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Minimal Faithful Representations for Trivial Central Extensions of Heisenberg Lie Superalgebras

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Abstract. Let the underlying field $\mathbb{F}$ be an algebraically closed field of characteristic 0. We obtain minimal dimensions of nice faithful representations and nilrepresentations for trivial central extensions of Heisenberg Lie superalgebras.

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Key Words: Heisenberg Lie superalgebra, minimal faithful representation, central extension.

1. Introduction

Ado's theorem shows that every finite-dimensional Lie (super)algebra over a field $\mathbb{F}$ of characteristic 0 has a finite-dimensional faithful representation (see [5]). Let $\mathfrak{g}$ be a Lie (super)algebra over $\mathbb{F}$. Let $\mu(\mathfrak{g})$ denote the minimal dimension of the faithful representations of $\mathfrak{g}$. Milnor's conjecture asserts that every solvmanifold admits a left-invariant affine structure (see [7]). Milnor's conjecture yields that $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$. However, Burde has obtained examples of nilmaniflods in the case where the Lie algebra of the Lie group is filiform nilpotent of dimension equal 10 or 11 without any affine structure (see [1]). It is of interest to determine the value $\mu(\mathfrak{g})$. Schur proved that $\mu(\mathfrak{a}_n) = \lceil 2\sqrt{n-1} \rceil$ for n-dimensional abelian Lie algebra $\mathfrak{a}_n$ over $\mathbb{C}$ (see [11]). Jacobson extended this result for arbitrary field (see [3]) and Mirzakhani gave a simple proof of Schur's theorem (see [8]). In 1998, Burde proved $\mu(\mathfrak{h}_m) = m+2$ for Heisenberg Lie algebra $\mathfrak{h}_m$ of dimension $2m+1$ (see [2]). In 2013, Rojas determined $\mu(\mathfrak{h}_m \oplus \mathfrak{a}_n) = m + \lceil 2\sqrt{n} \rceil$ for the Heisenberg Lie algebra $\mathfrak{h}_m$ with n-dimensional abelian factor $\mathfrak{a}_n$ (see [10]). The theory of faithful representations of Lie superalgebras also has experienced a vigorous development. In 2015, Chen and Liu determined the minimal dimension of Heisenberg Lie superalgebras (see [6]), that is,

$$\mu(\mathfrak{h}) = \begin{cases} m + \lceil \frac{n}{2} \rceil + 2 & \text{if } \mathfrak{h} = \mathfrak{h}_{m,n}, m > 0, \\ n + 2 & \text{if } \mathfrak{h} = \mathfrak{h}_n, \end{cases}$$

where $\mathfrak{h}_{m,n}$ and $\mathfrak{h}_n$ are Heisenberg Lie superalgebras of even center and odd center, respectively.

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Wang and Liu obtained

$$\mu(\mathfrak{g}) = \lceil 2\sqrt{\dim \mathfrak{g} - 1} \rceil \text{ or } \lceil 2\sqrt{\dim \mathfrak{g} - 1} \rceil + 1$$

for any finite-dimensional abelian Lie superalgebra $\mathfrak{g}$ (see [12]). Let

$$\widetilde{\mu}(\mathfrak{g}) = \min\{\dim V \mid (\rho, V) \text{ is a faithful nilrepresentation of } \mathfrak{g}\}.$$

In this paper, the field $\mathbb{F}$ is an algebraically closed field of characteristic 0. We determine the values of the minimal dimension of nice faithful representations and nilrepresentations for trivial central extensions of Heisenberg Lie superalgebras. Furthermore, we construct the nice faithful representations and nilrepresentations with the minimal dimension for trivial central extensions of Heisenberg Lie superalgebras.

2. Preliminaries

A Heisenberg Lie superalgebra is a two-step nilpotent Lie superalgebra with 1-dimensional center. All Heisenberg Lie superalgebras split into the following two types (see [9]).

(1) Write $\mathfrak{h}_{m,n}$ for the Heisenberg Lie superalgebra of even center, which has a $\mathbb{Z}_2$-homogeneous basis $\{x_1, \ldots, x_m, x'_1, \ldots, x'_m, z \mid y_1, \ldots, y_n\}$ with nontrivial multiplication given by

$$[x_i, x'_i] = -[x'_i, x_i] = z, [y_j, y_j] = z \text{ for all } 1 \le i \le m, 1 \le j \le n.$$

(2) Write $\mathfrak{h}_n$ for the Heisenberg Lie superalgebra of odd center, which has a $\mathbb{Z}_2$-homogeneous basis $\{x_1, \ldots, x_n \mid z, y_1, \ldots, y_n\}$ with nontrivial multiplication given by

$$[x_i, y_i] = -[y_i, x_i] = z \text{ for all } 1 \le i \le n,$$

Obviously, $\mathbb{F}z$ is the center of both $\mathfrak{h}_{m,n}$ and $\mathfrak{h}_n$.

Let $\mathfrak{a}_{s,t}$ be the abelian Lie superalgebra of superdimension (s,t), and its $\mathbb{Z}_2$-homogeneous basis be $\{A_1, \ldots, A_s \mid B_1, \ldots, B_t\}$. Then a trivial central extension of Heisenberg Lie superalgebras is $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, and

$$\{x_1, \ldots, x_m, x'_1, \ldots, x'_m, z, A_1, \ldots, A_s \mid y_1, \ldots, y_n, B_1, \ldots, B_t\} \tag{1}$$

or

$$\{x_1, \ldots, x_n, A_1, \ldots, A_s \mid z, y_1, \ldots, y_n, B_1, \ldots, B_t\} \tag{2}$$

is a $\mathbb{Z}_2$-homogeneous basis of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$ respectively. Obviously, $\operatorname{span}_{\mathbb{F}}\{z, A_1, \ldots, A_s \mid B_1, \ldots, B_t\}$ or $\operatorname{span}_{\mathbb{F}}\{A_1, \ldots, A_s \mid z, B_1, \ldots, B_t\}$ is the center of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, denoted by $Z(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t})$ or $Z(\mathfrak{h}_n \oplus \mathfrak{a}_{s,t})$, respectively.

Definition 2.1. Let $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$ and (ρ, V) be a representation of $\mathfrak{g}$ and

$$r_1 = \max\{\dim \rho(Z(\mathfrak{g}))v \mid v \in V_{\bar{0}}\}, \quad r_2 = \max\{\dim \rho(Z(\mathfrak{g}))v \mid v \in V_{\bar{1}}\}.$$

Then (ρ, V) is said to be *nice* if one of the following conditions holds:

(1) If $r_1 \ge r_2$, then $\rho(z)V_{\bar{0}} \ne 0$.

