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An Analogue of the Schur-Weyl Duality for the Automorphism Group of a II_1-Factor

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Abstract. An analogue of the Schur-Weyl duality for the automorphism group of the approximately finite dimensional (AFD) II_1-factor is produced.

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Key Words: Finite von Neumann factor, automorphism group of a factor, Schur-Weyl duality.

1. Introduction

Let M be a II_1-factor with the separable predual M_* and tr a unique normal trace on M such that $\mathrm{tr}(I) = 1$. The inner product $\langle a, b\rangle = \mathrm{tr}(b^*a)$ makes M a pre-Hilbert space. Denote by $L^2(M, \mathrm{tr})$ its completion. Let $\mathrm{Aut}\, M$ be the automorphism group of M and $U(M)$ the unitary subgroup of M. Every $u \in U(M)$ determines the *inner* automorphism $\mathrm{Ad}\, u$ of M, $\mathrm{Ad}\, u(x) = uxu^*$. Denote by $\mathrm{Inn}\, M$ the normal subgroup of $\mathrm{Aut}\, M$ formed by inner automorphisms.

One has a natural unitary representation $\mathfrak{N}$ of $\mathrm{Aut}\, M$ on the dense subspace M of $L^2(M, \mathrm{tr})$ given by

$$\mathfrak{N}(\theta)x = \theta(x), \qquad \theta \in \mathrm{Aut}\, M, \quad x \in M,$$

which is certainly extendable to a representation on $L^2(M, \mathrm{tr})$. Denote by $\mathfrak{N}_I$ the restriction of $\mathfrak{N}$ to the subgroup $\mathrm{Inn}\, M$.

$\mathrm{Aut}\, M$, being embedded as above into the algebra of bounded operators in $L^2(M, \mathrm{tr})$, becomes a topological group under the strong operator topology. The subspace $L_0 = \{v \in L^2(M, \mathrm{tr}) : \mathrm{tr}(v) = 0\}$ is $\mathfrak{N}$-invariant: $\mathfrak{N}(\theta)L_0 = L_0$ for all $\theta \in \mathrm{Aut}\, M$.

Theorem 1.1. *The restriction $\mathfrak{N}_I^0$ of the representation $\mathfrak{N}_I$ to the invariant subspace L_0 is irreducible.*

With an arbitrary II_1-factor M being replaced in the above settings by the algebra of complex $n \times n$ matrices, Theorem 1.1 reduces to the well known fact of classical representation theory (see [7], Ch. 3, §17.2, Theorem 2). Thus, in case of the

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approximately finite dimensional (AFD or hyperfinite) factor M, an argument based on approximation of II_1-factor M by finite dimensional factors is going to be applicable in proving Theorem 1.1. However, this theorem in its utmost generality requires a new approach.

Define a diagonal action $\mathfrak{N}^{\otimes k}$ of $\operatorname{Aut} M$ on $L^2(M,\mathrm{tr})^{\otimes k} = L^2\left(M^{\otimes k},\mathrm{tr}^{\otimes k}\right)$ by

$$\mathfrak{N}^{\otimes k}(\theta)\left(v_1\otimes v_2\otimes\cdots\otimes v_k\right) = \left(\mathfrak{N}(\theta)v_1\right)\otimes\left(\mathfrak{N}(\theta)v_2\right)\otimes\cdots\otimes\left(\mathfrak{N}(\theta)v_k\right).$$

Additionally, the symmetric group $\mathfrak{S}_k$ acts on $L^2\left(M^{\otimes k},\mathrm{tr}^{\otimes k}\right)$ by permutations

$$^{k}\mathcal{P}(s)\left(v_1\otimes v_2\otimes\cdots\otimes v_k\right) = v_{s^{-1}(1)}\otimes v_{s^{-1}(2)}\otimes\cdots\otimes v_{s^{-1}(k)}. \tag{1}$$

The representations $\mathfrak{N}^{\otimes k}$ and $^{k}\mathcal{P}$ are well defined in the case $k=\infty$ (see Theorem 1.5 below). Since the operators $\mathfrak{N}^{\otimes k}(\theta)$ and $^{k}\mathcal{P}(s)$ commute, we obtain a representation $\mathcal{F}_k$ of the group $\operatorname{Aut} M\times\mathfrak{S}_k$, $\mathcal{F}_k(\theta,s) = \mathfrak{N}^{\otimes k}(\theta)\cdot {}^{k}\mathcal{P}(s)$.
The left and right multiplication operators are

$$m\ni M \overset{\mathfrak{L}(u)}{\mapsto} um\in M \quad\text{and}\quad m\ni M \overset{\mathfrak{R}(u)}{\mapsto} mu^{*}\in M,$$

where $u\in U(M)$, define in $L^2(M,\mathrm{tr})^{\otimes k}$ representation T of $U(M)\times U(M)$. Namely, $T(u,v)=\mathfrak{L}(u)\cdot\mathfrak{R}(v)$. It is clear that under $k<\infty$ $^{k}\mathcal{P}(\mathfrak{S}_k)$ lies in the commutant of $T^{\otimes k}(U(M)\times U(M))$. In [12] an analogue of the classical Schur-Weyl duality was found (Theorem 2 (a)). It is proven there that the commutant of $T^{\otimes k}(U(M)\times U(M))$ coincides with the algebra generated by $^{k}\mathcal{P}(\mathfrak{S}_k)$.

Recall that the irreducible representations of $\mathfrak{S}_k$ are parameterized by the unordered partitions of k. Denote the set of all such partitions by Υ_k. Let $\lambda\in\Upsilon_k$ and let χ_λ be the character of the corresponding irreducible representation R_λ. Denote by $\dim\lambda$ the dimension of R_λ. The operator

$$P^{\lambda} = \frac{\dim\lambda}{k!}\sum_{s\in\mathfrak{S}_k}\chi_\lambda(s)\,{}^{k}\mathcal{P}(s) \tag{2}$$

is an orthogonal projection in the centre of the w^*-algebra generated by the operators $\left\{\mathcal{F}_k(\theta,s)\right\}_{(\theta,s)\in\operatorname{Aut} M\times\mathfrak{S}_k}$. Denote by $\mathfrak{F}_k^{\lambda}$ the representation $\mathcal{F}_k$ restricted to the subspace $H_0^{\lambda} = P^{\lambda}\left(L_0^{\otimes k}\right)$. Let $\mathfrak{N}_0^{\otimes k}$ and $^{k}\mathcal{P}_0$ be the restrictions of the representations $\mathfrak{N}^{\otimes k}$ and $^{k}\mathcal{P}$ to the subspace $L_0^{\otimes k}\subset L^2(M,\mathrm{tr})^{\otimes k}$.

Theorem 1.2. *Let M be an AFD II_1-factor. Then the commutant of the set $\mathfrak{N}_0^{\otimes k}(\operatorname{Aut} M)$ is generated by $^{k}\mathcal{P}_0(\mathfrak{S}_k)$.*

Corollary 1.3. *The representation $\mathfrak{F}_k^{\lambda}$ of $\operatorname{Aut} M\times\mathfrak{S}_k$ is irreducible. With different $\lambda,\zeta\in\Upsilon_k$, the restrictions of $\mathfrak{F}_k^{\lambda}$ and $\mathfrak{F}_k^{\zeta}$ to the subgroup $\operatorname{Aut} M$ are not quasi-equivalent.*

The representation $^{k}\mathcal{P}$ can be extended to a representation $^{k}\mathcal{P}^{\mathscr{I}_k}$ of *the symmetric inverse semigroup* $\mathscr{I}_k$, which can be realized as a semigroup of $\{0,1\}$-matrices $a=[a_{ij}]_{i,j=1}^{k}$ with the ordinary matrix multiplication in such a way that a has at most one nonzero entry in each row and each column. In this realization symmetric subgroup $\mathfrak{S}_k\subset\mathscr{I}_k$ consists of the $\{0,1\}$-matrices, which have exactly one nonzero entry in each row and each column; i.e. $\sum_{i=1}^{k}a_{ij}=\sum_{i=1}^{k}a_{ji}=1$ for all $i\in\{1,2,\ldots,k\}$.

We denote by ϵ_i a diagonal matrix $[a_{pq}]$ such that $a_{ii}=0$ and $a_{pq}=\delta_{pq}$, if $p\neq i$ or $q\neq i$. Of course, $\mathfrak{S}_k\subset\mathscr{I}_k$. Define operator ${}^k\mathcal{P}^{\mathscr{I}_k}(\epsilon_i)$ on $L^2\left(M^{\otimes k},\mathrm{tr}^{\otimes k}\right)$ as follows

$$ {}^k\mathcal{P}^{\mathscr{I}_k}(\epsilon_i)\left(\cdots v_{i-1}\otimes v_i\otimes v_{i+1}\cdots\right)=\mathrm{tr}(v_i)\left(\cdots v_{i-1}\otimes \mathrm{I}\otimes v_{i+1}\cdots\right). \tag{3} $$

We set ${}^k\mathcal{P}^{\mathscr{I}_k}(s)={}^k\mathcal{P}(s)$, if $s\in\mathfrak{S}_k$. Then ${}^k\mathcal{P}^{\mathscr{I}_k}$ is extended to a representation of the semigroup $\mathscr{I}_k$. Using Theorem 1.2, we prove in section 4 next statement, which establishes a link between the representation theory of the finite symmetric semigroups developed by Munn [9], [10], Grood [4], East [3], Popova [13] on the one hand, and the representation theory of the automorphism group of AFD II_1-factor.

Theorem 1.4. *If M is an AFD II_1-factor then the commutant of $\mathfrak{N}^{\otimes k}(\mathrm{Aut}\,M)$ is generated by ${}^k\mathcal{P}^{\mathscr{I}_k}(\mathscr{I}_k)$.*

Now we look at the qualitative differences between Theorem 1.4 and classical Schur-Weyl duality. Let $M=\mathbb{M}_n$, where $\mathbb{M}_n$ is the algebra of all complex $(n\times n)$-matrices. It is appropriate to recall that $\mathrm{Aut}\,\mathbb{M}_n=\mathrm{Inn}\,\mathbb{M}_n$. Denote by $\{e_{pq}\}_{p,q=1}^n$ the matrix units in $\mathbb{M}_n$. The group $\mathfrak{S}_k$ is embedded in $\mathbb{M}_n^{\otimes k}$ as follows

$$ \mathfrak{S}_k\ni s\overset{\mathfrak{i}}{\mapsto}\mathfrak{i}_s=\sum_{p_1,p_2,\ldots,p_k=1}^{n} e_{p_1p_{s(1)}}\otimes e_{p_2p_{s(2)}}\otimes\ldots\otimes e_{p_kp_{s(k)}}\in\mathbb{M}_n^{\otimes k}. $$

It is well known ([7], Theorem 2 on page 275) that algebra $\mathfrak{N}^{\otimes k}(\mathbb{M}_n)'$ of the intertwining operators for $\mathfrak{N}^{\otimes k}(\mathbb{M}_n)$ is generated by the operators $\mathcal{L}(s)$, $\mathcal{R}(s)$, where $s\in\mathfrak{S}_k$, and the convolutions C_{pq}, which act as follows

$$ \begin{gathered} \mathbb{M}_n^{\otimes k}\ni m\overset{\mathcal{L}(s)}{\mapsto}\mathfrak{i}_s\cdot m,\quad m\overset{\mathcal{R}(s)}{\mapsto} m\cdot(\mathfrak{i}_s)^*,\\ C_{pq}(e_{i_1j_1}\otimes e_{i_2j_2}\cdots\otimes e_{i_pj_p}\otimes\cdots\otimes\cdots e_{i_qj_q}\otimes\cdots\otimes e_{i_kj_k})\\ =\delta_{i_pj_q}\sum_{l=1}^{n} e_{i_1j_1}\otimes e_{i_2j_2}\cdots\otimes e_{lj_p}\otimes\cdots\otimes\cdots e_{i_ql}\otimes\cdots\otimes e_{i_kj_k}. \end{gathered} $$

Theorem 1.4 shows that in the case of II_1-factor commutant of $\mathfrak{N}^{\otimes k}(\mathrm{Aut}\,M)$ is generated by the analogous of the operators $\mathcal{L}(s)\cdot\mathcal{R}(s)$ and C_{pp}.

The Schur-Weyl duality for automorphism group of I_∞-factor was built by Kirillov [6]. In particular, he examined the tensor powers of the action by conjugation of the unitary group on the space of Hilbert-Schmidt operators. To formulate his result denote by $B_2(H)$ the two-sided $*$-ideal of Hilbert-Schmidt operators in the algebra $B(H)$ of all bounded operators on H. Let us introduce representation $\mathfrak{N}$ of the unitary subgroup $U(H)\subset B(H)$ as follows $B_2(H)\ni b\overset{\mathfrak{N}(u)}{\mapsto} ubu^*\in B_2(H)$.

If $\{e_j\}_{j=1}^\infty$ is orthonormal basis in H then the operators e_{kl}, which act by $e_{kl}\,e_j=\delta_{lj}\,e_k$, form an orthogonal basis in $B_2(H)$. Define the representations $\mathcal{L}$ and $\mathcal{R}$ of $\mathfrak{S}_n$ in $B_2(H)^{\otimes n}$ by

$$ \begin{gathered} \mathcal{L}(s)\,e_{i_1j_1}\otimes e_{i_2j_2}\otimes\cdots\otimes e_{i_nj_n}=e_{i_{s^{-1}(1)}j_1}\otimes e_{i_{s^{-1}(2)}j_2}\otimes\cdots\otimes e_{i_{s^{-1}(n)}j_n};\\ \mathcal{R}(s)\,e_{i_1j_1}\otimes e_{i_2j_2}\otimes\cdots\otimes e_{i_nj_n}=e_{i_1j_{s(1)}}\otimes e_{i_2j_{s(2)}}\otimes\cdots\otimes e_nj_{i_{s(n)}},\quad s\in\mathfrak{S}_n. \end{gathered} $$

Kirillov proved that commutant of $\mathfrak{N}^{\otimes n}(U(H))$ is generated by $\mathcal{L}(\mathfrak{S}_n)$ and $\mathcal{R}(\mathfrak{S}_n)$.

Now consider the irreducible representation π in the Hilbert space H of the $C^\star$-algebra $\mathcal{A}$. D. Beltita and K.-H. Neeb studied the unitary representation of the unitary subgroup $U(\mathcal{A})$ of $C^\star$-algebra $\mathcal{A}$ of the view

$$(B_2(H))^n \ni b_1 \otimes b_2 \otimes \cdots \otimes b_n \overset{\mathfrak{N}_\pi^{\otimes n}}{\mapsto} \pi(u)b_1\pi(u^*) \otimes \pi(u)b_2\pi(u^*) \otimes \cdots \otimes \pi(u)b_n\pi(u^*),$$

where $u \in U(\mathcal{A})$ [1]. They extended Kirillov's result above to the representations $\mathfrak{N}_\pi^{\otimes n}$. Using the embedding

$$L^2(M, \mathrm{tr})^{\otimes n} \ni m_1 \otimes \ldots \otimes m_n \mapsto m_1 \otimes \ldots \otimes m_n \otimes \mathrm{I} \in L^2(M, \mathrm{tr})^{\otimes(n+1)},$$

we identify $L^2(M, \mathrm{tr})^{\otimes n}$ with the subspace in $L^2(M, \mathrm{tr})^{\otimes(n+1)}$.
Denote by $L^2(M, \mathrm{tr})^{\otimes\infty}$ the completion of the pre-Hilbert space $\bigcup_{n=1}^{\infty} L^2(M, \mathrm{tr})^{\otimes n}$. It is convenient to consider $\bigcup_{n=1}^{\infty} L^2(M, \mathrm{tr})^{\otimes n}$ as the linear span of the vectors

$$v_1 \otimes \cdots \otimes v_n \otimes \mathrm{I} \otimes \mathrm{I} \otimes \cdots, \quad \text{where } v_j \in M.$$

At the same time, we will to identify $L^2(M, \mathrm{tr})^{\otimes n}$ with the closure of the linear span of all vectors $v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots$, where $v_i = \mathrm{I}$ for all $i > n$. Define the representation $\mathfrak{N}^{\otimes\infty}$ of group $\operatorname{Aut} M$ as follows

$$\mathfrak{N}^{\otimes\infty}(\theta)(v_1 \otimes \cdots \otimes v_n \otimes \cdots) = (\mathfrak{N}(\theta)v_1) \otimes \cdots \otimes (\mathfrak{N}(\theta)v_n) \otimes \cdots .$$

The infinite symmetric group $\mathfrak{S}_\infty$ acts on $L^2(M, \mathrm{tr})^{\otimes\infty}$ by permutations

$${}^{\infty}\mathcal{P}(s)(v_1 \otimes \cdots \otimes v_n \otimes \cdots) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)} \otimes \cdots, \quad s \in \mathfrak{S}_\infty.$$

We prove in Section 5 the following statement.

