

2-Local Derivations on the Centerless Ovsienko-Roger Algebra

Yan Liu, Yao Ma and Liangyun Chen*

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Abstract. We study 2-local derivations on the centerless Ovsienko-Roger algebra \mathfrak{L}_λ , which is the semi-direct product of the Witt algebra and its tensor density module. We prove that every 2-local derivation on \mathfrak{L}_λ is a derivation for $\lambda \in \mathbb{C} \setminus \{0, 1, 2\}$. We divide into two cases to consider 2-local derivations on \mathfrak{L}_λ depending on whether the parameter λ is an integer, that is for the case $\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}$ and the case $\lambda \notin \mathbb{Z}$.

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1. Introduction

The Witt algebra \mathcal{W} is an infinite-dimensional Lie algebra over the complex field \mathbb{C} with the basis $\{L_n \mid n \in \mathbb{Z}\}$ and the following relation

$$[L_n, L_m] = (m - n)L_{m+n}, \quad \forall m, n \in \mathbb{Z}.$$

For a parameter $\lambda \in \mathbb{C}$, the tensor density module of degree λ over \mathcal{W} is a vector space $\mathcal{F}_\lambda = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}W_m$, which was first introduced in [8, 12]. The action of \mathcal{W} on \mathcal{F}_λ is given by

$$L_n \cdot W_m = (m + \lambda n)W_{m+n}, \quad \forall m, n \in \mathbb{Z}.$$

Then, a class of infinite-dimensional Lie algebras $\mathfrak{L}_\lambda \simeq \mathcal{W} \ltimes \mathcal{F}_\lambda$ is constructed for $\lambda \in \mathbb{C}$. Obviously,

$$\mathfrak{L}_\lambda = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}L_m \oplus \bigoplus_{m \in \mathbb{Z}} \mathbb{C}W_m$$

satisfies the following brackets for all $m, n \in \mathbb{Z}$:

$$[L_n, L_m] = (m - n)L_{m+n}, \quad [L_n, W_m] = (m + \lambda n)W_{m+n}, \quad [W_n, W_m] = 0.$$

A central extension of \mathfrak{L}_λ was considered by Ovsienko and Roger [13, 14] to study matrix analogues of Sturm-Liouville operators [10, 11]. This extension was called the Ovsienko-Roger algebra in [9]. Hence, we call \mathfrak{L}_λ the centerless Ovsienko-Roger algebra in this paper. Note that \mathfrak{L}_λ includes some important Lie algebras as special

* Corresponding author.

cases. For example, \mathfrak{L}_0 is the centerless twisted Heisenberg-Virasoro algebra and \mathfrak{L}_{-1} is the centerless W -algebra $W(2, 2)$. More results of the structure theory for \mathfrak{L}_λ refer to [4, 5, 16, 20].

It is well-known that derivations play an important role in the study of the structure theory of an algebra. As a generalization, the concept of 2-local derivations was introduced in [15] when Šemrl studied the structure of $\mathcal{B}(H)$. For a given algebra, the main problem in studying 2-local derivations is to see whether they automatically become derivations. 2-Local derivations on many finite-dimensional and infinite-dimensional Lie algebras have been characterized. In [1], the authors have shown that every 2-local derivation on a semi-simple Lie algebra is a derivation and a finite-dimensional nilpotent Lie algebra with the dimension larger than 2 exists a 2-local derivation which is not a derivation. The authors have proved that all 2-local derivations on some Schrödinger algebras are derivations in [19, 21]. 2-Local derivations on some infinite-dimensional Lie algebras, such as the Witt algebra as well as the positive Witt algebra [2], the W -algebra $W(2, 2)$ [17], the twisted Heisenberg-Virasoro algebra [22], the Virasoro-like algebra [18], the Schrödinger-Virasoro algebra [7] and the planar Galilean conformal algebra [3] were determined. The authors have proved that every 2-local derivation on these infinite-dimensional Lie algebras mentioned above is a derivation. In addition, the authors have given an example of the infinite-dimensional Lie algebra, called the thin Lie algebra, such that it admits a lot of 2-local derivations which are not derivations in [2]. In particular, the author also gave a complete classification of 2-local derivations on the thin Lie algebra in [17].

These known results encourage us to consider 2-local derivations on the centerless Ovsienko-Roger algebra \mathfrak{L}_λ for $\lambda \in \mathbb{C}$. However, when $\lambda = 0, 1, 2$, there is more than one outer derivation on \mathfrak{L}_λ such that these cases are difficult to consider. Therefore, we only take into account the case for $\lambda \in \mathbb{C} \setminus \{0, 1, 2\}$. We divide into two cases to determine 2-local derivations on \mathfrak{L}_λ for $\lambda \in \mathbb{C} \setminus \{0, 1, 2\}$ depending on whether the parameter λ is an integer, that is for the case $\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}$ and the case $\lambda \notin \mathbb{Z}$.

The paper is organized as follows. In Section 2, we give some preliminaries concerning the centerless Ovsienko-Roger algebra \mathfrak{L}_λ for $\lambda \in \mathbb{C} \setminus \{0, 1, 2\}$ and the main results in this paper. In Section 3, we prove that every 2-local derivation on the algebra \mathfrak{L}_λ is a derivation for $\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}$. All 2-local derivations are also proved to be derivations on the algebra \mathfrak{L}_λ for $\lambda \notin \mathbb{Z}$ in Section 4.

Throughout this paper, we denote by \mathbb{C} , \mathbb{Z} , \mathbb{Z}^* , \mathbb{Z}_- and \mathbb{Z}_+ the sets of complex numbers, integers, nonzero integers, negative integers and positive integers, respectively. For convenience, $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0, 1, 2\}}$, $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}}$ and $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$ denote the algebra \mathfrak{L}_λ for $\lambda \in \mathbb{C} \setminus \{0, 1, 2\}$, the algebra \mathfrak{L}_λ for $\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}$ and the algebra \mathfrak{L}_λ for $\lambda \notin \mathbb{Z}$, respectively. All algebras and vector spaces are considered over the complex number field \mathbb{C} .

2. Preliminaries

In this section, we recall some known definitions and results for the study of 2-local derivations on $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0, 1, 2\}}$ and give the main results in this paper.

Definition 2.1. [6] Let \mathfrak{g} be a Lie algebra. A linear map $D : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a *derivation* of \mathfrak{g} if it satisfies the Leibniz law, that is,

$$D([x, y]) = [D(x), y] + [x, D(y)], \quad \text{for all } x, y \in \mathfrak{g}. \quad \blacksquare$$

It is easy to see that for $x \in \mathfrak{g}$, the linear map $\text{ad}(x)$ on \mathfrak{g} defined as $\text{ad}(x)(y) = [x, y]$ for all $y \in \mathfrak{g}$ is a derivation of \mathfrak{g} . The derivation of this form is called an inner derivation of \mathfrak{g} . Denote by $\text{Der}(\mathfrak{g})$ the space of all derivations of \mathfrak{g} , $\text{Inn}(\mathfrak{g})$ the space of all inner derivations of \mathfrak{g} , respectively.

Definition 2.2. [15] Let \mathfrak{g} be a Lie algebra. A map $\Delta : \mathfrak{g} \rightarrow \mathfrak{g}$ (not linear in general) is called *2-local derivation* of \mathfrak{g} if for every $x, y \in \mathfrak{g}$, there exists a derivation $D_{x,y} : \mathfrak{g} \rightarrow \mathfrak{g}$ (depending on x, y) such that

$$\Delta(x) = D_{x,y}(x), \quad \Delta(y) = D_{x,y}(y). \quad \blacksquare$$

In particular, for $\alpha \in \mathbb{C}$ and $x \in \mathfrak{g}$, we have

$$\Delta(\alpha x) = D_{x,\alpha x}(\alpha x) = \alpha D_{x,\alpha x}(x) = \alpha \Delta(x).$$

Lemma 2.3. [4] Let $\text{Der}(\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}})$ and $\text{Inn}(\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}})$ be the space of derivations of $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ and the space of inner derivations of $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$, respectively.

