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Generalized BGG Resolutions and Blattner's Formula in Type A

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Abstract. Consider the natural action of $\mathrm{GL}_n(\mathbb{C})$ on p covectors and q vectors; by Howe duality, the space of polynomial functions on this space decomposes multiplicity-free under the joint action of $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{gl}_{p+q}(\mathbb{C})$. When $n \geq p+q$ (which is known as the stable range), the $\mathfrak{gl}_{p+q}$-modules are generalized Verma modules (GVMs, introduced by Lepowsky), on which the unipotent radical of the Hermitian real form $\mathrm{U}(p,q)$ of $\mathfrak{gl}_{p+q}$ acts freely. When $n < p+q$, however, the structure of these modules is less transparent. Enright and Willenbring (2004) constructed resolutions for them in terms of GVMs. The goal of this paper is to exhibit a remarkable connection between these resolutions and a seemingly quite different situation, namely the K-type multiplicities in certain discrete series of $\mathrm{SU}(n, p+q)$. More precisely, we establish that the signed multiplicities of the GVMs in the resolution coincide with the values of Blattner's formula for the K-type multiplicities in appropriately chosen discrete series representations of $\mathrm{SU}(n, p+q)$.

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1. Introduction

The main result of this paper is a surprising dictionary between the following:

- the generalized BGG resolutions [5, 12] of the infinite-dimensional $\mathfrak{su}(p,q)$-modules arising in the Howe duality setting [9] ;
- Blattner's formula for the K-type multiplicities in certain discrete series representations of $\mathrm{SU}(n, p+q)$. (In this situation, K is the maximal compact subgroup $\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(p+q))$.)

The first of these two settings is the action of the general linear group $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, with defining representation $V = \mathbb{C}^n$, on the space $W := (V^*)^{\oplus p} \oplus V^{\oplus q}$. As a result of Howe duality, the polynomial functions on W decompose under the joint action of $\mathrm{GL}_n \times \mathfrak{gl}_{p+q}$:

$$\mathbb{C}[W] \cong \bigoplus_{\lambda} F_n^{\lambda} \otimes \widetilde{F}_{p,q}^{\lambda},$$

where the F_n^{λ} are irreducible rational representations of GL_n, and the $\widetilde{F}_{p,q}^{\lambda}$ are infinite-dimensional highest weight $\mathfrak{gl}_{p+q}$-modules. Recall that in the *stable range* $(n \geq p+q)$, these $\widetilde{F}_{p,q}^{\lambda}$ are generalized Verma modules [12].

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Although the modules $\widetilde{F}^{\lambda}_{p,q}$ have a complicated structure outside the stable range, they nonetheless have finite resolutions in terms of generalized Verma modules $N_{\mu,\nu}$. These resolutions are the *generalized BGG resolutions* constructed by Enright and Willenbring in [5]. The generalized BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$ is determined by the (non)occurrence of each possible $N_{\mu,\nu}$ along with its sign, which we will encode (see Definition 2.1) via the integer $\varepsilon^{\lambda}_{\mu,\nu} \in \{1, 0, -1\}$, since each $N_{\mu,\nu}$ occurs at most once.

The second setting is, seemingly, far removed from the first: namely, the discrete series of the special indefinite unitary group $\mathrm{SU}(n, p+q)$. The discrete series form the bedrock of Harish-Chandra's Plancherel Theorem for semisimple Lie groups (see Section 3 for more background). Harish-Chandra first conjectured (1954), and subsequently proved in [7], that a linear connected semisimple real Lie group G_0 has discrete series representations if and only if its rank is equal to that of a maximal compact subgroup K_0. It is natural to restrict a representation of G_0 to the action of the complexified Lie algebra $\mathfrak{k}$, and to study the multiplicities of irreducible $\mathfrak{k}$-modules ("$\mathfrak{k}$-types") in the resulting decomposition. A formula for these $\mathfrak{k}$-type multiplicities — an alternating sum over the Weyl group of $\mathfrak{k}$ — was (according to Harish-Chandra) conjectured by Robert J. Blattner, and this "Blattner's formula" was eventually proved by Hecht and Schmid in [8]. We will write $B(\delta, \eta)$ to denote the value of Blattner's formula for the $\mathfrak{k}$-type with highest weight δ inside the discrete series representation of G_0 with Blattner parameter η. (The Blattner parameter is the highest weight of the minimal $\mathfrak{k}$-type; see [11, p. 310].) Outside certain dominance and regularity conditions, the outputs of Blattner's formula do not correspond to actual multiplicities, and can assume negative values.

When we take $G_0 = \mathrm{SU}(n, p+q)$ with a certain choice of embedding of K_0 (see Figure 2), we obtain our main result (Theorem 4.1) uniting the two settings above, namely the identity

$$\varepsilon^{\lambda}_{\mu,\nu} = B\big(0, [\![\lambda, \mu, \nu]\!]\big),$$

with the $[\![-,-,-]\!]$ notation to be defined below in (14). The theoretical heart of the paper comprises Lemmas 5.1 and 5.2 (which spell out the explicit connection between $\mathbb{C}[W]$ and the generating function $\mathbf{b}(0) := \sum_{\eta} B(0,\eta) e^{\eta}$ introduced by Willenbring and Zuckerman in [14]), along with the character formula (10) and the argument leading to formula (17). Our result allows us (see Example A.2) to write down the BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$ — without any knowledge of the machinery used in [5] — simply by reading off the coefficient of $e^{[\![\lambda,0,0]\!]}$ in $\mathbf{b}(0)$.

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2. Howe duality in Type A

We begin by recalling the Howe duality setting in Type A (i.e., in the case where the complex classical group is the general linear group). We primarily follow the exposition in [10, Section 3]. Fix $n, p, q \in \mathbb{Z}^+$, and let

$$W = \mathrm{M}_{n,p} \oplus \mathrm{M}_{n,q} \tag{1}$$

where the matrices are over $\mathbb{C}$. Let $\mathbb{C}[W]$ denote the space of polynomial functions on W. Now consider the pair $(\mathrm{U}(n), \mathrm{U}(p,q))$. This is known as a *compact dual* *pair*, since the first element is a compact group, and since the two groups are the centralizers of each other as subgroups of $\mathrm{Sp}(2n(p+q),\mathbb{R})$.

In order to study the action of the compact dual pair $(\mathrm{U}(n),\mathrm{U}(p,q))$ on $\mathbb{C}[W]$, it will be convenient to consider instead the complexification of the first group, namely $\mathrm{GL}_n := \mathrm{GL}_n(\mathbb{C})$, and the complexified Lie algebra of the second group, namely $\mathfrak{gl}_{p+q} := \mathfrak{gl}_{p+q}(\mathbb{C})$; this will allow us to avoid technicalities involving a covering group of $\mathrm{U}(p,q)$. Hence, we consider the pair $(\mathrm{GL}_n, \mathfrak{gl}_{p+q})$ and its action on $\mathbb{C}[W]$. First, the group GL_n acts on W via

$$g \cdot (X,Y) = (g^{-T}X, gY) \tag{2}$$

for $g \in \mathrm{GL}_n$ and $(X,Y) \in W$, where the superscript $-T$ denotes the inverse transpose. The action (2) induces the usual action on $\mathbb{C}[W]$ via

$$g \cdot f(X,Y) = f(g^T X, g^{-1}Y).$$

This action of GL_n on $\mathbb{C}[W]$ induces an action by conjugation on the Weyl algebra $\mathcal{D}(W)$ of polynomial-coefficient differential operators on $\mathbb{C}[W]$; in particular, conjugation by GL_n normalizes $\mathcal{D}(W)$ inside $\mathrm{End}(\mathbb{C}[W])$. Let $\mathcal{D}(W)^{\mathrm{GL}_n}$ be the subalgebra containing those operators that commute with the GL_n-action on $\mathbb{C}[W]$. Then $\mathcal{D}(W)^{\mathrm{GL}_n}$ is generated (as an algebra) by a subset which is isomorphic to $\mathfrak{gl}_{p+q}$ as a Lie algebra; see [2, Section 2] for the explicit isomorphism. This fact implies the following Howe duality decomposition as a module for $\mathrm{GL}_n \times \mathfrak{gl}_{p+q}$:

$$\mathbb{C}[W] \cong \bigoplus_{\lambda} F_n^{\lambda} \otimes \widetilde{F}_{p,q}^{\lambda} \tag{3}$$

where

- λ ranges over a certain subset (to be specified below) of the highest weights indexing the rational representations of GL_n;
- F_n^{λ} is the irreducible rational representation of GL_n with highest weight λ;
- $\widetilde{F}_{p,q}^{\lambda}$ is an infinite-dimensional $\mathfrak{gl}_{p+q}$-module (which is unitarizable with respect to $\mathfrak{su}(p,q)$), uniquely determined by λ.

