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A Formula and Sharp Estimates for the Dunkl Kernel for the Root System A_2

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Abstract. We transform a formula for the A_2 Dunkl kernel by Béchir Amri. The resulting formula expresses the A_2 Dunkl kernel in terms of the A_1 Dunkl kernel involving only positive terms. This result allows us to derive sharp estimates for the A_2 Dunkl kernel. As an interesting by-product, we obtain sharp estimates for the corresponding heat kernel.

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Key Words: Dunkl kernel, sharp estimate, root system.

1. Introduction

Estimates of Dunkl kernel and of Dunkl heat kernel are a challenging problem. Sharp estimates of these kernels were proven only in rank 1, see [3]. In the Dunkl setting, estimates of the Dunkl kernel $E_k(X,\lambda)$ are equivalent to heat kernel estimates (refer to formula (6)). For a good introduction on rational Dunkl theory, the reader should consider the papers [2] and [11]. We provide here some details and notations on Dunkl analysis.

A *root system* Σ is a set of nonzero vectors (roots) of an Euclidean space (say, without loss of generality, $\mathbf{R}^d$ with the usual scalar product $\langle\cdot,\cdot\rangle$) satisfying the following conditions:

1. The roots span $\mathbf{R}^d$.
2. The only scalar multiple of $\alpha\in\Sigma$ is $-\alpha$.
3. If $\alpha,\ \beta\in\Sigma$ then $\sigma_\alpha(\beta)=\beta-2\,\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\,\beta$.
4. If $\alpha,\ \beta\in\Sigma$ then $2\,\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}$ is an integer.

The reflections σ_α, $\alpha\in\Sigma$ generate the Weyl group W.

Each root system splits into a disjoint union $\Sigma=\Sigma^+\cup(-\Sigma^+)$ separated by a hyperplane $H_\beta=\{x\in\mathbf{R}^d:\langle\beta,x\rangle=0\}$ with $\beta\notin\Sigma$. The reflecting hyperplanes $H_\alpha,\alpha\in\Sigma$ divide $\mathbf{R}^d$ into connected open components, called Weyl chambers. The open positive Weyl chamber is $C^+=\{x\in\mathbf{R}^d:\langle\alpha,x\rangle>0 \text{ for all } \alpha\in\Sigma^+\}$.

A function $k:\Sigma\to\mathbf{R}$ is called a *multiplicity function* if it is invariant under the action of W on Σ.

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Let ∂_ξ be the derivative in the direction of $\xi \in \mathbf{R}^d$. The *Dunkl operators* indexed by ξ are then given by

$$T_\xi(k)\, f(X) = \partial_\xi\, f(X) + \sum_{\alpha\in\Sigma_+} k(\alpha)\,\langle\alpha,\xi\rangle\,\frac{f(X)-f(\sigma_\alpha X)}{\langle\alpha,X\rangle}.$$

The T_ξ's, $\xi \in \mathbf{R}^d$, form a commutative family.

For fixed $Y \in \mathbf{R}^d$, the Dunkl kernel $E_k(\cdot,\cdot)$ is then the only real-analytic solution to the system

$$T_\xi(k)|_X\; E_k(X,Y) = \langle\xi,Y\rangle\, E_k(X,Y),\ \forall\xi\in\mathbf{R}^d$$

with $E_k(0,Y)=1$. In fact, E_k extends to a holomorphic function on $\mathbf{C}^d\times\mathbf{C}^d$.

Up to now, the best estimates of the Dunkl heat kernel for all root systems were recently proven by Dziubański and Hejna [8]. They improve earlier results obtained in [4]. The results of [8] imply immediately the estimates of the Dunkl kernel $E_k(X,\lambda)$ with the exponential factor of the form

$$e^{c\langle\lambda^+,X^+\rangle}$$

for any $c>1$ in the upper estimate and any $c<1$ in the lower estimate (here Z^+ denotes the projection of Z on the closed positive Weyl chamber $\overline{C^+}$, i.e. the value of wZ with $w\in W$ chosen in such a way that $wZ\in\overline{C^+}$). Our recent results [9] in the W-invariant case for the system A_n and the paper [3] devoted to the A_1 case suggest that sharp estimates of $E_k(X,\lambda)$ with the exponential factor $e^{\langle\lambda^+,X^+\rangle}$ hold true.

It is known since Dunkl's paper [6] that the root system A_2 allows some explicit or integral expressions for important objects of Dunkl analysis, in particular for the Dunkl kernel $E_k(X,\lambda)$. Some integral formulas for $E_k(X,\lambda)$ of the system A_2 were given by Amri [1].

In this paper, starting from a result by Amri ([1, Theorem 1.1]), we prove a new integral formula for the Dunkl kernel $E_k(X,\lambda)$ (Theorem 3.1) on A_2, relating it to the Dunkl kernel in rank 1. This formula allows us to prove sharp estimates of the Dunkl kernel $E_k(X,\lambda)$ on A_2 (Theorem 4.1) with the exponential factor $e^{\langle\lambda^+,X^+\rangle}$. The proof uses techniques analogous to those that we used in [9].

Remark 1.1. In 2017, Jacek Dziubański presented an upper estimate in a Weyl chamber ([7]) of a similar form to the ones we have obtained for the Dunkl kernel estimate (based on work by Jean-Philippe Anker and Bartosz Trojan). In June 2023, Anker and Trojan informed us that they continued their study and showed us their current achievement containing sharp lower and upper estimates for the Dunkl kernel in the case of root systems A_2 and B_2. Their method is completely different from ours (refer to [5]). ■

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2. Basic notation and definitions

We consider the root system A_2 on the space

$$\mathfrak{a} = \{X=(x_1,x_2,x_3)\in\mathbf{R}^3\colon x_1+x_2+x_3=0\}.$$

The Weyl group is S_3 acting as permutations on the x_i's. The positive Weyl chamber is $C^+ = \{(x_1,x_2,x_3)\in\mathfrak{a}\colon x_1>x_2>x_3\}$.

Given a domain D, $f(x) \asymp g(x)$ means that there exists $C > 0$ such that $C^{-1}\, g(x) \le f(x) \le C\, g(x)$ for all $x \in D$. Similarly, $f(x) \lesssim g(x)$ ($f(x) \gtrsim g(x)$) means that there exists $C > 0$ such that $f(x) \le C\, g(x)$ ($f(x) \ge C\, g(x)$) for all $x \in D$.

For $X = (x_1, x_2, x_3)$, we denote by X^+ the projection of X on C^+, i.e. the reordering of the entries of X in decreasing order. For example, if X is such that $x_2 \ge x_1 \ge x_3$, then $X^+ = (x_2, x_1, x_3)$.

3. A new formula for the Dunkl kernel on A_2

The Dunkl kernel $E_k^{\mathrm{rk}\,1}$ for the system A_1 has the integral representation [2]

$$E_k^{\mathrm{rk}\,1}(x,v) = \frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(k)} \int_{-1}^{1} e^{zxv}(1-z)^{k-1}(1+z)^k dz$$
$$= e^{vx}\frac{\Gamma(2k+1)}{k\Gamma(k)^2} \int_0^1 e^{-2xvz} z^{k-1}(1-z)^k dz \tag{1}$$

Let $\mathcal{J}_a$ denote the modified Bessel function of index a. Then

$$E_k^{\mathrm{rk}\,1}(x,v) = \mathcal{J}_{k-1/2}(vx) + \frac{vx}{2k+1}\mathcal{J}_{k+1/2}(vx). \qquad \text{[2, Remark 3.9]} \tag{2}$$

Let $X = (x_1, x_2, x_3)$ with $x_1 + x_2 + x_3 = 0$. We denote the positive roots of the system A_2 by $\alpha(X) = x_1 - x_2$, $\beta(X) = x_2 - x_3$ and $\gamma = \alpha + \beta$. For simplicity, we write $\alpha_X = \alpha(X)$ and similarly for $\beta_X, \gamma_X, \alpha_\lambda, \beta_\lambda, \gamma_\lambda$. We will assume that the multiplicity $k = k_\alpha = k_\beta = k_{\alpha+\beta}$ satisfies $k > 0$.

Let $W_k(\lambda,\nu) = ((\lambda_1-\nu_1)(\lambda_1-\nu_2)(\nu_1-\lambda_2)(\lambda_2-\nu_2)(\nu_1-\lambda_3)(\nu_2-\lambda_3))^{k-1}$.

In the next theorem we give a new integral formula relating the Dunkl kernel $E_k(X,\lambda)$ on A_2 with the Dunkl kernel $E_k^{\mathrm{rk}\,1}$ on A_1.