Then
$$\text{Der}(\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}) = \text{Inn}(\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}) \oplus \mathbb{C}D,$$

where D is an outer derivation of $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ and the action of D on $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ is defined as $D(L_m) = 0$ and $D(W_m) = W_m$ for all $m \in \mathbb{Z}$.

Lemma 2.4. Let Δ be a 2-local derivation on the algebra $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$. Then there exists a derivation $D_{x,y}$ of $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ satisfying

$$\Delta(x) = D_{x,y}(x), \quad \Delta(y) = D_{x,y}(y)$$

for any $x, y \in \mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$. Moreover, $D_{x,y}$ can be written the form as follows

$$D_{x,y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(x, y)L_k + q_k(x, y)W_k)\right) + \mu(x, y)D,$$

where p_k, q_k, μ are complex-valued functions on $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}} \times \mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ for $k \in \mathbb{Z}$ and the action of D on $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ is given in Lemma 2.3.

Proof. From Definition 2.2 and Lemma 2.3, the proof is obvious. \blacksquare

Now, we give the main result concerning 2-local derivations on $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ in this paper.

Theorem 2.5. Every 2-local derivation on the centerless Ovsienko-Roger algebra $\mathfrak{L}_{\lambda \in \mathbb{C} \setminus \{0,1,2\}}$ is a derivation.

We will divide the proof of Theorem 2.5 into two cases: the case $\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}$ in Section 3 and the case $\lambda \notin \mathbb{Z}$ in Section 4, respectively.

3. 2-Local derivations on \mathfrak{L}_λ for $\lambda \in \mathbb{Z} \setminus \{0, 1, 2\}$

2-Local derivations on the algebra $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ are determined in this section.

Lemma 3.1. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$. Take any but fixed $y \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$,*

(1) *for a given $m \in \mathbb{Z}$, if $\Delta(L_m) = 0$, then*

$$D_{L_m, y} = \text{ad}(p_m(L_m, y)L_m + q_{-\lambda m}(L_m, y)W_{-\lambda m}) + \mu(L_m, y)D,$$

(2) *if $\Delta(W_0) = 0$, then*

$$D_{W_0, y} = \text{ad}(p_0(W_0, y)L_0 + \sum_{k \in \mathbb{Z}} q_k(W_0, y)W_k),$$

where p_k, q_k, μ are complex-valued functions on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}} \times \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ for $k \in \mathbb{Z}$.

Proof. From Lemma 2.4, we have for a given $m \in \mathbb{Z}$ and any $y \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$,

$$D_{L_m, y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(L_m, y)L_k + q_k(L_m, y)W_k)\right) + \mu(L_m, y)D \quad (1)$$

and
$$D_{W_0, y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(W_0, y)L_k + q_k(W_0, y)W_k)\right) + \mu(W_0, y)D, \quad (2)$$

where p_k, q_k, μ are some complex-valued functions on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}} \times \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ for $k \in \mathbb{Z}$.

(1) If $\Delta(L_m) = 0$ for a given $m \in \mathbb{Z}$, by Eq. (1), we obtain

$$\begin{aligned} \Delta(L_m) &= D_{L_m, y}(L_m) \\ &= \sum_{k \in \mathbb{Z}} [p_k(L_m, y)L_k + q_k(L_m, y)W_k, L_m] + \mu(L_m, y)D(L_m) \\ &= \sum_{k \in \mathbb{Z}} ((m-k)p_k(L_m, y)L_{k+m} - (k+\lambda m)q_k(L_m, y)W_{k+m}) = 0. \end{aligned}$$

From the above equation, one has $(m-k)p_k(L_m, y) = 0$ and $(k+\lambda m)q_k(L_m, y) = 0$, which deduce that $p_k(L_m, y) = 0$ for $k \neq m$ and $q_k(L_m, y) = 0$ for $k \neq -\lambda m$. Hence, Eq. (1) becomes

$$D_{L_m, y} = \text{ad}(p_m(L_m, y)L_m + q_{-\lambda m}(L_m, y)W_{-\lambda m}) + \mu(L_m, y)D.$$

(2) When $\Delta(W_0) = 0$, together with Eq. (2), we have

$$\begin{aligned} \Delta(W_0) &= D_{W_0, y}(W_0) \\ &= \sum_{k \in \mathbb{Z}} [p_k(W_0, y)L_k + q_k(W_0, y)W_k, W_0] + \mu(W_0, y)D(W_0) \\ &= \sum_{k \in \mathbb{Z}} \lambda k p_k(W_0, y)W_k + \mu(W_0, y)W_0 = 0. \end{aligned}$$

This equation shows that $\lambda k p_k(W_0, y) = 0$ for $k \neq 0$, which implies $p_k(W_0, y) = 0$ for $k \neq 0$. If $k = 0$, we get $\mu(W_0, y) = 0$. So, Eq. (2) can be written as

$$D_{W_0, y} = \text{ad}(p_0(W_0, y)L_0 + \sum_{k \in \mathbb{Z}} q_k(W_0, y)W_k).$$

The proof is complete. ■

Lemma 3.2. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$. If $\Delta(L_0) = \Delta(L_1) = 0$, then $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$.*

Proof. Since $\Delta(L_0) = \Delta(L_1) = 0$ and by Lemma 3.1, we get for any $y \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$,

$$D_{L_0,y} = \text{ad}(p_0(L_0, y)L_0 + q_0(L_0, y)W_0) + \mu(L_0, y)D \tag{3}$$

and
$$D_{L_1,y} = \text{ad}(p_1(L_1, y)L_1 + q_{-\lambda}(L_1, y)W_{-\lambda}) + \mu(L_1, y)D, \tag{4}$$

where $p_0, p_1, q_0, q_{-\lambda}, \mu$ are some complex-valued functions on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}} \times \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$. For any $m \in \mathbb{Z}$, take $y = L_m$ in Eqs. (3) and (4), respectively. Then, we obtain

$$\begin{aligned} \Delta(L_m) &= D_{L_0,L_m}(L_m) \\ &= [p_0(L_0, L_m)L_0 + q_0(L_0, L_m)W_0, L_m] + \mu(L_0, L_m)D(L_m) \\ &= mp_0(L_0, L_m)L_m - \lambda m q_0(L_0, L_m)W_m \end{aligned}$$

and
$$\begin{aligned} \Delta(L_m) &= D_{L_1,L_m}(L_m) \\ &= [p_1(L_1, L_m)L_1 + q_{-\lambda}(L_1, L_m)W_{-\lambda}, L_m] + \mu(L_1, L_m)D(L_m) \\ &= (m - 1)p_1(L_1, L_m)L_{m+1} - \lambda(m - 1)q_{-\lambda}(L_1, L_m)W_{m-\lambda}. \end{aligned}$$

By comparing the two above equations, it is easy to see that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. The proof is complete. ■

Lemma 3.3. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ such that $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$. Then for any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, we have*

$$\Delta(x) = \mu_x \sum_{t \in \mathbb{Z}} \beta_t W_t, \quad \text{where } \mu_x \in \mathbb{C} \text{ only depends on } x.$$

Proof. Since we have $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$ and Lemma 3.1, we get for any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$:

$$D_{L_m,x} = \text{ad}(p_m(L_m, x)L_m + q_{-\lambda m}(L_m, x)W_{-\lambda m}) + \mu(L_m, x)D.$$

Then we have

$$\begin{aligned} \Delta(x) &= D_{L_m,x}(x) \\ &= [p_m(L_m, x)L_m + q_{-\lambda m}(L_m, x)W_{-\lambda m}, x] + \mu(L_m, x)D(x) \\ &= \sum_{t \in \mathbb{Z}} \alpha_t ((t - m)p_m(L_m, x)L_{m+t} - \lambda(t - m)q_{-\lambda m}(L_m, x)W_{-\lambda m+t}) \\ &\quad + \sum_{t \in \mathbb{Z}} \beta_t ((t + \lambda m)p_m(L_m, x)W_{m+t} + \mu(L_m, x)W_t). \end{aligned}$$

By taking enough different $m \in \mathbb{Z}$ in the above equation, and if necessary, let these m 's to be large enough, we have

$$\Delta(x) = \mu(L_m, x) \sum_{t \in \mathbb{Z}} \beta_t W_t.$$

Let $\mu_x = \mu(L_m, x)$, which is a constant complex number since it depends only on x and not on m . This finishes the proof. ■

Lemma 3.4. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ satisfying $\Delta(W_0) = 0$ and $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$. For any $j \in \mathbb{Z}^*$ and $y \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, then*

$$D_{L_j+W_{(1-\lambda)j},y} = \text{ad}(\xi_j^y L_j + \eta_{-\lambda j}^y W_{-\lambda j} + \xi_j^y W_{(1-\lambda)j}),$$

where $\xi_j^y, \eta_{-\lambda j}^y \in \mathbb{C}$ depend on y .