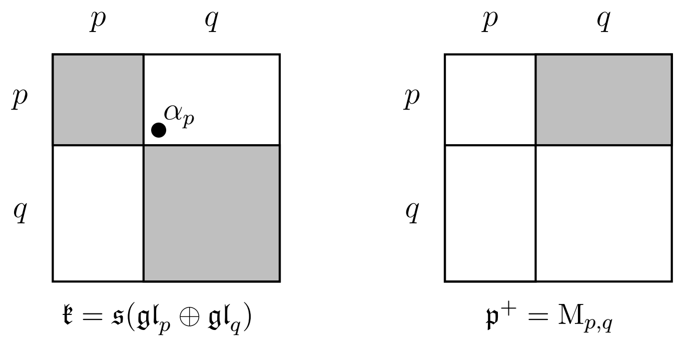


Figure 1: The block-diagonal embedding $\mathfrak{k} \subset \mathfrak{g} = \mathfrak{sl}_{p+q}$. Note the unique noncompact simple root α_p (see Section 3).

For the purposes of this paper, we restrict $\mathrm{U}(p,q)$ to the subgroup $G_0 = \mathrm{SU}(p,q)$. Therefore, writing $\mathfrak{g}_0 = \mathfrak{su}(p,q)$ and $\mathfrak{g} := (\mathfrak{g}_0)^{\mathbb{C}} = \mathfrak{sl}_{p+q} \subset \mathfrak{gl}_{p+q}$, we will regard $\widetilde{F}_{p,q}^{\lambda}$ as a $\mathfrak{g}$-module.

Consider the subalgebra $\mathfrak{k}_0 = \mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q)) \subset \mathfrak{g}_0$, embedded block-diagonally as in Figure 1. This $\mathfrak{k}_0$ is the Lie algebra of the maximal compact subgroup

$K_0 = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q)) \subset G_0$, for which reason we say that $\mathfrak{k}_0$ is a "maximal compact subalgebra" of $\mathfrak{g}_0$. (The symbols "S" and "$\mathfrak{s}$" denote the determinant-one and trace-zero conditions, respectively.) The complexification $\mathfrak{k} := (\mathfrak{k}_0)^{\mathbb{C}} = \mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q) \subset \mathfrak{g}$ acts on W by the derived action (up to a central character) of the complexified group $K = \mathrm{S}(\mathrm{GL}_p \times \mathrm{GL}_q)$:

$$(g, h) \cdot (X, Y) = (Xg^{-1}, Yh^{-1}), \tag{4}$$

where $g \in \mathrm{GL}_p$ and $h \in \mathrm{GL}_q$. Combining (2) and (4), we have an action by the product $M' = \mathrm{S}(\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q)$ upon W:

$$(g_n, g_p, g_q) \cdot (X, Y) = (g_n^{-T} X g_p^{-1}, g_n Y g_q^{-1}). \tag{5}$$

(We choose the notation M' because in Section 4 this group will play the role of a Levi factor in a different setting — a setting in which we will decorate all groups and Lie algebras with prime symbols in order to distinguish them from the present context.)

We now clarify which λ's appear in the decomposition (3). Recall that irreducible rational representations F_n^λ of GL_n are usually indexed by their highest weights, which (in standard coordinates) range over all weakly decreasing integer n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$. If the λ_i are all nonnegative, then λ is called a *partition*, and F_n^λ is a polynomial representation of GL_n. ("Rational" and "polynomial" refer to the matrix entry functions in the image of the representation, in terms of the coordinate functions on GL_n.) The number of positive components of a partition λ is called the *length* of λ, denoted by $\ell(\lambda)$. Note that any weakly decreasing n-tuple λ can be described by means of a pair $[\lambda^+, \lambda^-]$ of partitions λ^+ and λ^-, of lengths a and b respectively (where $a + b \le n$), by setting

$$\lambda = [\lambda^+, \lambda^-] := (\lambda_1^+, \ldots, \lambda_a^+, 0, \ldots, 0, -\lambda_b^-, \ldots, -\lambda_1^-).$$

For example, if $\lambda = (5, 2, 0, 0, -1, -3, -4)$, then $\lambda^+ = (5, 2)$ and $\lambda^- = (4, 3, 1)$. We will consider two partitions equivalent if they differ only by trailing zeros; thus $(5, 2)$ is the same as $(5, 2, 0, 0)$.

The sum in (3) ranges over all weakly decreasing integer n-tuples λ such that $\ell(\lambda^+) \le p$ and $\ell(\lambda^-) \le q$. These length conditions provide a context for a notion of "stability": following [10, p. 1609], we say that the parameters n, p, q lie in the *stable range* when $n \ge p + q$.

Inside the stable range, any pair of partitions (λ^+, λ^-) such that $\ell(\lambda^+) \le p$ and $\ell(\lambda^-) \le q$ determines a weakly decreasing n-tuple $\lambda = [\lambda^+, \lambda^-]$, i.e., the highest weight of the irreducible rational representation F_n^λ of GL_n. Furthermore, inside the stable range, upon restriction to $K = \mathrm{S}(\mathrm{GL}_p \times \mathrm{GL}_q)$, the module $\widetilde{F}_{p,q}^\lambda$ has the especially nice structure

$$\widetilde{F}_{p,q}^\lambda \cong \mathbb{C}[\mathrm{M}_{p,q}] \otimes (F_p^{\lambda^+} \otimes F_q^{\lambda^-}), \tag{6}$$

where K acts on the second tensor factor in the obvious way, and on the first tensor factor via

$$(g, h) \cdot f(X) = f(g^T X h). \tag{7}$$

The reader may recognize (6) as the generalized Verma module for $\mathfrak{g}$ induced from the irreducible $\mathfrak{k}$-module $F_p^{\lambda^+} \otimes F_q^{\lambda^-}$. In this context, we have $\mathbb{C}[\mathrm{M}_{p,q}] \cong S(\mathfrak{p}^-)$ as $\mathfrak{k}$-modules, given the Cartan decomposition $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$. (See Figure 1.)

Outside the stable range, $\widetilde{F}^{\lambda}_{p,q}$ has a more complicated structure than in (6), and admits a generalized BGG resolution, as shown by Enright and Willenbring in [5]. (This result is an extension of the Lepowsky resolution of a finite-dimensional module in terms of generalized Verma modules [12], which in turn generalized the BGG resolution of a finite-dimensional module in terms of Verma modules [1].) Each of the finitely many terms in the generalized BGG resolution is the direct sum of generalized Verma modules

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} L_{\mathfrak{k}}(\xi) \tag{8}$$

associated with the standard maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^{+}$, where $L_{\mathfrak{k}}(\xi)$ denotes the irreducible $\mathfrak{k}$-module with highest weight ξ, regarded as a $\mathfrak{q}$-module by letting $\mathfrak{p}^{+}$ act trivially. Since $\mathfrak{p}^{-}$ is abelian, it follows from the PBW theorem that (8) is isomorphic to $S(\mathfrak{p}^{-}) \otimes L_{\mathfrak{k}}(\xi)$ as a $\mathfrak{k}$-module. Therefore in the BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$, the generalized Verma modules restrict as $\mathfrak{k}$-modules to the form

$$N_{\mu,\nu} := \mathbb{C}[\mathrm{M}_{p,q}] \otimes (F^{\mu}_{p} \otimes F^{\nu}_{q}),$$

where μ and ν are partitions of lengths at most p and q, respectively. See Theorem 2 and the preceding discussion in [5] for the explicit construction of the individual terms in the resolution; for our purposes, we will appeal only to its existence, because our main result will actually show us how to read off its terms from a certain generating function (see Example A.2).