Theorem 3.1. *Let $\lambda \in C^+$. We have for any $X \in \mathfrak{a}$,*

$$\begin{aligned} E_k(X,\lambda) = \frac{3\,\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3} \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} &\{(\nu_1-\lambda_2)(\lambda_1-\nu_2)\, E_k^{rk\,1}((x_1-x_2)/2, (\nu_1-\nu_2)) \\ &+ (\lambda_1-\nu_1)(\lambda_2-\nu_2)\, E_k^{rk\,1}(-(x_1-x_2)/2, (\nu_1-\nu_2))\} \\ &(\nu_1-\lambda_3)(\nu_2-\lambda_3) e^{(x_1+x_2-2x_3)(\nu_1+\nu_2)/2}\, W_k(\lambda,\nu)\, d\nu_1 d\nu_2 \end{aligned} \tag{3}$$

and

$$\begin{aligned} E_k(X,\lambda) = \frac{3\,\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3} \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} &\{(\nu_1-\lambda_3)(\lambda_2-\nu_2)\, E_k^{rk\,1}((x_2-x_3)/2, (\nu_1-\nu_2)) \\ &+ (\nu_1-\lambda_2)(\nu_2-\lambda_3)\, E_k^{rk\,1}(-(x_2-x_3)/2, (\nu_1-\nu_2))\} \\ &(\lambda_1-\nu_1)(\lambda_1-\nu_2) e^{(x_2+x_3-2x_1)(\nu_1+\nu_2)/2}\, W_k(\lambda,\nu)\, d\nu_1 d\nu_2. \end{aligned} \tag{4}$$

Proof. By the integral formula for E_k of Amri [1, Theorem 1.1] and the fact that $-2(\nu_1\nu_2 + (\lambda_3/2)(\nu_1+\nu_2) + \lambda_1\lambda_2) = (\lambda_1-\lambda_2)(\nu_1-\nu_2) - 2(\lambda_1-\nu_1)(\lambda_2-\nu_2)$ when $\lambda_1+\lambda_2+\lambda_3 = 0$, we find

$$E_k(X,\lambda) = \frac{3\,\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3}\int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1}\Big\{(\lambda_1-\lambda_2)(\nu_1-\nu_2)\mathcal{J}_{k-1/2}\left(\frac{(x_1-x_2)(\nu_1-\nu_2)}{2}\right)$$
$$+[(\lambda_1-\lambda_2)(\nu_1-\nu_2)-2(\lambda_1-\nu_1)(\lambda_2-\nu_2)]\mathcal{J}'_{k-1/2}\left(\frac{(x_1-x_2)(\nu_1-\nu_2)}{2}\right)\Big\}$$
$$(\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{(x_1+x_2-2x_3)(\nu_1+\nu_2)/2}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2$$

Noting that $\mathcal{J}'_a(x) = (x/(2\,(a+1)))\,\mathcal{J}_{a+1}(x)$ ([1, (8)]) and using the formula (2), we obtain the following expression for the term $\{\dots\}$ in the last integral:

$$\{\dots\} = (\lambda_1-\lambda_2)(\nu_1-\nu_2)E_k^{\mathrm{rk}\,1}((x_1-x_2)/2,(\nu_1-\nu_2))$$
$$-(\lambda_1-\nu_1)(\lambda_2-\nu_2)\frac{(x_1-x_2)(\nu_1-\nu_2)}{2\,k+1}\mathcal{J}_{k+1/2}\left(\frac{(x_1-x_2)(\nu_1-\nu_2)}{2}\right).$$

Now, using the integral representation of the modified Bessel function and integration by parts, we get

$$-\frac{(x_1-x_2)(\nu_1-\nu_2)}{2\,k+1}\mathcal{J}_{k+1/2}\left(\frac{(x_1-x_2)(\nu_1-\nu_2)}{2}\right)$$
$$= -2\,\frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}\,\frac{(x_1-x_2)(\nu_1-\nu_2)}{4\,k}\int_{-1}^{1} e^{(x_1-x_2)(\nu_1-\nu_2)z/2}\,(1-z^2)^k\,dz$$
$$= -2\,\frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}\int_{-1}^{1} e^{(x_1-x_2)(\nu_1-\nu_2)z/2}\,z(1-z^2)^{k-1}\,dz$$
$$= -2\,\frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}\int_{-1}^{1} e^{(x_1-x_2)(\nu_1-\nu_2)z/2}\,(1+z-1)(1-z^2)^{k-1}\,dz$$
$$= -2\,\frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}\int_{-1}^{1} e^{(x_1-x_2)(\nu_1-\nu_2)z/2}\,(1+z)^k(1-z)^{k-1}\,dz$$
$$+2\,\frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}\int_{-1}^{1} e^{(x_1-x_2)(\nu_1-\nu_2)z/2}\,(1-z^2)^{k-1}\,dz$$
$$= -E_k^{\mathrm{rk}\,1}((x_1-x_2)/2,(\nu_1-\nu_2))$$
$$+\frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}\int_{-1}^{1} e^{(x_1-x_2)(\nu_1-\nu_2)z/2}\,\overbrace{[2(1-z^2)^{k-1}-(1+z)^k(1-z)^{k-1}]}^{(1+z)^{k-1}(1-z)^k}\,dz$$
$$= -E_k^{\mathrm{rk}\,1}((x_1-x_2)/2,(\nu_1-\nu_2))+E_k^{\mathrm{rk}\,1}(-(x_1-x_2)/2,(\nu_1-\nu_2))$$

(in the last equalities we used (1)). The formula (3) follows. Interchanging the roles of the roots α and β in (3), we obtain (4). ∎

Remark 3.2. It is important to note that in the formulas (3) and (4), all terms are positive (the formula of Amri [1, Theorem 1.1] has a difference of two terms). Taking into account the recursive formula for E_k^W for the system A_n given in [12], we conjecture that the formulas (3) and (4) generalize to recursive formulas for E_k for the system A_n. ∎

4. Sharp estimates of the Dunkl kernel and Dunkl heat kernel on A_2

For simplicity, we denote the Weyl chambers by $C_{ijk} = \{Y:\ y_i \ge y_j \ge y_k\}$. In particular $\overline{C^+} = C_{123}$.

Theorem 4.1. *We have the following estimates of the Dunkl kernel for the system A_2.*

(i) *For X in the closed positive Weyl chamber $\overline{C^+}$*

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X\rangle}}{(1+\alpha_\lambda\alpha_X)^k(1+\beta_\lambda\beta_X)^k(1+\gamma_\lambda\gamma_X)^k}$$

(ii) *For X in the Weyl chamber $\sigma_\alpha\overline{C^+} = C_{213}$*

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k+1}(1+\beta_\lambda\beta_{X^+})^k(1+\gamma_\lambda\gamma_{X^+})^k}.$$

For X in the Weyl chamber $\sigma_\beta\overline{C^+} = C_{132}$

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k}(1+\beta_\lambda\beta_{X^+})^{k+1}(1+\gamma_\lambda\gamma_{X^+})^k}$$

(iii) *For X in the Weyl chamber $\sigma_\alpha\sigma_\beta\overline{C^+} = \sigma_\beta\sigma_\gamma\overline{C^+} = \sigma_\gamma\sigma_\alpha\overline{C^+} = C_{231}$*

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k+1}(1+\beta_\lambda\beta_{X^+})^{k+1}(1+\gamma_\lambda\gamma_{X^+})^k} \quad \text{if } \alpha_\lambda \le \beta_\lambda$$

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k}(1+\beta_\lambda\beta_{X^+})^{k+1}(1+\gamma_\lambda\gamma_{X^+})^{k+1}} \quad \begin{cases} \text{if } \alpha_\lambda \ge \beta_\lambda, \ \textit{and} \\ \alpha_\lambda\alpha_{X^+} \ge \beta_\lambda\beta_{X^+} \end{cases}$$

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k+1}(1+\beta_\lambda\beta_{X^+})^{k}(1+\gamma_\lambda\gamma_{X^+})^{k+1}} \quad \begin{cases} \text{if } \alpha_\lambda \ge \beta_\lambda, \ \textit{and} \\ \alpha_\lambda\alpha_{X^+} \le \beta_\lambda\beta_{X^+}. \end{cases}$$

(iv) *for X in the Weyl chamber $\sigma_\beta\sigma_\alpha\overline{C^+} = \sigma_\gamma\sigma_\beta\overline{C^+} = \sigma_\alpha\sigma_\gamma\overline{C^+} = C_{312}$*

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k+1}(1+\beta_\lambda\beta_{X^+})^{k+1}(1+\gamma_\lambda\gamma_{X^+})^k} \quad \text{if } \beta_\lambda \le \alpha_\lambda$$

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k}(1+\beta_\lambda\beta_{X^+})^{k+1}(1+\gamma_\lambda\gamma_{X^+})^{k+1}} \quad \begin{cases} \text{if } \beta_\lambda \ge \alpha_\lambda, \ \textit{and} \\ \alpha_\lambda\alpha_{X^+} \ge \beta_\lambda\beta_{X^+} \end{cases}$$

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k+1}(1+\beta_\lambda\beta_{X^+})^{k}(1+\gamma_\lambda\gamma_{X^+})^{k+1}} \quad \begin{cases} \text{if } \beta_\lambda \ge \alpha_\lambda, \ \textit{and} \\ \alpha_\lambda\alpha_{X^+} \le \beta_\lambda\beta_{X^+}. \end{cases}$$

(v) *For X in the Weyl chamber $\sigma_\gamma\overline{C^+} = \sigma_\alpha\sigma_\beta\sigma_\alpha\overline{C^+} = \sigma_\beta\sigma_\alpha\sigma_\beta\overline{C^+} = C_{321}$,*

$$E_k(X,\lambda) \asymp \frac{e^{\langle \lambda, X^+\rangle}}{(1+\alpha_\lambda\alpha_{X^+})^{k}(1+\beta_\lambda\beta_{X^+})^{k}(1+\gamma_\lambda\gamma_{X^+})^{k+1}}$$

Theorem 4.1 and the estimates of $E_k(X,\lambda)$ obtained for the system A_1 in [3] support the next conjecture. Each Weyl chamber C is the image of C^+ by an element w of the Weyl group, i.e. $wC^+ = C$. Each element $w \in W$ decomposes (often non-uniquely) as a composition of symetries

$$w = \sigma_{\alpha_1}\dots\sigma_{\alpha_s} \tag{5}$$

We then call the sequence $(\sigma_{\alpha_1},\dots,\sigma_{\alpha_s})$ a *realization* of C. Let $\mathcal{S}hort(C) \subset W$ be the set of the shortest realizations of C, i.e. the number s of symetries in the decomposition (5) is minimised on elements of $\mathcal{S}hort(C) \subset W$. For example, for the system A_2, we have $\mathcal{S}hort(C_{231}) = \{(\sigma_\alpha,\sigma_\beta),(\sigma_\beta,\sigma_\gamma),(\sigma_\gamma,\sigma_\alpha)\}$.

