}(\mathcal{P}, G)^{\mu_{\mathbb{R}}-\mathrm{ss}},$$

and define the *twisted analytic representation variety* associated with $\pi$, $G$, and $X$ to be the holomorphic symplectic quotient

$$\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G) = \mu_{\mathbb{C}}^{-1}(X)^{\mu_{\mathbb{R}}-\mathrm{ss}}//G \cong (\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(X))/K. \tag{55}$$

Then the varieties $\mathrm{Rep}_{0,\mathrm{an}}(\Gamma, G)$ and $\mathrm{Rep}_{\mathrm{an}}(\pi, G)$ coincide. Theorem 3.27 implies the following.

**Theorem 5.1.** *The complex Lie group $G$, the invariant inner product on $\mathfrak{k}$, and the choice of $X \in \mathfrak{z}$ such that $\exp(X)$ lies in the center of $K$ and $\mu_{\mathbb{C}}^{-1}(X)$ is non-empty determine a stratified holomorphic symplectic structure on the twisted analytic representation variety $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$. The stratification is a complex Whitney stratification.* ■

**Remark 5.2.** The stratified holomorphic symplectic structure is independent of any complex structure on $\Sigma$. ■

**Remark 5.3.** Let $\varphi\colon \Gamma \to G$ be a representation which lies in $\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(X)$. Then $\varphi$ determines a $\pi$-module structure on $\mathfrak{g}$, and we denote this $\pi$-module by $\mathfrak{g}_\varphi$. One can show that right translation identifies an infinitesimal holomorphic symplectic slice at $\varphi$ as a point of $\mathcal{M}(\mathcal{P}, G)$ with $\mathrm{H}^1(\pi, \mathfrak{g}_\varphi) \cong \mathrm{H}^1(\Sigma, \mathfrak{g}_\varphi)$. In particular, at a regular point $[\varphi]$ of $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$, a choice of representative $\varphi$ in $[\varphi]$ induces an isomorphism from $\mathrm{H}^1(\Sigma, \mathfrak{g}_\varphi)$ to the tangent space $\mathrm{T}_{[\varphi]}(\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G))$ to $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ at the point $[\varphi]$. This kind of observation goes back to [74].

Let $[\pi] \in \mathrm{H}_2(\pi, \mathbb{Z}) \cong \mathbb{Z}$ denote a fundamental class (generator). Consider a general point $\varphi$ in $\mu_{\mathbb{R}}^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(X)$. The stabilizer $H_\varphi \subseteq G$ acts linearly on $\mathrm{H}^1(\pi, \mathfrak{g}_\varphi)$, the pairing

$$\omega_\varphi\colon \mathrm{H}^1(\pi, \mathfrak{g}_\varphi) \otimes \mathrm{H}^1(\pi, \mathfrak{g}_\varphi) \xrightarrow{\cdot \circ \cup} \mathrm{H}^2(\pi, \mathbb{C}) \xrightarrow{\cap[\pi]} \mathbb{C} \tag{56}$$

is skew and, in view of Poincaré duality, nondegenerate, i.e., a symplectic structure, necessarily $H_\varphi$-invariant. Moreover,

$$\mu_\varphi\colon \mathrm{H}^1(\pi, \mathfrak{g}_\varphi) \xrightarrow{\cup \circ [\,\cdot\,,\,\cdot\,]} \mathrm{H}^2(\pi, \mathfrak{g}_\varphi) \xrightarrow{\cong} \mathfrak{h}_\varphi^* \tag{57}$$

recovers the associated momentum mapping having the value zero at the origin. The resulting symplectic quotient $\mathrm{H}^1(\pi, \mathfrak{g}_\varphi)//H_\varphi$ is a local model for $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ near $[\varphi]$ as a stratified holomorphic symplectic space. We can interpret this as saying that $\mathrm{H}^1(\pi, \mathfrak{g}_\varphi)//H_\varphi$ yields generalized analytic Darboux coordinates for $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ near $[\varphi]$. In particular, at a regular point $[\varphi]$, this yields ordinary holomorphic Darboux coordinates for $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ near $[\varphi]$. See [35] and the literature there for details.