Proof. Since $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$, it follows from Lemma 3.3 that for $j \in \mathbb{Z}^*$,

$$\Delta(L_j + W_{(1-\lambda)j}) = \mu_{L_j+W_{(1-\lambda)j}} W_{(1-\lambda)j}, \tag{5}$$

where $\mu_{L_j+W_{(1-\lambda)j}} \in \mathbb{C}$. Again by $\Delta(W_0) = 0$ and Lemma 3.1 (2), we obtain

$$\begin{aligned} \Delta(L_j + W_{(1-\lambda)j}) &= D_{W_0, L_j+W_{(1-\lambda)j}}(L_j + W_{(1-\lambda)j}) \\ &= [p_0(W_0, L_j + W_{(1-\lambda)j})L_0 + \sum_{k \in \mathbb{Z}} q_k(W_0, L_j + W_{(1-\lambda)j})W_k, L_j + W_{(1-\lambda)j}] \\ &= jp_0(W_0, L_j + W_{(1-\lambda)j})L_j + (1 - \lambda)jp_0(W_0, L_j + W_{(1-\lambda)j})W_{(1-\lambda)j} \\ &\quad - \sum_{k \in \mathbb{Z}} (k + \lambda j)q_k(W_0, L_j + W_{(1-\lambda)j})W_{k+j}. \end{aligned}$$

This implies $p_0(W_0, L_j + W_{(1-\lambda)j}) = 0$ and $q_k(W_0, L_j + W_{(1-\lambda)j}) = 0$ for $k \neq -\lambda j$ in combination with Eq. (5), which leads to

$$\Delta(L_j + W_{(1-\lambda)j}) = 0, \quad \text{where } j \in \mathbb{Z}^*.$$

For any $y \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, we may assume by Lemma 2.3 that

$$\begin{aligned} D_{L_j+W_{(1-\lambda)j}, y} &= \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(L_j + W_{(1-\lambda)j}, y)L_k + q_k(L_j + W_{(1-\lambda)j}, y)W_k)\right) \\ &\quad + \mu(L_j + W_{(1-\lambda)j}, y)D. \end{aligned}$$

Then, we have

$$\begin{aligned} \Delta(L_j + W_{(1-\lambda)j}) &= D_{L_j+W_{(1-\lambda)j}, y}(L_j + W_{(1-\lambda)j}) \\ &= \left[\sum_{k \in \mathbb{Z}} (p_k(L_j + W_{(1-\lambda)j}, y)L_k + q_k(L_j + W_{(1-\lambda)j}, y)W_k), L_j + W_{(1-\lambda)j} \right] \\ &\quad + \mu(L_j + W_{(1-\lambda)j}, y)D(L_j + W_{(1-\lambda)j}) \\ &= \sum_{k \in \mathbb{Z}} ((j - k)p_k(L_j + W_{(1-\lambda)j}, y)L_{k+j} - (k + \lambda j)q_k(L_j + W_{(1-\lambda)j}, y)W_{k+j} \\ &\quad + ((1 - \lambda)j + \lambda k)p_k(L_j + W_{(1-\lambda)j}, y)W_{k+(1-\lambda)j}) + \mu(L_j + W_{(1-\lambda)j}, y)W_{(1-\lambda)j} \\ &= 0. \end{aligned} \tag{6}$$

From Eq. (6), we see that $(j - k)p_k(L_j + W_{(1-\lambda)j}, y) = 0$ for all $k \in \mathbb{Z}$, which implies that $p_k(L_j + W_{(1-\lambda)j}, y) = 0$ for $k \neq j$. Then, Eq. (6) becomes

$$\begin{aligned} jp_j(L_j + W_{(1-\lambda)j}, y)W_{(2-\lambda)j} - \sum_{k \in \mathbb{Z}} (k + \lambda j)q_k(L_j + W_{(1-\lambda)j}, y)W_{k+j} \\ + \mu(L_j + W_{(1-\lambda)j}, y)W_{(1-\lambda)j} = 0. \end{aligned}$$

By observing the coefficient of $W_{(2-\lambda)j}$ in the equation above, we have

$$jp_j(L_j + W_{(1-\lambda)j}, y) - jq_{(1-\lambda)j}(L_j + W_{(1-\lambda)j}, y) = 0,$$

which deduces that $p_j(L_j + W_{(1-\lambda)j}, y) = q_{(1-\lambda)j}(L_j + W_{(1-\lambda)j}, y)$. Next, we observe the coefficient of $W_{(1-\lambda)j}$, then $\mu(L_j + W_{(1-\lambda)j}, y) = 0$. Furthermore, by observing the coefficient of W_k for $k \neq (1 - \lambda)j, (2 - \lambda)j$, we obtain $q_k(L_j + W_{(1-\lambda)j}, y) = 0$ for $k \neq -\lambda j, (1 - \lambda)j$.

The proof is complete by taking $\xi_j^y = p_j(L_j + W_{(1-\lambda)j}, y) = q_{(1-\lambda)j}(L_j + W_{(1-\lambda)j}, y)$ and $\eta_{-\lambda j}^y = q_{-\lambda j}(L_j + W_{(1-\lambda)j}, y)$. ■

Lemma 3.5. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ satisfying $\Delta(W_0) = \Delta(L_0) = \Delta(L_1) = 0$. For any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, then $\Delta(x) = 0$.*

Proof. Since $\Delta(L_0) = \Delta(L_1) = 0$, from Lemma 3.2, we have for all $m \in \mathbb{Z}$,

$$\Delta(L_m) = 0.$$

For any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, this implies together with Lemma 3.3 that

$$\Delta(x) = \mu_x \sum_{t \in \mathbb{Z}} \beta_t W_t, \tag{7}$$

where μ_x is a complex number depending on x . For any $j \in \mathbb{Z}^*$, it follows from $\Delta(L_m) = 0$ and $\Delta(W_0) = 0$ that Lemma 3.4 holds, that is

$$D_{L_j + W_{(1-\lambda)j}, x} = \text{ad}(\xi_j^x L_j + \eta_{-\lambda j}^x W_{-\lambda j} + \xi_j^x W_{(1-\lambda)j}),$$

where $\xi_j^x, \eta_{-\lambda j}^x \in \mathbb{C}$. Taking any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, we get for $j \in \mathbb{Z}^*$,

$$\begin{aligned} \Delta(x) &= D_{L_j + W_{(1-\lambda)j}, x}(x) = [\xi_j^x L_j + \eta_{-\lambda j}^x W_{-\lambda j} + \xi_j^x W_{(1-\lambda)j}, x] \\ &= \sum_{t \in \mathbb{Z}} \alpha_t ((t-j)\xi_j^x L_{j+t} - \lambda(t-j)\eta_{-\lambda j}^x W_{-\lambda j+t} - ((1-\lambda)j + \lambda t)\xi_j^x W_{(1-\lambda)j+t}) \\ &\quad + \sum_{t \in \mathbb{Z}} \beta_t (t + \lambda j)\xi_j^x W_{j+t}. \end{aligned} \tag{8}$$

Case 1. $x = \sum_{t \in \mathbb{Z}} \beta_t W_t$.