Conjecture 4.2. For the root system A_n the following estimate of the Dunkl kernel holds. Let C a Weyl chamber. For all $\lambda \in C^+$ and $X \in C$ there exists $S \in Short(C)$ such that

$$E_k(X,\lambda) \asymp e^{\langle \lambda, X^+\rangle} F(X,\lambda) \quad \text{and} \quad F(X,\lambda) = \frac{1}{\prod_{\alpha>0}(1+\alpha_\lambda \alpha_{X^+})^{p(\alpha)}},$$

where $p(\alpha) = k+1$ if $\sigma_\alpha \in S$ and $p(\alpha) = k$ otherwise. In particular, for $X \in C^+$,

$$F(X,\lambda) = \frac{1}{\prod_{\alpha>0}(1+\alpha_\lambda \alpha_X)^k}$$

and, for $X \in \sigma_\beta C^+$,

$$F(X,\lambda) = \frac{1}{(1+\beta_\lambda\beta_X)^{k+1}\prod_{\alpha\in\Sigma^+\setminus\{\beta\}}(1+\alpha_\lambda\alpha_X)^k}.$$

Remark 4.3. The kernel $E_k(X,\lambda)$ is analytic and therefore continuous. This must be reflected in the estimates given in Theorem 4.1, in particular in the rational portion of the estimate as the projection $X \mapsto X^+$ is continuous.

When X traverses from C^+ to $\sigma_\alpha C^+$, $\alpha_{X^+} = 0$. This explains how the term $1+\alpha_\lambda\alpha_{X^+}$ can continuously change from exponent k to $k+1$. The same reasoning applies when X traverses from C^+ to $\sigma_\beta C^+$.

When X traverses from $\sigma_\alpha C^+$ (where $x_2 > x_1 > x_3$) to $\sigma_\beta\sigma_\alpha C^+$ (where $x_2 > x_3 > x_1$), $\beta_{X^+} = 0$. This explains how the term $1+\beta_\lambda\beta_{X^+}$ can continuously change from exponent k to $k+1$.

The same Weyl chamber may be represented in two different ways, as $\sigma_\gamma\sigma_\beta\overline{C^+} = \sigma_\alpha\sigma_\gamma\overline{C^+}$ and the same reasoning applies to explain the continuous change and appearance of powers $k+1$ of the two other terms.

When X traverses from $\sigma_\beta\sigma_\alpha C^+$ (where $x_2 > x_3 > x_1$) to $\sigma_\alpha\sigma_\beta\sigma_\alpha C^+$ (where $x_3 > x_2 > x_1$), $\alpha_{X^+} = 0$ and therefore $\gamma_{X^+} = \beta_{X^+}$. This explains how the term $1+\alpha_\lambda\alpha_{X^+}$ can continuously change from exponent $k+1$ to k and the terms $1+\beta_\lambda\beta_{X^+}$ and $1+\gamma_\lambda\gamma_{X^+}$ can continuously “exchange” exponents. ∎

Let $\{\xi_1,\ldots,\xi_d\}$ be any orthonormal basis of $\mathfrak{a}$. Recall that the Dunkl Laplacian is given as $\Delta_k = \sum_{k=1}^d T_\xi^2$ (this definition is independent of the choice of orthornormal basis). Let $w_k(Y) = \prod_{\alpha\in\Sigma} |\langle\alpha, Y\rangle|^{k_\alpha}$ be the weight function on $\mathfrak{a}$. The Dunkl heat kernel is then defined as the unique solution of the system

$$\begin{cases} \Delta_k|_X \, p_t(X,Y) & = \dfrac{\partial}{\partial t} p_t(X,Y), \\ \lim_{t\to 0^+} \int_{\mathfrak{a}^+} p_t(X,Y)\, f(Y)\, w_k(Y)\, dY & = f(X) \end{cases}$$

for every continuous function f on $\mathfrak{a}$ which vanishes at infinity.

Corollary 4.4. *Let $X \in C^+$, $Y \in \mathfrak{a}$. We have*

$$p_t(X,Y) \asymp \frac{t^{-4+k_\alpha+k_\beta+k_\gamma}\, e^{-|X-Y^+|^2/(4t)}}{(t+\alpha_X\alpha_{Y^+})^{k_\alpha}(t+\beta_X\beta_{Y^+})^{k_\beta}(t+\gamma_X\gamma_{Y^+})^{k_\gamma}}$$

where $w \in W$ is such that $Y = wY^+$ and

$$(k_\alpha, k_\beta, k_\gamma) = \begin{cases} (k,k,k) & \textit{if } w = id \\ (k+1,k,k) & \textit{if } w = \sigma_\alpha \\ (k,k+1,k) & \textit{if } w = \sigma_\beta \\ (k,k,k+1) & \textit{if } w = \sigma_\gamma \\ (k{+}1,k{+}1,k) \text{ or } (k{+}1,k,k{+}1) \text{ or } (k,k{+}1,k{+}1) & \textit{if } w = \sigma_\alpha\sigma_\beta \\ & \textit{or } w = \sigma_\beta\sigma_\alpha \end{cases}$$

When $w = \sigma_\alpha\sigma_\beta$ or $w = \sigma_\beta\sigma_\alpha$, the conditions for the appearance of one of three possible values of $(k_\alpha, k_\beta, k_\gamma)$ are given in Theorem 4.1.

Proof. The Dunkl heat kernel $p_t(X,Y)$ is given as

$$p_t(X,Y) = \frac{1}{2^{\gamma+d/2}c_k}\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{4t}}\, E_k\left(X, \frac{Y}{2t}\right), \tag{6}$$

where $\gamma = \sum_{\alpha>0} k(\alpha)$. It suffices then to use our estimates for $E_k(X,\lambda)$ ([10, Lemma 4.5]). ∎

5. Proof of the sharp estimate

In this Section we give the proof of Theorem 4.1.

5.1. Substitution of the estimates of $E_k^{rk\,1}$ in the integral formulas for E_k

In the beginning of the proof, in the formulas from Theorem 3.1, we replace the Dunkl kernels on A_1 by their estimates provided in [3]:

$$E_k^{\mathrm{rk}\,1}(x,y) \asymp \frac{e^{|xy|}}{(1+|xy|)^{k+p}} \quad \text{where } p=0 \text{ if } xy \geq 0 \text{ and } p=1 \text{ if } xy \leq 0. \tag{7}$$

Let $J(X,\lambda) = V(\lambda)^{2k}\, E_k(X,\lambda)$. Estimating $E_k(X,\lambda)$ is equivalent to estimating of $J(X,\lambda)$.

Lemma 5.1. *If $x_1 \geq x_2$, we have*

$$J(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\alpha_\lambda + \alpha_X\,(\nu_1-\lambda_2)\,(\lambda_1-\nu_2))}{(1+\alpha_X(\nu_1-\nu_2))^{k+1}} \\ (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{(x_1-x_3)\nu_1+(x_2-x_3)\nu_2}\, W_k(\lambda,\nu)\, d\nu_1 d\nu_2. \tag{8}$$

If $x_2 \geq x_1$, we have

$$J(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\alpha_\lambda - \alpha_X\,(\lambda_1-\nu_1)\,(\lambda_2-\nu_2))}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}} \\ (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{(x_2-x_3)\nu_1+(x_1-x_3)\nu_2}\, W_k(\lambda,\nu)\, d\nu_1 d\nu_2. \tag{9}$$

Proof. If $x_2 \geq x_1$, we have

$$J(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \{(\nu_1-\lambda_2)(\lambda_1-\nu_2)\, E_k^{\mathrm{rk}\,1}((x_1-x_2)/2, (\nu_1-\nu_2)) \\ + (\lambda_1-\nu_1)(\lambda_2-\nu_2)\, E_k^{\mathrm{rk}\,1}(-(x_1-x_2)/2, (\nu_1-\nu_2))\} \\ (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{(x_1+x_2-2x_3)(\nu_1+\nu_2)/2}\, W_k(\lambda,\nu)\, d\nu_1 d\nu_2.$$

$$\asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \{(\nu_1-\lambda_2)(\lambda_1-\nu_2)\,\frac{e^{(x_2-x_1)(\nu_1-\nu_2)/2}}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}} + (\lambda_1-\nu_1)(\lambda_2-\nu_2)\,\frac{e^{(x_2-x_1)(\nu_1-\nu_2)/2}}{(1-\alpha_X(\nu_1-\nu_2))^{k}}\} (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{(x_1+x_2-2x_3)(\nu_1+\nu_2)/2}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2$$

$$= \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\lambda_2)(\lambda_1-\nu_2)+(\lambda_1-\nu_1)(\lambda_2-\nu_2)\,(1-\alpha_X(\nu_1-\nu_2))}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}} (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{x_2\nu_1-\nu_1x_3+\nu_2x_1-x_3\nu_2}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2$$

$$= \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\alpha_\lambda-\alpha_X\,(\lambda_1-\nu_1)\,(\lambda_2-\nu_2))}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}} (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{x_2\nu_1-\nu_1x_3+\nu_2x_1-x_3\nu_2}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2.$$

If $x_1 \geq x_2$, we have

$$J(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\lambda_1-\lambda_2+\alpha_X\,(\nu_1-\lambda_2)\,(\lambda_1-\nu_2))}{(1+\alpha_X(\nu_1-\nu_2))^{k+1}} (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{x_1\nu_1-\nu_1x_3+x_2\nu_2-x_3\nu_2}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2. \qquad \blacksquare$$

Interchanging the roles of the roots α and β, we find

Lemma 5.2. *If $x_2 \geq x_3$, we have*

$$J(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\beta_\lambda+\beta_X\,(\nu_1-\lambda_3)\,(\lambda_2-\nu_2))}{(1+\beta_X(\nu_1-\nu_2))^{k+1}} (\lambda_1-\nu_1)(\lambda_1-\nu_2)e^{\nu_1(x_2-x_1)+\nu_2(x_3-x_1)}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2. \quad (10)$$

If $x_3 \geq x_2$, we have

$$J(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\beta_\lambda-\beta_X\,(\nu_1-\lambda_2)\,(\nu_2-\lambda_3))}{(1-\beta_X(\nu_1-\nu_2))^{k+1}} (\lambda_1-\nu_1)(\lambda_1-\nu_2)e^{\nu_1(x_3-x_1)+\nu_2(x_2-x_1)}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2. \quad (11)$$

5.2. Reduction to estimating $I(X,\lambda)=e^{-\langle\lambda,X^+\rangle}J(X,\lambda)$

Now, in the formulas from Lemmas 5.1 and 5.2, on each Weyl chamber, we will put in evidence the exponential factor $e^{\langle\lambda,X^+\rangle}$. Estimating $E_k(X,\lambda)$ is equivalent to estimating the integral $I(X,\lambda)$.