(2) If $r_1 \le r_2$, then $\rho(z)V_{\bar{1}} \ne 0$. ■

Let

$$\mu_{nic}(\mathfrak{g}) = \min\{\dim V \mid (\rho, V) \text{ is a nice faithful representation of } \mathfrak{g}\},$$
$$\widetilde{\mu}_{nic}(\mathfrak{g}) = \min\{\dim V \mid (\rho, V) \text{ is a nice faithful nilrepresentation of } \mathfrak{g}\}.$$

In this paper, we prove the following theorem.

Theorem 2.2. *Let $m, n, s, t \in \mathbb{N}$ and $m \neq 0$. Then*

$$\widetilde{\mu}_{nic}(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}) = m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil,$$
$$\widetilde{\mu}_{nic}(\mathfrak{h}_{n} \oplus \mathfrak{a}_{s,t}) = n + \lceil 2\sqrt{s+t+1} \rceil,$$
$$\mu_{nic}(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}) = m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil,$$
$$\mu_{nic}(\mathfrak{h}_{n} \oplus \mathfrak{a}_{s,t}) = n + \lceil 2\sqrt{s+t} \rceil.$$

3. The low bound for minimal faithful representations.

Lemma 3.1. *Let $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$. For any $x \in \mathfrak{g} \backslash Z(\mathfrak{g})$, there exists $y \in \mathfrak{g}$ such that $[x, y] = z$.*

Proof. (A) If $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$, then suppose that

$$x = \sum_{i=1}^{m} a_i x_i + \sum_{i=1}^{m} a_i' x_i' + cz + \sum_{j=1}^{n} b_j y_j + \sum_{k=1}^{s} p_k A_k + \sum_{l=1}^{t} q_l B_l,$$

where $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq k \leq s$, $1 \leq l \leq t$ and $c \in \mathbb{F}$. Since $x \notin Z(\mathfrak{g})$ and $Z(\mathfrak{g}) = \mathrm{span}_{\mathbb{F}}\{z, A_1, \ldots, A_s \mid B_1, \ldots, B_t\}$. Then a_i, a_i' and b_j are not zero simultaneously for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

(A.1) If $a_1, \ldots, a_m$ are not zero simultaneously, then let i_0 be minimal such that $a_{i_0} \neq 0$. Let $y = \frac{1}{a_{i_0}} x_{i_0}'$. Note that $[x, y] = z$, as desired.

(A.2) If $a_1 = \cdots = a_m = 0$ and $a_1', \ldots, a_m'$ are not zero simultaneously, then let i_0 be minimal such that $a_{i_0}' \neq 0$. Let $y = -\frac{1}{a_{i_0}'} x_{i_0}$. Note that $[x, y] = z$, as desired.

(A.3) If $a_1 = \cdots = a_m = a_1' = \cdots = a_m' = 0$ and $b_1, \ldots, b_n$ are not zero simultaneously, then let j_0 be minimal such that $b_{j_0} \neq 0$. Let $y = \frac{1}{b_{j_0}} y_{j_0}$. Note that $[x, y] = z$, proving the desired result.

(B) If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then suppose that

$$x = \sum_{i=1}^{n} a_i x_i + \sum_{k=1}^{s} p_k A_k + cz + \sum_{i=1}^{n} b_i y_i + \sum_{l=1}^{t} q_l B_l,$$

where $1 \leq i \leq n$, $1 \leq k \leq s$, $1 \leq l \leq t$ and $c \in \mathbb{F}$. Since $x \notin Z(\mathfrak{g})$ and $Z(\mathfrak{g}) = \mathrm{span}_{\mathbb{F}}\{A_1, \ldots, A_s \mid z, B_1, \ldots, B_t\}$. Then a_i and b_i are not zero simultaneously for all $1 \leq i \leq n$.

(B.1) If $a_1, \ldots, a_n$ are not zero simultaneously, then let i_0 be minimal such that $a_{i_0} \neq 0$. Let $y = \frac{1}{a_{i_0}} y_{i_0}$. It follows that $[x, y] = z$, as desired.

(B.2) If $a_1 = \cdots = a_n = 0$ and $b_1, \ldots, b_n$ are not zero simultaneously, then let i_0 be minimal such that $b_{i_0} \neq 0$. Let $y = -\frac{1}{b_{i_0}} x_{i_0}$. It follows that $[x, y] = z$, proving the desired result. ■

Lemma 3.2. *Let $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$. Suppose that $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ satisfying $z \notin \mathfrak{s}$. Then*

$$\dim \mathfrak{s} \leq \dim(\mathfrak{s} \cap Z(\mathfrak{g})) + \begin{cases} m + \lfloor \frac{n}{2} \rfloor, & \text{if } \mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}, \\ n, & \text{if } \mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}. \end{cases} \tag{3}$$

Proof. Let Z_1 and $\mathfrak{s}_1$ be complement superspaces of $\mathfrak{s} \cap Z(\mathfrak{g})$ in $Z(\mathfrak{g})$ and $\mathfrak{s}$, respectively. Define a linear function α of $Z(\mathfrak{g})$ by $\alpha|_{Z(\mathfrak{g}) \cap \mathfrak{s}} = 0$ and $\alpha(z) \neq 0$. Let $\mathfrak{s}'$ be a complement superspaces of $\mathfrak{s} \oplus Z_1$ in $\mathfrak{g}$, that is,

$$\mathfrak{g} = \mathfrak{s}' \oplus \mathfrak{s} \oplus Z_1.$$

Let B be a skew-supersymmetric bilinear form on $\mathfrak{s}' \oplus \mathfrak{s}_1$ by defining $B(X, Y) = \alpha([X, Y])$ for all $X, Y \in \mathfrak{s}' \oplus \mathfrak{s}_1$. Note that B is nondegenerate. Then $\mathfrak{s}_1$ is a B-isotropic subspace of $\mathfrak{s}' \oplus \mathfrak{s}_1$ by the definition of B.

It follows that $\dim \mathfrak{s}_1 \leq \dim(\mathfrak{s}' \oplus \mathfrak{s}_1)/2$. Hence

$$\dim \mathfrak{s} = \dim(\mathfrak{s} \cap Z(\mathfrak{g})) + \dim \mathfrak{s}_1 \leq \dim(\mathfrak{s} \cap Z(\mathfrak{g})) + \frac{\dim(\mathfrak{s}' \oplus \mathfrak{s}_1)}{2}.$$

Since $\mathfrak{g} = \mathfrak{s}' \oplus \mathfrak{s}_1 \oplus Z(\mathfrak{g})$. If $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$, then $\dim(\mathfrak{s}' \oplus \mathfrak{s}_1) = 2m + n$. If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then $\dim(\mathfrak{s}' \oplus \mathfrak{s}_1) = 2n$. By the above we have proven (3). ■

The following lemma is proved by methods similar to the ones employed in [13, Lemma 2.3].