Theorem 1.5. *If M is an AFD II_1-factor then the commutant of $\mathfrak{N}^{\otimes\infty}(\operatorname{Aut} M)$ is generated by ${}^{\infty}\mathcal{P}(\mathfrak{S}_\infty)$.*

It is clear that the representations $\mathfrak{N}^{\otimes k}$ are at the same time representations of $U(M)$: $U(M) \ni u \mapsto \mathfrak{N}^{\otimes k}(\operatorname{Ad} u)$. These examples are related to the class of *tame* representations of the group $U(M)$ by Definition 5 in [16]. Namely, $U(M)$ is a *Polish* group with the topology defined by the metric ρ as follows: $\rho(u,v) = \sqrt{1 - \Re\,\mathrm{tr}\,(uv^*)}$. There is also a topology $\mathfrak{t}$ on $U(M)$ with the neighbourhood base of the identity element

$$\mathcal{U}_{v,\epsilon} = \{u \in U(M) : \rho(uvu^*, v)\} < \epsilon. \tag{4}$$

The very property of continuity with respect to the topology $\mathfrak{t}$ distinguishes the class of tame representations of $U(M)$. Hypothetically, all tame representations of the group $U(M)$ in the case of AFD-factor M are of type I. It is quite probable that an arbitrary irreducible tame representation is contained in $\mathfrak{N}^{\otimes k}$ for some k.

If we replace uvu^* in (4) by $\theta(v)$, where $\theta(v) = \operatorname{Ad} u(v)$, then we obtain the *weak* topology on $\operatorname{Aut} M$. It is well known in the commutative case, when $L^\infty(X,\mu)$ is considered instead of the factor M. The tame representations of the group of automorphisms of a Lebesgue space (X,μ) that preserve the measure μ were studied in [11], where a complete classification of them was given and the corresponding version of Schur-Weyl duality was presented.

2. Proof of Theorem 1.1

Let M be a II_1-factor. Denote by $B\left(L^2(M,\mathrm{tr})\right)$ the algebra of all bounded operators on $L^2(M,\mathrm{tr})$. Recall that a w^*-subalgebra $\mathfrak{A} \subset M$ is called *MASA* (maximal Abelian subalgebra) if $(\mathfrak{A}' \cap M) = \mathfrak{A}$, where

$$\mathfrak{A}' = \left\{ b \in B\left(L^2(M,\mathrm{tr})\right) \middle|\ ba = ab \text{ for all } a \in \mathfrak{A} \right\}$$

is the commutant of $\mathfrak{A}$. Let $\mathcal{N}(\mathfrak{A}) = \{u \in U(M) : u\mathfrak{A}u^* = u^*\mathfrak{A}u = \mathfrak{A}\}$ be the *normalizer* of $\mathfrak{A}$. Let $\mathcal{N}(\mathfrak{A})''$ be the w^*-subalgebra generated by $\mathcal{N}(\mathfrak{A})$. A MASA $\mathfrak{A}$ is said to be *Cartan* if $\mathcal{N}(\mathfrak{A})'' = M$.

We need the following claim from [14] (p. 242).

Proposition 2.1. *There exist a MASA $\mathfrak{A}$ in M and an AFD-subfactor F of M containing $\mathfrak{A}$ such that $\mathfrak{A}$ is a Cartan subalgebra of M and $F' \cap M = \mathbb{C}I$.*

It is well known that, in the context of the latter proposition, one can readily find a family $\{K_n\}_{n=1}^{\infty}$ of pairwise commuting I_2-subfactors $K_n \subset F$ which generate F. Fix a system of matrix units $\{{}^{n}e_{ij}\}_{i,j=1}^{2} \subset K_n$. Denote by $\mathfrak{A}_K$ an Abelian w^*-subalgebra generated by $\{{}^{r}e_{11}, {}^{r}e_{22}\}_{r=1}^{\infty}$. It is easy to check that $\mathfrak{A}_K$ is a Cartan subalgebra in F. Since any two Cartan MASAS $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of F are conjugate, i.e. there exists $\theta \in \mathrm{Aut}\, F$ such that $\theta(\mathfrak{A}_1) = \mathfrak{A}_2$, we can assume without loss of generality that the MASA $\mathfrak{A}$ coincides with $\mathfrak{A}_K$.

Let E be a unique *conditional expectation* of M onto $\mathfrak{A}$ with respect to tr [15]. In particular, E is the orthogonal projection of the subspace L_0 onto the subspace

$$L_0^{\mathfrak{A}} = \left\{ x \in L^2(\mathfrak{A},\mathrm{tr}) :\ \mathrm{tr}(x) = 0 \right\}.$$

We claim that E belongs to the w^*-algebra generated by $\mathfrak{N}(\mathrm{Aut}\, M)$. To see this, consider a family $\{\Gamma_n\}$ of Abelian finite subgroups of $\mathrm{Aut}\, M$. Namely, Γ_n is generated by the inner automorphisms $\mathrm{Ad}\, u$, with the unitaries u belonging to the collection $\{{}^{r}e_{11} - {}^{r}e_{22}\}_{r=1}^{n}$. Since $\mathfrak{A}$ is a MASA in M, one has, in view of Proposition 2.1, that

$$\left(\{{}^{r}e_{11} - {}^{r}e_{22}\}_{r=1}^{\infty}\right)' = \mathfrak{A}. \tag{5}$$

Denote by E_n the orthogonal projection in $L^2(M,\mathrm{tr})$ determined by its values on the dense subset $M \subset L^2(M,\mathrm{tr})$

$$M \ni x \overset{E_n}{\mapsto} |\Gamma_n|^{-1} \sum_{\gamma \in \Gamma_n} \gamma(x). \tag{6}$$

Since $\Gamma_r \subset \Gamma_{r+1}$, inequality $E_r \geq E_{r+1}$ holds. Therefore, the sequence E_r converges in the strong operator topology. Let $\lim\limits_{r\to\infty} E_r = \widetilde{E}$. Hence, an application of (5) and (6) yields

$$\begin{aligned} &\widetilde{E}(x) \in \mathfrak{A}, \\ &\mathrm{tr}(\widetilde{E}(x)) = \mathrm{tr}(x) \qquad \text{for all } x \in M, \\ &\widetilde{E}(axb) = a\widetilde{E}(x)b \qquad \text{for all } a, b \in \mathfrak{A}, \quad x \in M. \end{aligned}$$

Therefore, $\widetilde{E}$ is the conditional expectation onto $\mathfrak{A}$. It follows that $\widetilde{E} = E$. Thus, in view of (6), E belongs to the w^*-algebra generated by $\mathfrak{N}(\mathrm{Inn}\, M)$.

Therefore, $$A'L_0^{\mathfrak{A}} \subset L_0^{\mathfrak{A}} \text{ for all } A' \in \left(\mathfrak{N}_I^0(\operatorname{Inn} M)\right)'.$$

The uniqueness of conditional expectation implies

$$\operatorname{Ad} u \circ E \circ \operatorname{Ad} u^* = E \text{ for all } u \in \mathcal{N}(\mathfrak{A}).$$

This is to be rephrased by claiming that the action of $\operatorname{Ad}\mathcal{N}(\mathfrak{A})$ leaves invariant $L_0^{\mathfrak{A}}$:

$$\operatorname{Ad} u\,(a) \in L_0^{\mathfrak{A}} \text{ for all } a \in L_0^{\mathfrak{A}}, \quad u \in \mathcal{N}(\mathfrak{A}). \tag{7}$$

Now to prove Theorem 1.1, it suffices to demonstrate the following:

(a) the action of $\mathcal{N}(\mathfrak{A})$, $u \mapsto \operatorname{Ad} u$, leaves no non-trivial closed subspace of $L_0^{\mathfrak{A}}$ invariant;

(b) the subspace $L_0^{\mathfrak{A}} \subset L_0$ is cyclic with respect to $\mathfrak{N}(\operatorname{Inn} M)$; i.e. the smallest closed subspace containing $\bigcup\limits_{\theta \in \operatorname{Inn} M} \mathfrak{N}(\theta) L_0^{\mathfrak{A}}$ is just L_0.

Let us start with proving (a). Consider an arbitrary unitary

$$u \in \{K_1, K_2, \ldots, K_n\}'',$$

to be expanded as

$$u = \sum_{j_1,k_1,j_2,k_2,\ldots,j_n,k_n=1}^{2} u_{j_1k_1\,j_2k_2\,\ldots\,j_nk_n}\ {}^1e_{j_1k_1}\,{}^2e_{j_2k_2}\,\ldots\,{}^ne_{j_nk_n},$$

where $u_{j_1k_1\,j_2k_2\,\ldots\,j_nk_n} \in \mathbb{C}$. Denote by $\mathfrak{S}_{2^n}$ the group of all bijections of the set $X_n = \{(i_1, i_2, \ldots, i_n),\ i_r \in \{1, 2\}\}$. Within our current argument, the symmetric group $\mathfrak{S}_{2^n}$ is about to be identified with the subgroup

$$\left\{u \in \{K_1, K_2, \ldots, K_n\}'' \cap U(M) : u_{j_1k_1\,j_2k_2\,\ldots\,j_nk_n} \in \{0, 1\}\right\} \subset \mathcal{N}(\mathfrak{A}),$$

in terms of the above expansion for $u \in \{K_1, K_2, \ldots, K_n\}''$. It is also convenient to denote by $\mathbf{i}_n$ the multi-index $(i_1, i_2, \ldots, i_n)$. Clearly, the collection of vectors $\{\mathfrak{e}_{\mathbf{i}_n} = {}^1e_{i_1i_1}\,{}^2e_{i_2i_2}\,\ldots\,{}^ne_{i_ni_n}\}$ forms an orthogonal basis of the subspace

$$\mathfrak{A}_n = \mathfrak{A} \cap \{K_1, K_2, \ldots, K_n\}''.$$

Let $\mathfrak{E}_n$ be the orthogonal projection of $L^2(\mathfrak{A}, \operatorname{tr})$ onto $\mathfrak{A}_n$, and consider a bounded operator $B' \in (\operatorname{Ad}\mathcal{N}(\mathfrak{A}))'$. It is clear that ${}^nB' \stackrel{\text{def}}{=} \mathfrak{E}_n B' \mathfrak{E}_n$ belongs to $(\operatorname{Ad}\mathfrak{S}_{2^n})'$ and

$$\lim_{n\to\infty} {}^nB' = B' \text{ in the strong operator topology.} \tag{8}$$

Hence, denoting the matrix element $\left({}^nB'\mathfrak{e}_{\mathbf{i}_n}, \mathfrak{e}_{\mathbf{j}_n}\right)$ by ${}^nB'_{\mathbf{i}_n\,\mathbf{j}_n}$, one has

$${}^nB'_{s(\mathbf{i}_n)\,s(\mathbf{j}_n)} = {}^nB'_{\mathbf{i}_n\,\mathbf{j}_n} \text{ for all } s \in \mathfrak{S}_{2^n}.$$

Therefore, there exist $\gamma, \delta \in \mathbb{C}$ such that

$${}^nB'_{\mathbf{i}_n\,\mathbf{j}_n} = \begin{cases} \gamma, & \text{if } \mathbf{i}_n \neq \mathbf{j}_n; \\ \delta, & \text{if } \mathbf{i}_n = \mathbf{j}_n. \end{cases}$$

It follows that ${}^nB'\eta = (\delta - \gamma)\eta$ for all $\eta \in L_0^{\mathfrak{A}} \cap \mathfrak{A}_n$.

Hence, applying (8), we obtain that $B'\eta = (\delta - \gamma)\eta$ for all $\eta \in L_0^{\mathfrak{A}}$. This proves (a).

We turn to proving (b). It suffices to demonstrate that, given a self-adjoint $B \in M$ and $\epsilon > 0$, there exist $A \in \mathfrak{A}$ and $U \in U(M)$ with the property

$$\|B - UAU^*\| < \epsilon, \text{ where } \|\cdot\| \text{ stands for the operator norm.} \tag{9}$$

Choose a positive integer $n > \frac{\|B\|}{\epsilon}$ and consider the set of reals

$$\Delta_l = \left\{ r \,\middle|\, \frac{2(l-1)\|B\|}{n} - \|B\| < r \leq \frac{2l\|B\|}{n} - \|B\| \right\}$$

for each $l = 0, 1, \ldots, n$. Let $E(\Delta_l)$ be the associated spectral projection related to the spectral decomposition of B. Under this setting, with

$$\alpha_l = \frac{(2l-1)\|B\|}{n} - \|B\|, \qquad B_n = \sum_{l=0}^{n} \alpha_l E(\Delta_l),$$

we conclude that

$$\|B - B_n\| \leq \epsilon. \tag{10}$$

One can readily find a family $(F_l)_{l=0}^n$ of pairwise orthogonal projections in $\mathfrak{A}$ such that $\operatorname{tr}(F_l) = \operatorname{tr}(E(\Delta_l))$. Thus we can also select partial isometries $u_l \in M$ with the properties $u_l u_l^* = E(\Delta_l)$ and $u_l^* u_l = F_l$ for all $l = 1, 2, \ldots, n$. It follows that $U = \sum_{l=0}^n u_l$ is a unitary operator, and with $A = \sum_{l=0}^n \alpha_l F_l$ the inequality (9) holds.

3. Proof of Theorem 1.2

Notice first that there exists a family $\{N_j\}_{j=1}^\infty$ of pairwise commuting type I_k subfactors $N_j \subset M$ generating M. Let $M_{jJ} = \left(\{N_l\}_{l=j}^J\right)''$. Fix a system of matrix units $\{{}^n e_{ij}\}_{i,j=1}^k \subset N_n$. Denote by $\mathfrak{A}$ an Abelian w^*-subalgebra generated by $\left\{{}^l e_{11}, {}^l e_{22}, \ldots, {}^l e_{kk}\right\}_{l=1}^\infty$. One can reproduce here the argument used at the beginning of Section 2 to demonstrate that $\mathfrak{A}$ is a Cartan MASA in M.

3.1. The conditional expectation from $M^{\otimes k}$ onto $\mathfrak{A}^{\otimes k}$

It is well known that there exists a unique conditional expectation ${}^k E$ from the II_1-factor $M^{\otimes k}$ onto the Cartan MASA $\mathfrak{A}^{\otimes k} \subset M^{\otimes k}$. Recall that ${}^k E$ is uniquely determined by the following properties (see [15]):

(1) ${}^k E$ is continuous with respect to the strong operator topology and ${}^k E\,\mathrm{I} = \mathrm{I}$;

(2) ${}^k E\,(a_1 m a_2) = a_1\, {}^k E(m) a_2$ for all $m \in M^{\otimes k}$ and $a_1, a_2 \in \mathfrak{A}^{\otimes k}$;

(3) $\operatorname{tr}^{\otimes k}({}^k E m) = \operatorname{tr}^{\otimes k}(m)$ for all $m \in M^{\otimes k}$.

We prove below that ${}^k E$ belongs to $\left(\mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M))\right)''$.

With $\mathbf{i}_J = (i_1, i_2, \ldots, i_J)$, let $\mathfrak{e}_{\mathbf{i}_J}$ stand for the minimal projection

$${}^1 e_{i_1 i_1}\, {}^2 e_{i_2 i_2} \cdots {}^J e_{i_J i_J}$$

of the algebra $M_{1J} \cap \mathfrak{A}$. Let ${}^n f$ be any embedding of the finite set

$$\mathfrak{I}_J = \{\mathbf{i}_J = (i_1, i_2, \ldots, i_J)\}_{i_1, i_2, \ldots, i_J = 1}^k$$

into $\{n+1, n+2, \ldots\}$; i.e. ${}^n f(\mathfrak{I}_J) \neq {}^n f(\mathfrak{I}'_J)$ for different $\mathfrak{I}_J$ and $\mathfrak{I}'_J$.