5.2.1. Formulas (8) and (9)

Let $Q_\alpha^+ = e^{\nu_1(x_1-x_3)+\nu_2(x_2-x_3)}$ and $Q_\alpha^- = e^{\nu_1(x_2-x_3)+\nu_2(x_1-x_3)}$. We have

1. For $X\in\overline{C^+}=C_{123}$: $Q_\alpha^+ = e^{\langle\lambda,X^+\rangle}\,e^{-(x_1-x_3)(\lambda_1-\nu_1)-(x_2-x_3)(\lambda_2-\nu_2)}$.
2. For $X\in\sigma_\gamma\overline{C^+}=C_{321}$: $Q_\alpha^- = e^{\langle\lambda,X^+\rangle}\,e^{-(x_3-x_2)(\nu_1-\lambda_2)-(x_3-x_1)(\nu_2-\lambda_3)}$.
3. For $X\in\sigma_\beta\sigma_\alpha\overline{C^+}=C_{231}$: $Q_\alpha^- = e^{\langle\lambda,X^+\rangle}\,e^{-(x_2-x_3)(\lambda_1-\nu_1)-(x_3-x_1)(\nu_2-\lambda_3)}$.
4. For $X\in\sigma_\alpha\sigma_\beta\overline{C^+}=C_{312}$: $Q_\alpha^+ = e^{\langle\lambda,X^+\rangle}\,e^{-(x_3-x_1)(\nu_1-\lambda_2)-(x_3-x_2)(\nu_2-\lambda_3)}$.
5. For $X\in\sigma_\alpha\overline{C^+}=C_{213}$: $Q_\alpha^- = e^{\langle\lambda,X^+\rangle}\,e^{-(x_2-x_3)(\lambda_1-\nu_1)-(x_1-x_3)(\lambda_2-\nu_2)}$.
6. For $X\in\sigma_\beta\overline{C^+}=C_{132}$: $Q_\alpha^+ = e^{\langle\lambda,X^+\rangle}\,e^{-(x_1-x_3)(\lambda_1-\nu_1)-(x_3-x_2)(\nu_2-\lambda_3)}$.

5.2.2. Formulas (10) **and** (11)

Let $Q_\beta^+ = e^{\nu_1(x_2-x_1)+\nu_2(x_3-x_1)}$ and $Q_\beta^- = e^{\nu_1(x_3-x_1)+\nu_2(x_2-x_1)}$. We have

1. For $X \in \overline{C^+} = C_{123}$: $Q_\beta^+ = e^{\langle\lambda,X^+\rangle}\, e^{-(x_1-x_2)(\nu_1-\lambda_2)-(x_1-x_3)(\nu_2-\lambda_3)}$.
2. For $X \in \sigma_\gamma\overline{C^+} = C_{321}$: $Q_\beta^- = e^{\langle\lambda,X^+\rangle}\, e^{-(x_3-x_1)(\lambda_1-\nu_1)-(x_2-x_1)(\lambda_2-\nu_2)}$.
3. For $X \in \sigma_\beta\sigma_\alpha\overline{C^+} = C_{231}$: $Q_\beta^+ = e^{\langle\lambda,X^+\rangle}\, e^{-(x_2-x_1)(\lambda_1-\nu_1)-(x_3-x_1)(\lambda_2-\nu_2)}$.
4. For $X \in \sigma_\alpha\sigma_\beta\overline{C^+} = C_{312}$: $Q_\beta^- = e^{\langle\lambda,X^+\rangle}\, e^{-(x_3-x_1)(\lambda_1-\nu_1)-(x_1-x_2)(\nu_2-\lambda_3)}$.
5. For $X \in \sigma_\alpha\overline{C^+} = C_{213}$: $Q_\beta^+ = e^{\langle\lambda,X^+\rangle}\, e^{-(x_2-x_1)(\lambda_1-\nu_1)-(x_1-x_3)(\nu_2-\lambda_3)}$.
6. For $X \in \sigma_\beta\overline{C^+} = C_{132}$: $Q_\beta^- = e^{\langle\lambda,X^+\rangle}\, e^{-(x_1-x_3)(\nu_1-\lambda_2)-(x_1-x_2)(\nu_2-\lambda_3)}$.

5.3. Reductions using symmetries between the Weyl chambers

Consider the following relations:

$$\begin{aligned}E((x_1,x_2,x_3),(\lambda_1,\lambda_2,\lambda_3)) &= E((-x_1,-x_2,-x_3),(-\lambda_1,-\lambda_2,-\lambda_3))\\ &= E((-x_3,-x_2,-x_1),(-\lambda_3,-\lambda_2,-\lambda_1))\end{aligned}$$

with $\tilde\lambda = (-\lambda_3,-\lambda_2,-\lambda_1) \in \mathfrak{a}^+$ since $\lambda \in \mathfrak{a}^+$. Write $\tilde X = (-x_3,-x_2,-x_1)$. This "symmetry" sends $X \in C_{123}$ to $\tilde X \in C_{123}$, $X \in C_{213}$ to $\tilde X \in C_{132}$, $X \in C_{231}$ to $\tilde X \in C_{312}$ and $X \in C_{321}$ to $\tilde X \in C_{321}$. Note also that $\alpha_{\tilde\lambda} = \beta_\lambda$ and $\beta_{\tilde\lambda} = \alpha_\lambda$. Thus, we reduce finding the estimates of E_k to the four Weyl chambers: C_{123}, C_{213}, C_{231} and C_{321}.

There are several other symmetries (such as exchanging the role between α and β or commuting X and λ) but they do not give rise to any other reduction.

5.4. Plan of the proof for a fixed Weyl chamber

Fix X in a Weyl chamber. The proof will be done separately for $\alpha_\lambda \le \beta_\lambda$ and for $\beta_\lambda \le \alpha_\lambda$.

Step 1. Choose a "starting" formula for $I(X,\lambda)$ either with α_X (i.e. using J expressed by the formula (8) or (9)) or with β_X (i.e. using J expressed by the formula formula (10) or (11)). A choice allowing the next Step 2 always exists. This depends on the form of the exponentials $Q_\alpha^\pm$ or $Q_\beta^\pm$ which is crucial in this choice.

Step 2. In the double integral for $I(X,\lambda)$ chosen in Step 1, choose a half-subintegral I_1 such that it may be proven that $I_1 \gtrsim I_2 = I - I_1$.

Step 3. Find a sharp estimate of I_1. It will be a sharp estimate of I.

5.5. Positive Weyl chamber

Suppose $\alpha_\lambda \ge \beta_\lambda$. We choose as the starting formula the formula (8) with α_X. Using the corresponding formula for Q_α^+ of Section 5.2, we have

$$I(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\alpha_\lambda+\alpha_X\,(\nu_1-\lambda_2)\,(\lambda_1-\nu_2))}{(1+\alpha_X(\nu_1-\nu_2))^{k+1}} \\ (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{-(x_1-x_3)(\lambda_1-\nu_1)-(x_2-x_3)(\lambda_2-\nu_2)}\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2.$$

We set $I_1=\int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1}[\ldots]$ and $I_2=I-I_1$. Denote by $\tilde{I}_i$, $i=1,2$, the integral in the variable ν_1. We have

$$\tilde{I}_1\geq e^{-\alpha_\lambda\gamma_X/3}\int_{(2\,\lambda_1+\lambda_2)/3}^{(3\,\lambda_1+\lambda_2)/4}\ldots d\nu_1 \gtrsim \frac{{\alpha_\lambda}^{1+1+k+2(k-1)+1}}{(1+\alpha_X\alpha_\lambda)^{k+1}}\,(1+\alpha_X\alpha_\lambda)e^{-\alpha_\lambda\gamma_X/3}$$
$$=\frac{{\alpha_\lambda}^{3k+1}}{(1+\alpha_X\alpha_\lambda)^k}e^{-\alpha_\lambda\gamma_X/3}$$

while, using $\lambda_1-\nu_2\lesssim\alpha_\lambda$ and $\nu_1-\lambda_2\leq\nu_1-\nu_2$

$$\tilde{I}_2\lesssim{\alpha_\lambda}^{k-1}e^{-\alpha_\lambda\gamma_X/2}\int_{\lambda_2}^{M_1}(\nu_1-\nu_2)\frac{\alpha_\lambda+\alpha_X(\nu_1-\lambda_2)(\lambda_1-\nu_2)}{(1+\alpha_X(\nu_1-\nu_2))^{k+1}}\,(\nu_1-\lambda_2)^{k-1}\,(\nu_1-\lambda_3)^k\,d\nu_1$$
$$\leq{\alpha_\lambda}^{k-1+1+k}e^{-\alpha_\lambda\gamma_X/2}\int_{\lambda_2}^{M_1}\frac{(\nu_1-\nu_2)(\nu_1-\lambda_2)^{k-1}}{(1+\alpha_X(\nu_1-\nu_2))^k}\,d\nu_1.$$