Lemma 3.3. *Let V be a finite dimensional superspace and $\mathfrak{F}$ a nonzero subsuperspace of* End(V). *Write*

$$r_1 = \max\{\dim \mathfrak{F}(v) \mid v \in V_{\bar{0}}\}, \; r_2 = \max\{\dim \mathfrak{F}(v) \mid v \in V_{\bar{1}}\}.$$

(1) *Suppose that a finite subset $\{f_1, \ldots, f_q\}$ of $\mathfrak{F}$ has the following property: $f_j(V_{\bar{0}}) \neq 0$ for all $1 \leq j \leq q$ if $r_1 \geq r_2$, or $f_j(V_{\bar{1}}) \neq 0$ for all $1 \leq j \leq q$ if $r_2 \geq r_1$. Then there exists $x \in V_{\bar{0}} \cup V_{\bar{1}}$ such that* $\dim \mathfrak{F}(x) = \max\{r_1, r_2\}$ *and $f_j(x) \neq 0$ for $1 \leq j \leq q$.*

(2) *There exists a linearly independent set $\{v_1, \ldots, v_k\} \subset V_{\bar{0}} \cup V_{\bar{1}}$ and a vector space decomposition*

$$\mathfrak{F} = \mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_k$$

such that

(a) $\dim \mathfrak{F}(v_1) = \max\{r_1, r_2\}$.

(b) *$f(v_i) \neq 0$ for all nonzero $f \in \mathfrak{F}_i$ and all $1 \leq i \leq k$.*

(c) *$\mathfrak{F}_j(v_i) = 0$ for $1 \leq i < j \leq k$.*

Suppose that a finite subset $\{f_1, \ldots, f_q\}$ of $\mathfrak{F}$ has the following property:
$f_j(V_{\bar{0}}) \neq 0$ for all $1 \leq j \leq q$ if $r_1 \geq r_2$, or $f_j(V_{\bar{1}}) \neq 0$ for all $1 \leq j \leq q$ if $r_2 \geq r_1$.
Then v_1 can be chosen so that $f_j(v_1) \neq 0$ for all $1 \leq j \leq q$.

Theorem 3.4. *Let* $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ *or* $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$. *Then*

$$\widetilde{\mu}_{nic}(\mathfrak{g}) \geq \begin{cases} m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil, & \text{if } \mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}, \\ n + \lceil 2\sqrt{s+t+1} \rceil, & \text{if } \mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}. \end{cases} \tag{4}$$

Proof. Suppose that (π, V) is a nice faithful nilrepresentation of $\mathfrak{g}$. Note that $\pi(Z(\mathfrak{g}))$ is a subsuperspace of $\mathrm{gl}(V)$ and satisfies the condition of Lemma 3.3 (1). By Lemma 3.3, there exists a linearly independent set $X = \{v_1, \ldots, v_k\} \subset V_{\bar{0}} \cup V_{\bar{1}}$ and a vector space decomposition

$$\pi(Z(\mathfrak{g})) = \mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_k$$

satisfying (a), (b) and (c). We define a linear mapping $\varphi : \mathfrak{g} \to V$ by means of $\varphi(x) = \pi(x)(v_1)$ for all $x \in \mathfrak{g}$. Then we obviously have $z \notin \ker\varphi$. Now we assert that $\mathrm{Im}\varphi \cap \mathbb{F}\{X\} = 0$. Suppose that there exists a nonzero element $\pi(x)(v_1) \in \mathrm{Im}\varphi \cap \mathbb{F}\{X\}$, we assume that $\pi(x)(v_1) = \sum_{i=1}^{k} a_i v_i$, where $x \in \mathfrak{g}$ and $a_i \in \mathbb{F}$, $1 \leq i \leq k$. Let i_0 be maximal such that $a_i \neq 0$. Then

$$\pi(x)(v_1) = \sum_{i=1}^{i_0} a_i v_i. \tag{5}$$

By Lemma 3.3, there exists $f_{i_0} \in \mathfrak{F}_{i_0} \subset \pi(Z(\mathfrak{g}))$ such that $f_{i_0}(v_{i_0}) \neq 0$. Application of f_{i_0} to the equation (5) yields

$$f_{i_0}(\pi(x)(v_1)) = \pi(x) f_{i_0}(v_1) = a_{i_0} f_{i_0}(v_{i_0}) \neq 0. \tag{6}$$

If $i_0 = 1$, then $\pi(x)(v_1) = a_1 v_1$ by (5). Then (6) shows that $a_1 \neq 0$. This contradicts the nilpotency of $\pi(x)$. Hence $i_0 > 1$ and $f_{i_0}(v_1) = 0$ by Lemma 3.3. We have get a contradiction for (6). Thus $\mathrm{Im}\varphi \cap \mathbb{F}\{X\} = 0$. It follows that

$$\dim V \geq k + \dim \mathrm{Im}\,\varphi \geq k + \dim \mathfrak{g} - \dim \ker \varphi.$$

Since $\ker\varphi$ is a subalgebra of $\mathfrak{g}$ and $z \notin \ker\varphi$, by Lemma 3.2, we have

$$\dim \ker \varphi \leq \dim(\ker\varphi \cap Z(\mathfrak{g})) + \begin{cases} m + \lfloor \frac{n}{2} \rfloor, & \text{if } \mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}, \\ n, & \text{if } \mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}. \end{cases}$$

The definition of φ shows that

$$\pi(\ker\varphi \cap Z(\mathfrak{g})) = \mathfrak{F}_2 \oplus \cdots \oplus \mathfrak{F}_k.$$

Since π is a faithful representation of $\mathfrak{g}$. Then we have

$$\dim(\ker\varphi \cap Z(\mathfrak{g})) = \dim Z(\mathfrak{g}) - \dim \mathfrak{F}_1.$$

Lemma 3.3 illustrates that

$$\dim \mathfrak{F}_i = \dim \mathfrak{F}_i(v_i) \leq \dim(\pi(Z(\mathfrak{g}))(v_1)) = \dim \mathfrak{F}_1(v_1) = \dim \mathfrak{F}_1,$$

for $i > 1$.