From Eqs. (7) and (8), we have for all $j \in \mathbb{Z}^*$,

$$\Delta(x) = \mu_x \sum_{t \in \mathbb{Z}} \beta_t W_t = \sum_{t \in \mathbb{Z}} \beta_t (t + \lambda j)\xi_j^x W_{j+t}.$$

By taking enough different $j \in \mathbb{Z}^*$ in the above equation, and if necessary, let these j 's to be large enough, then $\Delta(x) = 0$.

Case 2. $x = \alpha_{t_0} L_{t_0} + \sum_{t \in \mathbb{Z} \setminus \{t_0\}} \alpha_t L_t + \sum_{t \in \mathbb{Z}} \beta_t W_t$ with $t_0 \in \mathbb{Z}$ and $\alpha_{t_0} \neq 0$.

Take two different nonzero integers j_1 and j_2 such that $t_0 - j_1 \neq 0$ and $t_0 - j_2 \neq 0$. Then, it follows from Eqs. (7) and (8) that $\alpha_{t_0} (t_0 - j_i)\xi_{j_i}^x = 0$ for $i = 1, 2$, which implies that $\xi_{j_i}^x = 0$ for $i = 1, 2$. Therefore, Eq. (8) can be written as

$$\Delta(x) = - \sum_{t \in \mathbb{Z}} \alpha_t \lambda (t - j_i) \eta_{-\lambda j_i}^x W_{-\lambda j_i+t}, \quad i = 1, 2.$$

By taking j_1 and j_2 in the above equation such that j_1, j_2 and $j_1 - j_2$ large enough, we obtain $\Delta(x) = 0$. We finish the proof. ■

Theorem 3.6. *Every 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$ is a derivation.*

Proof. Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$. Then, there exists a derivation D_{L_0, L_1} such that

$$\Delta(L_0) = D_{L_0, L_1}(L_0), \quad \Delta(L_1) = D_{L_0, L_1}(L_1).$$

Set $\Delta_1 = \Delta - D_{L_0, L_1}$, which is a 2-local derivation satisfying $\Delta_1(L_0) = \Delta_1(L_1) = 0$ on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$. It follows from Lemma 3.2 that $\Delta_1(L_m) = 0$ for all $m \in \mathbb{Z}$.

This together with Lemma 3.3 gives us $\Delta_1(W_0) = \mu_{W_0}W_0$, where $\mu_{W_0} \in \mathbb{C}$. Next, let $\Delta_2 = \Delta_1 - \mu_{W_0}D$. Obviously, Δ_2 is a 2-local derivation on $\mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$. Moreover,

$$\begin{aligned} \Delta_2(L_0) &= \Delta_1(L_0) - \mu_{W_0}D(L_0) = 0 - 0 = 0, \\ \Delta_2(L_1) &= \Delta_1(L_1) - \mu_{W_0}D(L_1) = 0 - 0 = 0, \\ \Delta_2(W_0) &= \Delta_1(W_0) - \mu_{W_0}D(W_0) = \mu_{W_0}W_0 - \mu_{W_0}W_0 = 0. \end{aligned}$$

Then, by Lemma 3.5, we have $\Delta_2(x) = 0$ for any $x \in \mathfrak{L}_{\lambda \in \mathbb{Z} \setminus \{0,1,2\}}$, which shows that

$$\Delta_2 = \Delta - D_{L_0, L_1} - \mu_{W_0}D = 0.$$

Hence, $\Delta = D_{L_0, L_1} - \mu_{W_0}D$ is a derivation. The proof is completed. ■

Remark 3.7. 2-Local derivations on the centerless W -algebra $W(2, 2)$ have been determined in [17]. Our results recover and generalize results in [17]. ■

4. 2-Local derivations on \mathfrak{L}_λ for $\lambda \notin \mathbb{Z}$

In this section, we will consider all 2-local derivations on the algebra $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$.

Lemma 4.1. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$. Take any but fixed $y \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$,*

(1) *for a given $m \in \mathbb{Z}$, if $\Delta(L_m) = 0$, then*

$$D_{L_m, y} = \text{ad}(p_m(L_m, y)L_m + \delta_{\lambda m, \mathbb{Z}}q_{-\lambda m}(L_m, y)W_{-\lambda m}) + \mu(L_m, y)D,$$

(2) *if $\Delta(L_0 + W_0) = 0$, then*

$$D_{L_0+W_0, y} = \text{ad}(p_0(L_0 + W_0, y)L_0 + q_0(L_0 + W_0, y)W_0),$$

(3) *if $\Delta(W_0 + W_1) = 0$, then*

$$\left\{ \begin{array}{l} D_{W_0+W_1, y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} q_k(W_0 + W_1, y)W_k\right), \quad \text{when } \frac{1}{\lambda} \notin \mathbb{Z}^* \text{ or } \frac{1}{\lambda} \in \mathbb{Z}_+, \\ D_{W_0+W_1, y} = \text{ad}\left(\sum_{k=0}^{-\frac{1}{\lambda}} p_k(W_0 + W_1, y)L_k + \sum_{k \in \mathbb{Z}} q_k(W_0 + W_1, y)W_k\right), \quad \text{when } \frac{1}{\lambda} \in \mathbb{Z}_-, \end{array} \right.$$

where p_k, q_k, μ are complex-valued functions on $\mathfrak{L}_{\lambda \notin \mathbb{Z}} \times \mathfrak{L}_{\lambda \notin \mathbb{Z}}$ for $k \in \mathbb{Z}$ and $\delta_{k, \mathbb{Z}}$ is defined as 1 for $k \in \mathbb{Z}$ and as 0 for $k \notin \mathbb{Z}$. Moreover, when $\frac{1}{\lambda} \in \mathbb{Z}_-$, $p_0(W_0+W_1, y) = -\lambda p_1(W_0+W_1, y)$ and $p_k(W_0+W_1, y)$ is defined as for $1 \leq k \leq -\frac{1}{\lambda}$,

$$p_1(W_0 + W_1, y) = \frac{-2\lambda}{1 + \lambda} p_2(W_0 + W_1, y) = \dots = \frac{(-1)^{-\frac{1}{\lambda}-1} \prod_{i=2}^{-\frac{1}{\lambda}} (\lambda i)}{\prod_{i=1}^{-\frac{1}{\lambda}-1} (1 + \lambda i)} p_{-\frac{1}{\lambda}}(W_0 + W_1, y).$$

Proof. By Lemma 2.4, we obtain for a given $m \in \mathbb{Z}$ and any $y \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$,

$$D_{L_m, y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(L_m, y)L_k + q_k(L_m, y)W_k)\right) + \mu(L_m, y)D, \tag{9}$$

$$D_{L_0+W_0, y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(L_0+W_0, y)L_k + q_k(L_0+W_0, y)W_k)\right) + \mu(L_0+W_0, y)D \tag{10}$$

and

$$D_{W_0+W_1,y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(W_0+W_1,y)L_k + q_k(W_0+W_1,y)W_k)\right) + \mu(W_0+W_1,y)D, \quad (11)$$

where p_k, q_k, μ are some complex-valued functions on $\mathfrak{L}_{\lambda \notin \mathbb{Z}} \times \mathfrak{L}_{\lambda \notin \mathbb{Z}}$ for $k \in \mathbb{Z}$.