Now, if $k\leq 1$, since $\nu_1\mapsto\frac{\nu_1-\nu_2}{(1+\alpha_X(\nu_1-\nu_2))^k}$ is increasing, we have

$$\int_{\lambda_2}^{M_1}\frac{(\nu_1-\nu_2)(\nu_1-\lambda_2)^{k-1}}{(1+\alpha_X(\nu_1-\nu_2))^k}\,d\nu_1\lesssim\frac{\alpha_\lambda}{(1+\alpha_X\alpha_\lambda)^k}\int_{\lambda_2}^{M_1}(\nu_1-\lambda_2)^{k-1}d\nu_1$$
$$\asymp\frac{{\alpha_\lambda}^{k+1}}{(1+\alpha_X\alpha_\lambda)^k}.\tag{12}$$

If $k\geq 1$, since $(\nu_1-\lambda_2)^{k-1}\leq(\nu_1-\nu_2)^{k-1}$ and $\nu_1\mapsto\frac{\nu_1-\nu_2}{1+\alpha_X(\nu_1-\nu_2)}$ is increasing, we get

$$\int_{\lambda_2}^{M_1}\frac{(\nu_1-\nu_2)(\nu_1-\lambda_2)^{k-1}}{(1+\alpha_X(\nu_1-\nu_2))^k}\,d\nu_1\leq\frac{{\alpha_\lambda}^k}{(1+\alpha_X\alpha_\lambda)^k}\int_{\lambda_2}^{M_1}d\nu_1\asymp\frac{{\alpha_\lambda}^{k+1}}{(1+\alpha_X\alpha_\lambda)^k}.\tag{13}$$

We will use the same device (separating $k\leq 1$ and $k\geq 1$) in (15), (18) and in (20) without repeating the details. In both cases, $\tilde{I}_2\lesssim\tilde{I}_1$. Finally,

$$I_1\asymp\frac{{\alpha_\lambda}^{1+1+k+2(k-1)}}{(1+\alpha_X\alpha_\lambda)^k}\int_{M_1}^{\lambda_1}e^{-\gamma_X\,(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^{k-1}\,d\nu_1$$
$$\cdot\int_{\lambda_3}^{\lambda_2}e^{-\beta_X\,(\lambda_2-\nu_2)}\,(\lambda_2-\nu_2)^{k-1}(\nu_2-\lambda_3)^{k-1}\,d\nu_1$$
$$\asymp\frac{{\alpha_\lambda}^{3k}}{(1+\alpha_X\alpha_\lambda)^k}\frac{{\alpha_\lambda}^k}{(1+\gamma_X\alpha_\lambda)^k}\frac{{\beta_\lambda}^{2k}}{(1+\beta_X\beta_\lambda)^k}$$

In the first integral we make the change of variable $u=\gamma_X(\lambda_1-\nu_1)$ and we use

$$\int_0^x e^{-u}u^{k-1}du\asymp x^k(1+x)^{-k}.\tag{14}$$

We obtain $$\int_{M_1}^{\lambda_1}e^{-\gamma_X\,(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^{k-1}\,d\nu_1\asymp\frac{{\alpha_\lambda}^k}{(1+\gamma_X\alpha_\lambda)^k}.$$

In the second integral, we change $v=\lambda_2-\nu_2$ and next $v=\beta_\lambda t$. We obtain

$${\beta_\lambda}^{2k}\int_0^1 e^{-\beta_X\beta_\lambda t}t^{k-1}(1-t)^k dt$$

We use the integral representation (1) of the function $E^{\mathrm{rk}\,1}(\beta_X/2,\beta_\lambda)$ and we apply the estimate (7) which is the desired result since $\alpha_\lambda\asymp\alpha_\lambda+\beta_\lambda=\gamma_\lambda$.

Note that if we assume $\beta_\lambda\geq\alpha_\lambda$, a similar proof holds using the starting formula the formula (10) with β_X and we set $I_1=\int_{\lambda_3}^{M_2}\int_{\lambda_2}^{\lambda_1}[\ldots]$.

5.6. Weyl chamber C_{321}

Suppose $\beta_\lambda \geq \alpha_\lambda$. We choose (9) as the starting formula with α_X. By Section 5.2, the exponent $Q_\alpha^- = e^{\langle \lambda, X^+\rangle}\, e^{\beta_X(\nu_1-\lambda_2)+\gamma_X(\nu_2-\lambda_3)}$ and the starting formula is

$$I(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\alpha_\lambda - \alpha_X\,(\lambda_1-\nu_1)\,(\lambda_2-\nu_2))}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}} \\ (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{\beta_X(\nu_1-\lambda_2)+\gamma_X(\nu_2-\lambda_3)}\, W_k(\lambda,\nu)\, d\nu_1 d\nu_2.$$

We set $I_1 = \int_{\lambda_3}^{M_2}\int_{\lambda_2}^{\lambda_1}[\ldots]$ and $I_2 = I - I_1$. Denote by $\tilde I_i$, $i=1,2$, the integrals in the variable ν_2. We have,

$$\tilde I_1 \geq \int_{(3\,\lambda_3+\lambda_2)/4}^{(2\,\lambda_3+\lambda_2)/3} \ldots d\nu_2 \gtrsim e^{\gamma_X\beta_\lambda/3}\frac{{\beta_\lambda}^{1+k+2(k-1)+1}}{(1-\alpha_X\beta_\lambda)^{k+1}}\,(\alpha_\lambda - \alpha_X\beta_\lambda(\lambda_1-\nu_1)) \\ \gtrsim e^{-(x_3-x_1)\beta_\lambda/3}\frac{{\beta_\lambda}^{3k}}{(1-\alpha_X\beta_\lambda)^{k+1}}\,(\alpha_\lambda - \alpha_X\beta_\lambda(\lambda_1-\nu_1))$$

while

$$\tilde I_2 \lesssim e^{-(x_3-x_1)\beta_\lambda/2}{\beta_\lambda}^k \int_{M_2}^{\lambda_2} \frac{(\nu_1-\nu_2)(\alpha_\lambda-\alpha_X(\lambda_1-\nu_1)(\lambda_2-\nu_2))(\lambda_1-\nu_2)^{k-1}(\lambda_2-\nu_2)^{k-1}}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}}\,d\nu_2 \\ \leq e^{-(x_3-x_1)\beta_\lambda/2}\alpha_\lambda{\beta_\lambda}^k \int_{M_2}^{\lambda_2} \frac{(\nu_1-\nu_2)(\lambda_1-\nu_2)^{k-1}(\lambda_2-\nu_2)^{k-1}}{(1-\alpha_X(\nu_1-\nu_2))^{k}}\,d\nu_2 \\ \lesssim \frac{e^{-(x_3-x_1)\beta_\lambda/2}\alpha_\lambda{\beta_\lambda}^{3k}}{(1-\alpha_X\beta_\lambda)^k}. \tag{15}$$

We now proceed similarly as in (12) and (13). The case $k\geq 1$ is similar to C^+. For $k\leq 1$, we use

$$\int_{M_2}^{\lambda_2}(\lambda_1-\nu_2)^{k-1}(\lambda_2-\nu_2)^{k-1}d\nu_2 = \int_0^{\alpha_\lambda/2} u^{k-1}(\alpha_\lambda-u)^{k-1}\,du = {\alpha_\lambda}^{2k-1}\int_0^{1/2} v^{k-1}(1-v)^{k-1}\,dv.$$

Now, since $x_3-x_1 \geq -\alpha_X = x_2-x_1$, we have

$$\frac{\tilde I_1}{\tilde I_2} \gtrsim \frac{e^{(x_3-x_1)\beta_\lambda/6}\,(\alpha_\lambda - \alpha_X(\lambda_1-\nu_1))}{\alpha_\lambda(1-\alpha_X\beta_\lambda)} \gtrsim \frac{1+(x_3-x_1)\beta_\lambda/6}{1-\alpha_X\beta_\lambda} \geq 1/6.$$

Now, estimating I boils down to estimating the following three integrals:

$$I \asymp I_1 \asymp \frac{{\beta_\lambda}^{3k-1}}{(1-\alpha_X\beta_\lambda)^{k+1}}\int_{\lambda_3}^{M_2}\int_{\lambda_2}^{\lambda_1} e^{-(x_3-x_2)(\nu_1-\lambda_2)-(x_3-x_1)(\nu_2-\lambda_3)}(\alpha_\lambda - \alpha_X(\lambda_1-\nu_1)\beta_\lambda) \\ (\nu_2-\lambda_3)^k(\nu_1-\lambda_2)^{k-1}(\lambda_1-\nu_1)^{k-1}d\nu_1 d\nu_2 \\ = \frac{{\beta_\lambda}^{3k-1}}{(1-\alpha_X\beta_\lambda)^{k+1}}\left[\int_{\lambda_3}^{M_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)}(\nu_2-\lambda_3)^k d\nu_2\right] \\ \left[\alpha_\lambda\int_{\lambda_2}^{\lambda_1} e^{-(x_3-x_2)(\nu_1-\lambda_2)}(\nu_1-\lambda_2)^{k-1}(\lambda_1-\nu_1)^{k-1}d\nu_1 \right. \\ \left. -\alpha_X\beta_\lambda\int_{\lambda_2}^{\lambda_1} e^{-(x_3-x_2)(\nu_1-\lambda_2)}(\nu_1-\lambda_2)^{k-1}(\lambda_1-\nu_1)^{k}d\nu_1\right]$$