We have $k \dim \mathfrak{F}_1 \geq \dim(\pi(Z(\mathfrak{g}))$. If $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$, then we have

$$\begin{aligned}\dim V &\geq k + \dim \mathfrak{g} - \dim \ker \varphi \\ &\geq k + m + \lceil \frac{n}{2} \rceil + s + t + 1 - \dim(\ker \varphi \cap Z(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t})) \\ &\geq k + m + \lceil \frac{n}{2} \rceil + s + t + 1 - \dim Z(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}) + \dim \mathfrak{F}_1 \\ &\geq m + \lceil \frac{n}{2} \rceil + k + \dim \mathfrak{F}_1 \\ &\geq m + \lceil \frac{n}{2} \rceil + 2\sqrt{k \dim \mathfrak{F}_1} \\ &\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil.\end{aligned}$$

If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then we have

$$\begin{aligned}\dim V &\geq k + \dim \mathfrak{g} - \dim \ker \varphi \\ &\geq k + n + s + t + 1 - \dim(\ker \varphi \cap Z(\mathfrak{h}_n \oplus \mathfrak{a}_{s,t})) \\ &\geq k + n + s + t + 1 - \dim Z(\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}) + \dim \mathfrak{F}_1 \\ &\geq n + 2\sqrt{k \dim \mathfrak{F}_1} \\ &\geq n + \lceil 2\sqrt{s+t+1} \rceil.\end{aligned}$$

■

Theorem 3.5. *Let* $m \neq 0$. *Then*

$$\mu_{nic}(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}) \geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil$$

and

$$\mu_{nic}(\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}) \geq n + \lceil 2\sqrt{s+t} \rceil.$$

Proof. Let $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$ and (π, V) be a representation of $\mathfrak{g}$. Since $\mathfrak{g}_{\bar{0}}$ is a nilpotent Lie algebra. By Zassenhaus's theorem (see [4]) of $\pi(\mathfrak{g}_{\bar{0}})$, there exists a superspace decomposition of V such that

$$V = V_1 \oplus \cdots \oplus V_k,$$

where

$$V_i = \{v \in V \mid \forall x \in \mathfrak{g}_{\bar{0}},\ \exists r(x,v) \in \mathbb{N},\ (\pi(x) - \lambda_i(x)\mathrm{id})^{r(x,v)}(v) = 0\}$$

and $1 \leq i \leq k$. Note that

$$\lambda_i(x) = \frac{\mathrm{trac}(\pi(x)|_{V_i})}{\dim V_i} \tag{7}$$

for any $x \in \mathfrak{g}_{\bar{0}}$. This shows that λ_i is a homomorphism of $\mathfrak{g}_{\bar{0}}$. Since π is a faithful representation of $\mathfrak{g}$, we have $\pi(z) \neq 0$ and there exists $v \in V$ such that $\pi(z)(v) \neq 0$. Hence there exists $i_0 \in \mathbb{F}$ such that $\pi(z)(v_{i_0}) \neq 0$.

(a) If $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ and $m \neq 0$, then we have

$$\pi(z)(v_{i_0}) = \lambda_{i_0}(z)v_{i_0} + N_{i_0}(z)(v_{i_0}). \tag{8}$$

Suppose that $z = [x_1, x_1']$, where x_1 and x_1' are basis elements of $\mathfrak{g}_{\bar{0}}$ in (2). We obtain $\lambda_{i_0}(z) = 0$. It follows that $N_{i_0}(z)(v_{i_0}) \neq 0$. Let $f : \mathfrak{h}_{m,n} \to \mathrm{gl}(V_{i_0})$ be a linear mapping by defining

$$f(x) = N_{i_0}(x) \text{ and } f(y) = \pi(y),$$

for $x \in (\mathfrak{h}_{m,n})_{\bar{0}}$, $y \in (\mathfrak{h}_{m,n})_{\bar{1}}$.

For any $x \in (\mathfrak{h}_{m,n})_{\bar{0}}$ and $v \in V_{i_0}$, there exists $r(x,v) \in \mathbb{N}$ such that

$$(\pi(x) - \lambda_i(x)\mathrm{id})^{r(x,v)}(v) = 0.$$

Then we have

$$\begin{aligned}(\pi(x)-&\lambda_i(x)\mathrm{id})^{r(x,v)+1}(\pi(y)(v))\\ &= [(\pi(x) - \lambda_i(x)\mathrm{id})^{r(x,v)+1}, \pi(y)](v)\\ &= \sum_{j=0}^{r(x,v)} (\pi(x) - \lambda_i(x)\mathrm{id})^j [(\pi(x) - \lambda_i(x)\mathrm{id}), \pi(y)](\pi(x) - \lambda_i(x)\mathrm{id})^{r(x,v)-j}(v)\\ &= (r(x,v)+1)[\pi(x) - \lambda_i(x)\mathrm{id}, \pi(y)](\pi(x) - \lambda_i(x)\mathrm{id})^{r(x,v)}(v) = 0,\end{aligned}$$

where $y \in (\mathfrak{h}_{m,n})_{\bar{1}}$. It follows that $f(y)(v) = \pi(y)(v) \in V_{i_0}$ for any $v \in V_{i_0}$. Since λ_i is a homomorphism of $\mathfrak{g}_{\bar{0}}$. Then N_{i_0} is also a homomorphism of $\mathfrak{g}_{\bar{0}}$. By the definition of f, we have

$$\begin{aligned}[f(x), f(y)] = [N_{i_0}(x), \pi(y)] &= [N_{i_0}(x) + \lambda_{i_0}(x), \pi(y)]\\ &= [\pi(x), \pi(y)] = \pi([x,y]) = f([x,y]),\end{aligned}$$

for any $x \in (\mathfrak{h}_{m,n})_{\bar{0}}$, $y \in (\mathfrak{h}_{m,n})_{\bar{1}}$. Since $f(z) = N_{i_0}(z) \neq 0$. According to the above, (f, V_{i_0}) is a faithful representation of $\mathfrak{h}_{m,n}$. This yields $\ker N_{i_0} \cap \mathfrak{h}_{m,n} = 0$. Then

$$[x,y] \in [\mathfrak{g},\mathfrak{g}] = \mathbb{F}z \subseteq \mathfrak{h}_{m,n},$$

for any $x, y \in \ker N_{i_0}$. Hence $\ker N_{i_0}$ is an abelian Lie superalgebra. Let $\mathfrak{a}$ be a complement superspace of $\ker N_{i_0} \oplus \mathfrak{h}_{m,n}$ in $\mathfrak{g}_{\bar{0}}$, that is,

$$\mathfrak{g}_{\bar{0}} = \ker N_{i_0} \oplus \mathfrak{h}_{m,n} \oplus \mathfrak{a}.$$