Endow the (real) surface $\Sigma$ with a complex structure. Using the corresponding Hodge decomposition, one can put a complex structure on $\mathrm{H}^1(\Sigma, \mathfrak{g}_\varphi)$ distinct from that coming from the complex structure of $\mathfrak{g}$. As $\varphi$ varies, one can, perhaps, in this way recover the requisite complex analytic and holomorphic Poisson structures and prove that an analytic representation variety of the kind $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ acquires a stratified hyperkähler structure which in particular recovers the hyperkähler structure on the top stratum built in [29]. ■

## 6. Comparison of the twisted analytic and algebraic representation varieties

**Theorem 6.1.** *Let $X$ be a member of the center $\mathfrak{z}$ of $\mathfrak{k}$ such that* $\exp(X)$ *lies in the center of $K$ and that $\mu_{\mathbb{C}}^{-1}(X)$ is non-empty. The holomorphic map $\eta\colon \mathcal{M}(\mathcal{P}, G) \to G^{2\ell}$ induces an analytic isomorphism*

$$\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G) \longrightarrow \mathrm{Rep}_{X,\mathrm{alg}}(\Gamma, G). \tag{58}$$

Thus the twisted analytic representation varieties recover the Betti moduli spaces as analytic spaces.

**Proof.** As in the previous section, let $O$ be an open $G$-invariant neighborhood of $X$ in $\mathfrak g$ where the exponential map is a biholomorphism onto an open neighborhood $\widehat O$ of $\exp(X) \in G$. The diagram

$$\begin{array}{ccccc} \mu_{\mathbb C}^{-1}(X) & \xrightarrow{\subseteq} & \mathcal M(\mathcal P, G) & \xrightarrow{r_O} & O \\ \eta_| \big\downarrow & & \eta \big\downarrow & & \big\downarrow \exp \\ \mathrm{Hom}_X(\Gamma, G) & \xrightarrow{\subseteq} & \widehat{\mathcal M}(\mathcal P, G) & \xrightarrow[r]{} & \widehat O \end{array} \tag{59}$$

is commutative, $\exp\colon O \to \widehat O$ and $\eta$ are biholomorphisms, and $\eta$ restricts to an isomorphism $\eta_|\colon \mu_{\mathbb C}^{-1}(X) \to \mathrm{Hom}_X(\Gamma, G)$ of analytic sets.

Consider the momentum mapping $\mu_{\mathbb R}\colon G^{2\ell} \to \mathfrak k^*$. The zero locus $\mu_{\mathbb R}^{-1}(0) \subseteq G^{2\ell}$ is a *Kempf-Ness* set (in the algebraic sense) for the algebraic $G$-action on $G^{2\ell}$, and $\mu_{\mathbb R}^{-1}(0) \cap \mathrm{Hom}_X(\Gamma, G)$ is a Kempf-Ness set for the algebraic $G$-action on $\mathrm{Hom}_X(\Gamma, G)$. Hence the injection of $\mu_{\mathbb R}^{-1}(0) \cap \mathrm{Hom}_X(\Gamma, G)$ into $\mathrm{Hom}_X(\Gamma, G)$ induces a homeomorphism

$$(\mu_{\mathbb R}^{-1}(0) \cap \mathrm{Hom}_X(\Gamma, G))/K \longrightarrow \mathrm{Hom}_X(\Gamma, G)//G = \mathrm{Rep}_{X,\mathrm{alg}}(\Gamma, G). \tag{60}$$

See, e.g., [62] for details.

Relative to the momentum mapping $\mathcal M(\mathcal P, G) \xrightarrow{\eta} G^{2\ell} \xrightarrow{\mu_{\mathbb R}} \mathfrak k^*$—above we also used the notation $\mu_{\mathbb R}$ for it—, the zero locus $(\mu_{\mathbb R} \circ \eta)^{-1}(0) \subseteq \mathcal M(\mathcal P, G)$ is likewise a *Kempf-Ness* set (in the analytic sense) for the analytic $G$-action on $\mathcal M(\mathcal P, G)$, and $(\mu_{\mathbb R} \circ \eta)^{-1}(0) \cap \mu_{\mathbb C}^{-1}(X)$ is a Kempf-Ness set for the analytic $G$-action on $\mu_{\mathbb C}^{-1}(X)$. See [26, §1.2] for this notion of Kempf-Ness set. By [26, Intro §1.3 p. 289, §3.3 Theorem p. 295], the injection