We estimate the first integral using the estimate (14). In the second integral, up to a natural change of variables, we recognize the integral formula for the modified Bessel function $\mathcal{J}_{k-1/2}$. Recall that $\mathcal{J}_{k-1/2}(vx) = E_k^{W,rk\,1}(v,x)$. By the estimates of the function $\mathcal{J}_k$ (we can also use of the estimates of the Dunkl kernel in the W-invariant rank 1 case [9]), we have

$$\int_{\lambda_2}^{\lambda_1} e^{-(x_3-x_2)(\nu_1-\lambda_2)}(\lambda_1-\nu_1)^{k-1}(\nu_1-\lambda_2)^{k-1}d\nu_1 \asymp \frac{{\alpha_\lambda}^{2k-1}}{(1+(x_3-x_2)\alpha_\lambda)^k}. \tag{16}$$

The third integral, similarly as in C^+, is estimated by the formula (7) in the general rank 1 case. Finally, we obtain

$$\begin{aligned} I &\asymp \frac{{\beta_\lambda}^{3k-1}}{(1-\alpha_X\beta_\lambda)^{k+1}}\frac{{\beta_\lambda}^{k+1}}{(1+(x_3-x_1)\beta_\lambda)^{k+1}}\left[\frac{\alpha_\lambda{\alpha_\lambda}^{2k-1}}{(1+(x_3-x_2)\alpha_\lambda)^k}-\frac{\alpha_X\beta_\lambda{\alpha_\lambda}^{2k}}{(1+(x_3-x_2)\alpha_\lambda)^k}\right] \\ &= \frac{{\alpha_\lambda}^{2k}{\beta_\lambda}^{4k}(1-\alpha_X\beta_\lambda)}{(1-\alpha_X\beta_\lambda)^{k+1}(1+(x_3-x_2)\alpha_\lambda)^k(1+(x_3-x_1)\beta_\lambda)^{k+1}}. \end{aligned}$$

Note that if we assume $\alpha_\lambda \geq \beta_\lambda$, a similar proof holds using the formula (11) with $I_1 = \int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1}$ and $I_2 = I - I_1$.

5.7. Weyl chamber $\sigma_\beta\sigma_\alpha\overline{C^+}$ (i.e. $x_2 \geq x_3 \geq x_1$)

In the case $\alpha_\lambda \geq \beta_\lambda$, we use the formula (10) from Lemma 5.2. Discussions with Anker and Trojan were helpful in determining the right bounds in this chamber and in the chamber C_{312}. Let

$$\begin{aligned} I_1(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1} & e^{-(x_2-x_1)(\lambda_1-\nu_1)-(x_3-x_1)(\lambda_2-\nu_2)}\frac{(\nu_1-\nu_2)\,(\beta_\lambda+\beta_X\,(\nu_1-\lambda_3)\,(\lambda_2-\nu_2))}{(1+\beta_X(\nu_1-\nu_2))^{k+1}} \\ & (\lambda_1-\nu_1)(\lambda_1-\nu_2)\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2 \end{aligned}$$

and $I_2 = I - I_1$. We have

$$\tilde{I}_1 \geq \int_{(2\lambda_1+\lambda_2)/3}^{(3\lambda_1+\lambda_2)/4}\cdots \gtrsim \frac{{\alpha_\lambda}^{1+k+2(k-1)+1}e^{-(x_2-x_1)/3}\;(\beta_\lambda+\beta_X\alpha_\lambda\,(\lambda_2-\nu_2))}{(1+\beta_X\alpha_\lambda)^{k+1}}.$$

Next, for $\tilde{I}_2$, we proceed similarly as in (12) and (13).

$$\begin{aligned} \tilde{I}_2 &\lesssim {\alpha_\lambda}^k\,e^{-(x_2-x_1)\alpha_\lambda/2}\int_{\lambda_2}^{M_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)\beta_\lambda} \qquad (17) \\ &\qquad \frac{(\nu_1-\nu_2)\,(\nu_1-\lambda_2)^{k-1}(\nu_1-\lambda_3)^{k-1}\,(\beta_\lambda+\beta_X\,(\nu_1-\lambda_3)\,\beta_\lambda)}{(1+\beta_X(\nu_1-\nu_2))^{k+1}}d\nu_1 \\ &\leq {\alpha_\lambda}^k\beta_\lambda\,e^{-(x_2-x_1)\alpha_\lambda/2}\int_{\lambda_2}^{M_1}\frac{(\nu_1-\nu_2)\,(\nu_1-\lambda_2)^{k-1}(\nu_1-\lambda_3)^{k-1}}{(1+\beta_X(\nu_1-\nu_2))^k}d\nu_1 \\ &\lesssim \frac{{\alpha_\lambda}^{3k}\beta_\lambda\,e^{-(x_2-x_1)\alpha_\lambda/2}}{(1+\beta_X\alpha_\lambda)^k}. \qquad (18) \end{aligned}$$

Hence, $\dfrac{\tilde{I}_1}{\tilde{I}_2} \gtrsim \dfrac{e^{(x_2-x_1)\alpha_\lambda/6}\;(\beta_\lambda+\beta_X\alpha_\lambda\,(\lambda_2-\nu_2))}{\beta_\lambda(1+\beta_X\alpha_\lambda)} \geq \dfrac{1+(x_2-x_1)\,\alpha_\lambda/6}{1+\beta_X\alpha_\lambda} \geq 1/6$. Now,

$$\begin{aligned} I_1(X,\lambda) \asymp \frac{{\alpha_\lambda}^{1+k+2(k-1)}}{(1+\beta_X\alpha_\lambda)^{k+1}}\int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1} & e^{-(x_2-x_1)(\lambda_1-\nu_1)-(x_3-x_1)(\lambda_2-\nu_2)}\,(\beta_\lambda+\beta_X\alpha_\lambda\,(\lambda_2-\nu_2)) \\ & (\lambda_1-\nu_1)^k(\lambda_2-\nu_2)^{k-1}(\nu_2-\lambda_3)^{k-1}d\nu_1 d\nu_2 \end{aligned}$$

$$
\begin{aligned}
&\asymp \frac{{\alpha_\lambda}^{3k-1}}{(1+\beta_X\alpha_\lambda)^{k+1}} \left[\int_{M_1}^{\lambda_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^k d\nu_1\right] \\
&\quad\cdot\left[\beta_\lambda \int_{\lambda_3}^{\lambda_2} e^{-(x_3-x_1)(\lambda_2-\nu_2)}(\lambda_2-\nu_2)^{k-1}(\nu_2-\lambda_3)^{k-1}d\nu_2\right. \\
&\quad\left.+\beta_X\alpha_\lambda \int_{\lambda_3}^{\lambda_2} e^{-(x_3-x_1)(\lambda_2-\nu_2)}(\lambda_2-\nu_2)^{k}(\nu_2-\lambda_3)^{k-1}d\nu_2\right] \\
&\asymp \frac{{\alpha_\lambda}^{3k-1}}{(1+\beta_X\alpha_\lambda)^{k+1}}\frac{{\alpha_\lambda}^{k+1}}{(1-\alpha_X\alpha_\lambda)^{k+1}}\left[\beta_\lambda\frac{{\beta_\lambda}^{2k-1}}{(1+(x_3-x_1)\beta_\lambda)^k}+\beta_X\alpha_\lambda\frac{{\beta_\lambda}^{2k}}{(1+(x_3-x_1)\beta_\lambda)^{k+1}}\right] \\
&= \frac{{\alpha_\lambda}^{4k}{\beta_\lambda}^{2k}(1+(x_3-x_1)\beta_\lambda+\beta_X\alpha_\lambda)}{(1+\beta_X\alpha_\lambda)^{k+1}(1-\alpha_X\alpha_\lambda)^{k+1}(1+(x_3-x_1)\beta_\lambda)^{k+1}}.
\end{aligned}
$$

We conclude by considering two cases $(x_3-x_1)\beta_\lambda \geq \beta_X\alpha_\lambda$ and $(x_3-x_1)\beta_\lambda \leq \beta_X\alpha_\lambda$.

We now consider the case $\beta_\lambda \geq \alpha_\lambda$.

Step 1. We use the formula (9) from Lemma 5.1. We will estimate the integral $I(X,\lambda)$ given by

$$
I = I(X,\lambda) \asymp \int_{\lambda_3}^{\lambda_2}\int_{\lambda_2}^{\lambda_1} \frac{(\nu_1-\nu_2)\,(\alpha_\lambda-\alpha_X\,(\lambda_1-\nu_1)\,(\lambda_2-\nu_2))}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}} \tag{19}
$$

$$
[1mm] \qquad (\nu_1-\lambda_3)(\nu_2-\lambda_3)e^{-(x_2-x_3)(\lambda_1-\nu_1)-(x_3-x_1)(\nu_2-\lambda_3)}\,W_k(\lambda,\nu)d\nu_1 d\nu_2.
$$

If $x_3-x_1 \geq x_2-x_3$ the proof is also similar to the previous cases since then $x_2-x_1 \asymp x_3-x_1$.

The case $\beta_\lambda \geq \alpha_\lambda$ and $\beta_{X^+} \leq \alpha_{X^+}$ remains. The proof is more involved than in preceding cases.