Write $\mathfrak{h}'_{0,n} = \mathfrak{h}_{0,n} \setminus \mathbb{F}\{z\}$. Let $\mathfrak{b}$ be a complement superspace of $\mathfrak{h}'_{0,n} \oplus \ker_{i_0}\pi$ in $\mathfrak{g}_{\bar{1}}$, where $\ker_{i_0}\pi = \{x \in \mathfrak{a}_{0,t} \mid \pi(x)(V_{i_0}) = 0\}$, that is,

$$\mathfrak{g}_{\bar{1}} = \mathfrak{h}'_{0,n} \oplus \ker_{i_0}\pi \oplus \mathfrak{b}.$$

We expand f as a homomorphism of $\mathfrak{h}_{m,n} \oplus \mathfrak{a} \oplus \mathfrak{b}$ by defining

$$f(x) = N_{i_0}(x),\ f(y) = \pi(y), \forall x \in \mathfrak{a}, y \in \mathfrak{b}.$$

It is easy to see that (f, V_{i_0}) is a faithful nilrepresentation of $\mathfrak{h}_{m,n} \oplus \mathfrak{a} \oplus \mathfrak{b}$. By Theorem 3.4, we have

$$\dim V_{i_0} \geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + 1} \rceil \tag{9}$$

(b) If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then $z \in \mathfrak{g}_{\bar{1}}$. It follows that $\pi(z)$ is a nilpotent transformation. Suppose that $f' : \mathfrak{h}_n \to \mathrm{gl}(V_{i_0})$ is a linear mapping by defining

$$f'(x) = N_{i_0}(x) \text{ and } f'(y) = \pi(y),$$

for $x \in (\mathfrak{h}_n)_{\bar{0}}$, $y \in (\mathfrak{h}_n)_{\bar{1}}$. We can prove that (f', V_{i_0}) is a nilrepresentation of $\mathfrak{h}_n$. By [6, Lemma 2.1], we obtain (f', V_{i_0}) is a faithful nilrepresentation of $\mathfrak{h}_n$.

Hence $\ker N_{i_0} \cap \mathfrak{h}_n = 0$. It follows that $\ker N_{i_0}$ is an abelian Lie superalgebra. Let $\mathfrak{a}'$ be a complement superspace of $\ker N_{i_0} \oplus (\mathfrak{h}_n)_{\bar{0}}$ in $\mathfrak{g}_{\bar{0}}$, that is,

$$\mathfrak{g}_{\bar{0}} = \ker N_{i_0} \oplus (\mathfrak{h}_n)_{\bar{0}} \oplus \mathfrak{a}'.$$

Let $\mathfrak{b}'$ be a complement superspace of $(\mathfrak{h}_n)_{\bar{1}} \oplus \ker_{i_0}\pi$ in $\mathfrak{g}_{\bar{1}}$, where

$$\ker_{i_0}\pi = \{x \in \mathfrak{a}_{0,t} \mid \pi(x)(V_{i_0}) = 0\}, \quad \text{that is,} \quad \mathfrak{g}_{\bar{1}} = (\mathfrak{h}_n)_{\bar{1}} \oplus \ker_{i_0}\pi \oplus \mathfrak{b}'.$$

We can obtain that f' is a faithful nilrepresentation of $\mathfrak{h}_n \oplus \mathfrak{a}' \oplus \mathfrak{b}'$ by defining

$$f'(x) = N_{i_0}(x), \ f'(y) = \pi(y), \forall x \in \mathfrak{a}, y \in \mathfrak{b}.$$

Hence, by Theorem 3.4,

$$\dim V_{i_0} \geq n + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + 1} \rceil.$$

Since $\ker N_{i_0}$ is an abelian Lie superalgebra. $(\ker N_{i_0} \cap \ker \lambda_{i_0}) \oplus \ker_{i_0}\pi$ is obviously an abelian Lie superalgebra, which is denoted by A. Note that

$$\dim(\ker N_{i_0} \cap \ker \lambda_{i_0}) = \dim \ker N_{i_0} - 1.$$

We propose to prove $(\pi|_A, \oplus_{i \neq i_0} V_i)$ is a faithful representation of A. Since

$$\pi(x)(v) = \pi(x_0 + x_1)(v) = (\lambda_{i_0}(x_0) + N_{i_0}(x_0))(v) + \pi(x_1)(v) = 0,$$

where $x \in A$, $x_i \in A_{\bar{i}}$, $i = 0, 1$ and $v \in V_{i_0}$. Then $\pi(x)(V) \subseteq \oplus_{i \neq i_0} V_i$. Hence $\pi(x)$ is a homomorphism of $\oplus_{i \neq i_0} V_i$ for any $x \in A$. Suppose that $x' \in \ker \pi|_A$. Then $\pi(x')(\oplus_{i \neq i_0} V_i) = 0$. Hence

$$\pi(x')(V) = \pi(x')(V_{i_0} \oplus \bigoplus_{i \neq i_0} V_i) = 0,$$

that is, $x' \in \ker \pi$. Faithfulness of π implies that $x' = 0$. Then we obtain $(\pi|_A, \oplus_{i \neq i_0} V_i)$ is a faithful representation of A. By [12, Theorem 5], we have

$$\dim(\oplus_{i \neq i_0} V_i) \geq \lceil 2\sqrt{\dim A - 1} \rceil.$$

Hence if $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$, then

$$\begin{aligned} \dim V &= \dim V_{i_0} + \dim(\oplus_{i \neq i_0} V_i) \\ &\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + 1} \rceil + \lceil 2\sqrt{\dim A - 1} \rceil \\ &\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + \dim A + 1} \rceil \\ &\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + \dim \ker N_{i_0} + \dim \ker_{i_0}\pi} \rceil \\ &\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil. \end{aligned}$$

If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then

$$\begin{aligned} \dim V &= \dim V_{i_0} + \dim(\oplus_{i \neq i_0} V_i) \\ &\geq n + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + 1} \rceil + \lceil 2\sqrt{\dim A - 1} \rceil \\ &\geq n + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + \dim A + 1} \rceil \\ &\geq n + \lceil 2\sqrt{\dim(\mathfrak{a} \oplus \mathfrak{b}) + \dim \ker N_{i_0} + \dim \ker_{i_0}\pi} \rceil \\ &\geq n + \lceil 2\sqrt{s+t} \rceil. \end{aligned}$$

■

4. The upper bound for minimal faithful representations

In this section, we propose to establish faithful representations of the desired dimension for $\mathfrak{g}$. We write $e_{i,j}$ for a matrix that has 1 in the (i,j) cell and 0 in all the other cells.