We now assign to each generalized Verma module $N_{\mu,\nu}$ an integer $\varepsilon^{\lambda}_{\mu,\nu} \in \{-1, 0, 1\}$ which describes its occurrence in the BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$. (It is apparent from the construction in [5] that each $N_{\mu,\nu}$ can appear at most once in the resolution.) Let $\mathrm{ch}_{\mathfrak{k}}(-)$ denote the character of a $\mathfrak{k}$-module. Let

$$\mathcal{X} := \left\{ \mathrm{ch}_{\mathfrak{k}}(N_{\mu,\nu}) \;\middle|\; \ell(\mu) \le p \text{ and } \ell(\nu) \le q \right\},$$

which is a basis for its integer span $\mathbb{Z}\mathcal{X}$. Hence $\mathrm{ch}_{\mathfrak{k}}(\widetilde{F}^{\lambda}_{p,q})$ has a unique expansion as an element of $\mathbb{Z}\mathcal{X}$, which we can regard as the Euler characteristic of the BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$.

Definition 2.1. For fixed n, p, q, suppose λ, μ, ν satisfy the following conditions:

$$\lambda \text{ a weakly decreasing } n\text{-tuple}; \quad \ell(\lambda^{+}), \ell(\mu) \le p; \quad \ell(\lambda^{-}), \ell(\nu) \le q. \tag{9}$$

Then we define $\varepsilon^{\lambda}_{\mu,\nu} :=$ the coefficient of $\mathrm{ch}_{\mathfrak{k}}(N_{\mu,\nu})$ in $\mathrm{ch}_{\mathfrak{k}}(\widetilde{F}^{\lambda}_{p,q})$, where $\mathrm{ch}_{\mathfrak{k}}(\widetilde{F}^{\lambda}_{p,q})$ is expanded as an element of $\mathbb{Z}\mathcal{X}$. ■

Whether or not we are in the stable range, we thus have a $\mathfrak{k}$-character of the form

$$\mathrm{ch}_{\mathfrak{k}}(\widetilde{F}^{\lambda}_{p,q}) = \mathrm{ch}_{\mathfrak{k}}(\mathbb{C}[\mathrm{M}_{p,q}]) \cdot \sum_{\mu,\nu} \varepsilon^{\lambda}_{\mu,\nu} \cdot \mathrm{ch}_{\mathfrak{k}}(F^{\mu}_{p} \otimes F^{\nu}_{q}), \tag{10}$$

where $\varepsilon^{\lambda}_{\mu,\nu}$ is nonzero for only finitely many μ, ν. Note that inside the stable range, (6) implies that

$$\varepsilon^{\lambda}_{\mu,\nu} = \delta_{(\lambda^{+},\lambda^{-}),\,(\mu,\nu)}$$

where δ is the Kronecker delta.

We now write out explicitly the character of $\mathbb{C}[W]$ as a representation of the group $M' = \mathrm{S}(\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q)$, under the action in (5). Combining (3) and (10), we have the factorization

$$\operatorname{ch}\mathbb{C}[W] = \operatorname{ch}\mathbb{C}[\mathrm{M}_{p,q}] \cdot \left(\sum_{\lambda,\mu,\nu} \varepsilon^{\lambda}_{\mu,\nu} \cdot \operatorname{ch}(F^{\lambda}_n \otimes F^{\mu}_p \otimes F^{\nu}_q) \right) \tag{11}$$

as a character of M' and therefore of $\mathfrak{m}' = \mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_p \oplus \mathfrak{gl}_q)$. This time the sum is infinite, since λ ranges over an infinite set. We remark that at this point, the integers $\varepsilon^{\lambda}_{\mu,\nu}$ are still quite mysterious, since they depend upon understanding the BGG resolution of each module $\widetilde{F}^{\lambda}_{p,q}$.

3. Discrete series and Blattner's formula

An irreducible unitary representation of a Lie group G_0 is a *discrete series* representation if it is isomorphic to a representation of G_0 acting on a translation-invariant subspace of $L^2(G_0)$. Equivalently, a discrete series representation has positive point mass with respect to the Plancherel measure on $\widehat{G}_0$. The discrete series plays a central role in Harish-Chandra's Plancherel Theorem for semisimple Lie groups.

Harish-Chandra [7] proved that a connected semisimple group G_0 has discrete series representations if and only if $\operatorname{rank} G_0 = \operatorname{rank} K_0$, where $K_0 \subset G_0$ is a maximal compact subgroup. In other words, there is a maximal torus $H_0 \subset K_0$ which is a Cartan subgroup of G_0. Let $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ denote the complexified Lie algebras. Let $\Phi := \Phi(\mathfrak{g}, \mathfrak{h})$ denote the root system for $\mathfrak{g}$ with respect to $\mathfrak{h}$. We call a root $\alpha \in \Phi$ a *compact root* if its corresponding root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}$; otherwise, we call the root *noncompact*. We denote the sets of compact and noncompact roots by Φ_{c} and Φ_{nc}, respectively. Thus we have $\mathfrak{k} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_{\mathrm{c}}} \mathfrak{g}_\alpha$. We write $\mathfrak{p} := \bigoplus_{\alpha\in\Phi_{\mathrm{nc}}} \mathfrak{g}_\alpha$ for the sum of the noncompact root spaces, and so we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Each choice of positive roots $\Phi^+ \subset \Phi$ determines a set $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ of simple roots, along with a positive Weyl chamber in $\mathfrak{h}^*$. Set $\Pi_{\mathrm{c}} := \Pi \cap \Phi_{\mathrm{c}}$ and $\Pi_{\mathrm{nc}} := \Pi \cap \Phi_{\mathrm{nc}}$. We emphasize that the number of noncompact simple roots depends on this choice of positive Weyl chamber. (The case where $|\Pi_{\mathrm{nc}}| = 1$, as in Figure 1, is especially important in the theory of maximal parabolic subalgebras, and in this case we say that (G_0, K_0) is a *Hermitian symmetric pair*. See [4] for a detailed exposition of this theory. Moreover, in this case the discrete series is said to be *holomorphic*. These were the first examples constructed by Harish-Chandra.) The analytic properties of the discrete series of G_0 also depend on this choice of positive Weyl chamber. This viewpoint can be restated equivalently as follows (which will be more convenient in this paper): if one first fixes Φ^+, then the size of Π_{nc}, along with the analytic properties of the discrete series, depend on the choice of embedding $\mathfrak{k} \subset \mathfrak{g}$. Since this dependence plays a major role in our main result, we illustrate with an example below.

Example 3.1. Let $G_0 = \mathrm{SU}(3,2)$, so that $K_0 = \mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2))$. Then $\mathfrak{g} = \mathfrak{sl}_5$, and we let $\mathfrak{h} \subset \mathfrak{g}$ consist of the diagonal matrices, with positive roots $\varepsilon_i - \varepsilon_j$, for $1 \le i < j \le 5$ (in standard Bourbaki notation). Then the four simple roots are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, for $1 \le i \le 4$. Each root space $\mathfrak{g}_{\varepsilon_i - \varepsilon_j}$ is the span of the matrix E_{ij} (containing a 1 in the (i,j) position and 0's elsewhere). In this example, we show

how the number of noncompact simple roots varies with the choice of embedding of $\mathfrak{k} = \mathfrak{s}(\mathfrak{gl}_3 \oplus \mathfrak{gl}_2)$. Below we give examples of embeddings (where the asterisks represent arbitrary entries) with one, two, three, and four noncompact simple roots (which we depict with a boldface $\mathbf{0}$ in the corresponding root space):

$$\begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & \mathbf{0} & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} \qquad \begin{bmatrix} * & * & 0 & 0 & * \\ * & * & \mathbf{0} & 0 & * \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & \mathbf{0} \\ * & * & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & \mathbf{0} & * & * & 0 \\ 0 & * & \mathbf{0} & 0 & * \\ * & 0 & * & * & 0 \\ * & 0 & * & * & \mathbf{0} \\ 0 & * & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & \mathbf{0} & * & 0 & * \\ 0 & * & \mathbf{0} & * & 0 \\ * & 0 & * & \mathbf{0} & * \\ 0 & * & 0 & * & \mathbf{0} \\ * & 0 & * & 0 & * \end{bmatrix}$$

An important technique in representation theory is to decompose representations of G_0 under restriction to the action of the maximal compact subgroup K_0. In particular, passing to the complexified Lie algebra $\mathfrak{k}$, it is natural to study the multiplicities of the finite-dimensional simple $\mathfrak{k}$-modules (i.e., $\mathfrak{k}$-*types*) in the decomposition of a representation of G_0. In the case of the discrete series, there is a known formula for the $\mathfrak{k}$-type multiplicities, named after Robert J. Blattner. Before presenting Blattner's formula, we first explain the necessary notation below, following that used by Willenbring and Zuckerman in [14].