Set ${}^p u = {}^p e_{k1} + \sum\limits_{l=1}^{k-1} {}^p e_{l\,l+1} \in N_p$.

Lemma 3.1. *Consider the unitary* ${}^{J}U_n = \sum_{\mathbf{i}_J \in \mathfrak{I}_J} \mathfrak{e}_{\mathbf{i}_J} \cdot {}^{p}u$*, where* $p = {}^{n}f\left(\mathbf{i}_J\right)$ *and* $n > J$*. Then for any* $m \in M$ *the sequence* $\mathfrak{N}\left(\mathrm{Ad}\left({}^{J}U_n\right)\right) m$ *converges in the weak operator topology so that* $\lim_{n\to\infty} \mathfrak{N}\left(\mathrm{Ad}({}^{J}U_n)\right) m = E_J(m)$*, with*

$$E_J(m) = \sum_{\mathbf{i}_J \in \mathfrak{I}_J} \mathfrak{e}_{\mathbf{i}_J} \cdot m \cdot \mathfrak{e}_{\mathbf{i}_J} \in \mathfrak{A}' \cap M_{1J}. \tag{11}$$

In particular, E_J *belongs to the* w^**-algebra generated by* $\mathfrak{N}\left(\mathrm{Ad}\, U(M)\right)$*.*

Proof. Since the algebra $\bigcup_{Q=1}^{\infty} M_{1Q}$ is dense in M in the strong operator topology, one can assume without loss of generality that $m \in M_{1L}$, where $L > J$. Under this assumption, we have with $n > L$

$${}^{J}U_n \cdot m \cdot {}^{J}U_n^* = \sum_{\mathbf{i}_J, \mathbf{r}_J \in \mathfrak{I}_J} \mathfrak{e}_{\mathbf{i}_J} \cdot m \cdot \mathfrak{e}_{\mathbf{r}_J} \cdot {}^{p}u \cdot {}^{q}u^*,$$

where $p = {}^{n}f\left(\mathbf{i}_J\right)$, $q = {}^{n}f\left(\mathbf{r}_J\right)$. Note that with $\mathbf{i}_J \neq \mathbf{r}_J$ one has

$$\lim_{n\to\infty} {}^{p}u \cdot {}^{q}u^* = \mathrm{tr}\left({}^{p}u \cdot {}^{q}u^*\right) \mathrm{I} = 0$$

in the weak operator topology. Therefore, $\lim_{n\to\infty} {}^{J}U_n \cdot m \cdot {}^{J}U_n^* = E_J(m)$. ■

Remark 3.2. Clearly, E_J is an orthogonal projection in $L^2\left(M, \mathrm{tr}\right)$. Also, one readily observes that $E_J \geq E_{J+1}$ for all J. Hence for any $m \in L^2(M, \mathrm{tr})$ there exists

$$\lim_{J\to\infty} E_J(m) = E(m).$$

In particular,
$$E(m) = E_J(m) \text{ for all } m \in M_{1J}. \tag{12}$$

It is easy to verify that E is the unique *conditional expectation* of M onto $\mathfrak{A}$ with respect to tr [15]. On the other hand, (1)–(3) are valid also for the projection $E^{\otimes k}$. The uniqueness of conditional expectation now implies

$${}^{k}E\left(m_1 \otimes m_2 \otimes \cdots \otimes m_k\right) = E(m_1) \otimes E(m_2) \otimes \cdots \otimes E(m_k) \tag{13}$$

for all $m_1, m_2, \ldots, m_k \in M$.

Proposition 3.3. ${}^{k}E \in \left(\mathfrak{N}^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)''$.

Proof. Let $E_J^{\otimes k}(m_1 \otimes m_2 \otimes \cdots \otimes m_k) \stackrel{\mathrm{def}}{=} E_J(m_1) \otimes E_J(m_2) \otimes \cdots \otimes E_J(m_k)$. By Lemma 3.1,

$$E_J^{\otimes k} \in \left(\mathfrak{N}^{\otimes k}\left(\mathrm{Ad}\, U(M)\,\right)\right)''. \tag{14}$$

$E_J^{\otimes k}$ is an orthogonal projection in $L^2\left(M^{\otimes k}, \mathrm{tr}^{\otimes k}\right)$ and $E_J^{\otimes k} \geq E_L^{\otimes k}$ for all $L > J$. It follows that for any $m \in L^2\left(M^{\otimes k}, \mathrm{tr}^{\otimes k}\right)$ there exists

$$\lim_{J\to\infty} E_J^{\otimes k}(m) \stackrel{\mathrm{def}}{=} \widetilde{E}(m) \in M^{\otimes k} \cap \left(\mathfrak{A}^{\otimes k}\right)'.$$

Therefore, $\widetilde{E} \in \left(\mathfrak{N}^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)''$. An application of (11) allows one to verify that **1**) – **3**) are valid for $\widetilde{E}$. Since $\mathfrak{A}^{\otimes k}$ is a MASA in $M^{\otimes k}$, we conclude that $\widetilde{E}\left(M^{\otimes k}\right) = \mathfrak{A}^{\otimes k}$. Therefore, $\widetilde{E}$ is a conditional expectation from $M^{\otimes k}$ onto $\mathfrak{A}^{\otimes k}$, hence $\widetilde{E} = {}^{k}E = E^{\otimes k}$ by (13). ■

3.2. The operators ${}^{k}E \cdot \mathfrak{N}^{\otimes k}(u) \cdot {}^{k}E$ on $L^2\left(\mathfrak{A}^{\otimes k}, \mathrm{tr}^{\otimes k}\right)$

If we have $\mathbf{i}_J = (i_1, i_2, \ldots, i_J)$, $\mathbf{i}'_J = (i'_1, i'_2, \ldots, i'_J)$, we denote the partial isometry ${}^{1}e_{i_1 i'_1}\, {}^{2}e_{i_2 i'_2} \cdots {}^{J}e_{i_J i'_J} \in M_{1J}$ by $\mathfrak{e}_{\mathbf{i}_J \mathbf{i}'_J}$. Given a collection ${}^{l}x \in M_{1J}$, $1 \le l \le k$, we use below the expansion

$$ {}^{l}x = \sum_{\mathrm{i}_J, \mathrm{i}'_J \in \mathfrak{I}_J} {}^{l}c_{\mathbf{i}_J \mathbf{i}'_J} \mathfrak{e}_{\mathbf{i}_J \mathbf{i}'_J} \in M_{1J}, \text{ where } {}^{l}c_{\mathbf{i}_J \mathbf{i}'_J} \in \mathbb{C}. $$

In view of (13) one has

$$ {}^{k}E\left({}^{1}x \otimes {}^{2}x \otimes \cdots \otimes {}^{k}x\right) = E_J({}^{1}x) \otimes E_J({}^{2}x) \otimes \cdots \otimes E_J({}^{k}x) \tag{15} $$

$$ [1mm] = \Big(\sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{1}c_{\mathbf{i}_J \mathbf{i}_J}\, \mathfrak{e}_{\mathbf{i}_J \mathbf{i}_J}\Big) \otimes \Big(\sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{2}c_{\mathbf{i}_J \mathbf{i}_J}\, \mathfrak{e}_{\mathbf{i}_J \mathbf{i}_J}\Big) \otimes \cdots \otimes \Big(\sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{k}c_{\mathbf{i}_J \mathbf{i}_J}\, \mathfrak{e}_{\mathbf{i}_J \mathbf{i}_J}\Big). \tag{16} $$

Note that in Subsection 3.1 another notation $\mathfrak{e}_{\mathbf{i}_J}$ was used for $\mathfrak{e}_{\mathbf{i}_J \mathbf{i}_J}$.
Consider a unitary $u = \sum_{\mathrm{i}_J, \mathrm{i}'_J \in \mathfrak{I}_J} u_{\mathbf{i}_J \mathbf{i}'_J} \cdot \mathfrak{e}_{\mathbf{i}_J \mathbf{i}'_J} \in M_{1J}$ and a collection

$$ {}^{l}a = \sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{l}a_{\mathbf{i}_J} \cdot \mathfrak{e}_{\mathbf{i}_J \mathbf{i}_J} \in M_{1J} \cap \mathfrak{A},\ 1 \le l \le k, $$

where $u_{\mathbf{i}_J \mathbf{i}'_J}, {}^{l}a_{\mathbf{i}_J} \in \mathbb{C}$. Since

$$ \begin{aligned} &{}^{k}E\left(\mathfrak{N}^{\otimes k}(\mathrm{Ad}\, u)\left({}^{1}a \otimes {}^{2}a \otimes \cdots \otimes {}^{k}a\right)\right) \\ &= {}^{k}E\left(u \cdot {}^{1}a \cdot u^* \otimes u \cdot {}^{2}a \cdot u^* \otimes \cdots \otimes u \cdot {}^{k}a \cdot u^*\right), \end{aligned} $$

an application of (12) and (13) yields

$$ \begin{gathered} {}^{k}E\left(\mathfrak{N}^{\otimes k}(\mathrm{Ad}\, u)\right)\left({}^{1}a \otimes {}^{2}a \otimes \cdots \otimes {}^{k}a\right) = {}^{1}b \otimes {}^{2}b \otimes \cdots \otimes {}^{k}b, \text{ where} \\ {}^{l}b = \sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{l}b_{\mathrm{i}_J} \cdot \mathfrak{e}_{\mathrm{i}_J \mathrm{i}_J} \in M_{1J} \cap \mathfrak{A} \text{ and } {}^{l}b_{\mathrm{i}_J} = \sum_{\mathfrak{k}_J \in \mathfrak{I}_J} |u_{\mathrm{i}_J \mathfrak{k}_J}|^2 \cdot {}^{l}a_{\mathfrak{k}_J}. \end{gathered} \tag{17} $$

This way the map

$$ \mu : M_{1J} \cap U(M) \to M_{1J}; \qquad \sum_{\mathrm{i}_J, \mathrm{i}'_J \in \mathfrak{I}_J} u_{\mathbf{i}_J \mathbf{i}'_J} \cdot \mathfrak{e}_{\mathbf{i}_J \mathbf{i}'_J} \mapsto \sum_{\mathrm{i}_J, \mathrm{i}'_J \in \mathfrak{I}_J} \left|u_{\mathbf{i}_J \mathbf{i}'_J}\right|^2 \cdot \mathfrak{e}_{\mathbf{i}_J \mathbf{i}'_J}. $$

is introduced. It is to be studied and used in what follows.
Note that $|u_{\mathrm{i}_J \mathfrak{k}_J}|^2$ form a *doubly stochastic* matrix (see Section 6), hence

$$ \sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{l}a_{\mathbf{i}_J} = \sum_{\mathrm{i}_J \in \mathfrak{I}_J} {}^{l}b_{\mathbf{i}_J} \text{ for all } l. \tag{18} $$

3.2.1. Some properties of the map μ

Set $n = k^J$. To simplify the notation, it is custom (and really convenient) to identify $m = \sum_{\mathrm{i}_J, \mathrm{i}'_J \in \mathfrak{I}_J} m_{\mathbf{i}_J \mathbf{i}'_J} \cdot \mathfrak{e}_{\mathbf{i}_J \mathbf{i}'_J} \in M_{1J}$ with the associated matrice $\left[m_{\mathbf{i}_J \mathbf{i}'_J}\right]$. Let $M_{1J}(\mathbb{R})$ be the subset of real matrices in M_{1J}. Denote also by $GL(n, \mathbb{R})$ the subgroup of all invertible elements of $M_{1J}(\mathbb{R})$. A matrix $m = \left[m_{\mathbf{i}_J \mathbf{i}'_J}\right] \in M_{1J}$ is said to be *doubly stochastic* if its elements satisfy

$$ \begin{aligned} &m_{\mathbf{i}_J \mathbf{i}'_J} \ge 0 \text{ for all } \mathbf{i}_J\, \mathbf{i}'_J, \\ &\sum_{\mathbf{i}_J \in \mathfrak{I}_J} m_{\mathbf{i}_J \mathbf{i}'_J} = 1 \text{ for all } \mathbf{i}'_J \qquad \text{and} \qquad \sum_{\mathbf{i}'_J \in \mathfrak{I}_J} m_{\mathbf{i}_J \mathbf{i}'_J} = 1 \text{ for all } \mathbf{i}_J. \end{aligned} $$

The set of doubly stochastic matrices is a convex polytope known as Birkhoff's polytope [2]. Denote by $\mathcal{DS}_n$ this polytope. Set $p = \left[p_{\mathbf{i}_J\,\mathbf{i}'_J}\right]$, where $p_{\mathbf{i}_J\,\mathbf{i}'_J} = \frac{1}{n}$ for all $\mathbf{i}_J$, $\mathbf{i}'_J$. A routine verification demonstrates that p is a *minimal orthogonal projection* from M_{1J}. If $m = \left[m_{\mathbf{i}_J\,\mathbf{i}'_J}\right] \in \mathcal{DS}_n$ then

$$mp = pm = p \ \text{ and } \ m = p + (I-p)m(I-p). \tag{19}$$

A natural method of producing a doubly stochastic matrix is to start with a unitary matrix $u = [u_{\mathbf{i}_J\mathfrak{k}_J}]$ and then to set $\mu(u) = \left[|u_{\mathbf{i}_J\mathfrak{k}_J}|^2\right] \in \mathcal{DS}_n$. The matrices of the form $\mu(u)$ with u unitary are called *unistochastic*.

It is well known that for $n > 3$ there are doubly stochastic matrices that are not unistochastic [8].

Let the notation G stand for the set of those $g = \left[g_{\mathbf{i}_J\,\mathbf{i}'_J}\right] \in GL(n,\mathbb{R})$ which satisfy $\sum\limits_{\mathbf{i}_J\in\mathfrak{I}_J} g_{\mathbf{i}_J\,\mathbf{i}'_J} = 1$ for all $\mathbf{i}'_J \in \mathfrak{I}_J$ and $\sum\limits_{\mathbf{i}'_J\in\mathfrak{I}_J} g_{\mathbf{i}_J\,\mathbf{i}'_J} = 1$ for all $\mathbf{i}_J \in \mathfrak{I}_J$. The latter relations are obviously equivalent to the vector $\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$ being invariant under both g and the transpose g^t with respect to matrix multiplication, hence G is a subgroup. One can clearly reproduce (19) for $g \in G$:

$$g = p + (I-p)g(I-p). \tag{20}$$

Consider the one parameter family ${}^{\theta}U = \left[{}^{\theta}U_{\mathbf{i}_J\,\mathbf{i}'_J}\right]$ of unitary matrices, where

$${}^{\theta}U_{\mathbf{i}_J\,\mathbf{i}'_J} = \delta_{\mathbf{i}_J\,\mathbf{i}'_J} + \frac{\theta-1}{n}, \qquad \theta \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}. \tag{21}$$

Now we are in a position to apply the above idea of the present Section 3.2 in order to introduce the map $\mu : \operatorname{Inn} M \to \mathcal{DS}_n$ given by

$$\operatorname{Ad} U \mapsto \left[\left|U_{\mathbf{i}_J\,\mathbf{i}'_J}\right|^2\right], \text{ where } U = \left[U_{\mathbf{i}_J\,\mathbf{i}'_J}\right].$$

An easy calculation demonstrates that

$$\mu\left({}^{\theta}U\right) = p + \left(1 - \frac{|\theta-1|^2}{n}\right)(I-p). \tag{22}$$

We need below the following claim which is proved in Section 6.