Step 2. We define $I_1 = \int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1}\dots$ and $I_2 = I - I_1$. As usual, we consider instead $\tilde{I}_1$ and $\tilde{I}_2$. We will show that $\tilde{I}_1 \geq \tilde{I}_2$. We have, using $\nu_1-\nu_2 \asymp \lambda_1-\nu_2$,

$$
\tilde{I}_1 \geq \int_{(2\,\lambda_1+\lambda_2)/3}^{(3\,\lambda_1+\lambda_2)/4} [\dots] \gtrsim e^{-(x_2-x_3)\alpha_\lambda/3}\,{\alpha_\lambda}^{2\,(k-1)+2}{\beta_\lambda}^k(\lambda_1-\nu_2)\frac{(1-\alpha_X\,(\lambda_2-\nu_2))}{(1-\alpha_X(\lambda_1-\nu_2))^{k+1}}
$$

while, similarly as in (12) and (13),

$$
\tilde{I}_2 \lesssim e^{-(x_2-x_3)\alpha_\lambda/2}\,{\alpha_\lambda}^{k-1}{\beta_\lambda}^k \int_{\lambda_2}^{M_1} \frac{(\nu_1-\nu_2)(\nu_1-\lambda_2)^{k-1}}{(1-\alpha_X(\nu_1-\nu_2))^{k+1}}(\alpha_\lambda-\alpha_X\alpha_\lambda(\lambda_2-\nu_2))\,d\nu_1
$$

$$
\lesssim e^{-(x_2-x_3)\alpha_\lambda/2}\,{\alpha_\lambda}^{1+(k-1)}{\beta_\lambda}^k \int_{\lambda_2}^{M_1} \frac{(\nu_1-\nu_2)(\nu_1-\lambda_2)^{k-1}}{(1-\alpha_X(\nu_1-\nu_2))^{k}}\,d\nu_1 \lesssim \frac{{\alpha_\lambda}^{2k}{\beta_\lambda}^k(\lambda_1-\nu_2)}{(1-\alpha_X\alpha_\lambda)^k}. \tag{20}
$$

$$
\frac{\tilde{I}_1}{\tilde{I}_2} \gtrsim \frac{e^{(x_2-x_3)\alpha_\lambda/6}\,(1-\alpha_X\,(\lambda_2-\nu_2))}{(1-\alpha_X(\lambda_1-\nu_2))} \gtrsim \frac{(1+(x_2-x_3)\,\alpha_\lambda/6)\,(1-\alpha_X\,(\lambda_2-\nu_2))}{(1-\alpha_X(\lambda_1-\nu_2))}
$$

$$
\gtrsim \frac{(1/6-\alpha_X\alpha_\lambda/6)\,(1-\alpha_X\,(\lambda_2-\nu_2))}{(1-\alpha_X(\lambda_1-\nu_2))} \geq \frac{(1/6-\alpha_X(\alpha_\lambda+\lambda_2-\nu_2)/6)}{(1-\alpha_X(\lambda_1-\nu_2))} = 1/6.
$$

since $x_2-x_3 \asymp -\alpha_X$.

Step 3. (Estimation of I_1) By Step 2, in order to estimate I, it is enough to estimate I_1. We have

$$
\begin{aligned}
I \asymp I_1 = {\alpha_\lambda}^{k-1}{\beta_\lambda}^k \Bigg\{ &\left[\alpha_\lambda \int_{M_1}^{\lambda_1} e^{-(x_2-x_3)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^{k-1}\,d\nu_1\right] \\
&\left[\int_{\lambda_3}^{\lambda_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^k\,(\lambda_2-\nu_2)^{k-1}\,(\nu_2-\lambda_3)^{k+1}}{(1-\alpha_X(\lambda_1-\nu_2))^{k+1}} d\nu_2\right] \\
&-\alpha_X \left[\alpha_\lambda \int_{M_1}^{\lambda_1} e^{-(x_2-x_3)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^{k}\,d\nu_1\right] \\
&\left[\int_{\lambda_3}^{\lambda_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^k\,(\lambda_2-\nu_2)^{k}\,(\nu_2-\lambda_3)^{k+1}}{(1-\alpha_X(\lambda_1-\nu_2))^{k+1}} d\nu_2\right]\Bigg\} \\
\asymp {\alpha_\lambda}^{k-1}{\beta_\lambda}^k \Bigg\{ &\left[\alpha_\lambda \int_{M_1}^{\lambda_1} e^{-(x_2-x_3)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^{k-1}\,d\nu_1\right] \\
&\left[\frac{{\beta_\lambda}^{2k-1}}{(1-\alpha_X\beta_\lambda)^{k+1}} \int_{\lambda_3}^{M_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)}(\nu_2-\lambda_3)^k\,d\nu_2\right. \\
&\left. +{\beta_\lambda}^k \int_{M_2}^{\lambda_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^k\,(\lambda_2-\nu_2)^{k-1}}{(1-\alpha_X(\lambda_1-\nu_2))^{k+1}} d\nu_2\right] \\
&-\alpha_X \left[\alpha_\lambda \int_{M_1}^{\lambda_1} e^{-(x_2-x_3)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^{k}\,d\nu_1\right] \\
&\left[\frac{{\beta_\lambda}^{2k}}{(1-\alpha_X\beta_\lambda)^{k+1}} \int_{\lambda_3}^{M_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)}(\nu_2-\lambda_3)^{k+1}\,d\nu_2\right. \\
&\left. +{\beta_\lambda}^k \int_{M_2}^{\lambda_2} e^{-(x_3-x_1)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^k\,(\lambda_2-\nu_2)^{k}}{(1-\alpha_X(\lambda_1-\nu_2))^{k+1}} d\nu_2\right]\Bigg\} \\
= \frac{{\alpha_\lambda}^{2k}{\beta_\lambda}^k}{(1+(x_2-x_3)\alpha_\lambda)^k} &\left[\frac{{\beta_\lambda}^{3k}}{(1-\alpha_X\beta_\lambda)^{k+1}(1+(x_3-x_1)\beta_\lambda)^{k+1}} + {\alpha_\lambda}^k{\beta_\lambda}^k\,U(k-1)\right. \\
&\left. + {\beta_\lambda}^k\,U(2k-1)\right] \\
+ \frac{-\alpha_X{\alpha_\lambda}^{2k}{\beta_\lambda}^k}{(1+(x_2-x_3)\alpha_\lambda)^{k+1}} &\left[\frac{{\beta_\lambda}^{3k+1}}{(1-\alpha_X\beta_\lambda)^{k+1}(1+(x_3-x_1)\beta_\lambda)^{k+1}}\right. \\
&\left. + {\alpha_\lambda}^k{\beta_\lambda}^k\,U(k) + {\beta_\lambda}^k\,U(2k)\right]
\end{aligned}
$$

using $(\lambda_1-\nu_2)^k = (\alpha_\lambda+\lambda_2-\nu_2)^k \asymp {\alpha_\lambda}^k + (\lambda_2-\nu_2)^k$ and where

$$U(p) = \int_{M_2}^{\lambda_2} \frac{e^{\gamma_X(v-\lambda_3)}(\lambda_2-v)^p}{(1-\alpha_X\alpha_\lambda-\alpha_X(\lambda_2-v))^{k+1}}\,dv.$$

We then obtain $I(X,\lambda) \asymp I_1 \asymp c_1(t_1+t_2+t_3) + c_2(T_1+T_2+T_3)$, where

$$
\begin{aligned}
c_1 &= \frac{{\alpha_\lambda}^{2k}{\beta_\lambda}^k}{(1+\beta_X\alpha_\lambda)^k}, \qquad c_2 = \frac{-\alpha_X{\alpha_\lambda}^{2k}{\beta_\lambda}^k}{(1+\beta_X\alpha_\lambda)^{k+1}}, \\
t_1 &= \frac{{\beta_\lambda}^{3k}}{(1-\alpha_X\beta_\lambda)^{k+1}(1-\gamma_X\beta_\lambda)^{k+1}}, \qquad t_2 = {\alpha_\lambda}^k{\beta_\lambda}^k U(k-1), \qquad t_3 = {\beta_\lambda}^k U(2k-1) \\
T_1 &= \frac{{\beta_\lambda}^{3k+1}}{(1-\alpha_X\beta_\lambda)^{k+1}(1-\gamma_X\beta_\lambda)^{k+1}}, \qquad T_2 = {\alpha_\lambda}^k{\beta_\lambda}^k U(k), \qquad T_3 = {\beta_\lambda}^k U(2k).
\end{aligned}
$$

The rest of the proof is straightforward but tedious. We will only give its outline and omit the details.

In order to estimate the four needed integrals of the form $U(p)$, with $p = k-1$, $2k-1$, k, $2k$, it is useful to consider four cases:

(C1) $-\alpha_X\alpha_\lambda \le 1, -\gamma_X\beta_\lambda \le 1$,

(C2) $-\alpha_X\alpha_\lambda \ge 1, -\gamma_X\beta_\lambda \le 1$,

(C3) $-\alpha_X\alpha_\lambda \le 1, -\gamma_X\beta_\lambda \ge 1$,

(C4) $-\alpha_X\alpha_\lambda \ge 1, -\gamma_X\beta_\lambda \ge 1$.

We work separately with $k > 1$, $k = 1$ and $k < 1$.

In the cases (C3) and (C4), we use the estimate

$$\int_1^a e^x x^M dx \asymp e^a a^M,$$

for $a \ge 2$. Otherwise, the integrals are easily estimated. In all cases, one notes that ${\alpha_\lambda}^k U(k-1) \lesssim U(2k-1)$ and ${\alpha_\lambda}^k U(k) \lesssim U(2k)$.

Note that for $p = k$ the estimate of $U(k)$ has a logarithmic factor, which is also the case when $k = 1$ for $U(2k-1) = U(1) = U(k)$. However, these terms are always dominated by other non-logarithmic terms, thanks to the inequality $\ln x \le x^k$, for x large enough.