As above, let Φ^+ denote a choice of positive roots and let $\Phi^- = -\Phi^+$ denote the set of negative roots. We denote the sets of positive/negative compact/noncompact roots as follows:

$$\Phi^+_{\mathrm{c}} = \Phi^+ \cap \Phi_{\mathrm{c}}, \quad \Phi^+_{\mathrm{nc}} = \Phi^+ \cap \Phi_{\mathrm{nc}}, \quad \Phi^-_{\mathrm{c}} = \Phi^- \cap \Phi_{\mathrm{c}}, \quad \Phi^-_{\mathrm{nc}} = \Phi^- \cap \Phi_{\mathrm{nc}}.$$

Then we have $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where

$$\mathfrak{p}^+ = \bigoplus_{\alpha \in \Phi^+_{\mathrm{nc}}} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{p}^- = \bigoplus_{\alpha \in \Phi^-_{\mathrm{nc}}} \mathfrak{g}_\alpha.$$

The terminology below follows any standard reference, such as [6, Ch. 3].

Let $\mathcal{W}_\mathfrak{g}$ and $\mathcal{W}_\mathfrak{k}$ be the Weyl groups for $\mathfrak{g}$ and $\mathfrak{k}$ and let $\ell(w) := |w(\Phi^+_{\mathrm{c}}) \cap \Phi^-_{\mathrm{c}}|$ denote the length of a Weyl group element $w \in \mathcal{W}_\mathfrak{k}$. Let $(-,-)$ denote the Killing form on $\mathfrak{g}$, which restricts to a nondegenerate form on $\mathfrak{h}$ and thus allows us to identify $\mathfrak{h}$ with $\mathfrak{h}^*$. Under this identification, $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ is identified with the simple coroot in $\mathfrak{h}$ corresponding to α, for each $\alpha \in \Pi$.

A weight $\xi \in \mathfrak{h}^*$ is called an *integral weight* for $\mathfrak{g}$ if $(\xi, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Pi$; the set of $\mathfrak{g}$-integral weights is denoted by $P(\mathfrak{g})$. The same condition defines the $\mathfrak{k}$-integral weights $P(\mathfrak{k})$ if we replace Π by Π_{c}. Moreover, ξ is said to be $\mathfrak{g}$-*dominant* if $(\xi, \alpha) \geq 0$ for all $\alpha \in \Pi$, and likewise $\mathfrak{k}$-dominant if $(\xi, \alpha) \geq 0$ for all $\alpha \in \Pi_{\mathrm{c}}$; the sets of dominant integral weights are denoted by $P_+(\mathfrak{g})$ and $P_+(\mathfrak{k})$. We say that $\xi \in \mathfrak{h}^*$ is $\mathfrak{g}$-*regular* if $(\xi, \alpha) \neq 0$ for all $\alpha \in \Phi$. For $\delta \in P_+(\mathfrak{k})$ we let $L_\mathfrak{k}(\delta)$ denote the irreducible representation of $\mathfrak{k}$ with highest weight δ. We define

$$\rho_{\mathrm{c}} := \frac{1}{2} \sum_{\alpha \in \Phi^+_{\mathrm{c}}} \alpha \qquad \text{and} \qquad \rho_{\mathrm{nc}} := \frac{1}{2} \sum_{\beta \in \Phi^+_{\mathrm{nc}}} \beta.$$

Finally, define $Q(\xi)$ to be the number of ways of writing $\xi \in P(\mathfrak{k})$ as a sum of noncompact positive roots; in other words, Q is the Φ^+_{nc}-partition function. Note that $Q(\xi)$ is zero in case ξ is not positive integral.

For $\delta, \eta \in P(\mathfrak{k})$, *Blattner's formula* is given by

$$B(\delta, \eta) = \sum_{w \in \mathcal{W}_{\mathfrak{k}}} (-1)^{\ell(w)} Q\big(w(\delta + \rho_{\mathrm{c}}) - \rho_{\mathrm{c}} - \eta\big). \tag{12}$$

Under certain assumptions on δ and η (see Theorem 3.2 below), the output of Blattner's formula gives a $\mathfrak{k}$-type multiplicity in a discrete series representation of G_0. In order to index these representations, we appeal to Vogan's theory [13] of the *minimal* $\mathfrak{k}$*-type*, i.e., the unique $\mathfrak{k}$-type which appears with multiplicity 1 in the $\mathfrak{k}$-decomposition of a discrete series representation. For $\eta \in P_+(\mathfrak{k})$, we write $D(\eta)$ to denote the discrete series representation of G_0 whose minimal $\mathfrak{k}$-type is $L_{\mathfrak{k}}(\eta)$. Following [11, p. 310], we call this η the *Blattner parameter* of $D(\eta)$. We now state the precise interpretation of Blattner's formula in terms of $\mathfrak{k}$-type multiplicities, as proved in [8].

Theorem 3.2 (Hecht and Schmid, 1975)**.** *Assume $\delta, \eta \in P_+(\mathfrak{k})$ such that the weight $\eta + \rho_{\mathrm{c}} - \rho_{\mathrm{nc}}$ is $\mathfrak{g}$-dominant regular. Then $B(\delta, \eta)$ equals the multiplicity of $L_{\mathfrak{k}}(\delta)$ in the discrete series representation $D(\eta)$ of G_0 with Blattner parameter η.*

In [14], Willenbring and Zuckerman introduce the formal power series

$$\mathbf{b}(\delta) := \sum_{\eta \in \mathfrak{h}^*} B(\delta, \eta) e^{\eta}$$

where e is a formal indeterminate. In words, $\mathbf{b}(\delta)$ encodes the values of Blattner's formula when the first argument $\delta \in P_+(\mathfrak{k})$ is held fixed. The main result in [14] is the formula in Proposition 2:

$$\mathbf{b}(\delta) = \operatorname{ch} L_{\mathfrak{k}}(\delta) \cdot \frac{\prod_{\alpha \in \Phi_{\mathrm{c}}^+} 1 - e^{-\alpha}}{\prod_{\beta \in \Phi_{\mathrm{nc}}^+} 1 - e^{-\beta}},$$

which specializes to

$$\mathbf{b}(0) = \frac{\prod_{\alpha \in \Phi_{\mathrm{c}}^+} 1 - e^{-\alpha}}{\prod_{\beta \in \Phi_{\mathrm{nc}}^+} 1 - e^{-\beta}}. \tag{13}$$

In proving our main result in Section 4, we will reinterpret $\mathbf{b}(0)$ in terms of the character of $\mathbb{C}[W]$, with W as defined above in (1).

4. Main result

We should point out that in the Howe duality decomposition (3), in the stable range $n \geq p + q$, the modules $\widetilde{F}^{\lambda}_{p,q}$ are themselves holomorphic discrete series representations (or limits thereof) of a covering group of $\mathrm{SU}(p,q)$. (See [10] for details.) These modules are also just generalized Verma modules, so no BGG resolution is needed. Recall from Figure 1 that in the Howe duality setting (3), the positive Weyl chamber was chosen such that $\mathfrak{g} = \mathfrak{sl}_{p+q}$ had exactly one noncompact root.

It turns out that the generalized BGG resolutions of the modules $\widetilde{F}^{\lambda}_{p,q}$ are related, in a very different way, to certain (non-holomorphic) discrete series representations of a different group $\mathrm{SU}(n, p+q)$. (This is true even inside the stable range, where the resolutions are trivial.) The thrust of our main result is that we can read off the character theory of $\mathbb{C}[W]$ – in particular the integers $\varepsilon^{\lambda}_{\mu,\nu}$ and hence the desired BGG resolutions – from the generating function $\mathbf{b}(0)$ for Blattner's formula for the discrete series of $\mathrm{SU}(n, p+q)$, where the positive Weyl chamber is chosen such that there are *two* noncompact simple roots.

To distinguish this new setting from the setting of Section 2, we will now decorate all of our notation with prime symbols; a complete summary is found in Table 1, accompanied by illustrations in Figure 2.