$$(\mu_{\mathbb R} \circ \eta)^{-1}(0) \cap \mu_{\mathbb C}^{-1}(X) \longrightarrow \mu_{\mathbb C}^{-1}(X) \tag{61}$$

induces a homeomorphism

$$((\mu_{\mathbb R} \circ \eta)^{-1}(0) \cap \mu_{\mathbb C}^{-1}(X))/K \longrightarrow \mu_{\mathbb C}^{-1}(X)//G = \mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G). \tag{62}$$

However, $\eta$ also induces a homeomorphism

$$((\mu_{\mathbb R} \circ \eta)^{-1}(0) \cap \mu_{\mathbb C}^{-1}(X))/K \longrightarrow (\mu_{\mathbb R}^{-1}(0) \cap \mathrm{Hom}_X(\Gamma, G))/K. \tag{63}$$

This implies the claim. ∎

## 7. Parabolic structures

Retain the notation established before. Thus $G$ is a complex reductive Lie group. Consider the standard presentation

$$\mathcal P = \langle x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n; r\rangle, \quad r = \Pi[x_j, y_j] z_1 \cdots z_n, \tag{64}$$

of the fundamental group $\pi$ of a (real) compact surface of genus $\ell \geq 0$ with $n \geq 0$ boundary circles, and suppose $n \geq 3$ when $\ell = 0$, cf. [19, (2.1) p. 381]. (An assumption of the kind $\ell + n > 0$ avoids inconsistencies, and the case $(\ell, n) = (0, 2)$ is special and not interesting.) In the literature, it is also common to consider punctures rather than boundary circles.

As in [19, Section 5 and thereafter], let $F$ be the free group on the generators $x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n$. Let $\mathbf{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ be a family of $n$ conjugacy classes in $G$, and let $\mathrm{Hom}(F, G)_{\mathbf{C}}$ denote the space of homomorphisms from $F$ to $G$ for which the value of the generator $z_j$ lies in $\mathcal{C}_j$ $(1 \leq j \leq n)$. The choice of generators $x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n$ of $\pi$ induces an identification

$$\mathrm{Hom}(F, G)_{\mathbf{C}} \longrightarrow G^{2\ell} \times \mathcal{C}_1 \times \ldots \times \mathcal{C}_n. \tag{65}$$

Accordingly, let $\mathrm{Hom}(\pi, G)_{\mathbf{C}}$ denote the space of homomorphisms from $\pi$ to $G$ for which the value of the generator $z_j$ lies in $\mathcal{C}_j$ $(1 \leq j \leq n)$. The canonical projection $F \to \pi$ induces an embedding $\mathrm{Hom}(\pi, G)_{\mathbf{C}} \to \mathrm{Hom}(F, G)_{\mathbf{C}}$, and this embedding realizes $\mathrm{Hom}(\pi, G)_{\mathbf{C}}$ as an affine algebraic variety in $\mathrm{Hom}(F, G)_{\mathbf{C}}$. As in Section 2, the relator $r$ induces a complex algebraic map

$$r \colon G^{2\ell} \times \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \longrightarrow G, \tag{66}$$

and $\mathrm{Hom}(\pi, G)_{\mathbf{C}}$ coincides with the pre-image $r^{-1}(e)$ of the identity element of $G$. In the construction in Section 2, substitute (64) for (2) and define the space $\mathcal{H}(\mathcal{P}, G)_{\mathbf{C}}$ by requiring that

$$\begin{array}{ccc} \mathcal{H}(\mathcal{P}, G)_{\mathbf{C}} & \xrightarrow{(r_O, \overline{z}_1, \ldots, \overline{z}_n)} & O \times \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \\ {\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle \exp \times \mathrm{Id} \times \ldots \times \mathrm{Id}} \\ G^{2\ell} & \xrightarrow[(r, z_1, \ldots, z_n)]{} & G \times \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \end{array} \tag{67}$$

be a pullback diagram; here the notation $\eta$, $r_O$, and $\overline{z}_1, \ldots, \overline{z}_n$ refers to the induced maps. The construction of the closed 2-form $\omega_{c,\mathcal{P},\mathbf{C}}$ on $\mathcal{H}(\mathcal{P}, G)$ in [19, Theorem 7.1.1] carries over more or less verbatim to the present situation. The reader will notice that, for $n = 0$, we are in the situation of Section 2.