Lemma 4.1. *Let $\mathfrak{g}$ be a nilpotent Lie superalgebra over $\mathbb{F}$. Suppose that (ρ, V) is a representation of $\mathfrak{g}$. Then (ρ, V) is faithful if and only if $(\rho|_{Z(\mathfrak{g})}, V)$ is faithful on $Z(\mathfrak{g})$.*

Proof. One implication is trivial. Suppose that $(\rho|_{Z(\mathfrak{g})}, V)$ is faithful. If $\ker\rho \neq 0$, then $\ker\rho \cap Z(\mathfrak{g}) \neq 0$ by Engel theorem (see [5]). It follows that $\ker\rho|_{Z(\mathfrak{g})} \neq 0$, contradicting the faithfulness of $\rho|_{Z(\mathfrak{g})}$. Hence (ρ, V) is faithful. ■

We may choose a $\mathbb{Z}_2$-homogeneous basis of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ as follows.

(1) If $n = 2k,\ k \in \mathbb{N}$, then let

$$\{x_1, \ldots, x_m, x'_1, \ldots, x'_m, z, A_1, \ldots, A_s \mid Y_1, \ldots, Y_k, Y'_1, \ldots, Y'_k, B_1, \ldots, B_t\}$$

be a basis of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ with multiplication

$$[x_i, x'_i] = z \text{ and } [Y_j, Y'_j] = z,\ i = 1, \ldots, m,\ j = 1, \ldots, k.$$

(2) If $n = 2k+1,\ k \in \mathbb{N}$, then let

$$\{x_1, \ldots, x_m, x'_1, \ldots, x'_m, z, A_1, \ldots, A_s \mid Y_1, \ldots, Y_k, Y'_1, \ldots, Y'_k, Y_{k+1}, B_1, \ldots, B_t\}$$

be a basis of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ with multiplication

$$[x_i, x'_i] = z,\ [Y_j, Y'_j] = z \text{ and } [Y_{k+1}, Y_{k+1}] = z,\ i = 1, \ldots, m,\ j = 1, \ldots, k.$$

Suppose that $s = rb + l$ and $t = r'b + l'$, where $r, r', b, a', l, l' \in \mathbb{N}$ and $l, l' < b$. Let $\rho : \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t} \to \mathrm{gl}(\mathbb{F}^{m+a+b+\lceil \frac{n}{2} \rceil + a'})$ be an even mapping given by

$$\begin{aligned}
\rho(x_i) &= e_{1,a+i}, & \rho(x'_i) &= e_{a+i,a+m+1},\ 1 \le i \le m,\\
\rho(Y_j) &= e_{1,\widetilde{j}}, & \rho(Y'_j) &= e_{\widetilde{j},a+m+1},\ 1 \le j \le k,\\
\rho(z) &= e_{1,a+m+1}, & \rho(Y_{k+1}) &= \frac{1}{2}e_{1,\widetilde{k+1}} + e_{\widetilde{k+1},a+m+1},\\
\rho(A_1) &= e_{1,a+m+2}, & \rho(B_1) &= e_{\widetilde{\lceil \frac{n}{2} \rceil + 1},a+m+1},\\
&\ \vdots & &\ \vdots\\
\rho(A_{b-1}) &= e_{1,a+m+b}, & \rho(B_b) &= e_{\widetilde{\lceil \frac{n}{2} \rceil + 1},a+m+b},\\
&\ \vdots & &\ \vdots\\
\rho(A_s) &= e_{r+1,a+m+l+1}, & \rho(B_t) &= e_{\widetilde{\lceil \frac{n}{2} \rceil + r' + 1},a+m+l'},
\end{aligned}$$

where $\widetilde{p} = m + a + b + p$ for $p \in \mathbb{N}$. A straightforward computation shows that $(\rho, \mathbb{F}^{m+a+b+\lceil \frac{n}{2} \rceil + a'})$ is a nilrepresentation of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$. By Lemma 4.1, we obtain $(\rho, \mathbb{F}^{m+a+b+\lceil \frac{n}{2} \rceil + a'})$ is faithful if and only if $ab \ge s+1$ and $a'b \ge t$.

Suppose that $a, b, d \in \mathbb{N}$. Write $i' = a + n + i$, for $i \in \mathbb{N}$.
Let $\sigma : \mathfrak{h}_n \oplus \mathfrak{a}_{s,t} \to \mathrm{gl}(\mathbb{F}^{a+n+b+d})$ be an even mapping given by

$$\begin{aligned}
\sigma(x_i) &= e_{1,a+i}, & \sigma(y_i) &= e_{a+i,1'},\\
\sigma(z) &= e_{1,1'}, & &\\
\sigma(A_1) &= e_{(b+1)',1'}, & \sigma(B_1) &= e_{1,2'},\\
&\vdots & &\vdots\\
\sigma(A_b) &= e_{(b+1)',b'}, & \sigma(B_{b-1}) &= e_{1,b'},\\
\sigma(A_{b+1}) &= e_{(b+2)',1'}, & \sigma(B_b) &= e_{2,1'},\\
&\vdots & &\vdots\\
\sigma(A_s) &= e_{(b+r+1)',l'}, & \sigma(B_t) &= e_{r_1+1,(l_1+1)'},
\end{aligned}$$

where $s = rb + l$, $t = r_1 b + l_1$ and $l, l_1 < b$. By Lemma 4.1, we obtain $(\sigma, \mathbb{F}^{a+n+b+d})$ is faithful if and only if $ab \geq t+1$ and $bd \geq s$.

Theorem 4.2. *Let $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$. Then*

$$\widetilde{\mu}_{nic}(\mathfrak{g}) \leq \begin{cases} m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil, & \text{if } \mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t},\\ n + \lceil 2\sqrt{s+t+1} \rceil, & \text{if } \mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}. \end{cases}$$

Proof. If $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$, then let $(\rho, \mathbb{F}^{m+a+b+\lceil \frac{n}{2} \rceil + a'})$ defined above be a faithful nilrepresentation of $\mathfrak{g}$. Then $ab \geq s+1$ and $a'b \geq t$. Hence

$$\begin{aligned}
\dim(\mathbb{F}^{m+a+b+\lceil \frac{n}{2} \rceil + a'}) &= m + a + b + \lceil \frac{n}{2} \rceil + a'\\
&\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{b(a+a')} \rceil\\
&\geq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil.
\end{aligned}$$