Proposition 3.4. *With $\theta \in \mathbb{T} \setminus \{-1,1\}$ and $n > 4$, there exists an open neighborhood $\mathcal{U}$ of ${}^{\theta}U$ such that $\mu(\mathcal{U})$ is open in G.*

3.3. The commutant of ${}^kE \cdot \mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M)) \cdot {}^kE$

Let us start with observing that, in view of (13), ${}^kE\left(L_0^{\otimes k}\right) = \left(L_0^{\mathfrak{A}}\right)^{\otimes k}$. It follows that ${}^kE \cdot \mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M))\left(L_0^{\mathfrak{A}}\right)^{\otimes k} \subset \left(L_0^{\mathfrak{A}}\right)^{\otimes k}$. Thus we can view ${}^kE \cdot \mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M)) \cdot {}^kE$ as a family of operators on $\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$. Finally, let us restrict the representation ${}^k\mathcal{P}$ from 1 of $\mathfrak{S}_k$ to the subspace $\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$, to be denoted by ${}^k\mathcal{P}_0^{\mathfrak{A}}$. Let $\mathcal{N}_0$ be the w^*-algebra generated by the operators ${}^kE \cdot \mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M)) \cdot {}^kE$ in $\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$.

Proposition 3.5. *$\mathcal{N}_0$ coincides with* $\left({}^k\mathcal{P}_0^{\mathfrak{A}}(\mathfrak{S}_k)\right)'$.

We need an auxiliary

Lemma 3.6. *Let ${}^k\mathfrak{E}_J^p$ $(p < J)$ be the conditional expectation of $M^{\otimes k}$ onto the I_N-subfactor $M_{pJ}^{\otimes k} = \left(\left(\{N_l\}_{l=p}^J\right)''\right)^{\otimes k}$ with respect to $\mathrm{tr}^{\otimes k}$, where $L = k^{J-p+1}$. Then ${}^k\mathfrak{E}_J^p$ belongs to the w^*-algebra generated by $\mathfrak{N}^{\otimes k}(\mathrm{Ad}\,u)$ with u spanning the unitary group of w^*-algebra $\mathfrak{N}\{N_1 N_2 \cdots N_{p-1} N_{J+1} N_{J+2} \cdots\}''$.*

Proof. Notice first that

$$M_{pJ}' \cap M = \{N_1 N_2 \cdots N_{p-1} N_{J+1} N_{J+2} \cdots\}''. \tag{23}$$

Every $x \in M$ can be written in the form $x = \sum_{r,q=1}^N a_{rq}\, x_{rq}'$, where $a_{rq} \in M_{pJ}$, $x_{rq}' \in M_{pJ}'$. Set $\mathfrak{E}_J^p(x) = \sum_{r,q=1}^N \mathrm{tr}\left(x_{rq}'\right) a_{rq}$. The uniqueness of conditional expectations implies

$${}^k\mathfrak{E}_J^p\left({}^1x \otimes {}^2x \otimes \cdots \otimes {}^kx\right) = \mathfrak{E}_J^p({}^1x) \otimes \mathfrak{E}_J^p({}^2x) \otimes \cdots \otimes \mathfrak{E}_J^p({}^kx) \tag{24}$$

for any ${}^1x, {}^2x, \ldots, {}^kx \in M$. Let $\{j_l\}$ and $\{J_l\}$ be two increasing sequences of positive integers with the property

$$J_{l+1} - j_{l+1} > \max\{J_l, J\} \text{ for all } l. \tag{25}$$

By (23), there exists a sequence $\{U_l\}$ of unitaries from $M_{pJ}' \cap M$ such that

$$U_l \in M_{pJ}' \cap M_{1J_{l+1}} \text{ and } \mathrm{Ad}\,U_l\left(M_{pJ}' \cap M_{1J_l}\right) \subset M_{j_{l+1}\,J_{l+1}}. \tag{26}$$

Therefore, $\operatorname*{w-lim}_{n\to\infty} \mathrm{Ad}\,U_n(x) = \mathrm{tr}(x)I$ for each $x \in \bigcup_{r=1}^\infty M_{1r} \cap M_{pJ}'$, where $\operatorname*{w-lim}_{n\to\infty} x_n$ denotes the limit of the sequence $x_n \in M$ in the weak operator topology. Since $\bigcup_{r=1}^\infty M_{1r}$ is dense in M with respect to the strong operator topology, one has

$$\operatorname*{w-lim}_{n\to\infty} \mathrm{Ad}\,U_n(x) = \mathrm{tr}(x)I \text{ for each } x \in M_{pJ}' \cap M.$$

Now, in view of the above observations, with $x = \sum_{r,q=1}^L a_{pq}\, x_{rq}' \in M$, $a_{rq} \in M_{pJ}$, $x_{rq}' \in M_{pJ}' \cap M$, one establishes that

$$\operatorname*{w-lim}_{n\to\infty} \mathrm{Ad}\,U_n(x) = \sum_{r,q=1}^L \mathrm{tr}\left(x_{rq}'\right) a_{rq} = \mathfrak{E}_J^p(x) \in M_{pJ}.$$

Hence

$$\operatorname*{w-lim}_{n\to\infty} \mathfrak{N}^{\otimes k}(\mathrm{Ad}\,U_n)\left({}^1x \otimes {}^2x \otimes \cdots \otimes {}^kx\right) = \mathfrak{E}_J^p\left({}^1x\right) \otimes \mathfrak{E}_J^p\left({}^2x\right) \otimes \cdots \otimes \mathfrak{E}_J^p\left({}^kx\right).$$

Now combine the latter with (24) and (26) to establish the claim of Lemma 3.6. ∎

Proof of Proposition 3.5. Note first that the conditional expectations ${}^kE$ and ${}^k\mathfrak{E}_J^p$ commute and

$$\lim_{J\to\infty} {}^k\mathfrak{E}_J^1 \circ {}^kE = {}^kE. \tag{27}$$

To simplify the notation, we substitute below F_J for ${}^k\mathfrak{E}_J^1 \circ {}^kE$. The projection F_J is just the conditional expectation of $M^{\otimes k}$ onto $\mathfrak{A}^{\otimes k} \cap M_{1J}^{\otimes k}$ with respect to $\mathrm{tr}^{\otimes k}$.

Since ${}^kE\left(L_0^{\otimes k}\right)\subset\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$ and ${}^k\mathfrak{E}_J^1\left(L_0^{\otimes k}\right)=L_0^{\otimes k}\cap M_{1J}^{\otimes k}$, one deduces that

$$F_J\left(L_0^{\otimes k}\right)\subset M_{1J}^{\otimes k}\cap\left(L_0^{\mathfrak{A}}\right)^{\otimes k}=\left(M_{1J}\cap L_0^{\mathfrak{A}}\right)^{\otimes k}. \tag{28}$$

By Proposition 3.3 and Lemma 3.6,

$$F_J\in\left(\mathfrak{N}^{\otimes k}\left(\operatorname{Ad}U(M)\right)\right)''. \tag{29}$$

We are about to use the notation $T_J(u)$ for the operator $F_J\cdot\mathfrak{N}^{\otimes k}(\operatorname{Ad}u)\cdot F_J$. It follows from (28) that

$$T_J(u)\left(M_{1J}^{\otimes k}\cap\left(L_0^{\mathfrak{A}}\right)^{\otimes k}\right)\subset M_{1J}^{\otimes k}\cap\left(L_0^{\mathfrak{A}}\right)^{\otimes k}\quad\text{for each unitary } u\in M_{1J}. \tag{30}$$

The above observations imply that the action of $T_J(u)$ on $M_{1J}^{\otimes k}\cap\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$ is determined by (17).

Denote by $\mathfrak{L}$ an auxiliary representation of the general linear group $GL(n,\mathbb{R})$, with $n=k^J=|\mathfrak{I}_J|$, which coincides with the natural action of $GL(n,\mathbb{R})$ on the complex n-dimensional space $M_{1J}\cap\mathfrak{A}$; more precisely, with $g=\left[g_{\mathrm{i}_J\,\mathrm{i}'_J}\right]_{\mathrm{i}_J\,\mathrm{i}'_J\in\mathfrak{I}_J}\in GL(n,\mathbb{R})$ one has

$$\mathfrak{L}(g)\left(\sum_{\mathrm{i}_J\in\mathfrak{I}_J}a_{\mathrm{i}_J}\cdot\mathfrak{e}_{\mathrm{i}_J\,\mathrm{i}_J}\right)=\sum_{\mathrm{i}_J\in\mathfrak{I}_J}\sum_{\mathrm{i}'_J\in\mathfrak{I}_J}g_{\mathrm{i}_J\,\mathrm{i}'_J}\,a_{\mathrm{i}'_J}\cdot\mathfrak{e}_{\mathrm{i}_J\,\mathrm{i}_J}. \tag{31}$$

Let us introduce the subgroup ${}^IGL(n,\mathbb{R})$ formed by such $g\in GL(n,\mathbb{R})$ that $\mathfrak{L}(g)\mathrm{I}=\mathrm{I}$ and $\mathfrak{L}(g^t)\mathrm{I}=\mathrm{I}$, where the vector $\mathrm{I}=\sum_{\mathrm{i}_J\in\mathfrak{I}_J}\mathfrak{e}_{\mathrm{i}_J\,\mathrm{i}_J}$ is just the unit of the algebra $M_{1J}\cap\mathfrak{A}$, and the superscript t stands for passage to the transpose. Given a unitary $u=\sum\limits_{\mathrm{i}_J,\mathrm{i}'_J\in\mathfrak{I}_J}u_{\mathrm{i}_J\,\mathrm{i}'_J}\cdot\mathfrak{e}_{\mathrm{i}_J\,\mathrm{i}'_J}\in M_{1J}$, the matrix $\mu(u)=\left[\left|u_{\mathrm{i}_J\,\mathrm{i}'_J}\right|^2\right]$ is doubly stochastic. In the case $\mu(u)$ is also invertible one easily deduces from (31) that $\mu(u)\in{}^IGL(n,\mathbb{R})$, and in view of (17) one has

$$T_J(u)=\mathfrak{L}(\mu(u)). \tag{32}$$

${}^IGL(n,\mathbb{R})$ is the intersection of stationary subgroups of a vector I with respect to the left action $g\mapsto\mathfrak{L}(g)$ and to the right action $g\mapsto\mathfrak{L}(g^t)$ on $M_{1J}\cap\mathfrak{A}$. Hence it is isomorphic to $GL(n-1,\mathbb{R})$, and

$$\mathfrak{L}(g)\left(M_{1J}\cap L_0^{\mathfrak{A}}\right)=M_{1J}\cap L_0^{\mathfrak{A}}\quad\text{for all } g\in{}^IGL(n,\mathbb{R}). \tag{33}$$

By (32) and (33), the restrictions $T_J^0(u)$ and $\mathfrak{L}_0(g)$ of $T_J(u)$ and $\mathfrak{L}(g)$, respectively, to $M_{1J}\cap L_0^{\mathfrak{A}}$ are well defined. We are about to prove that

$$\left\{T_J^0(u),\ u\in M_{1J}\cap U(M)\right\}''=\left\{\mathfrak{L}_0^{\otimes k}\left({}^IGL(n,\mathbb{R})\right)\right\}''. \tag{34}$$

Once the latter relation is established, an application of the well known results of classical Schur-Weyl duality (see, for example, [?], Lecture 6) allows one to obtain

$$\left\{\mathfrak{L}_0^{\otimes k}\left({}^IGL(n,\mathbb{R})\right)\right\}''=\left\{F_J^0\ {}^k\mathcal{P}^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\ F_J^0\right\}',$$

and then to deduce that

$$\left\{T_J^0(u),\ u\in M_{1J}\cap U(M)\right\}''=\left\{F_J^0\ {}^k\mathcal{P}^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\ F_J^0\right\}', \tag{35}$$

where F_J^0 is the restriction of F_J to $L_0^{\otimes k}$ (see (28)).

Now we turn to proving (34).

In view of the inclusion $\left(\mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M))\right)'' \subset \left({}^{k}\mathcal{P}(\mathfrak{S}_k)\right)'$ and (29) one concludes that $F_J \in \left(\mathfrak{N}^{\otimes k}(\operatorname{Ad} U(M))\right)''$, and it follows that

$$F_J^0 \in \mathcal{N}_0 \subset \left({}^{k}\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\right)'. \tag{36}$$

This implies that for each J the operators $F_J^0 \ {}^{k}\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\ F_J^0$ determine a unitary representation of $\mathfrak{S}_k$.

In consequence of Proposition 3.4 there exists an open neighborhood $\mathcal{U} \in U(n)$ of ${}^{0}U$ such that $\mu(\mathcal{U})$ is an open subset in ${}^{I}GL(n,\mathbb{R}) \cong GL(n-1,\mathbb{R})$. Hence, an application of (32) yields

$$T_J^0(\mathcal{U}) = \mathfrak{L}_0^{\otimes k}(\mu(\mathcal{U})) \subset \left\{T_J^0(u)\middle|\ u \in M_{1J} \cap U(M)\right\}''.$$

Therefore, with $\mathcal{U}\cdot\mathcal{U}^{-1}$ being a neighborhood of the identity in $U(n)$,

$$\mathfrak{L}_0^{\otimes k}\left(\mu(\mathcal{U})\cdot\mu(\mathcal{U})^{-1}\right) \subset \left\{T_J^0(u)\middle|\ u \in M_{1J} \cap U(M)\right\}''. \tag{37}$$

Denote by ${}^{I}\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{gl}(n-1,\mathbb{R})$ the Lie algebras of ${}^{I}GL(n,\mathbb{R})$ and $GL(n-1,\mathbb{R})$, respectively.

A representation $\mathfrak{L}_0^{\otimes k}$ restricted to the neighborhood $\mu(\mathcal{U})\cdot\mu(\mathcal{U})^{-1}$ of unit in ${}^{I}GL(n,\mathbb{R}) \cong GL(n-1,\mathbb{R})$ determines a representation $\mathfrak{l}_0^{\otimes k}$ of the Lie algebra ${}^{I}\mathfrak{gl}(n,\mathbb{R}) \cong \mathfrak{gl}(n-1,\mathbb{R})$ in the $(n-1)^k$-dimensional vector space $M_{1J}^{\otimes k} \cap \left(L_0^{\mathfrak{A}}\right)^{\otimes k}$.

$$By (37), \mathfrak{l}_0^{\otimes k}\left({}^{I}\mathfrak{gl}(n,\mathbb{R})\right) \subset \left\{T_J^0(u),\ u \in M_{1J} \cap U(M)\right\}''.$$

This implies (34).

Consider a bounded operator $B' \in \mathcal{N}_0'$ together with its action on $\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$. It follows from (36) that $F_J^0 B' = B' F_J^0$. Therefore $B'_J \stackrel{\text{def}}{=} F_J^0 B' F_J^0$ belongs to

$$\left\{T_J^0(u)\middle|\ u \in M_{1J} \cap U(M)\right\}'.$$

Let R_λ, $\lambda \in \Upsilon_k$, be an irreducible representation of $\mathfrak{S}_k$ and χ_λ its character. Then the operator

$$P_0^\lambda = \frac{\dim\lambda}{k!}\sum_{s\in\mathfrak{S}_k}\chi_\lambda(s)\mathcal{P}_k^{\mathfrak{A}}(s)$$

is an orthogonal projection that belongs to the center of $\left({}^{k}\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\right)'$.

One can readily find such positive integer N that for all $J > N$ one has $F_J P_0^\lambda \neq 0$. Only such J are to be considered below.

It is clear that $P_0^\lambda \in \mathcal{N}_0'$. In view of (35),

$$\begin{aligned} &B'_J = \sum_{g\in\mathfrak{S}_k} c_J(g)\ F_J^0\ {}^{k}\mathcal{P}^{\mathfrak{A}}(g)\ F_J^0, \text{ where } c_J(g) \in \mathbb{C}, \text{ and} \\ &P_0^\lambda\, B'_J = B'_J\, P_0^\lambda \text{ for all sufficiently large } J. \end{aligned} \tag{38}$$

It also follows from (35) that $(F_J^0 \mathcal{N}_0 F_J^0)' = F_J^0 \left\{{}^{k}\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\right\}'' F_J^0$.