Proceeding much the same way as before, we arrive at an analytic $G$-quotient $\mathrm{Rep}_{\mathrm{an}}(\pi, G)_{\mathbf{C}}$ of $\mathrm{Hom}(\pi, G)_{\mathbf{C}}$ as a stratified holomorphic symplectic space. In view of the nonabelian Hodge correspondence for Higgs bundles with parabolic structure, cf. [6], [65], this recovers the entire moduli space of corresponding Higgs bundles as a stratified holomorphic symplectic space. We spare the reader and ourselves the trouble of spelling out details.

**Remark 7.1.** Return to the presentation (2) of the fundamental group $\pi$ of a closed Riemann surface of genus $\ell \geq 1$ and consider the presentation

$$\langle x_1, y_1, \ldots, x_\ell, y_\ell, z; r \rangle, \quad r = \Pi[x_j, y_j] z, \tag{68}$$

of the fundamental group $\widehat{\pi}$ of that surface with 1 boundary circle. In the situation of Theorem 5.1, let $\mathcal{C} = \{\exp(X)\} \subseteq G$, the conjugacy class of the member $\exp(X)$ of the center of $G$. The canonical projection $\widehat{\pi} \to \Gamma$ induces an isomorphism

$$\mathrm{Hom}_X(\Gamma, G) \longrightarrow \mathrm{Hom}(\widehat{\pi}, G)_{\{\mathcal{C}\}}$$

of algebraic varieties and hence, between the quotients, an algebraic (and therefore analytic) isomorphism

$$\mathrm{Rep}_X(\Gamma, G) \longrightarrow \mathrm{Rep}(\widehat{\pi}, G)_{\{\mathcal{C}\}}.$$

The construction of the stratified holomorphic symplectic structure on $\mathrm{Rep}_X(\Gamma, G)$ is considerably less involved than that on a space of the kind $\mathrm{Rep}(\widehat{\pi}, G)_{\{\mathcal{C}\}}$, however. ■

## Appendix I: Quasi Hamiltonian approach

Relative to a compact group, [2] reworks the extended moduli space concept in terms of quasi Hamiltonian spaces; [39, Section 4] contains a detailed comparison. In particular, by [39, Conclusions 4.22 and 4.23], the reduced Poisson structure arising form an extended moduli space is equivalent to that arising from a corresponding quasi Hamiltonian space.