	Section 2		Sections 4 and 5	
Group	G_0	$\mathrm{SU}(p,q)$	G'_0	$\mathrm{SU}(n,p+q)$
Compl. Lie algebra	$\mathfrak{g}$	$\mathfrak{sl}_{p+q}$	$\mathfrak{g}'$	$\mathfrak{sl}_{p+n+q}$
Noncomp. simple roots	Π_{nc}	$\{\alpha_p\}$	Π'_{nc}	$\{\alpha_p, \alpha_{p+n}\}$
Compl. max. compact	$\mathfrak{k}$	$\mathfrak{s}(\mathfrak{gl}_p \oplus \mathfrak{gl}_q)$	$\mathfrak{k}'$	$\mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_{p+q})$
$\bigoplus$ pos nc root spaces	$\mathfrak{p}^+$	$\mathrm{M}_{p,q}$	$\mathfrak{p}'^+$	$\mathrm{M}_{p,n} \oplus \mathrm{M}_{n,q}$
$\bigoplus$ neg nc root spaces	$\mathfrak{p}^-$	$\mathrm{M}_{q,p}$	$\mathfrak{p}'^-$	$\mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n}$
Levi subalgebra	–	–	$\mathfrak{m}'$	$\mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_p \oplus \mathfrak{gl}_q)$

Table 1: Summary of notation in our two settings.

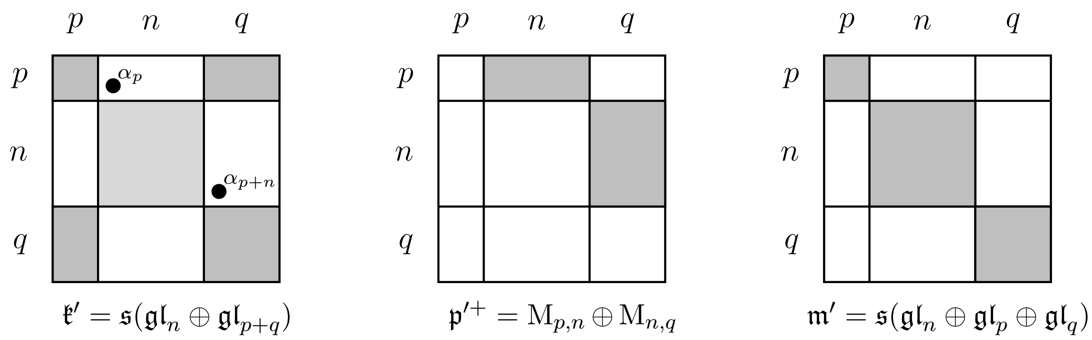


Figure 2: A visual companion to the notation in Table 1 for Sections 4 and 5; regard as subspaces of $\mathfrak{g}' = \mathfrak{sl}_{p+n+q}$.

Let $G'_0 = \mathrm{SU}(n, p+q)$, whose complexified Lie algebra is $\mathfrak{g}' = \mathfrak{sl}_{p+n+q}$. Crucially, we choose a set of positive roots in such a way that $\mathfrak{g}'$ has exactly two noncompact simple roots. Equivalently, given the standard choice of positive roots for $\mathfrak{g}'$, we embed $\mathfrak{k}' = \mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_{p+q})$ in such a way that $\Pi'_{\mathrm{nc}} = \{\alpha_p, \alpha_{p+n}\}$, as shown in Figure 2: roughly speaking, $\mathfrak{gl}_n$ is embedded in the "middle" and $\mathfrak{gl}_{p+q}$ is embedded in the "four corners."

Hence the direct sum $\mathfrak{p}'^+$ of the positive noncompact root spaces is embedded in $\mathfrak{g}'$ as the two blocks in the upper-right, so that $\mathfrak{p}'^+ \cong \mathrm{M}_{p,n} \oplus \mathrm{M}_{n,q}$ as a vector space. Likewise, the sum $\mathfrak{p}'^-$ of the negative noncompact root spaces is embedded in the two corresponding blocks in the lower-left, so that $\mathfrak{p}'^- \cong \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n}$ as a vector space. An important role will be played by the Levi subalgebra $\mathfrak{m}' = \mathfrak{s}(\mathfrak{gl}_n \oplus \mathfrak{gl}_p \oplus \mathfrak{gl}_q) \subset \mathfrak{k}'$, which is the Lie algebra of $M' = \mathrm{S}(\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q)$.

We will write a weight of $\mathfrak{g}'$ as a triple

$$[\![\lambda, \mu, \nu]\!] := (\underbrace{-\mu_p, \ldots, -\mu_1}_{\substack{\mu \\ \text{negated and} \\ \text{reversed}}}, \underbrace{\lambda_1, \ldots, \lambda_n}_{\lambda}, \underbrace{\nu_1, \ldots, \nu_q}_{\nu}) \tag{14}$$

where $\lambda \in P(\mathfrak{gl}_n)$, $\mu \in P(\mathfrak{gl}_p)$, and $\nu \in P(\mathfrak{gl}_q)$. The resulting $(p+n+q)$-tuple on the right-hand side of (14) is written in terms of the standard coordinates $\varepsilon_i : \mathrm{diag}[h_1, \ldots, h_{p+n+q}] \longmapsto h_i$. Notice that the order of the three weights written in the shorthand $[\![\lambda, \mu, \nu]\!]$ follows the alphabetical order n, p, q, whereas the explicit tuple switches λ and μ in order to respect the embedding of $\mathfrak{m}'$ in $\mathfrak{g}'$. The reason for negating and reversing the coordinates of μ will soon become apparent, in the proof of Lemma 5.1: in order to line up the actions of M' in the Howe duality setting and in the Blattner setting, we will need to regard $\mathfrak{gl}_p$ as being embedded in $\mathfrak{g}'$ with a twist (i.e., negative transpose). A weight $[\![\lambda, \mu, \nu]\!] \in P(\mathfrak{g}')$ is $\mathfrak{m}'$-dominant if and only if $\lambda \in P_+(\mathfrak{gl}_n)$ and $\mu \in P_+(\mathfrak{gl}_p)$ and $\nu \in P_+(\mathfrak{gl}_q)$. Note that for any λ, μ, ν satisfying the conditions (9), the weight $[\![\lambda, \mu, \nu]\!]$ is automatically $\mathfrak{m}'$-dominant.

We now state our main result, connecting the BGG resolutions of the $\mathfrak{su}(p,q)$-modules in Section 2 with Blattner's formula for $G'_0 = \mathrm{SU}(n, p+q)$ in the present section. We reserve the proof for Section 5.

Theorem 4.1. *Let λ, μ, ν satisfy the conditions* (9)*, with $\varepsilon^{\lambda}_{\mu,\nu}$ as in Definition* 2.1*. Let $B(\,,\,)$ denote Blattner's formula as defined in* (12)*, for the special case determined by the rightmost column of Table* 1*, and with $[\![-,-,-]\!]$ as defined in* (14)*. Then*

$$\varepsilon^{\lambda}_{\mu,\nu} = B\big(0,\ [\![\lambda, \mu, \nu]\!]\big).$$

For some explicit examples of using Theorem 4.1 to compute BGG resolutions of the modules $\widetilde{F}^{\lambda}_{p,q}$, we direct the reader to the appendix of this paper.

5. Proving Theorem 4.1

All notation remains as in Section 4. We will prove Theorem 4.1 by rewriting the closed form for $\mathbf{b}(0)$ in (13), in terms of the character theory of $\mathbb{C}[W]$ from Section 2. To this end, we partition the positive compact roots Φ'^{+}_{c} into two subsets, as in Figures 3a and 3b below. Let $\Phi_{\mathfrak{m}'} = \{\alpha \in \Phi'^{+}_{\mathrm{c}} \mid \mathfrak{g}'_\alpha \subset \mathfrak{m}'\}$, which contains those positive compact roots whose root spaces span the three triangular regions in Figure 3a. Let $\overline{\Phi_{\mathfrak{m}'}}$ denote the complement of $\Phi_{\mathfrak{m}'}$ in Φ'^{+}_{c}, which contains those positive compact roots whose root spaces span the upper-right $p \times q$ block in Figure 3b.