Hence, since P_0^λ is central in $\left({}^{k}\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\right)'$ and commutes with $F_J^0 \in \mathcal{N}_0$, one has

$$\left(P_0^\lambda F_J^0 \mathcal{N}_0\ F_J^0 P_0^\lambda\right)' = F_J^0 P_0^\lambda \left\{{}^{k}\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\right\}'' P_0^\lambda F_J^0.$$

Therefore, $\left(P_0^\lambda F_J^0 \mathcal{N}_0 P_0^\lambda F_J^0\right)'$ is a finite $\mathrm{I}_{\dim\lambda}$-factor for all J large enough. This implies that the map

$$F_{\widehat{J}}^0 P_0^\lambda \left\{ {}^k\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k) \right\}'' P_0^\lambda F_{\widehat{J}}^0 \ni A \mapsto F_J^0 A F_J^0 \in F_J^0 P_0^\lambda \left\{ {}^k\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k) \right\}'' P_0^\lambda F_J^0$$

is an isomorphism for $\widehat{J} > N$. Hence an application of (38) yields

$$P_0^\lambda B'_{\widehat{J}} = P_0^\lambda \sum_{g\in\mathfrak{S}_k} c_J(g)\ F_{\widehat{J}}^0\ {}^k\mathcal{P}^{\mathfrak{A}}(g)\ F_{\widehat{J}}^0.$$

Now, using (27), after the passage to the limit $\widehat{J} \to \infty$ we obtain

$$P_0^\lambda B' = P_0^\lambda \sum_{g\in\mathfrak{S}_k} c_J(g)\ {}^k\mathcal{P}^{\mathfrak{A}}(g) \text{ for all } \lambda \in \Upsilon_k.$$

Therefore, $B' = \sum_{g\in\mathfrak{S}_k} c_J(g)\ {}^k\mathcal{P}^{\mathfrak{A}}(g) \in \left({}^k\mathcal{P}^{\mathfrak{A}}(\mathfrak{S}_k)\right)''$, which completes the proof of Proposition 3.5. ■

3.4. The cyclicity of $\mathfrak{N}^{\otimes k}(\mathrm{Inn}\, M)\left(\left(L_0^{\mathfrak{A}}\right)^{\otimes k}\right)$ in $L_0^{\otimes k}$

Denote by $\mathcal{H}$ the closure of the linear span of $\mathfrak{N}^{\otimes k}(\mathrm{Inn}\, M)\left(\left(L_0^{\mathfrak{A}}\right)^{\otimes k}\right)$ in $L_0^{\otimes k}$. Our claim to be proved below is that $\mathcal{H}$ coincides with $L_0^{\otimes k}$.

Let us keep the notation $\{N_l\}_{l=1}^\infty$ introduced at the beginning of Section 3; let also $\{{}^n e_{ij}\}_{i,j=1}^k \subset N_n$ stand for the collection of matrix units of N_n. Denote by ${}^n p_1^s$, $s \in \mathfrak{S}_k$, the projection

$${}^k\mathcal{P}(s)\left({}^n e_{11} \otimes {}^n e_{22} \otimes \ldots \otimes {}^n e_{kk}\right) \in M^{\otimes k} \subset L^2(M^{\otimes k}, \mathrm{tr}^{\otimes k}).$$

Set ${}^n E_1 = \sum_{s\in\mathfrak{S}_k} {}^n p_1^s$ and ${}^n p_2^s = (\mathrm{I} - {}^n E_1) \cdot {}^{(n+1)} p_1^s$. Proceed with this construction by introducing ${}^n p_{i+1}^s = (\mathrm{I} - {}^n E_i) \cdot {}^{(n+i)} p_i^s$ and ${}^n E_{i+1} = {}^n E_i + \sum_{s\in\mathfrak{S}_k} {}^n p_{i+1}^s$. It is clear that the projections ${}^n p_m^s$ are pairwise orthogonal. Introduce

$${}^n E_m = \sum_{j=1}^m \sum_{s\in\mathfrak{S}_k} {}^n p_j^s,$$

and $\tau_i = \mathrm{tr}^{\otimes k}\left({}^n E_i\right)$, which is certainly an increasing sequence. One can readily compute that $\tau_{i+1} = \tau_i + (1-\tau_i)\frac{k!}{k^k}$, whence

$$\lim_{i\to\infty} \mathrm{tr}^{\otimes k}\left({}^n E_i\right) = 1.$$

This implies

$$\sum_{j=1}^\infty \sum_{s\in\mathfrak{S}_k} {}^n p_j^s = I. \tag{39}$$

due to the faithfulness of the trace $\mathrm{tr}^{\otimes k}$.

Lemma 3.7. *Let $A_1, A_2, \ldots, A_k$ be a family of selfadjoint operators in M_{1J}. Set $A = A_1 \otimes A_2 \otimes \cdots \otimes A_k$. Then for any pair of positive integers m, n with $n > J$, and any $s \in \mathfrak{S}_k$ there exists a unitary $U \in M$ such that $\mathrm{Ad}\, U\left(A\, {}^n p_m^s\right) \in \mathfrak{A}^{\otimes k}$.*

Proof. Note that

$$A \cdot {}^{n}p_m^s = \left(\mathrm{I} - {}^{n}E_{m-1}\right)\left(B_1 \otimes B_2 \otimes \cdots \otimes B_k\right), \text{ where} \quad B_i = A_i \cdot {}^{(n+m-1)}e_{s^{-1}(i)\, s^{-1}(i)}. \tag{40}$$

There exists a unitary $U_i \in M_{1J}$ such that

$$U_i\, A_i\, U_i^* \in \mathfrak{A} \cap M_{1j}. \tag{41}$$

Since $n > J$, the operator ${}^{n}U_m^s = \sum\limits_{i=1}^{k} U_i \cdot {}^{(n+m-1)}e_{s^{-1}(i)\, s^{-1}(i)}$ is unitary. By (40) and (41), $\mathfrak{N}^{\otimes k}(\mathrm{Ad}\, {}^{n}U_m^s)\,(A \cdot {}^{n}p_m^s) \in \mathfrak{A}^{\otimes k}$. ∎

Corollary 3.8. *Let A be the same as in Lemma 3.7. Then A belongs to the closed linear span of the collection of operators $\left\{\mathfrak{N}^{\otimes k}(\mathrm{Ad}\, u)\left(\mathfrak{A}^{\otimes k}\right)\right\}_{u \in U(M)}$ with respect to the norm topology of the space $L^2\left(M^{\otimes k}, \mathrm{tr}^{\otimes k}\right)$.*

Proof. One deduces from (39) that

$$A = \sum_{j=1}^{\infty} \sum_{s \in \mathfrak{S}_k} A \cdot {}^{n}p_j^s.$$

Hence, an application of Lemma 3.7 proves our claim. ∎

3.5. Proof of Theorem 1.2

Let $\mathfrak{A}$ be a Cartan MASA in M introduced at the beginning of Section 3. For convenience, we recall the notations used above:

$$L_0 = \left\{v \in L^2(M, \mathrm{tr}) : \mathrm{tr}(v) = 0\right\}, \quad L_0^{\mathfrak{A}} = \left\{x \in L^2\left(\mathfrak{A}, \mathrm{tr}\right) : \mathrm{tr}(x) = 0\right\}.$$

We denote by $\mathfrak{N}_0^{\otimes k}$ the restriction of $\mathfrak{N}^{\otimes k}$ to $L_0^{\otimes k}$. Conditional expectation ${}^{k}E$ introduced in section 3.1 is at the same time an orthogonal projection of $L^2(M^{\otimes k}, \mathrm{tr}^{\otimes k})$ onto $L^2\left(\mathfrak{A}^{\otimes k}, \mathrm{tr}^{\otimes k}\right)$ and

$${}^{k}E\, L_0^{\otimes k} = \left(L_0^{\mathfrak{A}}\right)^{\otimes k} \tag{42}$$

By Proposition 3.5,

$$\left({}^{k}E \cdot \mathfrak{N}_0^{\otimes k}\left(\mathrm{Ad}\, U(M)\right) \cdot {}^{k}E\right)' = \left({}^{k}\mathcal{P}_0^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\right)'', \tag{43}$$

where ${}^{k}\mathcal{P}_0^{\mathfrak{A}}$ is the restriction of the representation ${}^{k}\mathcal{P}$ (see (1)) to the subspace $\left(L_0^{\mathfrak{A}}\right)^{\otimes k}$. Take any operator $B' \in \left(\mathfrak{N}_0^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)'$. It follows from Proposition 3.3 that ${}^{k}E \in \left(\mathfrak{N}^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)''$. Hence, using (43), we have

$${}^{k}E \cdot B' \cdot {}^{k}E = B' \cdot {}^{k}E = {}^{k}E \cdot B' \in \left({}^{k}\mathcal{P}_0^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\right)''. \tag{44}$$

It follows from Corollary 3.8 that the maps

$$\left(\mathfrak{N}_0^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)' \ni X' \overset{\Theta}{\mapsto} {}^{k}E\, X' \in \left(\mathfrak{N}_0^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)' {}^{k}E,$$
$$\left({}^{k}\mathcal{P}_0^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\right)'' \ni X' \overset{\Phi}{\mapsto} {}^{k}E\, X' \in \left({}^{k}\mathcal{P}_0^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\right)''$$

are isomorphisms. Hence, using the equality

$$\left(\mathfrak{N}_0^{\otimes k}\left(\mathrm{Ad}\, U(M)\right)\right)' {}^{k}E \overset{(43)}{=} \left({}^{k}\mathcal{P}_0^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\right)'',$$

we get that $B' \in \left({}^{k}\mathcal{P}_0^{\mathfrak{A}}\left(\mathfrak{S}_k\right)\right)''$. Theorem 1.2 is proven. ∎

4. The Schur-Weyl duality for the automorphism group of a factor and the symmetric inverse semigroup

The symmetric inverse semigroup $\mathscr{I}_k$ is formed by all the partial *bijections* from the set $X_k = \{1, 2, \ldots, k\}$ to itself, with the natural definition of multiplication. An element $\mathbf{b} \in \mathscr{I}_m$ is conveniently written as $\mathbf{b} = \begin{pmatrix} i_1 & i_2 & \ldots & i_r \\ j_1 & j_2 & \ldots & j_r \end{pmatrix}$, where $\{i_1, i_2, \ldots, i_r\} \subset X_k$, $\{j_1, j_2, \ldots, j_r\} \subset X_k$ and i_l maps to j_l. The number r involved here is denoted by $\operatorname{rank} \mathbf{b}$. There exists a natural involution on $\mathscr{I}_k$: $\mathbf{b}^* = \begin{pmatrix} j_1 & j_2 & \cdots & j_r \\ i_1 & i_2 & \ldots & i_r \end{pmatrix}$. Denote by $\mathrm{id}_{\mathcal{A}} \in \mathscr{I}_m$ the partial bijection obtained by restricting the identity map to $\mathcal{A} \subset X_k$. We introduce also the abbreviation $\epsilon_j = \mathrm{id}_{(X_k \setminus \{j\})}$. The subcollection $\{\mathbf{b} \in \mathscr{I}_k : \operatorname{rank} \mathbf{b} = k\}$ is just the ordinary symmetric group $\mathfrak{S}_k$.

Let $\{s_i\}_{i=1}^{k-1}$ be the collection of Coxeter generators of $\mathfrak{S}_k$, where $s_i = (i\ i+1)$ is the transposition of i and $i+1$. The following claim is due to L. Popova [13]. A more up-to-date exposition of her results is given in [3].

Theorem 4.1. (A description of $\mathscr{I}_m$ in terms of the generators and the relations) *The semigroup $\mathscr{I}_k$ is generated by $\{s_i\}_{i=1}^{k-1}$ and ϵ_1 with the relations as follows:*

(a) *the Coxeter relations for $\{s_i\}_{i=1}^{k-1}$;*

(b) $s_i\, \epsilon_1 = \epsilon_1\, s_i$ *for all* $i > 1$*;*

(c) $(s_1\, \epsilon_1)^2 = (\epsilon_1\, s_1)^2 = \epsilon_1\, s_1\, \epsilon_1$.

This implies that one can realize $\mathscr{I}_k$ as a semigroup of $\{0,1\}$-matrices $a = [a_{ij}]$ with the ordinary matrix multiplication in such a way that a has at most one nonzero entry in each row and each column. The matrix $a = [a_{ij}]$, where $a_{11} = 0$ and $a_{ij} = \delta_{ij}$, if $i \neq 1$ or $j \neq 1$, corresponds to ϵ_1 under this realization.

Let $\mathbb{C}\,[\mathfrak{S}_k]$ be the complex group algebra of the symmetric group $\mathfrak{S}_k$. This algebra as well as the group algebra of every finite group, is semisimple. The complex semigroup algebra $\mathbb{C}\,[\mathscr{I}_k]$ of the inverse symmetric semigroup is semisimple too. Namely, Munn proved the next statement.

Theorem 4.2. [10] *The algebra $\mathbb{C}\,[\mathscr{I}_k]$ has the decomposition*

$$\mathbb{C}\,[R_k] = \bigoplus_{l=0}^{k} \mathbb{M}_{\binom{k}{l}}\left(\mathbb{C}\,[\mathfrak{S}_l]\right),$$

where $\mathbb{M}_j(A)$ is the algebra of all $j \times j$-matrices over an algebra A.

Recall that we denote by Υ_m the set of all unordered partitions of a positive integer $m \leq k$. It follows from previous theorem that the set of the irreducible representations of the semigroup $\mathscr{I}_k$ can be naturally indexed by the set $\bigcup_{m=0}^{k} \Upsilon_m$.

4.1. The action of $\mathscr{I}_k$ on $L^2\left(M^{\otimes k}, \mathrm{tr}^{\otimes k}\right)$

Set ${}^{k}\mathcal{P}^{\mathscr{I}}(s) = {}^{k}\mathcal{P}(s)$ with $s \in \mathfrak{S}_k$, see (1). Theorem 4.1 implies that the operators ${}^{k}\mathcal{P}^{\mathscr{I}}(s)$ and ${}^{k}\mathcal{P}^{\mathscr{I}}(\epsilon_i)$ from (3) define the representation of $\mathscr{I}_k$. One has the following obvious result:

Proposition 4.3. $\left(\mathfrak{N}^{\otimes k}\,(\operatorname{Aut} M)\right)'' \subset \left({}^{k}\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)\right)'$.

Below we prove Theorem 1.4, which is the analogue of the Schur-Weyl duality for $\operatorname{Aut} M$ and $\mathscr{I}_k$.

Theorem 4.4. $\left(\mathfrak{N}^{\otimes k}(\operatorname{Aut} M)\right)'' = \left({}^k\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)\right)'.$

Remark 4.5. The operator ${}^k\mathcal{P}^{\mathscr{I}}(\epsilon_i)$ is an orthogonal projection in $L^2(M,\operatorname{tr})^{\otimes k}$ and

$$\begin{aligned}&\prod_{i=1}^{k}\left(\mathrm{I}-{}^k\mathcal{P}^{\mathscr{I}}(\epsilon_i)\right)L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)\\&=\left\{v\in L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right):\ {}^k\mathcal{P}^{\mathscr{I}}(\epsilon_i)v=0\ \text{ for all }\ i=1,2,\ldots,k\right\}=L_0^{\otimes k}.\end{aligned}$$

Let $\wp_m(X_k)$ be the collection[1] of all non-ordered m-element subsets of X_k. With $\mathcal{A}\in\wp_m(X_k)$, let us introduce the pairwise orthogonal projections ${}^kP_{\mathcal{A}}$ as follows

$${}^kP_{\mathcal{A}}=\prod_{j\in X_k\setminus\mathcal{A}}{}^k\mathcal{P}^{\mathscr{I}}(\epsilon_j)\cdot\prod_{j\in\mathcal{A}}\left(\mathrm{I}-{}^k\mathcal{P}^{\mathscr{I}}(\epsilon_j)\right).$$

Hence

$$\begin{aligned}&{}^k\mathcal{P}^{\mathscr{I}}(\epsilon_j)\,{}^kP_{\mathcal{A}}=0 &&\text{for all } j\in\mathcal{A},\\&{}^k\mathcal{P}^{\mathscr{I}}(\epsilon_j)\,{}^kP_{\mathcal{A}}={}^kP_{\mathcal{A}} &&\text{for all } j\in X_k\setminus\mathcal{A}.\end{aligned}\tag{45}$$

Since the projections ${}^kP_{\mathcal{A}}$ and ${}^kP_{\mathcal{B}}$ are orthogonal for different $\mathcal{A}$ and $\mathcal{B}$, then operator ${}^kP_m=\sum_{\mathcal{A}\in\wp_m(X_k)}{}^kP_{\mathcal{A}}$ is an orthogonal projection. It is clear that

$${}^kP_k\,L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)=L_0^{\otimes k},\quad {}^kP_k\,L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)=\mathbb{C}\mathrm{I}^{\otimes k}$$

and

$$\sum_{m=0}^{k}{}^kP_m\,L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)=L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right).$$

Let $m\le k$ and $\mathfrak{S}_m=\{s\in\mathfrak{S}_k : s(j)=j \text{ for all } j\in X_k\setminus X_m\}$, where we define $X_m=\{1,2,\ldots,m\}\subset X_k$. Denote by χ_γ the character of the irreducible representation T_γ of $\mathfrak{S}_m$, corresponding to $\gamma\in\Upsilon_m$, such that its value on the unit is equal to the dimension of T_γ. Then Young projection

$$P^\gamma=\frac{\dim\gamma}{m!}\sum_{s\in\mathfrak{S}_m}\chi_\gamma(s)\ {}^k\mathcal{P}^{\mathscr{I}}(s)$$

lies in the center of the $*$-algebra generated by ${}^k\mathcal{P}^{\mathscr{I}}(\mathfrak{S}_m)$. Since ${}^kP_{X_m}$ belongs to ${}^k\mathcal{P}^{\mathscr{I}}(\mathfrak{S}_m)'$, then ${}^kP^\gamma_{X_m}={}^kP_{X_m}\cdot P^\gamma$ is an orthogonal projection from ${}^k\mathcal{P}^{\mathscr{I}}(\mathfrak{S}_m)'$. Denote by ${}^k\mathcal{H}^\gamma_m$ the closure of the linear span of the set

$$\left\{{}^k\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)\,{}^kP^\gamma_{X_m}\,L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)\right\}$$

with respect to the norm topology of the space $L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)$. By Proposition 4.3, the ${}^k\mathcal{P}^{\mathscr{I}}$-invariant subspace ${}^k\mathcal{H}^\gamma_m$ is $\mathfrak{N}^{\otimes k}(\operatorname{Aut} M)$-invariant too.