(1) Since $\Delta(L_m) = 0$ for a given $m \in \mathbb{Z}$ and Eq. (9), similar to the proof of Lemma 3.1 (1), we have

$$\Delta(L_m) = D_{L_m,y}(L_m) = \sum_{k \in \mathbb{Z}} ((m-k)p_k(L_m,y)L_{k+m} - (k+\lambda m)q_k(L_m,y)W_{k+m}) = 0.$$

From the equation above, we obtain $(m-k)p_k(L_m,y) = 0$ and $(k+\lambda m)q_k(L_m,y) = 0$ for all $k \in \mathbb{Z}$, which imply that $p_k(L_m,y) = 0$ for $k \neq m$ and if $\lambda m \in \mathbb{Z}$, $q_k(L_m,y) = 0$ for $k \neq -\lambda m$ or if $\lambda m \notin \mathbb{Z}$, $q_k(L_m,y) = 0$ for all $k \in \mathbb{Z}$. Thus,

$$D_{L_m,y} = \text{ad}(p_m(L_m,y)L_m + \delta_{\lambda m, \mathbb{Z}} q_{-\lambda m}(L_m,y)W_{-\lambda m}) + \mu(L_m,y)D,$$

where $\delta_{k, \mathbb{Z}}$ is defined as 1 for $k \in \mathbb{Z}$ and as 0 for $k \notin \mathbb{Z}$.

(2) By $\Delta(L_0 + W_0) = 0$ and Eq. (10), we get

$$\begin{aligned} \Delta(L_0 + W_0) &= D_{L_0+W_0,y}(L_0 + W_0) \\ &= \left[\sum_{k \in \mathbb{Z}} (p_k(L_0 + W_0,y)L_k + q_k(L_0 + W_0,y)W_k), L_0 + W_0 \right] \\ &\quad + \mu(L_0 + W_0,y)D(L_0 + W_0) \\ &= \sum_{k \in \mathbb{Z}} (-kp_k(L_0 + W_0,y)L_k + \lambda kp_k(L_0 + W_0,y)W_k \\ &\quad - kq_k(L_0 + W_0,y)W_k) + \mu(L_0 + W_0,y)W_0 = 0. \end{aligned} \quad (12)$$

Eq. (12) shows that $kp_k(L_0 + W_0,y) = 0$ for all $k \in \mathbb{Z}$, from which we deduce that $p_k(L_0 + W_0,y) = 0$ for $k \in \mathbb{Z}^*$. Then, Eq. (12) becomes

$$-\sum_{k \in \mathbb{Z}} kq_k(L_0 + W_0,y)W_k + \mu(L_0 + W_0,y)W_0 = 0.$$

Considering the coefficient of W_0 in the equation above, we have $\mu(L_0 + W_0,y) = 0$. Moreover, by observing the coefficient of W_k for $k \in \mathbb{Z}^*$, we obtain $q_k(L_0+W_0,y) = 0$.

Hence, $D_{L_0+W_0,y} = \text{ad}(p_0(L_0 + W_0,y)L_0 + q_0(L_0 + W_0,y)W_0)$.

(3) Since $\Delta(W_0 + W_1) = 0$, it follows from Eq. (11) that

$$\begin{aligned} \Delta(W_0 + W_1) &= D_{W_0+W_1,y}(W_0 + W_1) \\ &= \left[\sum_{k \in \mathbb{Z}} (p_k(W_0 + W_1,y)L_k + q_k(W_0 + W_1,y)W_k), W_0 + W_1 \right] \\ &\quad + \mu(W_0 + W_1,y)D(W_0 + W_1) \\ &= \sum_{k \in \mathbb{Z}} (\lambda kp_k(W_0 + W_1,y)W_k + (1 + \lambda k)p_k(W_0 + W_1,y)W_{k+1}) \\ &\quad + \mu(W_0 + W_1,y)(W_0 + W_1) \\ &= \sum_{k \in \mathbb{Z}} (\lambda(k+1)p_{k+1}(W_0 + W_1,y) + (1 + \lambda k)p_k(W_0 + W_1,y))W_{k+1} \\ &\quad + \mu(W_0 + W_1,y)(W_0 + W_1) = 0. \end{aligned} \quad (13)$$

Eq. (13) shows that

$$\lambda(k+1)p_{k+1}(W_0+W_1, y) + (1+\lambda k)p_k(W_0+W_1, y) = 0, \quad \forall k \neq -1, 0, \quad (14)$$

$$(1-\lambda)p_{-1}(W_0+W_1, y) + \mu(W_0+W_1, y) = 0 \quad (15)$$

and $\lambda p_1(W_0+W_1, y) + p_0(W_0+W_1, y) + \mu(W_0+W_1, y) = 0. \quad (16)$

It is obvious that Eq. (14) can be written as

$$p_{k+1}(W_0+W_1, y) = -\frac{(1+\lambda k)}{\lambda(k+1)}p_k(W_0+W_1, y), \quad \forall k \neq -1, 0, \quad (17)$$

or

$$p_k(W_0+W_1, y) = -\frac{\lambda(k+1)}{1+\lambda k}p_{k+1}(W_0+W_1, y), \quad \forall k \neq -1, 0, -\frac{1}{\lambda} \text{ (if } \frac{1}{\lambda} \in \mathbb{Z}^*). \quad (18)$$

Case 1. $\frac{1}{\lambda} \notin \mathbb{Z}^*$.

Since $\frac{1}{\lambda} \notin \mathbb{Z}^*$, it is obvious that $1+\lambda k \neq 0$ for all $k \in \mathbb{Z}$. For $k \leq -2$, by Eq. (17), we have

$$p_{-1}(W_0+W_1, y) = \frac{1-2\lambda}{\lambda}p_{-2}(W_0+W_1, y) = \dots = \frac{\prod_{i=2}^{-k}(1-\lambda i)}{\prod_{i=1}^{-k-1}(\lambda i)}p_k(W_0+W_1, y) = \dots .$$

From Eq. (18), it is easy to see that for $k \geq 2$,

$$p_1(W_0+W_1, y) = \frac{-2\lambda}{1+\lambda}p_2(W_0+W_1, y) = \dots = \frac{(-1)^{k-1}\prod_{i=2}^k(\lambda i)}{\prod_{i=1}^{k-1}(1+\lambda i)}p_k(W_0+W_1, y) = \dots .$$

Since the set $\{p_k(W_0+W_1, y) \mid p_k(W_0+W_1, y) \neq 0, k \in \mathbb{Z}^*\}$ is finite, we conclude that $p_k(W_0+W_1, y) = 0$ for all $k \in \mathbb{Z}^*$ by the relations above. Then, Eqs. (15) and (16) show that $p_0(W_0+W_1, y) = \mu(W_0+W_1, y) = 0$. Hence,

$$D_{W_0+W_1, y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} q_k(W_0+W_1, y)W_k\right).$$

Case 2. $\frac{1}{\lambda} \in \mathbb{Z}_+$.

When $\frac{1}{\lambda} \in \mathbb{Z}_+$, it is easy to see that $\frac{1}{\lambda} \geq 2$. By taking $k = -\frac{1}{\lambda}$ in Eq. (17), we have $p_{-\frac{1}{\lambda}+1}(W_0+W_1, y) = 0$, which implies that $p_k(W_0+W_1, y) = 0$ for $-\frac{1}{\lambda}+1 \leq k \leq -1$. Moreover, again by Eq. (17), for $k \leq -\frac{1}{\lambda}-1$, we get

$$p_{-\frac{1}{\lambda}}(W_0+W_1, y) = -\lambda p_{-\frac{1}{\lambda}-1}(W_0+W_1, y) = \dots = \frac{\prod_{i=\frac{1}{\lambda}+1}^{-k}(1-\lambda i)}{\prod_{i=\frac{1}{\lambda}}^{-k-1}(\lambda i)}p_k(W_0+W_1, y) = \dots .$$

Similar to Case 1, we have for $k \geq 2$,

$$p_1(W_0+W_1, y) = \frac{-2\lambda}{1+\lambda}p_2(W_0+W_1, y) = \dots = \frac{(-1)^{k-1}\prod_{i=2}^k(\lambda i)}{\prod_{i=1}^{k-1}(1+\lambda i)}p_k(W_0+W_1, y) = \dots .$$

Since the sets $\{p_k(W_0+W_1, y) \mid p_k(W_0+W_1, y) \neq 0, k \geq 1\}$

and $\{p_k(W_0+W_1, y) \mid p_k(W_0+W_1, y) \neq 0, k \leq -\frac{1}{\lambda}\}$

are finite, we obtain $p_k(W_0+W_1, y) = 0$ for $k \geq 1$ and $k \leq -\frac{1}{\lambda}$. Therefore, for all $k \in \mathbb{Z}^*$, $p_k(W_0+W_1, y) = 0$.