Since $\min\{\dim(\mathbb{F}^{m+a+b+\lceil \frac{n}{2} \rceil + a'}) \mid ab \geq s+1, a'b \geq t\} = m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil$. Then $(\rho, \mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil})$ is a faithful nilrepresentation of $\mathfrak{g}$. By the definition of $(\rho, \mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil})$, we have $\rho(Z(\mathfrak{g}))(w) = 0$ for any $w \in \mathbb{F}_{\bar{1}}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil}$. This yields $r_1 \geq r_2 = 0$. Since

$$0 \neq \rho(z)(\varepsilon_{a+m+1}) \in \rho(z)(\mathbb{F}_{\bar{0}}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil}),$$

where ε_{a+m+1} stand for the column vector with 1 in the $(a+m+1)$-th entry and 0 elsewhere in $\mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil}$. Hence $(\rho, \mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil})$ is a nice faithful nilrepresentation of $\mathfrak{g}$. It follows that

$$\widetilde{\mu}_{nic}(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}) \leq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t+1} \rceil.$$

For any $x \in \mathfrak{g}$, if $n = 2k$, $k \in \mathbb{N}$, then suppose that

$$x = \sum_{i=1}^{m} a_i x_i + \sum_{i=1}^{m} a_i' x_i' + cz + \sum_{j=1}^{k} b_j Y_j + \sum_{j=1}^{k} b_j' Y_j' + \sum_{k=1}^{s} p_k A_k + \sum_{l=1}^{t} q_l B_l.$$

Then $\rho(x)$ is presented under the standard basis of $\mathbb{F}^{m+a+b+\lceil\frac{n}{2}\rceil+a'}$ as shown in Matrix 1.

$$
\begin{array}{c}
\begin{array}{ccccc} \overbrace{\qquad}^{a} & \overbrace{\qquad\qquad}^{m} & \overbrace{\qquad\qquad\qquad\qquad}^{b} & \overbrace{\qquad\qquad}^{k} & \overbrace{\qquad}^{a'} \end{array} \\
\left(\begin{array}{c ccc cccc|ccc c}
 & a_1 & \cdots & a_m & c & p_1 & \cdots & p_{b-1} & b_1 & \cdots & b_k & \\
 & & & & p_b & p_{b+1} & \cdots & p_{2b-1} & & & & \\
 & & & & \vdots & \ddots & \ddots & \vdots & & & & \\
 & & & & p_{rb} & \cdots & p_s & 0 & & & & \\
 & & & & 0 & \cdots & 0 & 0 & & & & \\
 & & & & a'_1 & & & & & & & \\
 & & & & \vdots & & & & & & & \\
 & & & & a'_m & & & & & & & \\
 & & & & & & & & & & & \\
\hline
 & & & & b'_1 & & & & & & & \\
 & & & & \vdots & & & & & & & \\
 & & & & b'_k & & & & & & & \\
 & & & & q_1 & q_2 & \cdots & q_b & & & & \\
 & & & & q_{b+1} & q_{b+2} & \cdots & q_{2b} & & & & \\
 & & & & \vdots & \ddots & \ddots & \vdots & & & & \\
 & & & & q_{r'b+1} & \cdots & q_t & 0 & & & & \\
 & & & & 0 & 0 & \cdots & 0 & & & &
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} a \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m \\
\left.\begin{array}{c} \\ \end{array}\right\} b \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} k \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} a'
\end{array}
\end{array}
$$

Matrix 1

If $n = 2k+1,\ k\in\mathbb{N}$, then suppose that

$$x = \sum_{i=1}^{m} a_i x_i + \sum_{i=1}^{m} a'_i x'_i + cz + \sum_{j=1}^{k} b_j Y_j + \sum_{j=1}^{k} b'_j Y'_j + bY_{k+1} + \sum_{k=1}^{s} p_k A_k + \sum_{l=1}^{t} q_l B_l.$$

Then $\rho(x)$ is presented under the standard basis of $\mathbb{F}^{m+a+b+\lceil\frac{n}{2}\rceil+a'}$ as shown in Matrix 2.

If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then let $(\sigma, \mathbb{F}^{a+n+b+d})$ defined above be a faithful nilrepresentation of $\mathfrak{g}$. Then $ab \geq t+1$ and $bd \geq s$. Hence

$$\dim \mathbb{F}^{a+n+b+d} = a+n+b+d \geq n + \lceil 2\sqrt{b(a+d)}\rceil \geq n + \lceil 2\sqrt{s+t+1}\rceil.$$

Since

$$\min\{\dim(\mathbb{F}^{a+n+b+d}) \mid ab \geq t+1, bd \geq s\} = n + \lceil 2\sqrt{s+t+1}\rceil$$

we conclude that $(\sigma, \mathbb{F}^{n+\lceil 2\sqrt{s+t+1}\rceil})$ is a faithful nilrepresentation of $\mathfrak{g}$. By the definition of $(\sigma, \mathbb{F}^{n+\lceil 2\sqrt{s+t+1}\rceil})$, we have $\sigma(Z(\mathfrak{g}))(v) = 0$ for any $v \in \mathbb{F}_{\bar{0}}^{n+\lceil 2\sqrt{s+t+1}\rceil}$.

$$
\begin{array}{c}
\overbrace{\quad}^{a}\overbrace{\qquad\qquad}^{m}\overbrace{\qquad\qquad\qquad\qquad}^{b}\overbrace{\qquad\qquad\quad}^{k+1}\overbrace{\qquad}^{a'}\\
\left(\begin{array}{c ccc cccc|cccc c}
 & a_1 & \cdots & a_m & c & p_1 & \cdots & p_{b-1} & b_1 & \cdots & b_k & \frac{1}{2}b & \\
 & & & & p_b & p_{b+1} & \cdots & p_{2b-1} & & & & & \\
 & & & & \vdots & \ddots & \ddots & \vdots & & & & & \\
 & & & & p_{rb} & \cdots & p_s & 0 & & & & & \\
 & & & & 0 & \cdots & 0 & 0 & & & & & \\
 & & & & a'_1 & & & & & & & & \\
 & & & & \vdots & & & & & & & & \\
 & & & & a'_m & & & & & & & & \\
 & & & & & & & & & & & & \\
\hline
 & & & & b'_1 & & & & & & & & \\
 & & & & \vdots & & & & & & & & \\
 & & & & b'_k & & & & & & & & \\
 & & & & b & & & & & & & & \\
 & & & & q_1 & q_2 & \cdots & q_b & & & & & \\
 & & & & q_{b+1} & q_{b+2} & \cdots & q_{2b} & & & & & \\
 & & & & \vdots & \ddots & \ddots & \vdots & & & & & \\
 & & & & q_{r'b+1} & \cdots & q_t & 0 & & & & & \\
 & & & & 0 & 0 & \cdots & 0 & & & & &
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right\} a\\
\left.\begin{array}{c}\\ \\ \end{array}\right\} m\\
\left.\begin{array}{c}\\ \\ \end{array}\right\} b\\
\left.\begin{array}{c}\\ \\ \\ \end{array}\right\} k+1\\
\left.\begin{array}{c}\\ \\ \\ \\ \\ \end{array}\right\} a'
\end{array}
\end{array}
$$