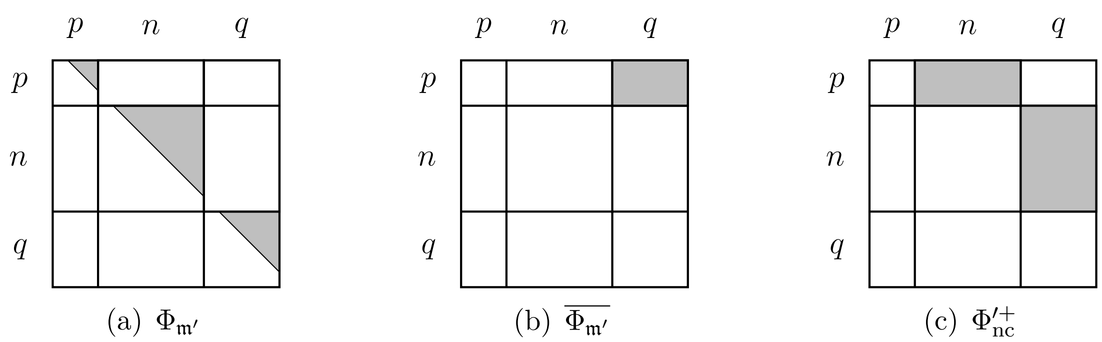


Figure 3: Root spaces corresponding to three subsets of Φ'^{+}.

Now define the products

$$\Delta_{\mathfrak{m}'} = \prod_{\alpha \in \Phi_{\mathfrak{m}'}} 1 - e^{-\alpha}, \qquad \overline{\Delta_{\mathfrak{m}'}} = \prod_{\alpha \in \overline{\Phi_{\mathfrak{m}'}}} 1 - e^{-\alpha}, \qquad \Delta_{\mathfrak{p}'^{+}} = \prod_{\beta \in \Phi'^{+}_{\mathrm{nc}}} 1 - e^{-\beta}. \tag{15}$$

Using these definitions, we can rewrite the formula (13) for $\mathbf{b}(0)$ as

$$\mathbf{b}(0) = \frac{\prod_{\alpha \in \Phi_c^+} 1 - e^{-\alpha}}{\prod_{\beta \in \Phi_{nc}^+} 1 - e^{-\beta}} = \frac{\Delta_{\mathfrak{m}'} \cdot \overline{\Delta_{\mathfrak{m}'}}}{\Delta_{\mathfrak{p}'^+}}. \tag{16}$$

The following two lemmas capture the connection between the expression for $\mathbf{b}(0)$ in (16) and the Howe duality decomposition (3).

Lemma 5.1. *Consider* $\mathbb{C}[\mathrm{M}_{p,q}]$ *with the action given in* (7)*, and let* $\overline{\Delta_{\mathfrak{m}'}}$ *be as defined in* (15)*. Then we have* $\overline{\Delta_{\mathfrak{m}'}} = \left(\operatorname{ch}\mathbb{C}[\mathrm{M}_{p,q}]\right)^{-1}$ *as a character of* $\mathfrak{m}'$*.*

Proof. Define the block upper-right embedding

$$\psi : \mathrm{M}_{p,q} \longrightarrow \bigoplus_{\alpha \in \overline{\Phi_{\mathfrak{m}'}}} \mathfrak{g}'_\alpha, \quad E_{i,j} \longmapsto E_{i,p+n+j}$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. (Recall that $E_{i,j}$ denotes the matrix with 1 in the (i,j) position and 0's elsewhere.) Clearly ψ is a vector space isomorphism; we claim that ψ is in fact an isomorphism of $\mathfrak{m}'$-modules.

To see this, recall the action of $K = \mathrm{S}(\mathrm{GL}_p \times \mathrm{GL}_q)$ on $\mathrm{M}_{p,q}$ given in (7) in the Howe duality setting, where $(g,h) \cdot X = g^{-T} X h^{-1}$. Here in the Blattner setting, we also have $K \subset M' \subset G' = \mathrm{SL}_{p+n+q}$ embedded block-diagonally with GL_p in the upper-left and GL_q in the lower-right. In this embedding, $\mathrm{im}(\psi) \subset \mathfrak{g}'$ is a K-module via the adjoint action, and the explicit action is $(g,h) \cdot X = gXh^{-1}$. Therefore, the K-actions in the Howe duality setting and in the Blattner setting are the same, up to a twist in the GL_p-action. (This is remedied by embedding GL_p in the upper-left via its inverse transpose, and at the Lie algebra level, by embedding $\mathfrak{gl}_p$ via its negative transpose.) Extending this K-action to M' by letting the factor GL_n act trivially, we conclude that $\mathrm{M}_{p,q} \cong \mathrm{im}(\psi)$ as modules for M', and thus for $\mathfrak{m}'$, which proves the claim.

Now, $\mathrm{im}(\psi)^T := \{X^T \mid X \in \mathrm{im}(\psi)\}$ is the embedding of $\mathrm{M}_{p,q}$ via the transpose into the lower-left block of $\mathfrak{g}'$. Furthermore, $\mathrm{im}(\psi)^T \cong \mathrm{im}(\psi)^*$ as an M'-module, which is clear from the adjoint action of $M' \subset G'$ on $\mathfrak{g}'$. Therefore

$$S\big(\mathrm{im}(\psi)^T\big) \cong \mathbb{C}[\mathrm{im}(\psi)] \cong \mathbb{C}[\mathrm{M}_{p,q}]$$

as $\mathfrak{m}'$-modules. Now we can conclude that

$$\overline{\Delta_{\mathfrak{m}'}} = \prod_{\alpha \in \overline{\Phi_{\mathfrak{m}'}}} (1 - e^{-\alpha}) = \Big(\operatorname{ch} S\big(\mathrm{im}(\psi)^T\big)\Big)^{-1} = \left(\operatorname{ch}\mathbb{C}[\mathrm{M}_{p,q}]\right)^{-1}$$

as a character of $\mathfrak{m}'$. ■

Lemma 5.2. *Let* $\Delta_{\mathfrak{p}'^+}$ *be as defined in* (15)*.*
Then we have $(\Delta_{\mathfrak{p}'^+})^{-1} = \operatorname{ch}\mathbb{C}[W]$ *as a character of* $\mathfrak{m}'$*.*

Proof. Let $\varphi : W \longrightarrow \mathfrak{p}'^+$ be given by $\varphi(X,Y) = (X^T, Y)$, with W as given in (1). We claim that φ is an isomorphism of $\mathfrak{m}'$-modules.

To see this, note that an element of W is of the form (X,Y), while an element of $\mathfrak{p}'^+$ is of the form (X^T, Y), where $X \in \mathrm{M}_{n,p}$ and $Y \in \mathrm{M}_{n,q}$. Recall the action of

M' on W given in (5), from the Howe duality setting. Here in the Blattner setting (as explained in the proof of Lemma 5.1), regard GL_p as being embedded in the upper-left block of G' via inverse transpose; then the adjoint action of $M' \subset G'$ on $\mathfrak{p}'^+ \subset \mathfrak{g}'$ is given by

$$(g_n, g_p, g_q) \cdot (X^T, Y) = (g_p^{-T} X^T g_n^{-1},\ g_n Y g_q^{-1}).$$

For $g = (g_n, g_p, g_q) \in M'$, we must show that $g \circ \varphi = \varphi \circ g$. But

$$\begin{aligned} g \circ \varphi(X, Y) &= g(X^T, Y) = (g_p^{-T} X^T g_n^{-1},\ g_n Y g_q^{-1}) \\ &= \varphi(g_n^{-T} X g_p^{-1},\ g_n Y g_q^{-1}) = \varphi \circ g(X, Y), \end{aligned}$$

by (5), which proves the claim. Now, observing from the adjoint M'-action that $\mathfrak{p}'^+ \cong (\mathfrak{p}'^-)^*$ as M'-modules, we have

$$S(\mathfrak{p}'^-) \cong \mathbb{C}[W]$$

as modules for M', and therefore for $\mathfrak{m}'$. Therefore we have, as a character of $\mathfrak{m}'$:

$$(\Delta_{\mathfrak{p}'^+})^{-1} = \prod_{\beta \in \Phi_{\mathrm{nc}}^+} \frac{1}{1 - e^{-\beta}} = \mathrm{ch}\, S(\mathfrak{p}'^-) = \mathrm{ch}\, \mathbb{C}[W]. \qquad \blacksquare$$