Then, we immediately obtain $p_0(W_0 + W_1, y) = \mu(W_0 + W_1, y) = 0$ from Eqs. (15) and (16). So, $D_{W_0+W_1,y}$ can also be written as the form

$$D_{W_0+W_1,y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} q_k(W_0 + W_1, y)W_k\right).$$

Case 3. $\frac{1}{\lambda} \in \mathbb{Z}_-$.

In fact, $\frac{1}{\lambda} \leq -2$ is obvious. It follows from Eq. (17) that for $k \leq -2$,

$$p_{-1}(W_0 + W_1, y) = \frac{1 - 2\lambda}{\lambda} p_{-2}(W_0 + W_1, y) = \dots = \frac{\prod_{i=2}^{-k} (1 - \lambda i)}{\prod_{i=1}^{-k-1} (\lambda i)} p_k(W_0 + W_1, y) = \dots$$

By Eq. (18), we have

$$\begin{aligned} p_1(W_0 + W_1, y) &= \frac{-2\lambda}{1 + \lambda} p_2(W_0 + W_1, y) = \dots \\ &= \frac{(-1)^{-\frac{1}{\lambda}-1} \prod_{i=2}^{-\frac{1}{\lambda}} (\lambda i)}{\prod_{i=1}^{-\frac{1}{\lambda}-1} (1 + \lambda i)} p_{-\frac{1}{\lambda}}(W_0 + W_1, y) \end{aligned} \tag{19}$$

and for $k \geq -\frac{1}{\lambda} + 2$,

$$\begin{aligned} p_{-\frac{1}{\lambda}+1}(W_0 + W_1, y) &= \frac{1 - 2\lambda}{\lambda} p_{-\frac{1}{\lambda}+2}(W_0 + W_1, y) = \dots \\ &= \frac{(-1)^{k+\frac{1}{\lambda}-1} \prod_{i=-\frac{1}{\lambda}+2}^k (\lambda i)}{\prod_{i=-\frac{1}{\lambda}+1}^{k-1} (1 + \lambda i)} p_k(W_0 + W_1, y) = \dots \end{aligned}$$

Since the sets $\{p_k(W_0 + W_1, y) \mid p_k(W_0 + W_1, y) \neq 0, k \geq -\frac{1}{\lambda} + 1\}$

and $\{p_k(W_0 + W_1, y) \mid p_k(W_0 + W_1, y) \neq 0, k \leq -1\}$

are finite, we have $p_k(W_0 + W_1, y) = 0$ for $k \geq -\frac{1}{\lambda} + 1$ and $k \leq -1$. From Eqs. (15) and (16), we have $\mu(W_0 + W_1, y) = 0$ and $p_0(W_0 + W_1, y) = -\lambda p_1(W_0 + W_1, y)$. Therefore,

$$D_{W_0+W_1,y} = \text{ad}\left(\sum_{k=0}^{-\frac{1}{\lambda}} p_k(W_0 + W_1, y)L_k + \sum_{k \in \mathbb{Z}} q_k(W_0 + W_1, y)W_k\right),$$

where $p_0(W_0 + W_1, y) = -\lambda p_1(W_0 + W_1, y)$ and $p_k(W_0 + W_1, y)$ satisfies Eq. (19) for $1 \leq k \leq -\frac{1}{\lambda}$. This completes the proof. ■

Lemma 4.2. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$. If $\Delta(L_1) = \Delta(L_0 + W_0) = 0$, we have $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$.*

Proof. Since $\Delta(L_1) = \Delta(L_0 + W_0) = 0$, by Lemma 4.1, we get for any $y \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$,

$$D_{L_1,y} = \text{ad}(p_1(L_1, y)L_1) + \mu(L_1, y)D$$

and $D_{L_0+W_0,y} = \text{ad}(p_0(L_0 + W_0, y)L_0 + q_0(L_0 + W_0, y)W_0)$,

where p_0, p_1, q_0, μ are complex-valued functions on $\mathfrak{L}_{\lambda \notin \mathbb{Z}} \times \mathfrak{L}_{\lambda \notin \mathbb{Z}}$. Then, for any $m \in \mathbb{Z}$, we obtain

$$\begin{aligned} \Delta(L_m) &= D_{L_1,L_m}(L_m) = [p_1(L_1, L_m)L_1, L_m] + \mu(L_1, L_m)D(L_m) \\ &= (m - 1)p_1(L_1, L_m)L_{m+1} \end{aligned} \tag{20}$$

and
$$\begin{aligned} \Delta(L_m) &= D_{L_0+W_0, L_m}(L_m) = [p_0(L_0 + W_0, L_m)L_0 + q_0(L_0 + W_0, L_m)W_0, L_m] \\ &= mp_0(L_0 + W_0, L_m)L_m - \lambda mq_0(L_0 + W_0, L_m)W_m. \end{aligned} \tag{21}$$

By comparing with Eqs. (20) and (21), it is obvious that $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$. The proof is complete. ■

Lemma 4.3. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$ such that $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$. For any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$, then*

$$\Delta(x) = \mu_x \sum_{t \in \mathbb{Z}} \beta_t W_t,$$

where μ_x is a complex number depending on x .

Proof. Since $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$, it follows from Lemma 4.1 (1) that for any $x \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$,

$$D_{L_m, x} = \text{ad}(p_m(L_m, x)L_m + \delta_{\lambda m, \mathbb{Z}} q_{-\lambda m}(L_m, x)W_{-\lambda m}) + \mu(L_m, x)D,$$

where $\delta_{k, \mathbb{Z}}$ is given in Lemma 4.1 for $k \in \mathbb{Z}$. For any $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$, we have

$$\begin{aligned} \Delta(x) &= D_{L_m, x}(x) = [p_m(L_m, x)L_m + \delta_{\lambda m, \mathbb{Z}} q_{-\lambda m}(L_m, x)W_{-\lambda m}, x] + \mu(L_m, x)D(x) \\ &= \sum_{t \in \mathbb{Z}} \alpha_t ((t - m)p_m(L_m, x)L_{m+t} - \lambda(t - m)\delta_{\lambda m, \mathbb{Z}} q_{-\lambda m}(L_m, x)W_{-\lambda m+t}) \\ &\quad + \sum_{t \in \mathbb{Z}} \beta_t ((t + \lambda m)p_m(L_m, x)W_{m+t} + \mu(L_m, x)W_t). \end{aligned}$$

By taking enough different $m \in \mathbb{Z}$ in the equation above such that $\lambda m \notin \mathbb{Z}$, and if necessary, let these m 's to be large enough, we have

$$\Delta(x) = \mu(L_m, x) \sum_{t \in \mathbb{Z}} \beta_t W_t.$$

Now, let $\mu_x = \mu(L_m, x) \in \mathbb{C}$, which is independent on m but dependent on x . This finishes the proof. ■

Lemma 4.4. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$ such that $\Delta(L_0 + W_0) = 0$ and $\Delta(L_m) = 0$ for all $m \in \mathbb{Z}$. Then for any $j \in \mathbb{Z}^*$ and any $y \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$, we have*

$$D_{L_j+W_{2j}, y} = \text{ad}(\xi_j^y L_j + \delta_{\lambda j, \mathbb{Z}} \eta_{-\lambda j}^y W_{-\lambda j} + \epsilon_j^y W_j + \xi_j^y W_{2j}) + (\lambda + 1)j\epsilon_j^y D,$$

where $\xi_j^y, \eta_{-\lambda j}^y, \epsilon_j^y \in \mathbb{C}$ depend on y and $\delta_{k, \mathbb{Z}}$ is defined as 1 for $k \in \mathbb{Z}$ and as 0 for $k \notin \mathbb{Z}$.