4.2. Decomposing $\mathfrak{N}^{\otimes k}$ into factor-components

Set ${}^k\mathcal{H}_{X_m}={}^kP_{X_m}L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)$. By Proposition 4.3, ${}^k\mathcal{H}_{X_m}$ is $\mathfrak{N}^{\otimes k}$-invariant. Let $\mathfrak{N}^{\otimes k}_{X_m}$ be the restriction of $\mathfrak{N}^{\otimes k}$ to ${}^k\mathcal{H}_{X_m}$. Here $m\le k$ and we consider $X_m=\{1,2\ldots,m\}$ as a subset of X_k.

[1] $\wp_0(X_k)$ is the unique empty subset.

Clearly, ${}^{k}\mathcal{H}_{X_m}$ is invariant under the operators ${}^{k}\mathcal{P}(s)$, where $s \in \mathfrak{S}_m \subset \mathfrak{S}_k$, and, more generally,

$$ {}^{k}\mathcal{P}(s) \cdot {}^{k}P_{\mathcal{A}} \cdot {}^{k}\mathcal{P}(s^{-1}) = {}^{k}P_{s(\mathcal{A})} \text{ for all } s \in \mathfrak{S}_k \text{ and } \mathcal{A} \in \wp_m(X_k). \tag{46} $$

Consider the Young subgroup $\mathfrak{S}_{m\,(k-m)} = \{s \in \mathfrak{S}_k : sX_m = X_m\}$. Let s_1, s_2, …, s_r be a full set of the representatives in $\mathfrak{S}_k$ of the left cosets $\mathfrak{S}_k \diagup \mathfrak{S}_{m\,(k-m)}$, where $r = |\mathfrak{S}_k \diagup \mathfrak{S}_{m\,(k-m)}|$. Then the projections ${}^{k}P_{s_j(X_m)}$ are pairwise orthogonal and

$$ {}^{k}P_m = \sum_{j=1}^{r} {}^{k}P_{s_j(X_m)}. \tag{47} $$

By (45),
$$ \mathfrak{N}^{\otimes k}(\theta)\ {}^{k}\mathcal{P}^{\mathscr{I}}(s)\ {}^{k}P_m = {}^{k}P_m\, \mathfrak{N}^{\otimes k}(\theta)\ {}^{k}\mathcal{P}^{\mathscr{I}}(s) \tag{48} $$

for all $\theta \in \operatorname{Aut} M$ and $s \in \mathscr{I}_k$. We emphasize again that ${}^{k}P_{X_m}\ {}^{k}\mathcal{P}^{\mathscr{I}}(\epsilon_j) = 0$ for all $j \in X_m$. Therefore,

$$ \left({}^{k}P_{X_m}\ {}^{k}\mathcal{P}^{\mathscr{I}}(\mathscr{I}_m)\right)'' = \left({}^{k}P_{X_m}\ {}^{k}\mathcal{P}^{\mathscr{I}}(\mathfrak{S}_m)\right)''. \tag{49} $$

Let $\gamma \in \Upsilon_m$ be an unordered partition of m and let χ_γ be the character of the corresponding irreducible representation of $\mathfrak{S}_m$. Set

$$ P^{\gamma} = \frac{\dim \gamma}{m!} \sum_{s \in \mathfrak{S}_m} \chi_\gamma(s)\ {}^{k}\mathcal{P}^{\mathscr{I}}(s). \tag{50} $$

Since the projections $\left\{{}^{k}P_{s_j(X_m)}\right\}_{j=1}^{r}$ are pairwise orthogonal and

$$ {}^{k}P_{X_m} \in \left({}^{k}\mathcal{P}^{\mathscr{I}}(\mathfrak{S}_m)\right)' \text{ then } {}^{k}P^{\gamma}_{X_m} = P^{\gamma} \cdot {}^{k}P_{X_m} $$

is an orthogonal projection from the center of w^*-algebra, generated by the operators ${}^{k}P_{X_m}\mathfrak{N}^{\otimes k}(\operatorname{Aut} M)$ and ${}^{k}P_{X_m} \cdot {}^{k}\mathcal{P}^{\mathscr{I}}(\mathfrak{S}_m)$. Therefore, the operator

$$ {}^{k}P^{\gamma}_{m} = \sum_{j=1}^{r} {}^{k}\mathcal{P}(s_j) \cdot {}^{k}P^{\gamma}_{X_m} \cdot {}^{k}\mathcal{P}(s_j^{-1}) \tag{51} $$

is an orthogonal projection too. Moreover, the projections ${}^{k}P^{\gamma}_{m}$ and ${}^{k}P^{\widetilde{\gamma}}_{m}$ are orthogonal for different $\gamma, \widetilde{\gamma} \in \Upsilon_m$ and the following equality holds

$$ {}^{k}P_m = \sum_{\gamma \in \Upsilon_m} {}^{k}P^{\gamma}_{m}. \tag{52} $$

The next statement follows from Theorem 1.2.

Lemma 4.6. *The family of the operators* $\left\{{}^{k}P_{X_m}\ {}^{k}\mathcal{P}^{\mathscr{I}}(s)\ {}^{k}P_{X_m}\right\}_{s \in \mathfrak{S}_m}$ *define the unitary representation* ${}^{k}\mathcal{P}^{\mathscr{I}}_{X_m}$ *of the group* $\mathfrak{S}_m$ *in the subspace* ${}^{k}\mathcal{H}_{X_m}$ *and one has* $\left(\mathfrak{N}^{\otimes k}_{X_m}(\operatorname{Aut} M)\right)'' = \left({}^{k}\mathcal{P}^{\mathscr{I}}_{X_m}(\mathfrak{S}_m)\right)'$.

Define the representation ${}^{k}\Pi$ of the semigroup $(\operatorname{Aut} M) \times \mathscr{I}_k$ as follows

$$ {}^{k}\Pi(\theta, s) = \mathfrak{N}^{\otimes k}(\theta) \cdot {}^{k}\mathcal{P}^{\mathscr{I}}(s), \text{ where } \theta \in \operatorname{Aut} M, s \in \mathscr{I}_k. \tag{53} $$

Lemma 4.7. *The projection* ${}^{k}P_m^{\gamma}$ *belongs to* w^**-algebra* $\left({}^{k}\Pi\left((\operatorname{Aut} M)\times\mathscr{I}_k\right)\right)'$ *and the restriction of* ${}^{k}\Pi$ *to the subspace* ${}^{k}P_m^{\gamma}\,L^2\left(M^{\otimes k},\operatorname{tr}^{\otimes k}\right)$ *is the irreducible representation of the semigroup* $(\operatorname{Aut} M)\times\mathscr{I}_k$*.*

Proof. Let us prove that

$$ {}^{k}P_m^{\gamma}\in\left({}^{k}\Pi\left((\operatorname{Aut} M)\times\mathscr{I}_k\right)\right)' \qquad \text{(see (51)).} \tag{54}$$

Each $t\in\mathfrak{S}_k$ defines a bijection $\mathfrak{b}_t$ of the set $\{s_1,s_2,\ldots,s_r\}$, where $r=|\mathfrak{S}_k\diagup\mathfrak{S}_{m\,(k-m)}|$, as follows

$$\mathfrak{b}_t(s_j)=s_{j_t},\ \text{ where } ts_j\in s_{j_t}\mathfrak{S}_{m\,(k-m)}.$$

Hence, since ${}^{k}P_m^{\gamma}=\sum\limits_{j=1}^{|\mathfrak{S}_k\diagup\mathfrak{S}_{m\,(k-m)}|}{}^{k}\mathcal{P}(s_j)\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}(s_j^{-1})$, then

$$ {}^{k}\mathcal{P}(t)\cdot{}^{k}P_m^{\gamma}\cdot{}^{k}\mathcal{P}(t^{-1})=\sum_{j=1}^{|\mathfrak{S}_k\diagup\mathfrak{S}_{m\,(k-m)}|}{}^{k}\mathcal{P}(ts_j)\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}(s_j^{-1}t^{-1})$$

$$=\sum_{j=1}^{|\mathfrak{S}_k\diagup\mathfrak{S}_{m\,(k-m)}|}{}^{k}\mathcal{P}(\mathfrak{b}_t(s_j)\,h_j)\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}\left(h_j^{-1}\,(\mathfrak{b}_t(s_j))^{-1}\right),\ \text{ where } h_j\in\mathfrak{S}_m.$$

Now, using the equality ${}^{k}\mathcal{P}(h_j)\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}\left(h_j^{-1}\right)={}^{k}P_{X_m}^{\gamma}$, we obtain

$$ {}^{k}\mathcal{P}(t)\cdot{}^{k}P_m^{\gamma}\cdot{}^{k}\mathcal{P}(t^{-1})=\sum_{j=1}^{|\mathfrak{S}_k\diagup\mathfrak{S}_m|}{}^{k}\mathcal{P}(\mathfrak{b}_t(s_j))\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}\left((\mathfrak{b}_t(s_j))^{-1}\right).$$

Since $\mathfrak{b}_t$ is the bijection, then

$$\sum_{j=1}^{|\mathfrak{S}_k\diagup\mathfrak{S}_m|}{}^{k}\mathcal{P}(\mathfrak{b}_t(s_j))\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}\left((\mathfrak{b}_t(s_j))^{-1}\right)$$
$$=\sum_{j=1}^{|\mathfrak{S}_k\diagup\mathfrak{S}_{m\,(k-m)}|}{}^{k}\mathcal{P}(s_j)\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}\left(s_j^{-1}\right).$$

Thus
$$ {}^{k}\mathcal{P}(t)\cdot{}^{k}P_m^{\gamma}\cdot{}^{k}\mathcal{P}(t^{-1})={}^{k}P_m^{\gamma}\ \text{for all } t\in\mathfrak{S}_k. \tag{55}$$

Set $\mathcal{A}_i=\left\{j\in\{1,2,\ldots,|\mathfrak{S}_k\diagup\mathfrak{S}_{m\,(k-m)}|\}:s_j^{-1}(i)\notin X_m\right\}$.

Since ${}^{k}P_{X_m}^{\gamma}=P^{\gamma}\cdot{}^{k}P_{X_m}={}^{k}P_{X_m}\cdot P^{\gamma}$, then, using (45) and (46), we have

$$ {}^{k}\mathcal{P}^{\mathscr{I}}(\epsilon_i)\cdot{}^{k}P_m^{\gamma}={}^{k}P_m^{\gamma}\cdot{}^{k}\mathcal{P}^{\mathscr{I}}(\epsilon_i)=\sum_{j\in\mathcal{A}_i}{}^{k}\mathcal{P}(s_j)\cdot{}^{k}P_{X_m}^{\gamma}\cdot{}^{k}\mathcal{P}\left(s_j^{-1}\right).$$

Now we conclude from (55) that ${}^{k}P_m^{\gamma}\in{}^{k}\mathcal{P}^{\mathscr{I}}\left(\mathscr{I}_k\right)'$. Hence, applying Proposition 4.3, we obtain (54).

Therefore, the operators ${}^{k}\Pi_m^{\gamma}(\theta,s)={}^{k}P_m^{\gamma}\cdot{}^{k}\Pi(\theta,s)$, where $\theta\in\operatorname{Aut} M$, $s\in\mathscr{I}_k$, define a $*$-representation of semigroup $\operatorname{Aut} M\times\mathscr{I}_k$.

Let us prove that ${}^{k}\Pi_m^{\gamma}$ is an irreducible representation; i.e.

$$ {}^{k}\Pi_m^{\gamma}\left(\operatorname{Aut} M\times\mathscr{I}_k\right)'=\mathbb{C}\cdot{}^{k}P_m^{\gamma}.$$

First, we notice that ${}^kP^{\gamma}_{X_m} \in {}^kP^{\gamma}_m \cdot {}^k\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)'' \subset {}^k\Pi^{\gamma}_m(\operatorname{Aut} M \times \mathscr{I}_k)''$. Therefore, if $B' \in {}^k\Pi^{\gamma}_m(\operatorname{Aut} M \times \mathscr{I}_k)'$ then

$$B' \cdot {}^kP^{\gamma}_{X_m} \in {}^kP^{\gamma}_{X_m} \cdot {}^k\Pi^{\gamma}_m(\operatorname{Aut} M \times \mathscr{I}_k)' \cdot {}^kP^{\gamma}_{X_m}.$$

Hence, applying Lemma 4.6, we see that

$$B' \cdot {}^kP^{\gamma}_{X_m} = c \cdot {}^kP^{\gamma}_{X_m}, \text{ where } c \in \mathbb{C}.$$

Now, using (51), we obtain $B' = B' \cdot {}^kP^{\gamma}_m = c \cdot {}^kP^{\gamma}_m$. ∎

4.3. The proof of Theorem 4.4

Take any B' from $\left(\mathfrak{N}^{\otimes k}(\operatorname{Aut} M)\right)'$. For the matrix ${}^{\theta}U = \left[{}^{\theta}U_{\mathbf{i}_J\,\mathbf{i}'_J}\right]$ (see (21)), we denote by ${}^{\theta}\mathbf{U}$ an element from M_{1J} of the form

$${}^{\theta}\mathbf{U} = \sum_{\mathrm{i}_J,\mathrm{i}'_J \in \mathfrak{I}_J} {}^{\theta}U_{\mathbf{i}_J\,\mathbf{i}'_J} \cdot \mathfrak{e}_{\mathbf{i}_J\,\mathbf{i}'_J}.$$

Let $a \in M_{1J} \cap \mathfrak{A}$. Using (17) and (22), we obtain

$${}^kE \circ \mathfrak{N}^{\otimes k}(\operatorname{Ad}{}^{\theta}\mathbf{U})({}^kP_m(a)) = \left(1 - \frac{|\theta-1|^2}{n}\right)^m {}^kP_m(a).$$

It follows that

$$\begin{aligned} &{}^kE \circ \mathfrak{N}^{\otimes k}(\operatorname{Ad}{}^{\theta}\mathbf{U}) \circ {}^kE \\ &= \sum_{j=0}^{k} \left(1 - \frac{|\theta-1|^2}{n}\right)^j {}^kE \circ {}^kP_j \in \left(\mathfrak{N}^{\otimes k}(\operatorname{Aut} M)\right)''. \end{aligned}$$