At this point, we observe that for $\xi \in P_+(\mathfrak{m}')$, the Weyl character formula [6, Thm. 7.1.1] can be written as

$$\mathrm{ch}\, L_{\mathfrak{m}'}(\xi) = \frac{\sum_{w \in \mathcal{W}_{\mathfrak{m}'}} (-1)^{\ell(w)} e^{w(\xi+\rho)}}{e^{\rho} \prod_{\alpha \in \Phi_{\mathfrak{m}'}} 1 - e^{-\alpha}} = \frac{\sum_{w \in \mathcal{W}_{\mathfrak{m}'}} (-1)^{\ell(w)} e^{w(\xi+\rho)-\rho}}{\Delta_{\mathfrak{m}'}},$$

where $\mathcal{W}_{\mathfrak{m}'}$ is the Weyl group for $\mathfrak{m}'$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Phi_{\mathfrak{m}'}} \alpha$. Multiplying both sides of the equation by $\Delta_{\mathfrak{m}'}$ gives an equation that exhibits the product $\Delta_{\mathfrak{m}'} \cdot \mathrm{ch}\, L_{\mathfrak{m}'}(\xi)$ as an alternating sum of terms of the form $e^{w(\xi+\rho)-\rho}$. But the weight $w(\xi+\rho) - \rho$ lies in $P_+(\mathfrak{m}')$ if and only if $w = 1$, which means that

$$\Delta_{\mathfrak{m}'} \cdot \mathrm{ch}\, L_{\mathfrak{m}'}(\xi) = e^{\xi} + \text{signed sum of } e^{\text{nondominant } \mathfrak{m}'\text{-weight}}\text{s}.$$

More generally, consider an arbitrary $\mathfrak{m}'$-module $L = \bigoplus_{\xi} m_\xi L_{\mathfrak{m}'}(\xi)$, ranging over $\xi \in P_+(\mathfrak{m}')$, with multiplicities $m_\xi \in \mathbb{N}$. Then we have

$$\Delta_{\mathfrak{m}'} \cdot \mathrm{ch}\, L = \sum_{\xi} m_\xi e^{\xi} + \text{signed sum of } e^{\text{nondominant } \mathfrak{m}'\text{-weight}}\text{s}. \tag{17}$$

The upshot is that multiplying the character of an $\mathfrak{m}'$-module L by $\Delta_{\mathfrak{m}'}$ produces a linear combination of formal weights (i.e., terms of the form e^{γ}, where $\gamma \in P(\mathfrak{m}')$), in which the $\mathfrak{m}'$-dominant weights are precisely the highest weights of the irreducible $\mathfrak{m}'$-modules in the decomposition of L; moreover, the coefficient of each formal $\mathfrak{m}'$-dominant weight e^{ξ} is the multiplicity of the module $L_{\mathfrak{m}'}(\xi)$ in L.

Proof of Theorem 4.1. From (16), we have (using the notation of (15)) that

$$\mathbf{b}(0) = \frac{\Delta_{\mathfrak{m}'} \cdot \overline{\Delta_{\mathfrak{m}'}}}{\Delta_{\mathfrak{p}'^+}}$$

which, by Lemmas 5.1 and 5.2, becomes

$$\mathbf{b}(0) = \Delta_{\mathfrak{m}'} \cdot (\mathrm{ch}\, \mathbb{C}[\mathrm{M}_{p,q}])^{-1} \cdot \mathrm{ch}\, \mathbb{C}[W].$$

Substituting for $\operatorname{ch}\mathbb{C}[W]$ via (11), we have

$$\mathbf{b}(0) = \Delta_{\mathfrak{m}'} \cdot (\operatorname{ch}\mathbb{C}[\mathrm{M}_{p,q}])^{-1} \cdot \operatorname{ch}\mathbb{C}[\mathrm{M}_{p,q}] \cdot \sum_{\lambda,\mu,\nu} \varepsilon^{\lambda}_{\mu,\nu} \cdot \operatorname{ch}(F^{\lambda}_n \otimes F^{\mu}_p \otimes F^{\nu}_q),$$

in which the two instances of $\operatorname{ch}\mathbb{C}[\mathrm{M}_{p,q}]$ cancel each other out, giving

$$\Delta_{\mathfrak{m}'} \cdot \sum_{\lambda,\mu,\nu} \varepsilon^{\lambda}_{\mu,\nu} \cdot \operatorname{ch}(F^{\lambda}_n \otimes F^{\mu}_p \otimes F^{\nu}_q). \tag{18}$$

The sum in (18) is a virtual $\mathfrak{m}'$-character, and so by (17), we have

$$\mathbf{b}(0) = \sum_{\lambda,\mu,\nu} \varepsilon^{\lambda}_{\mu,\nu} \cdot e^{[\![\lambda,\mu,\nu]\!]} + \sum_{\xi \notin P_+(\mathfrak{m}')} c_\xi e^{\xi} \tag{19}$$

with $c_\xi \in \mathbb{Z}$. This completes the proof of Theorem 4.1. ■

A. Examples using Theorem 4.1

Example A.1. We work out the case when $n = p = q = 1$ in full detail. (Note that although this is outside the stable range $n \geq p + q$, nevertheless the modules $\widetilde{F}^{\lambda}_{1,1}$ appearing in (3) are still generalized Verma modules, as we will see below.) In this case, $\mathbb{C}[W] = \mathbb{C}[x, y]$, and the action of $M' = \mathrm{S}(\mathrm{GL}_1 \times \mathrm{GL}_1 \times \mathrm{GL}_1)$ upon a typical monomial follows from (5):

$$(g_n, g_p, g_q) \cdot x^a y^b = (g_n g_p x)^a \left(\frac{g_q}{g_n} y\right)^b = \left(g_n^{a-b} g_p^a g_q^b\right) x^a y^b,$$

where $g_n, g_p, g_q \in \mathbb{C}^\times$. Hence each monomial spans a one-dimensional subrepresentation of M'. Note that by setting $\lambda = a - b$ and $c = \min(a, b)$, we can rewrite the formal sum of these monomials as

$$\begin{aligned}\sum_{a,b \in \mathbb{N}} x^a y^b &= \sum_{c=0}^{\infty} (xy)^c \left(\sum_{\lambda \geq 0} x^{\lambda} + \sum_{\lambda < 0} y^{-\lambda}\right) \\ &= \sum_{\lambda \geq 0} x^{\lambda} \sum_{c=0}^{\infty} (xy)^c + \sum_{\lambda < 0} y^{-\lambda} \sum_{c=0}^{\infty} (xy)^c.\end{aligned}$$

When $n = p = q = 1$, each rational representation of GL_n and GL_p and GL_q is one-dimensional, indexed by a single integer λ (nonnegative if the representation is polynomial), where the group action is $g \cdot z = g^{\lambda} z$. Hence we will write $\mathbb{C}_\lambda$ for the representation F^{λ}_1 of $\mathrm{GL}_1 \cong \mathbb{C}^\times$. Comparing the two calculations above, we see that the Howe decomposition (3) in this case is

$$\mathbb{C}[x, y] = \bigoplus_{\lambda \in \mathbb{Z}} \mathbb{C}_\lambda \otimes \widetilde{F}^{\lambda}_{1,1}$$

where $$\widetilde{F}^{\lambda}_{1,1} = \begin{cases} \mathbb{C}[xy] \otimes (\mathbb{C}_\lambda \otimes \mathbb{C}_0) = N_{\lambda,0}, & \lambda \geq 0, \\ \mathbb{C}[xy] \otimes (\mathbb{C}_0 \otimes \mathbb{C}_{-\lambda}) = N_{0,-\lambda}, & \lambda < 0. \end{cases}$$