Proof. By $\Delta(L_m) = 0$ for any $m \in \mathbb{Z}$ and Lemma 4.3 it is obvious that for any $j \in \mathbb{Z}^*$,

$$\Delta(L_j + W_{2j}) = \mu_{L_j+W_{2j}} W_{2j}, \tag{22}$$

where $\mu_{L_j+W_{2j}} \in \mathbb{C}$. Meanwhile, by Lemma 4.1 (2), we have

$$\begin{aligned} \Delta(L_j + W_{2j}) &= D_{L_0+W_0, L_j+W_{2j}}(L_j + W_{2j}) \\ &= [p_0(L_0 + W_0, L_j + W_{2j})L_0 + q_0(L_0 + W_0, L_j + W_{2j})W_0, L_j + W_{2j}] \\ &= jp_0(L_0 + W_0, L_j + W_{2j})L_j + 2jp_0(L_0 + W_0, L_j + W_{2j})W_{2j} \\ &\quad - \lambda jq_0(L_0 + W_0, L_j + W_{2j})W_j. \end{aligned} \tag{23}$$

A comparison with Eqs. (22) and (23) shows that

$$p_0(L_0 + W_0, L_j + W_{2j}) = q_0(L_0 + W_0, L_j + W_{2j}) = 0.$$

Therefore, $\Delta(L_j + W_{2j}) = 0$ for any $j \in \mathbb{Z}^*$.

By Lemma 2.4, we have for any $y \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$,

$$D_{L_j+W_{2j},y} = \text{ad}\left(\sum_{k \in \mathbb{Z}} (p_k(L_j+W_{2j},y)L_k + q_k(L_j+W_{2j},y)W_k)\right) + \mu(L_j+W_{2j},y)D. \quad (24)$$

Then, it follows from Eq. (24) that for any $j \in \mathbb{Z}^*$ and $y \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$,

$$\begin{aligned} \Delta(L_j + W_{2j}) &= D_{L_j+W_{2j},y}(L_j + W_{2j}) \\ &= \left[\sum_{k \in \mathbb{Z}} (p_k(L_j + W_{2j}, y)L_k + q_k(L_j + W_{2j}, y)W_k), L_j + W_{2j} \right] \\ &\quad + \mu(L_j + W_{2j}, y)D(L_j + W_{2j}) \\ &= \sum_{k \in \mathbb{Z}} ((j - k)p_k(L_j + W_{2j}, y)L_{k+j} + (2j + \lambda k)p_k(L_j + W_{2j}, y)W_{k+2j} \\ &\quad - (k + \lambda j)q_k(L_j + W_{2j}, y)W_{k+j}) + \mu(L_j + W_{2j}, y)W_{2j} = 0. \end{aligned} \quad (25)$$

From Eq. (25), we immediately get $(j - k)p_k(L_j + W_{2j}, y) = 0$ for all $k \in \mathbb{Z}$, which deduces that $p_k(L_j + W_{2j}, y) = 0$ for $k \neq j$. Then, Eq. (25) becomes

$$(2 + \lambda)jp_j(L_j + W_{2j}, y)W_{3j} - \sum_{k \in \mathbb{Z}} (k + \lambda j)q_k(L_j + W_{2j}, y)W_{k+j} + \mu(L_j + W_{2j}, y)W_{2j} = 0.$$

By observing the coefficient of W_{3j} in the equation above, we have

$$(2 + \lambda)jp_j(L_j + W_{2j}, y) - (2 + \lambda)jq_{2j}(L_j + W_{2j}, y) = 0,$$

which implies that $p_j(L_j + W_{2j}, y) = q_{2j}(L_j + W_{2j}, y)$. Next, we consider the coefficient of W_{2j} , we obtain $-(1 + \lambda)jq_j(L_j + W_{2j}, y) + \mu(L_j + W_{2j}, y) = 0$, which implies that $(1 + \lambda)jq_j(L_j + W_{2j}, y) = \mu(L_j + W_{2j}, y)$. For the coefficient of W_k , $k \neq 2j, 3j$, we get $(k + \lambda j)q_k(L_j + W_{2j}, y) = 0$. Therefore, if $\lambda j \in \mathbb{Z}$, $q_k(L_j + W_{2j}, y) = 0$ for $k \neq -\lambda j, j, 2j$ or if $\lambda j \notin \mathbb{Z}$, $q_k(L_j + W_{2j}, y) = 0$ for $k \neq j, 2j$. Hence,

$$\begin{aligned} D_{L_j+W_{2j},y} &= \text{ad}(p_j(L_j + W_{2j}, y)L_j + \delta_{\lambda j, \mathbb{Z}}q_{-\lambda j}(L_j + W_{2j}, y)W_{-\lambda j} + q_j(L_j + W_{2j}, y)W_j \\ &\quad + p_j(L_j + W_{2j}, y)W_{2j}) + (1 + \lambda)jq_j(L_j + W_{2j}, y)D, \end{aligned}$$

where $\delta_{k, \mathbb{Z}}$ is defined as 1 for $k \in \mathbb{Z}$ and as 0 for $k \notin \mathbb{Z}$.

Let $\xi_j^y = p_j(L_j + W_{2j}, y)$, $\eta_{-\lambda j}^y = q_{-\lambda j}(L_j + W_{2j}, y)$ and $\epsilon_j^y = q_j(L_j + W_{2j}, y)$. Obviously, ξ_j^y , $\eta_{-\lambda j}^y$ and ϵ_j^y are complex numbers depending on y . The proof is finished. ■

Lemma 4.5. *Let Δ be a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$ such that $\Delta(L_0 + W_0) = \Delta(L_1) = 0$. For any nonzero $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$, then $\Delta(x) = 0$.*

Proof. Since $\Delta(L_0 + W_0) = \Delta(L_1) = 0$, by Lemma 4.2, we have for any $m \in \mathbb{Z}$,

$$\Delta(L_m) = 0.$$

Then, Lemma 4.3 implies that for any nonzero $x = \sum_{t \in \mathbb{Z}} (\alpha_t L_t + \beta_t W_t) \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$ we get

$$\Delta(x) = \mu_x \sum_{t \in \mathbb{Z}} \beta_t W_t \quad \text{where } \mu_x \in \mathbb{C}. \tag{26}$$

Case 1. $x = \sum_{t \in \mathbb{Z}} \beta_t W_t$.

It is easy to see that

$$\begin{aligned} \Delta(x) &= D_{L_0+W_0,x}(x) = [p_0(L_0 + W_0, x)L_0 + q_0(L_0 + W_0, x)W_0, x] \\ &= \sum_{t \in \mathbb{Z}} \beta_t t p_0(L_0 + W_0, x)W_t. \end{aligned} \tag{27}$$

Subcase 1. $x = \beta_{t_1} W_{t_1} + \beta_{t_2} W_{t_2} + \sum_{t \in \mathbb{Z} \setminus \{t_1, t_2\}} \beta_t W_t$ with $t_1 \neq t_2$ for $t_1, t_2 \in \mathbb{Z}$ and $\beta_{t_1} \beta_{t_2} \neq 0$.

Comparing with Eqs. (26) and (27), we obtain

$$t_1 p_0(L_0 + W_0, x) = \mu_x, \quad \text{and} \quad t_2 p_0(L_0 + W_0, x) = \mu_x.$$

Then, $(t_1 - t_2)p_0(L_0 + W_0, x) = 0$. This implies that $p_0(L_0 + W_0, x) = \mu_x = 0$. Hence, $\Delta(x) = 0$.

Subcase 2. $x = \beta_{t_0} W_{t_0}$ with $t_0 \in \mathbb{Z}$ and $\beta_{t_0} \neq 0$.