Therefore,

$$\sum_{j=0}^{k} \left(1 - \frac{|\theta-1|^2}{n}\right)^j B' \circ {}^kE \circ {}^kP_j = \sum_{j=0}^{k} \left(1 - \frac{|\theta-1|^2}{n}\right)^j {}^kE \circ {}^kP_j \circ B'.$$

Hence, thanks to the relation ${}^kP_l \circ {}^kP_m = \delta_{ml}\,{}^kP_l$, we have

$$\begin{aligned} &\left(1 - \frac{|\theta-1|^2}{n}\right)^m {}^kP_l \circ B' \circ {}^kE \circ {}^kP_m \\ &= \sum_{j=0}^{k} \left(1 - \frac{|\theta-1|^2}{n}\right)^j {}^kP_l \circ {}^kE \circ {}^kP_j \circ B' \circ {}^kP_m. \end{aligned}$$

Now we conclude from Propositions 3.3 and 4.3 that

$$\left(1 - \frac{|\theta-1|^2}{n}\right)^m {}^kP_l \circ B' \circ {}^kE \circ {}^kP_m = \left(1 - \frac{|\theta-1|^2}{n}\right)^l {}^kP_l \circ {}^kE \circ B' \circ {}^kP_m$$

and

$$\left(1 - \frac{|\theta-1|^2}{n}\right)^m {}^kP_l \circ B' \circ {}^kE \circ {}^kP_m = \left(1 - \frac{|\theta-1|^2}{n}\right)^l {}^kP_l \circ B' \circ {}^kE \circ {}^kP_m.$$

Therefore, ${}^{k}P_l \circ B' \circ {}^{k}E \circ {}^{k}P_m = \delta_{lm}\, {}^{k}P_m \circ B' \circ {}^{k}E \circ {}^{k}P_m$. Now, using the relation $\sum\limits_{j=0}^{k} {}^{k}P_j = \mathrm{I}$, we have

$$B' \circ {}^{k}E = {}^{k}E \circ B' = \sum_{m=0}^{k} {}^{k}P_m \circ B' \circ {}^{k}E \circ {}^{k}P_m.$$

Hence, applying Corollary 3.8, we conclude

$$B' = \sum_{m=0}^{k} {}^{k}P_m \circ B' \circ {}^{k}P_m. \tag{56}$$

Let us prove that $B'_m \stackrel{\text{def}}{=} {}^{k}P_m \circ B' \circ {}^{k}P_m$ lies in $*$-algebra ${}^{k}P_m\, {}^{k}\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)''\, {}^{k}P_m$ (see (52) and Lemma 4.6).

Since ${}^{k}P_m = \sum\limits_{\mathcal{A}\in\wp_m(X_k)} {}^{k}P_{\mathcal{A}}$, then $B'_m = \sum\limits_{\mathcal{A},\mathcal{B}\in\wp_m(X_k)} {}^{k}P_{\mathcal{A}} \circ B'_m \circ {}^{k}P_{\mathcal{B}}$. There exist $s_{\mathcal{A}}$, $s_{\mathcal{B}} \in \mathfrak{S}_k$ such that

$$s_{\mathcal{A}}(X_m) = \mathcal{A} \text{ and } s_{\mathcal{B}}(X_m) = \mathcal{B}. \tag{57}$$

Hence, using (46), we have

$${}^{k}P_{\mathcal{A}} \circ B'_m \circ {}^{k}P_{\mathcal{B}} = {}^{k}\mathcal{P}(s_{\mathcal{A}}) \circ {}^{k}P_{X_m} \circ {}^{k}\mathcal{P}(s_{\mathcal{A}}^{-1}) \circ B'_m \circ {}^{k}\mathcal{P}(s_{\mathcal{B}}) \circ {}^{k}P_{X_m} \circ {}^{k}\mathcal{P}(s_{\mathcal{B}}^{-1}).$$

It follows from Lemma 4.6 that ${}^{k}P_{X_m} \circ {}^{k}\mathcal{P}(s_{\mathcal{A}}^{-1}) \circ B'_m \circ {}^{k}\mathcal{P}(s_{\mathcal{B}}) \circ {}^{k}P_{X_m}$ lies in the algebra ${}^{k}P_{X_m} \circ {}^{k}\mathcal{P}(\mathfrak{S}_m)'' \circ {}^{k}P_{X_m}$. Therefore,

$${}^{k}P_{\mathcal{A}} \circ B'_m \circ {}^{k}P_{\mathcal{B}} \in \left({}^{k}\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)\right)''.$$

Thus $B' = \sum\limits_{m=0}^{k} \sum\limits_{\mathcal{A},\mathcal{B}\in\wp_m(X_k)} {}^{k}P_{\mathcal{A}} \circ B'_m \circ {}^{k}P_{\mathcal{B}}$ lies in $\left({}^{k}\mathcal{P}^{\mathscr{I}}(\mathscr{I}_k)\right)''$. This completes the proof of Theorem 4.4.

5. The Schur-Weyl duality for Aut M and the infinite symmetric group

Let $\overline{\mathfrak{S}}_\infty$ be the group of all bijections of the set $\mathbb{Z}_{>0} = \{1, 2, \ldots\}$.

Set $\mathfrak{S}_n = \left\{s \in \overline{\mathfrak{S}}_\infty : s(k) = k \text{ for all } k > n\right\}$.

Further we will consider $L^2(M, \mathrm{tr})^{\otimes n}$ as the subspace of $L^2(M, \mathrm{tr})^{\otimes(n+1)}$, using the embedding

$$L^2(M, \mathrm{tr})^{\otimes n} \ni m_1 \otimes \ldots \otimes m_n \mapsto m_1 \otimes \ldots \otimes m_n \otimes \mathrm{I} \in L^2(M, \mathrm{tr})^{\otimes(n+1)}.$$

Let $L^2(M, \mathrm{tr})^{\otimes\infty}$ be the completion of the pre-Hilbert space $\bigcup_{n=1}^{\infty} L^2(M, \mathrm{tr})^{\otimes n}$. It is convenient to consider $\bigcup_{n=1}^{\infty} L^2(M, \mathrm{tr})^{\otimes n}$ as the linear span of the vectors

$$v_1 \otimes \cdots \otimes v_n \otimes \mathrm{I} \otimes \mathrm{I} \otimes \cdots,$$

where $v_j \in M$. At the same time, we will identify $L^2(M, \mathrm{tr})^{\otimes n}$ with the closure of the linear span of all vectors $v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \otimes \cdots$, where $v_i = \mathrm{I}$ for all $i > n$.

Then the elements $\theta \in \operatorname{Aut} M$ and $s \in \overline{\mathfrak{S}}_\infty$ act on $L^2(M, \mathrm{tr})^{\otimes\infty}$ as follows

$$\mathfrak{N}^{\otimes\infty}(\theta)\,(v_1 \otimes \cdots \otimes v_n \otimes \cdots) = (\mathfrak{N}(\theta)v_1) \otimes \cdots \otimes (\mathfrak{N}(\theta)v_n) \otimes \cdots ;$$
$${}^{\infty}\mathcal{P}(s)\,(v_1 \otimes \cdots \otimes v_n \otimes \cdots) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)} \otimes \cdots .$$

We now have:

Theorem 5.1. $\{\mathfrak{N}^{\otimes\infty}(\operatorname{Aut} M)\}' = \left\{{}^{\infty}\mathcal{P}\left(\overline{\mathfrak{S}}_\infty\right)\right\}''$.

Proof. Let $(k\ l)$ be a transposition that swaps k and l. We denote by $\overline{\mathfrak{S}}_{n,\infty}$ the subgroup $\left\{s \in \overline{\mathfrak{S}}_\infty : s(k) = k \text{ for all } k \in \{1, 2, \dots, n\}\right\}$. Let us prove that

$$L^2(M, \mathrm{tr})^{\otimes n} = \left\{v \in L^2(M, \mathrm{tr})^{\otimes\infty} : \ {}^{\infty}\mathcal{P}(s)v = v \text{ for all } \ s \in \overline{\mathfrak{S}}_{n,\infty}\right\}. \tag{58}$$

Fix any $\mathbf{v} \in L^2(M, \mathrm{tr})^{\otimes\infty}$ such that ${}^{\infty}\mathcal{P}(s)\mathbf{v} = \mathbf{v}$ for all $s \in \overline{\mathfrak{S}}_{n,\infty}$.

Take orthonormal basis $\{e_k\}_{k=0}^{\infty}$ in $L^2(M, \mathrm{tr})$, where $e_0 = \mathrm{I}$ and $e_k \in M$ for all k. Denote by $\mathfrak{K}$ a set of all sequences $\mathfrak{k} = \{k_i\}_{i=1}^{\infty}$, $k_i \in \{0, 1, \dots\}$ with the property: there exists some natural $N(\mathfrak{k})$ such, that $k_i = 0$ for all $i > N(\mathfrak{k})$. For convenience, we set $N(\mathfrak{k}) = \min\{m : k_i = 0 \ \text{ for all } \ i > m\}$. Then the set $\left\{\mathbf{e}_{\mathfrak{k}} = e_{k_1} \otimes e_{k_2} \otimes \dots e_{k_{N(\mathfrak{k})}} \otimes \mathrm{I} \otimes \mathrm{I} \otimes \dots\right\}_{\mathfrak{k}\in\mathfrak{K}}$ is an orhonormal basis in $L^2(M, \mathrm{tr})^{\otimes\infty}$.

Set
$$\mathbf{v} = \sum_{\mathfrak{k}\in\mathfrak{K}} c_{\mathfrak{k}}(\mathbf{v})\mathbf{e}_{\mathfrak{k}} \ \text{ where } \ c_{\mathfrak{k}}(\mathbf{v}) \in \mathbb{C}.$$

To prove (58) it is sufficient to establish that $c_{\mathfrak{k}}(\mathbf{v}) = 0$ if $N(\mathfrak{k}) > n$.

Consider an orthogonal projection O_m in $L^2(M, \mathrm{tr})^{\otimes\infty}$ that is defined as follows

$$\begin{aligned} &O_m\left(\dots \otimes e_{k_{m-1}} \otimes e_{k_m} \otimes e_{k_{m+1}} \otimes \dots e_{k_{N(\mathfrak{k})}} \otimes \mathrm{I} \otimes \mathrm{I} \otimes \dots\right) \\ &\quad = \mathrm{tr}(e_{k_m})\left(\dots \otimes e_{k_{m-1}} \otimes \mathrm{I} \otimes e_{k_{m+1}} \otimes \dots e_{k_{N(\mathfrak{k})}} \otimes \mathrm{I} \otimes \mathrm{I} \otimes \dots\right). \end{aligned} \tag{59}$$

It is easily seen that the sequence $\left\{{}^{\infty}\mathcal{P}((m\ l))\right\}_{l=1}^{\infty}$ converges in the week operator topology to $O_m = \mathrm{w} - \lim\limits_{l\to\infty} {}^{\infty}\mathcal{P}((m\ l))$. Therefore,

$$O_m \in \left({}^{\infty}\mathcal{P}(\overline{\mathfrak{S}}_\infty)\right)'' \quad \text{for all } \ m, \ \text{ and } \ O_m\mathbf{v} = \mathbf{v} \ \text{ for all } \ m > n. \tag{60}$$

Hence, applying (59), we have $c_{\mathfrak{k}}(\mathbf{v}) = 0$ for all $\mathfrak{k}$ such that $N(\mathfrak{k}) > n$. This proves equality (58).

According to (59), we have that the operator $\mathfrak{P}_{n,N} = O_{n+1}O_{n+2}\cdots O_N$, where $N > n$ is an orthogonal projection. Since $\mathfrak{P}_{n,m} \geq \mathfrak{P}_{n,m+1}$ for all $m > n$, there exists the orthogonal projection $\mathfrak{P}_n = \lim\limits_{m\to\infty} \mathfrak{P}_{n,m}$. By (60), $\mathfrak{P}_n$ belongs to $\left({}^{\infty}\mathcal{P}(\overline{\mathfrak{S}}_{n,\infty})\right)''$. Using (59), we obtain

$$\begin{aligned} &\mathfrak{P}_n\,(v_1 \otimes v_2 \otimes \otimes \dots \otimes v_n \otimes v_{n+1} \otimes \dots \otimes v_j \otimes \dots) \\ &\quad = \left(\prod_{j=n+1}^{\infty} \mathrm{tr}(v_j)\right)(v_1 \otimes v_2 \otimes \dots \otimes v_n \otimes \mathrm{I} \otimes \dots \otimes \mathrm{I} \otimes \dots). \end{aligned} \tag{61}$$

Therefore, $\mathfrak{P}_n\left(L^2(M, \mathrm{tr})^{\otimes\infty}\right) = L^2(M, \mathrm{tr})^{\otimes n}$.

Consider the operator $B' \in \{\mathfrak{N}^{\otimes\infty}(\mathrm{Aut}\, M)\}'$. Since projection $\mathfrak{P}_n \in \left({}^{\infty}\mathcal{P}(\overline{\mathfrak{S}}_{n,\infty})\right)''$ and $\left({}^{\infty}\mathcal{P}(\overline{\mathfrak{S}}_{n,\infty})\right)'' \subset \{\mathfrak{N}^{\otimes\infty}(\mathrm{Aut}\, M)\}'$, the operator $B'_n = \mathfrak{P}_n B' \mathfrak{P}_n$ belongs to $\{\mathfrak{N}^{\otimes\infty}(\mathrm{Aut}\, M)\}'$, too. It follows from Section 4 that

$$\begin{aligned} \mathfrak{P}_n \mathfrak{N}^{\otimes\infty}(\theta) \mathfrak{P}_n &= \mathfrak{N}^{\otimes n}(\theta), \quad \theta \in \mathrm{Aut}\, M, \\ \mathfrak{P}_n {}^{\infty}\mathcal{P}(s) \mathfrak{P}_n &= {}^{n}\mathcal{P}(s), \quad \text{for all } s \in \mathfrak{S}_n, \\ \mathfrak{P}_n O_i \mathfrak{P}_n &= {}^{k}\mathcal{P}^{\mathscr{I}}(\epsilon_i), \quad i = 1, 2, \ldots, n. \end{aligned}$$

Hence, applying Theorem 4.4, we obtain that B'_n belongs to $\left({}^{\infty}\mathcal{P}(\overline{\mathfrak{S}}_{\infty})\right)''$ (see (60)). Since $B' = \lim\limits_{n\to\infty} B'_n$ in the strong operator topology, operator B' lies in $\left({}^{\infty}\mathcal{P}(\overline{\mathfrak{S}}_{\infty})\right)''$, too. This completes the proof of Theorem 5.1. ■

6. A mapping from unitary to doubly stochastic matrices

Recall that a $n \times n$-matrix $P = [P_{ij}]$ is called *doubly stochastic* if $\sum\limits_{i=1}^{n} P_{ij} = 1$, $\sum\limits_{j=1}^{n} P_{ij} = 1$ and $P_{ij} \geq 0$ for all i, j. The property of P being doubly stochastic is obviously equivalent to the vector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ being invariant both for P and the transpose P^t. Recall that $\mathcal{DS}_n$ stands for the set of all doubly stochastic $n \times n$ matrices. There exists an orthogonal matrix $O = [O_{ij}]$ such that for any $P \in \mathcal{DS}_n$ one has $(OPO^{-1})_{1j} = \delta_{1j}$ and $(OPO^{-1})_{j1} = \delta_{j1}$ $(j = 1, 2, \ldots, n)$, where δ_{kl} is the Kronecker delta. Let us fix such matrix O.

Lemma 6.1. *Let ${}^{1}_{\gamma}\mathbb{M}_n(\mathbb{R})$ be the set of all real $n \times n$ matrices of the form* $\begin{bmatrix} \gamma & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{bmatrix}$. *Suppose that a doubly stochastic matrix $P = [P_{ij}]$ has only nonzero entries. Then there exists $\kappa > 0$ such that the matrix $P + O^{-1}BO$ is doubly stochastic for any matrix $B = [B_{ij}] \in {}^{1}_{0}\mathbb{M}_n(\mathbb{R})$ such that $|B_{ij}| < \kappa$ for all i, j.*

By the above lemma, each doubly stochastic matrix P with positive entries is an interior point of $\mathcal{DS}_n$, and the real dimension of the tangent space $T_P \mathcal{DS}_n$ at this point is $(n-1)^2$. In addition, we have a linear one-to-one map between $T_P \mathcal{DS}_n$ and ${}^{1}_{0}\mathbb{M}_n(\mathbb{R})$.