The paper [7] offers a construction of algebraic "wild character varieties" from complex algebraic quasi Hamiltonian spaces. In a nutshell, the procedure is as follows: Let $G$ be a complex algebraic group with an Ad-invariant non-degenerate symmetric bilinear form on its Lie algebra $\mathfrak{g}$. Let $(\sigma, \Phi)$ be an algebraic $G$-quasi Hamiltonian structure on an affine complex algebraic $G$-variety $M$. That is to say, $\sigma$ is an algebraic 2-form on $M$ and $\Phi\colon M \to G$ a $G$-equivariant algebraic map, and $\sigma$ and $\Phi$ are subject to the quasi Hamiltonian constraints. Let $\mathbb{C}[M]$ denote the affine coordinate ring of $M$. The $G$-quasi Hamiltonian structure on $M$ determines a Poisson bracket on the invariants $\mathbb{C}[M]^G$, that is, by construction, on the coordinate ring of the affine algebraic quotient $M//G$. This is the content of [7, Proposition 2.8]. To establish this fact, one must notice that, in the proof of [2, Proposition 4.6] (for the case of a compact group), one can get rid of the constraint that the corresponding 2-form on the Lie algebra be positive. See also [39, Proposition 4.6, Theorem 4.8]. Now, for a conjugacy class $\mathcal{C} \subseteq G$, the affine algebraic quotient $\Phi^{-1}(\mathcal{C})//G$ embeds canonically into $M//G$ as a subvariety. The "wild character varieties" in [7] are of that kind. When $M//G$ and $\Phi^{-1}(\mathcal{C})//G$ are non-singular, the quotient $\Phi^{-1}(\mathcal{C})//G$ is a symplectic leaf in $M//G$ with respect to the Poisson structure on $\mathbb{C}[M//G] \cong \mathbb{C}[M]^G$ and hence acquires a Poisson structure. While quasi Hamiltonian reduction is available in the regular case, in the presence of singularities, the reasoning in [7] in terms of Thm. 1.1 and Prop. 2.8 of that paper does not explain whether and how the Poisson structure descends to one on the coordinate ring $\mathbb{C}[\Phi^{-1}(\mathcal{C})//G]$ of $\Phi^{-1}(\mathcal{C})//G$. To achieve this, one must prove that the vanishing ideal of $\Phi^{-1}(\mathcal{C})$ in $\mathbb{C}[M//G]$ is a Poisson ideal. The argument for Theorem 3.13 settles this issue, that is, the Poisson bracket on $\mathbb{C}[M]^G$ actually induces a Poisson bracket on $\mathbb{C}[\Phi^{-1}(\mathcal{C})//G]$ in the general case. Theorem 5.18 in [39] yields an analytic proof thereof. It is, perhaps, worthwhile noting that the singular case is the typical case, so insisting on the significance of understanding the singular situation is not academic.

By construction, the quotient $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ embeds into the affine algebraic quotient $G^{2\ell}//G$ as an affine algebraic variety. Let $\ell \geq 1$, and consider the standard $G$-quasi Hamiltonian structure $(\sigma, \Phi)$ on $G^{2\ell}$. For reasons just explained, [7] does not enable us to conclude that the resulting algebraic Poisson bracket on $\mathbb{C}[G^{2\ell}]^G$ induces a Poisson bracket on the (affine algebraic) coordinate ring of $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ unless both varieties involved are non-singular whereas the argument for Theorem 3.13 and [39, Theorem 5.18] yields this fact in the singular case as well. By the general comparison result recalled at the beginning of this appendix, the analytification of this affine algebraic Poisson algebra yields the Poisson algebra which arises from the corresponding extended moduli space, that is, the Poisson algebra of functions on $\mathrm{Rep}_{X,\mathrm{an}}(\Gamma, G)$ which we exploit in the present paper.

The analytic approach in terms of extended moduli spaces is much more straightforward, however, than the algebraic approach, and analyticity is essential for another reason: There is no machinery in sight which enables us to concoct, in the algebraic setting, a symplectic slice theorem and to accordingly build an algebraic local model of the kind explained in the holomorphic setting in Subsection 3.4 above; in particular, there is no algebraic Darboux-Weinstein theorem, cf. Proposition 3.7, crucial for the proof of Proposition 3.6. Hence we cannot naively construct an algebraic variant of the stratified holomorphic symplectic structure. In fact, trying to carry out such a program in the etale setting might be an interesting endeavor.

## Appendix II

We profit from the opportunity to correct a minor technical flaw in [19]. We are indebted to Suhyoung Choi for having isolated this flaw.

The reasoning in [19, p. 402] relies on an identity of the kind

$$- < c, v \cup u > = < c, u \cup v >$$

but there is no reason for such an identity to be valid since $u$ and $v$ are (parabolic) 1-cocycles on $\pi$, and parabolicity does not entail such an identity.

To fix this problem, in the statement of [19, Key Lemma 8.4 p. 397], replace identity (8.4.2) with

$$\omega_V([v],[u]) = \tfrac{1}{2}(< c, u \cup v - v \cup u > + \sum (X_j \cdot z_j Y_j - Y_j \cdot z_j X_j)). \qquad (69)$$

The proof of [19, Theorem 8.3 p. 397] works fine with this identity.

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Johannes Huebschmann
Université de Lille – Sciences et Technologies
Département de Mathématiques
CNRS-UMR 8524, Labex CEMPI (ANR-11-LABX-0007-01)
59655 Villeneuve d'Ascq Cedex, France
Johannes.Huebschmann@univ-lille.fr

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