Matrix 2

This yields $0 = r_1 \leq r_2$. Since

$$0 \neq \sigma(z)(\varepsilon_{a+n+1}) \in \sigma(z)(\mathbb{F}_1^{n+\lceil 2\sqrt{s+t+1}\rceil}),$$

where ε_{a+n+1} stand for the column vector with 1 in the $(a+n+1)$-th entry and 0 elsewhere in $\mathbb{F}^{n+\lceil 2\sqrt{s+t+1}\rceil}$. Hence $(\sigma, \mathbb{F}^{n+\lceil 2\sqrt{s+t+1}\rceil})$ is a nice faithful nilrepresentation of $\mathfrak{g}$. It follows that

$$\widetilde{\mu}_{nic}(\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}) \leq n + \lceil 2\sqrt{s+t+1}\rceil.$$

For any $x \in \mathfrak{g}$, suppose that

$$x = \sum_{i=1}^{n} a_i x_i + \sum_{k=1}^{s} p_k A_k + cz + \sum_{i=1}^{n} b_i y_i + \sum_{l=1}^{t} q_l B_l,$$

where $1 \leq i \leq n$, $1 \leq k \leq s$, $1 \leq l \leq t$ and $c \in \mathbb{F}$. Then $\sigma(x)$ is presented under the standard basis of $\mathbb{F}^{a+n+b+d}$ as shown in Matrix 3. ■

Theorem 4.3. *Let $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ or $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$. Then*

$$\mu_{nic}(\mathfrak{g}) \leq \begin{cases} m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t}\rceil, & \text{if } \mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}, \\ n + \lceil 2\sqrt{s+t}\rceil, & \text{if } \mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}. \end{cases}$$

$$
\begin{array}{c}
\begin{array}{cccc} \overbrace{\qquad}^{a} & \overbrace{\qquad\qquad}^{n} & \overbrace{\qquad\qquad\qquad\qquad}^{b} & \overbrace{\qquad}^{d} \end{array} \\
\left(\begin{array}{cccc|ccccc}
 & a_1 & \cdots & a_n & c & q_1 & \cdots & q_{b-1} & \\
 & & & & q_b & q_{b+1} & \cdots & q_{2b-1} & \\
 & & & & \vdots & \ddots & \ddots & \vdots & \\
 & & & & q_{r_1 b} & \cdots & q_t & 0 & \\
 & & & & 0 & \cdots & \cdots & 0 & \\
 & & & & b_1 & & & & \\
 & & & & \vdots & & & & \\
 & & & & b_n & & & & \\
\hline
 & & & & & & & & \\
 & & & & p_1 & p_2 & \cdots & p_b & \\
 & & & & p_{b+1} & p_{b+2} & \cdots & p_{2b} & \\
 & & & & \vdots & \ddots & \ddots & \vdots & \\
 & & & & p_{rb+1} & \cdots & p_s & 0 & \\
 & & & & 0 & \cdots & \cdots & 0 &
\end{array}\right)
\begin{array}{l}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} a \\
\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} n \\
\left.\vphantom{\begin{array}{c}1\end{array}}\right\} b \\
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} d
\end{array}
\end{array}
$$

Matrix 3

Proof. If $\mathfrak{g} = \mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$, then let $(\widetilde{\rho}, \mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil})$ be an even mapping given by

$$\widetilde{\rho}(x) = \rho(x), \ \forall x \in \mathfrak{g}/\mathbb{F}\{A_s\},$$
$$\widetilde{\rho}(A_s) = e_{2,a+1} - e_{2,a+m+1}.$$

We can obtain that $\widetilde{\rho}$ is a faithful representation of $\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}$ and $\widetilde{\rho}(Z(\mathfrak{g}))(w) = 0$ for any $w \in \mathbb{F}_{\bar{1}}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil}$. This yields $r_1 \geq r_2 = 0$. Since

$$0 \neq \widetilde{\rho}(z)(\varepsilon_{a+m+1}) \in \widetilde{\rho}(z)(V_{\bar{0}}),$$

where ε_{a+m+1} stand for the column vector with 1 in the $(a+m+1)$-th entry and 0 elsewhere in $\mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil}$. Hence $(\widetilde{\rho}, \mathbb{F}^{m+\lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil})$ is a nice faithful representation of $\mathfrak{g}$. It follows that

$$\mu_{nic}(\mathfrak{h}_{m,n} \oplus \mathfrak{a}_{s,t}) \leq m + \lceil \frac{n}{2} \rceil + \lceil 2\sqrt{s+t} \rceil.$$

If $\mathfrak{g} = \mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$, then let $(\widetilde{\sigma}, \mathbb{F}^{n+\lceil 2\sqrt{s+t} \rceil})$ be an even mapping given by

$$\widetilde{\sigma}(x) = \sigma(x), \forall x \in \mathfrak{g}/\mathbb{F}\{B_t\},$$
$$\widetilde{\sigma}(B_t) = e_{a+1,2'} - e_{1,2'}.$$

We can obtain that $\widetilde{\sigma}$ is a faithful representation of $\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}$ and $\widetilde{\sigma}(Z(\mathfrak{g}))(v) = 0$ for any $v \in \mathbb{F}_{\bar{0}}^{n+\lceil 2\sqrt{s+t} \rceil}$. This yields $0 = r_1 \leq r_2$. Since

$$0 \neq \widetilde{\sigma}(z)(\varepsilon_{a+n+1}) \in \widetilde{\sigma}(z)(V_{\bar{1}}),$$

where ε_{a+n+1} stand for the column vector with 1 in the $(a+n+1)$-th entry and 0 elsewhere in $\mathbb{F}^{n+\lceil 2\sqrt{s+t} \rceil}$. Hence $(\widetilde{\sigma}, \mathbb{F}^{n+\lceil 2\sqrt{s+t} \rceil})$ is a nice faithful representation of $\mathfrak{g}$. It follows that

$$\mu_{nic}(\mathfrak{h}_n \oplus \mathfrak{a}_{s,t}) \leq n + \lceil 2\sqrt{s+t} \rceil. \qquad \blacksquare$$

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