We need in the sequel the obvious claim as follows.

Proposition 6.2. *Let $\mathcal{U}$ be an open subset in $\mathcal{DS}_n$, and $GL(n, \mathbb{R})$ stand for the group of real invertible $n \times n$ matrices. Identify the group $GL(n-1, \mathbb{R})$ with the subgroup $(O^{-1} \cdot {}^{1}_{1}\mathbb{M}_n(\mathbb{R}) \cdot O) \cap GL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$. Then the topological component of the identity in $GL(n-1, \mathbb{R})$ is contained in*

$$\bigcup_{j=1}^{\infty} \left((\mathcal{U} \cap GL(n, \mathbb{R})) \cdot (\mathcal{U} \cap GL(n, \mathbb{R}))^{-1} \right)^{j}.$$

Denote by $U(n)$ a group of unitary $n\times n$-matrices. We will consider $U(n)$ and $\mathcal{DS}_n$ as real manifolds of the dimension n^2 and $(n-1)^2$ respectively. Now let $f: U(n)\mapsto \mathcal{DS}_n$ be a smooth map and let $\mathrm{d}f_u$ be a differential of f at the point u. The mapping $\mathrm{d}f_u$ is the linear operator from the tangent space $T_uU(n)$ at u to the tangent space $T_{f(u)}\mathcal{DS}_n$. Function f is a *submersion* at a point $u\in U(n)$ if $\mathrm{d}f_u\, T_uU(n)=T_{f(u)}\mathcal{DS}_n$. In connection with formula (17) we will find the unitary matrices u such that the map

$$U(n)\ni u=[u_{ij}]\overset{\mu}{\mapsto}\left[|u_{ij}|^2\right]\in\mathcal{DS}_n \text{ is submersion at the point } u. \tag{62}$$

Hence it follows that there exists the open neighborhood $\mathcal{U}$ of the point u such that $\mu(\mathcal{U})\subset\mathcal{DS}_n$ is open subset.

We adopt below the results of A. Karabegov [5] to make them applicable to proving Proposition 3.5.

Denote by $\mathcal{SH}_n$ the set of all skew-Hermitian $n\times n$-matrices. It is clear that the dimension of $U(n)$, as a real manifold, is equal n^2. Considering a smooth one-parameter family $U(t)=[U_{kl}(t)]\subset U(n)$ and using the equality $U(t)^*\cdot U(t)=\mathrm{I}_n$, we obtain

$$U(0)^*\cdot U'(0)+U'(0)^*\cdot U(0)=0,\ \text{where } U'(0)=[U'_{kl}(0)]\,.$$

Hence
$$\begin{aligned}&U'(0)\cdot U(0)^*+U(0)\cdot U'(0)^*\\&=U(0)\left(U(0)^*\cdot U'(0)+U'(0)^*\cdot U(0)\right)(U(0))^*=0.\end{aligned}\tag{63}$$

This implies that $U'(0)\in T_uU(n)$ is identified with the skew Hermitian matrix $X=u^*\cdot U'(0)\in T_{\mathrm{I}_n}U(n)$ treated as an element of the Lie algebra $\mathcal{SH}_n$ of $U(n)$. Here $u=[u_{kl}]=U(0)$.

Applying (62), we see that $\mathrm{d}\mu_u: T_uU(n)\mapsto T_{\mu(u)}\mathcal{DS}_n$ acts as follows

$$\mathrm{d}\mu_u\ \left(U'(0)\right)=\left[u_{kl}\overline{U'_{kl}(0)}+U'_{kl}(0)\overline{u_{kl}}\right]\in T_{\mu(u)}\mathcal{DS}_n.$$

Let us introduce the operator ${}^u\mathrm{d}\mu_u: T_{\mathrm{I}_n}U(n)\mapsto T_{\mu(u)}\mathcal{DS}_n$ which acts by

$${}^u\mathrm{d}\mu_u(A)=\mathrm{d}\mu_u(uA),\ A\in T_{\mathrm{I}_n}U(n),\ uA\in T_uU(n). \tag{64}$$

Therefore,

$${}^u\mathrm{d}\mu_u\left(u^*U'(0)\right)=\left[u_{kl}\overline{U'_{kl}(0)}+U'_{kl}(0)\overline{u_{kl}}\right]\in T_{\mu(u)}\mathcal{DS}_n.$$

Hence, assuming that all entries of $u=U(0)=[u_{kl}]$ are nonzero, we obtain

$${}^u\mathrm{d}\mu_u\left(u^*U'(0)\right)=\left[\left(\frac{U'_{kl}(0)}{u_{kl}}+\frac{\overline{U'_{kl}(0)}}{\overline{u_{kl}}}\right)|u_{kl}|^2\right]. \tag{65}$$

Now we can rewrite the equality (63) as follows

$$\sum_{j=1}^n u_{kj}\frac{U'_{kj}(0)}{u_{kj}}\overline{u_{lj}}+\sum_{j=1}^n u_{kj}\frac{\overline{U'_{lj}(0)}}{\overline{u_{lj}}}\overline{u_{lj}}=0. \tag{66}$$

Consider the family ${}^{\theta}U = \left[{}^{\theta}U_{kl}\right]$ of the unitary matrices, where

$$ {}^{\theta}U_{kl} = \delta_{kl} + \frac{\theta - 1}{n}, \quad \theta \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}. \tag{67} $$

On the space $\mathbb{M}_n$ of all complex $n \times n$-matrices define two inner products

$$ \langle A, B\rangle_\theta = \sum_{k,l-1}^{n} A_{kl}\overline{B_{kl}}\left|{}^{\theta}U_{kl}\right|^2, \; A = [A_{kl}], B = [B_{kl}], $$
$$ \langle A, B\rangle_{\mathrm{Tr}} = \mathrm{Tr}\,(AB^*), \text{ where Tr is an ordinary trace on } \mathbb{M}_n. $$

Denote by $\mathbb{M}_n^{\theta}$ and $\mathbb{M}_n^{\mathrm{Tr}}$ the corresponding Hilbert spaces.
Now we introduce two operators $\mathbf{C}_\theta$ and $\mathbf{D}_\theta$ as follows

$$ \mathbb{M}_n^{\theta} \ni f = [f_{kl}] \overset{\mathbf{C}_\theta}{\mapsto} Y = [Y_{kl}] \in \mathbb{M}_n^{\mathrm{Tr}}, \text{ where } Y_{kl} = \sum_{j=1}^{n} {}^{\theta}U_{kj} f_{kj}\overline{{}^{\theta}U_{lj}}; $$
$$ \mathbb{M}_n^{\theta} \ni g = [g_{kl}] \overset{\mathbf{D}_\theta}{\mapsto} Z = [Z_{kl}] \in \mathbb{M}_n^{\mathrm{Tr}}, \text{ where } Z_{kl} = \sum_{j=1}^{n} {}^{\theta}U_{kj} g_{lj}\overline{{}^{\theta}U_{lj}}. $$

Hence, using the orthogonality relations between ${}^{\theta}U_{kj}$, we obtain the formulas for the inverse operators

$$ \left(\mathbf{C}_\theta^{-1}Y\right)_{kq} = {}^{\theta}U_{kq}^{-1} \sum_{j=1}^{n} Y_{kj}\,{}^{\theta}U_{jq} \text{ and } \left(\mathbf{D}_\theta^{-1}Y\right)_{kq} = \overline{{}^{\theta}U}_{kq}^{-1} \sum_{j=1}^{n} Y_{jk}\,\overline{{}^{\theta}U}_{jq}. \tag{68} $$

Set $u = U(0) = {}^{\theta}U$, $X = u^*U'(0)$, $f_{kj} = \frac{U'_{kj}(0)}{u_{kj}}$ and $\overline{f} = \left[\,\overline{f_{kj}}\,\right]$. Then

$$ uXu^* = U'(0) \cdot u^* = \mathbf{C}_\theta f \text{ and } uX^*u^* = u \cdot U'(0)^* = \mathbf{D}_\theta\overline{f}. \tag{69} $$

Hence, applying (66), we have

$$ \mathbf{C}_\theta f = uXu^*, \mathbf{D}_\theta\overline{f} = -uXu^*. \tag{70} $$

It easy to check that the next statement holds.

Proposition 6.3. (Proposition 2.1 [5]) *If $\theta \notin \{-1, 1\}$ then the mappings $\mathbf{C}_\theta$ and $\mathbf{D}_\theta$ are unitary isomorphisms between the Hilbert spaces $\mathbb{M}_n^{\theta}$ and $\mathbb{M}_n^{\mathrm{Tr}}$.*

Furthermore, using (65) and (70), we obtain for $X = u^*U'(0)$ and $u = {}^{\theta}U$

$$ \left({}^{u}\mathrm{d}\mu_u\, X\right)_{kl} = \left(\mathbf{C}_\theta^{-1}(uXu^*) - \mathbf{D}_\theta^{-1}(uXu^*)\right)_{kl} \cdot |u_{kl}|^2. \tag{71} $$

Now we will prove the next statement.

Theorem 6.4. (Theorem 5.1 [5]) *Let $u = {}^{\theta}U$, where $\theta \notin \{-1, 1\}$. Then the dimension of the kernel of the operator $\left(\mathbf{C}_\theta^{-1} - \mathbf{D}_\theta^{-1}\right)$ is equal to $2n - 1$.*

Since the real dimensions of $T_uU(n)$ and $T_{\mu(u)}\mathcal{DS}_n$ are equal n^2 and $(n-1)^2$, applying (71), we obtain the next statemant.

Corollary 6.5. *If $\theta \notin \{-1, 1\}$ then the spaces $\mathrm{d}\mu_u\,(T_u U(n))$ and $T_{\mu(u)}\mathcal{DS}_n$ coincide.*

Proof of Theorem 6.4. Let $\mathfrak{D}_n$ be the set of all diagonal matrices in $\mathcal{SH}_n$ and let K_n be a real subspace of $\mathcal{SH}_n$, generated by $\mathfrak{D}_n$ and $u\mathfrak{D}_n u^*$. The ordinary calculations shows that

$$\mathbf{C}_\theta^{-1}\eta = \mathbf{D}_\theta^{-1}\eta \text{ for all } \eta \in \mathrm{K}_n \text{ and } \dim \mathrm{K}_n = 2n - 1. \tag{72}$$

Define the entries of the matrix ${}^k_l B = \left[{}^k_l B_{pq}\right]$ as follows

$${}^k_l B_{pq} = \begin{cases} 0, & \text{if } p = q \text{ or } (p \notin \{k,l\}) \bigwedge (q \notin \{k,l\}); \\ -1 & \text{if } p = k, q = l; \\ 1, & \text{if } p = l, q = k; \\ \frac{n+\overline{\theta}-1}{(\overline{\theta}-1)(n-2)}, & \text{if } q = l, p \neq k \text{ and } p \neq l; \\ \frac{n+\theta-1}{(\theta-1)(n-2)}, & \text{if } p = k, q \neq l \text{ and } q \neq k; \\ -\frac{n+\overline{\theta}-1}{(\overline{\theta}-1)(n-2)}, & \text{if } q = k, p \neq k \text{ and } p \neq l; \\ -\frac{n+\theta-1}{(\theta-1)(n-2)}, & \text{if } p = l, q \neq l \text{ and } q \neq k. \end{cases} \tag{73}$$

Let B_n be a real subspace of $\mathcal{SH}_n$, generated by the matrices ${}^k_l B$, where $k, l = 1, 2, \ldots, n$. By calculations it can be checked that the subspaces K_n and B_n are mutually orthogonal and

$$\mathbf{C}_\theta^{-1}\eta = -\frac{n+\overline{\theta}-1}{n+\theta-1}\mathbf{D}_\theta^{-1}\eta \text{ for all } \eta \in \mathrm{B}_n. \tag{74}$$

It easy to check that the matrices ${}^1_2 B$, ${}^2_3 B$, $\ldots$, ${}^{(n-1)}{}_n B$ are linearly independent. Therefore,

$$\dim \mathrm{B}_n \geq n - 1. \tag{75}$$

Let O_n be one dimensional subspace $\mathbb{R}\, iO \subset \mathcal{SH}_n$, where $O = [O_{kl}] = [\delta_{kl} - 1]$. By calculations we see that K_n and B_n are orthogonal to O_n and

$$\mathbf{C}_\theta^{-1}\, O = -\theta\frac{n+\overline{\theta}-1}{n+\theta-1}\mathbf{D}_\theta^{-1}\, O. \tag{76}$$

Denote by IS_n the real subspace of the matrices $A = [A_{kl}] \in \mathcal{SH}_n$ with the purely imaginary entries such that

$$A_{kk} = 0 \text{ and } \sum_{l=1}^{n} A_{kl} = 0 \text{ for all } k = 1, 2, \ldots, n. \tag{77}$$

Hence, using (68), we obtain

$$\mathbf{C}_\theta^{-1}\, A = -\overline{\theta}\,\mathbf{D}_\theta^{-1}\, A \text{ for all } A \in \mathrm{IS}_n. \tag{78}$$

At last we introduce the real subspace RS_n of the matrices $A = [A_{kl}] \in \mathcal{SH}_n$ with the real entries which satisfy (77). It follows, by similar calculations, that

$$\mathbf{C}_\theta^{-1}\, A = \overline{\theta}\,\mathbf{D}_\theta^{-1}\, A \text{ for all } A \in \mathrm{RS}_n. \tag{79}$$

Applying (77), we obtain

$$\dim \mathrm{IS}_n = \Big(\sum_{j=1}^{n-1}(n-j)\Big) - n = \frac{n(n-3)}{2}. \tag{80}$$

Analogously,

$$\dim \mathrm{RS}_n = \Big(\sum_{j=1}^{n-1}(n-j)\Big) - (n-1) = \frac{(n-1)(n-2)}{2}. \tag{81}$$

By ordinary calculations it can be shown that subspaces K_n, B_n, O_n, IS_n, RS_n are pairwise orthogonal. Hence, applying (72), (75), (80) and (81), we have

$$\dim(\mathrm{K}_n \oplus \mathrm{B}_n \oplus \mathrm{O}_n \oplus \mathrm{B}_n \oplus \mathrm{IS}_n \oplus \mathrm{IR}_n) \geq n^2.$$

Therefore, $\mathrm{K}_n \oplus \mathrm{B}_n \oplus \mathrm{O}_n \oplus \mathrm{B}_n \oplus \mathrm{IS}_n \oplus \mathrm{IR}_n = \mathcal{SH}_n$. Thus any $\Psi \in \mathcal{SH}_n$ can be written as follows $\Psi = \Psi_K + \Psi_B + \Psi_O + \Psi_{IS} + \Psi_{RS}$, where $\Psi_\star$ lies in the corresponding orthogonal component. If Ψ lies in the kernel of the operator $\mathrm{d}\mu_u = \big(\mathbf{C}_\theta^{-1} - \mathbf{D}_\theta^{-1}\big)$ then, using (72), (74), (76), (78) and (79), we obtain

$$D_\theta \circ \mathrm{d}\mu_u \Psi = \Big(-\frac{n+\overline{\theta}-1}{n+\theta-1} - 1 + \Big)\Psi_B + \Big(-\theta\frac{n+\overline{\theta}-1}{n+\theta-1} - 1 + \Big)\Psi_O$$
$$- (\theta+1)\Psi_{IS} + (\overline{\theta}-1)\Psi_{RS}.$$

Since $\theta \notin \{-1,1\}$, then $\Psi_B = \Psi_O = \Psi_{IS} = \Psi_{RS} = 0$. Therefore, $\Psi = \Psi_K \in \mathrm{K}_n$. ■

The next statement follows from Corollary 6.5.

Corollary 6.6. *If $\theta \notin \{-1,1\}$ then $\mathrm{d}\mu_u$ is a submersion at the point $u = {}^\theta U$. Therefore, there exists an open subset $\mathcal{U}$ such that $u \in \mathcal{U}$ and $\mu(\mathcal{U})$ is an open subset in $\mathcal{DS}_n$.*

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