For the form $x = \beta_{t_0} W_{t_0} \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$, Eq. (27) becomes

$$\Delta(\beta_{t_0} W_{t_0}) = \beta_{t_0} t_0 p_0(L_0 + W_0, \beta_{t_0} W_{t_0}) W_{t_0}.$$

If $t_0 = 0$, we have $\Delta(\beta_0 W_0) = 0$. Next, we consider $\Delta(\beta_{t_0} W_{t_0})$ for $t_0 \in \mathbb{Z}^*$. From Subcase 1, we know that $\Delta(W_0 + W_1) = 0$. If $\frac{1}{\lambda} \notin \mathbb{Z}^*$ or $\frac{1}{\lambda} \in \mathbb{Z}_+$, from Lemma 4.1 (3), we have

$$\Delta(\beta_{t_0} W_{t_0}) = D_{W_0+W_1, \beta_{t_0} W_{t_0}}(\beta_{t_0} W_{t_0}) = \left[\sum_{k \in \mathbb{Z}} q_k(W_0 + W_1, \beta_{t_0} W_{t_0}) W_k, \beta_{t_0} W_{t_0} \right] = 0.$$

If $\frac{1}{\lambda} \in \mathbb{Z}_-$, again by Lemma 4.1 (3), we have

$$\begin{aligned} \Delta(\beta_{t_0} W_{t_0}) &= D_{W_0+W_1, \beta_{t_0} W_{t_0}}(\beta_{t_0} W_{t_0}) \\ &= \left[\sum_{k=0}^{-\frac{1}{\lambda}} p_k(W_0 + W_1, \beta_{t_0} W_{t_0}) L_k + \sum_{k \in \mathbb{Z}} q_k(W_0 + W_1, \beta_{t_0} W_{t_0}) W_k, \beta_{t_0} W_{t_0} \right] \\ &= \beta_{t_0} \sum_{k=0}^{-\frac{1}{\lambda}} (t_0 + \lambda k) p_k(W_0 + W_1, \beta_{t_0} W_{t_0}) W_{k+t_0}, \end{aligned} \tag{28}$$

where $p_0(W_0 + W_1, \beta_{t_0} W_{t_0}) = -\lambda p_1(W_0 + W_1, \beta_{t_0} W_{t_0})$ and $p_k(W_0 + W_1, \beta_{t_0} W_{t_0})$ for $1 \leq k \leq -\frac{1}{\lambda}$ satisfies the following relation

$$\begin{aligned} p_1(W_0 + W_1, \beta_{t_0} W_{t_0}) &= \frac{-2\lambda}{1+\lambda} p_2(W_0 + W_1, \beta_{t_0} W_{t_0}) = \dots \\ &= \frac{(-1)^{-\frac{1}{\lambda}-1} \prod_{i=2}^{-\frac{1}{\lambda}} (\lambda i)}{\prod_{i=1}^{-\frac{1}{\lambda}-1} (1+\lambda i)} p_{-\frac{1}{\lambda}}(W_0 + W_1, \beta_{t_0} W_{t_0}). \end{aligned}$$

From Eq. (26), we have $\Delta(\beta_{t_0} W_{t_0}) = \mu_{\beta_{t_0} W_{t_0}} \beta_{t_0} W_{t_0}$, where $\mu_{\beta_{t_0} W_{t_0}} \in \mathbb{C}$. This together with Eq. (28) shows that $(t_0 + \lambda) p_1(W_0 + W_1, \beta_{t_0} W_{t_0}) = 0$, which deduces that $p_1(W_0 + W_1, \beta_{t_0} W_{t_0}) = 0$. Obviously, $p_0(W_0 + W_1, \beta_{t_0} W_{t_0}) = 0$. Hence, $\Delta(\beta_{t_0} W_{t_0}) = 0$ for $t_0 \in \mathbb{Z}^*$.

Case 2. $x = \alpha_{t_0}L_{t_0} + \sum_{t \in \mathbb{Z} \setminus \{t_0\}} \alpha_t L_t + \sum_{t \in \mathbb{Z}} \beta_t W_t$ with $t_0 \in \mathbb{Z}$ and $\alpha_{t_0} \neq 0$.

From Lemma 4.4, we have for $j \in \mathbb{Z}^*$,

$$\begin{aligned} \Delta(x) &= D_{L_j+W_{2j},x}(x) \\ &= [\xi_j^x L_j + \delta_{\lambda j, \mathbb{Z}} \eta_{-\lambda j}^x W_{-\lambda j} + \epsilon_j^x W_j + \xi_j^x W_{2j}, x] + (\lambda + 1)j \epsilon_j^x D(x) \\ &= \sum_{t \in \mathbb{Z}} \alpha_t ((t-j)\xi_j^x L_{j+t} - \lambda(t-j)\delta_{\lambda j, \mathbb{Z}} \eta_{-\lambda j}^x W_{-\lambda j+t} - (j+\lambda t)\epsilon_j^x W_{j+t} - (2j+\lambda t)\xi_j^x W_{2j+t}) \\ &\quad + \sum_{t \in \mathbb{Z}} \beta_t ((t+\lambda j)\xi_j^x W_{j+t} + (\lambda+1)j \epsilon_j^x W_t), \end{aligned} \tag{29}$$

where $\xi_j^x, \eta_{-\lambda j}^x, \epsilon_j^x \in \mathbb{C}$ and $\delta_{k, \mathbb{Z}}$ is defined as 1 for $k \in \mathbb{Z}$ and as 0 for $k \notin \mathbb{Z}$. There exists a nonzero term $\alpha_{t_0}L_{t_0}$. Comparing Eqs. (26) and (29), the coefficient of L_k is 0 for all $k \in \mathbb{Z}$. By taking enough different $j \in \mathbb{Z}^*$ in Eq. (29) such that $t_0 - j \neq 0$ and $\lambda j \notin \mathbb{Z}$, if necessary, let these j 's to be large enough, we obtain $\xi_j^x = 0$. Then, Eq. (29) becomes

$$\Delta(x) = - \sum_{t \in \mathbb{Z}} \alpha_t (j + \lambda t) \epsilon_j^x W_{j+t} + \sum_{t \in \mathbb{Z}} \beta_t (\lambda + 1) j \epsilon_j^x W_t.$$

When j is large enough, we have $\epsilon_j^x = 0$. Then, $\Delta(x) = 0$. We finish the proof. ■

Theorem 4.6. *Every 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$ is a derivation.*

Proof. Assume that Δ is a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$. Then there exists a derivation $D_{L_0+W_0, L_1}$ such that we have $\Delta(L_0 + W_0) = D_{L_0+W_0, L_1}(L_0 + W_0)$ and $\Delta(L_1) = D_{L_0+W_0, L_1}(L_1)$. Let $\Delta_1 = \Delta - D_{L_0+W_0, L_1}$, which is also a 2-local derivation on $\mathfrak{L}_{\lambda \notin \mathbb{Z}}$ obviously. Then,

$$\begin{aligned} \Delta_1(L_0 + W_0) &= \Delta(L_0 + W_0) - D_{L_0+W_0, L_1}(L_0 + W_0) = 0 - 0 = 0, \\ \Delta_1(L_1) &= \Delta(L_1) - D_{L_0+W_0, L_1}(L_1) = 0 - 0 = 0. \end{aligned}$$

From Lemma 4.5, we have $\Delta_1(x) = \Delta(x) - D_{L_0+W_0, L_1}(x) = 0$ for any $x \in \mathfrak{L}_{\lambda \notin \mathbb{Z}}$. Obviously, $\Delta_1 = \Delta - D_{L_0+W_0, L_1} = 0$, which implies that $\Delta = D_{L_0+W_0, L_1}$. This shows that Δ is a derivation. The proof is completed. ■

Remark 4.7. It follows from Theorems 3.6 and 4.6 that Theorem 2.5 is obtained immediately. ■

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Yan Liu, Yao Ma, Liangyun Chen, School of Mathematics and Statistics,
 Northeast Normal University, Changchun, China;
 liuy726@nenu.edu.cn, may703@nenu.edu.cn, chenly640@nenu.edu.cn.

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