When $k > 1$, after estimating the integrals $U(p)$, we check that $t_1 \ge t_2 + t_3$ and $T_1 \ge T_2 + T_3$ so that $I(\lambda, X) \asymp c_1 t_1 + c_2 T_1$ and the estimate follows by putting the sum on a common denominator.

When $k \le 1$, we consider two following cases. When $-\gamma_X\alpha_\lambda \ge 1$, the proof is similar as for $k > 1$.

When $-\gamma_X\alpha_\lambda \le 1$, it is easy to see that $t_3 \ge t_2$ and that $T_1 \ge T_3 \ge T_2$. Next, we check that $c_2T_1/(c_1t_1) \ge 1$ and $c_2T_1/(c_1t_3) \ge 1$. Finally, $I(X, \lambda) \asymp c_2 T_1$ and the estimate follows.

5.8. Weyl chamber C_{213}

If we assume $\alpha_\lambda \ge \beta_\lambda$ and we use formula (9), we easily prove the relevant estimate using the same approach as for, say, C_{321}. If $\beta_\lambda \ge \alpha_\lambda$ and $x_1 - x_3 \ge x_2 - x_1$, the same approach works with formula (10) since $x_3 - x_1 \asymp x_2 - x_3$. We then consider the case $\beta_\lambda \ge \alpha_\lambda$ and $x_1 - x_3 \le x_2 - x_1$ so that $x_2 - x_1 \asymp x_2 - x_3$. Using formula (10), we have

$$I_1 \asymp \int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)-(x_1-x_3)(\nu_2-\lambda_3)} \frac{(\nu_1-\nu_2)\,(\beta_\lambda + \beta_X\,(\nu_1-\lambda_3)\,(\lambda_2-\nu_2))}{(1+\beta_X(\nu_1-\nu_2))^{k+1}} (\lambda_1-\nu_1)(\lambda_1-\nu_2)\,W_k(\lambda,\nu)\,d\nu_1 d\nu_2$$

and $I_2 = I - I_1$. Now, since $\lambda_1 - \nu_2 \asymp \nu_1 - \nu_2$, we have:

$$\tilde{I}_1 \ge \int_{(2\,\lambda_1+\lambda_2)/3}^{(3\,\lambda_1+\lambda_2)/4} \cdots \gtrsim \frac{{\alpha_\lambda}^{k+(k-1)+1}{\beta_\lambda}^{k-1} e^{-(x_2-x_1)\alpha_\lambda/3} (\lambda_1-\nu_2)\,(\beta_\lambda + \beta_X\beta_\lambda\,(\lambda_2-\nu_2))}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}}.$$

Now we proceed similarly as in (12) and (13).

$$\begin{aligned}\tilde{I}_2 &\lesssim {\alpha_\lambda}^k {\beta_\lambda}^{k-1} e^{-(x_2-x_1)\alpha_\lambda/2} \int_{\lambda_2}^{M_1} \frac{(\nu_1-\nu_2)\,(\beta_\lambda+\beta_X\beta_\lambda\,(\lambda_2-\nu_2))}{(1+\beta_X(\nu_1-\nu_2))^{k+1}} (\nu_1-\lambda_2)^{k-1}\,d\nu_1 \\ &\le {\alpha_\lambda}^k {\beta_\lambda}^{k} e^{-(x_2-x_1)\alpha_\lambda/2} \int_{\lambda_2}^{M_1} \frac{(\nu_1-\nu_2)}{(1+\beta_X(\nu_1-\nu_2))^{k}} (\nu_1-\lambda_2)^{k-1}\,d\nu_1 \\ &\lesssim \frac{{\alpha_\lambda}^{2k}{\beta_\lambda}^k(\lambda_1-\nu_2)e^{-(x_2-x_1)\alpha_\lambda/2}}{(1+\beta_X(\lambda_1-\nu_2))^k}. \end{aligned} \tag{21}$$

Hence,

$$\begin{aligned}\frac{\tilde{I}_1}{\tilde{I}_2} &\gtrsim \frac{e^{(x_2-x_1)\alpha_\lambda/6}\;\;(1+\beta_X\,(\lambda_2-\nu_2))}{1+\beta_X(\lambda_1-\nu_2)} \gtrsim \frac{(1+\beta_X\alpha_\lambda/6)\;(1+\beta_X\,(\lambda_2-\nu_2))}{1+\beta_X(\lambda_1-\nu_2)} \\ &\ge \frac{1+\beta_X(\lambda_1-\nu_2)/6}{1+\beta_X(\lambda_1-\nu_2)} \ge 1/6.\end{aligned}$$

We then have $I \asymp I_1$ and since $\lambda_1-\nu_2 \asymp \nu_1-\nu_2$,

$$\begin{aligned} I_1 &\asymp {\alpha_\lambda}^{k-1}{\beta_\lambda}^k \int_{\lambda_3}^{\lambda_2}\int_{M_1}^{\lambda_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)-(x_1-x_3)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^{k+1}\,(1+\beta_X\,(\lambda_2-\nu_2))}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}} \\ &\qquad (\lambda_1-\nu_1)^k(\lambda_2-\nu_2)^{k-1}(\nu_2-\lambda_3)^{k-1}\,d\nu_1 d\nu_2 \\ &= {\alpha_\lambda}^{k-1}{\beta_\lambda}^k \left[\int_{M_1}^{\lambda_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^k\,d\nu_1\right] \\ &\quad \cdot \left[\int_{\lambda_3}^{\lambda_2} e^{-(x_1-x_3)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^{k+1}\,(\lambda_2-\nu_2)^{k-1}(\nu_2-\lambda_3)^{k-1}}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}}\,d\nu_2 \right.\\ &\qquad \left. +\,\beta_X \int_{\lambda_3}^{\lambda_2} e^{-(x_1-x_3)(\nu_2-\lambda_3)} \frac{(\lambda_1-\nu_2)^{k+1}\,(\lambda_2-\nu_2)^{k}(\nu_2-\lambda_3)^{k-1}}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}}\,d\nu_2\right]. \end{aligned}$$

Now, let $J_1=\int_{\lambda_3}^{M_2}\dots$ and $J_2=\int_{M_2}^{\lambda_2}\dots$ in the first integral involving ν_2. We have

$$J_1 \ge \int_{(3\lambda_3+\lambda_2)/4}^{(2\lambda_3+\lambda_2)/3} \gtrsim \frac{{\beta_\lambda}^{(k+1)+2(k-1)+1}e^{-(x_1-x_3)\beta_\lambda/3}}{(1+\beta_X\beta_\lambda)^{k+1}}$$

while

$$\begin{aligned} J_2 &\lesssim e^{-(x_1-x_3)\beta_\lambda/2} \int_{M_2}^{\lambda_2} \frac{(\lambda_1-\nu_2)^{k+1}\,(\lambda_2-\nu_2)^k(\nu_2-\lambda_3)^{k-1}}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}}\,d\nu_2 \\ &\lesssim e^{-(x_1-x_3)\beta_\lambda/2}\frac{(\lambda_1-\lambda_3)^{k+1}}{(1+\beta_X(\lambda_1-\lambda_3))^{k+1}} \int_{M_2}^{\lambda_2} (\lambda_2-\nu_2)^k(\nu_2-\lambda_3)^{k-1}\,d\nu_2 \\ &\asymp \frac{{\beta_\lambda}^{3k+1}e^{-(x_1-x_3)\beta_\lambda/2}}{(1+\beta_X\beta_\lambda)^{k+1}} \lesssim J_1. \end{aligned}$$

We proceed in the same manner for the second integral in ν_2. We have

$$
\begin{aligned}
I_1 \asymp {\alpha_\lambda}^{k-1}{\beta_\lambda}^k &\left[\int_{M_1}^{\lambda_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^k\,d\nu_1\right] \\
&\cdot \left[\int_{\lambda_3}^{M_2} e^{-(x_1-x_3)(\nu_2-\lambda_3)}\,\frac{(\lambda_1-\nu_2)^{k+1}\,(\lambda_2-\nu_2)^{k-1}(\nu_2-\lambda_3)^{k-1}}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}}\,d\nu_2\right. \\
&\qquad \left. + \beta_X \int_{\lambda_3}^{M_2} e^{-(x_1-x_3)(\nu_2-\lambda_3)}\,\frac{(\lambda_1-\nu_2)^{k+1}\,(\lambda_2-\nu_2)^{k}(\nu_2-\lambda_3)^{k-1}}{(1+\beta_X(\lambda_1-\nu_2))^{k+1}}\,d\nu_2\right] \\
\asymp {\alpha_\lambda}^{k-1}{\beta_\lambda}^k &\left[\int_{M_1}^{\lambda_1} e^{-(x_2-x_1)(\lambda_1-\nu_1)}\,(\lambda_1-\nu_1)^k\,d\nu_1\right] \\
&\cdot \left[\frac{{\beta_\lambda}^{(k+1)+(k-1)}}{(1+\beta_X\beta_\lambda)^{k+1}}+\frac{{\beta_\lambda}^{(k+1)+k}}{(1+\beta_X\beta_\lambda)^{k+1}}\right]\int_{\lambda_3}^{M_2} e^{-(x_1-x_3)(\nu_2-\lambda_3)}(\nu_2-\lambda_3)^{k-1}\,d\nu_2 \\
\asymp {\alpha_\lambda}^{k-1}{\beta_\lambda}^k &\frac{{\alpha_\lambda}^{k+1}}{(1+(x_2-x_1)\alpha_\lambda)^{k+1}}\,\frac{{\beta_\lambda}^{2k}(1+\beta_X\beta_\lambda)}{(1+\beta_X\beta_\lambda)^{k+1}}\,\frac{{\beta_\lambda}^{k}}{(1+(x_3-x_1)\beta_\lambda)^{k}}
\end{aligned}
$$

which gives the correct estimate.

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