Therefore each $\widetilde{F}^{\lambda}_{1,1}$ is a generalized Verma module, and so the BGG resolution of $\widetilde{F}^{\lambda}_{1,1}$ contains only one term. In particular, for $\lambda, \mu, \nu \in \mathbb{Z}$ we have

$$\varepsilon^{\lambda}_{\mu,\nu} = \begin{cases} 1, & \lambda = \mu \geq 0 \text{ and } \nu = 0, \\ 1, & \lambda = -\nu < 0 \text{ and } \mu = 0, \\ 0 & \text{otherwise,} \end{cases}$$

so that the triples (λ,μ,ν) satisfying the first two cases are of the form

$$\mathbb{N}(1,1,0) \text{ and } \mathbb{N}(-1,0,1). \tag{20}$$

Now we check this answer using Theorem 4.1, which implies that $\varepsilon^{\lambda}_{\mu,\nu}$ equals the coefficient of $e^{[\![\lambda,\mu,\nu]\!]}$ in $\mathbf{b}(0)$. Therefore we compute $\mathbf{b}(0)$ in order to check it against (20). We have $\mathfrak{g}' = \mathfrak{sl}_3$, with $\Pi'_{\text{nc}} = \{\alpha_1,\alpha_2\} = \Pi'$. Then $\Phi'^{+}_{\text{c}} = \{\varepsilon_1-\varepsilon_3\}$ and $\Phi'^{+}_{\text{nc}} = \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3\}$. Recall that the triple $[\![\ell,m,n]\!]$ is the $\mathfrak{g}'$-weight $(-m,\ell,n)$ in standard ε-coordinates. Using (13) and setting $t_i = e^{\varepsilon_i}$ for $i=1,2,3$, we have

$$\begin{aligned}\mathbf{b}(0) &= \frac{1-\frac{t_3}{t_1}}{\left(1-\frac{t_2}{t_1}\right)\left(1-\frac{t_3}{t_2}\right)} = \frac{1}{1-\frac{t_2}{t_1}} + \frac{\frac{t_3}{t_2}}{1-\frac{t_3}{t_2}} = \sum_{k=0}^{\infty} e^{k(\varepsilon_2-\varepsilon_1)} + \sum_{k=1}^{\infty} e^{k(\varepsilon_3-\varepsilon_2)} \\ &= \sum_{k=0}^{\infty} e^{k(-1,1,0)} + \sum_{k=1}^{\infty} e^{k(0,-1,1)} = \sum_{k=0}^{\infty} e^{k[\![1,1,0]\!]} + \sum_{k=1}^{\infty} e^{k[\![-1,0,1]\!]},\end{aligned}$$

coinciding exactly with (20) and Theorem 4.1. (See Figure 4 for a visualization of $\mathbf{b}(0)$, in which we plot the support of $B(0,-)$ on the weight lattice of $\mathfrak{sl}_3$.)

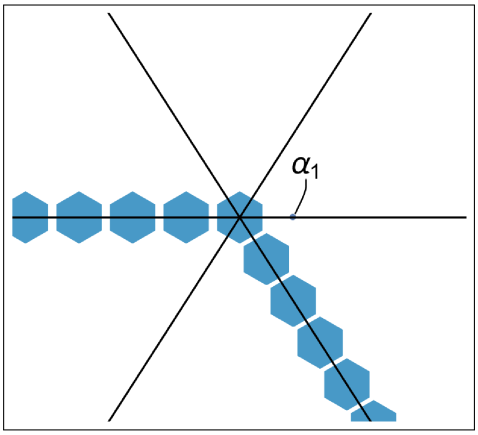


Figure 4: A visualization of $\mathbf{b}(0)$ from Example A.1. By programming Blattner's formula (12) directly in Mathematica, we plot a hexagon at each weight $\eta \in P(\mathfrak{sl}_3)$ such that $B(0,\eta)=1$. Note that these weights are the nonnegative multiples of $-\alpha_1 = (-1,1,0)$ and of $-\alpha_2 = (0,-1,1)$. (All other weights return 0.)

Before presenting a more interesting example, we outline how Theorem 4.1 allows us to write down explicitly the BGG resolution for $\widetilde{F}^{\lambda}_{p,q}$. Note that the theorem could just as well be rewritten as follows (in fact, this was the last step (19) of the proof):

$$\begin{aligned}\mathbf{b}(0) &= \sum_{\substack{\lambda,\mu,\nu \\ \text{as in (9)}}} \varepsilon^{\lambda}_{\mu,\nu}\, e^{[\![\lambda,\mu,\nu]\!]} + \sum_{\xi \notin P_+(\mathfrak{m}')} c_\xi e^{\xi}, \qquad c_\xi \in \mathbb{Z} \\ &= \sum_{\lambda} e^{[\![\lambda,0,0]\!]} \underbrace{\left(\sum_{\mu,\nu} \varepsilon^{\lambda}_{\mu,\nu} \cdot e^{[\![0,\mu,\nu]\!]}\right)}_{\text{coefficient of } e^{[\![\lambda,0,0]\!]}} + \sum_{\xi \notin P_+(\mathfrak{m}')} c_\xi e^{\xi}. \end{aligned} \tag{21}$$

Therefore we begin by computing $\mathbf{b}(0)$, expanding up to a sufficiently high order, and then ignoring all terms corresponding to nondominant $\mathfrak{m}'$-weights. Then given λ, we should collect all the remaining terms in $\mathbf{b}(0)$ of the form $e^{[\![\lambda,*,*]\!]}$ (where the stars are arbitrary), and then factor out $e^{[\![\lambda,0,0]\!]}$ to obtain the multi-term coefficient of $e^{[\![\lambda,0,0]\!]}$ indicated in (21). The terms inside this coefficient tell us exactly which generalized Verma modules $N_{\mu,\nu}$ appear in the BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$, along with appropriate signs depending on the parity of the term in the resolution.

In order to recover the complete data of the BGG resolution, one must supplement the method outlined above so as to determine the exact term (rather than just the parity) in which each generalized Verma module occurs. To this end, we observe that it follows from the construction in [5] that as the index of the terms increases, the partitions μ and ν strictly increase in size. This fact allows us to easily recover the full resolution once we have found the coefficient of $e^{[\![\lambda,0,0]\!]}$.

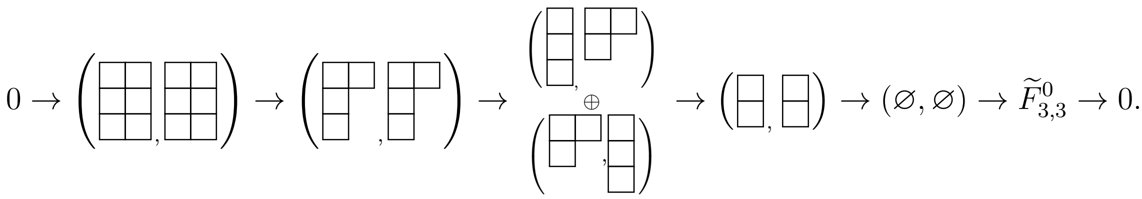
Example A.2. We use software to present an example of the method outlined above. On one hand, we will compute the terms of the BGG resolution of $\widetilde{F}^{\lambda}_{p,q}$ directly, using Maple code. The author thanks Dr. Jeb Willenbring for providing this code. We will then expand $\mathbf{b}(0)$ and isolate the coefficient of $e^{[\![\lambda,0,0]\!]}$ using Mathematica code written by the author.

Let $n = 1$, with $p = q = 3$. Set $\lambda = 0$, the empty partition; then $\widetilde{F}^{0}_{3,3} \cong \mathbb{C}[W]^{\mathrm{GL}_1}$ is the invariant subalgebra of $\mathbb{C}[W]$ in the setting of Section 2. This example is of special interest because, by the second fundamental theorem of classical invariant theory, $\mathbb{C}[W]^{\mathrm{GL}_1}$ is isomorphic to the coordinate ring of the determinantal variety consisting of matrices in $\mathrm{M}_3(\mathbb{C})$ with rank at most 1. (This is the associated variety of the first Wallach representation of $\mathfrak{su}(3,3)$; see Enright and Hunziker's paper [3] on minimal resolutions for the Wallach representations.) Our present example is exactly Example 7.11 in [4] where $p = 3$.

In Maple, we compute the following resolution for $\widetilde{F}^{0}_{3,3}$:

$$\begin{aligned} 0 \to N_{(2,2,2),(2,2,2)} \to N_{(2,1,1),(2,1,1)} \to N_{(1,1,1),(2,1,0)} \oplus N_{(2,1,0),(1,1,1)} \\ \to N_{(1,1,0),(1,1,0)} \to N_{(0,0,0),(0,0,0)} \to \widetilde{F}^{0}_{3,3} \to 0. \end{aligned}$$

We can also write this more compactly with Young diagrams:



Now using Mathematica, we will verify Theorem 4.1 by defining the generating function $\mathbf{b}(0)$ directly from the definition (13), setting $x_i = e^{\varepsilon_i}$ for $i = 1,2,3$, and $w = e^{\varepsilon_4}$, and $y_i = e^{\varepsilon_{i+4}}$ for $i = 1,2,3$. Note that the alphabetical order w, x, y mirrors that of n, p, q as we visualize the embedding $\mathfrak{m}' \subset \mathfrak{g}'$; in this way, the exponent vector for the variables x_i encodes a weight of $\mathfrak{gl}_p = \mathfrak{gl}_3$, the exponent vector for the y_i encodes a weight of $\mathfrak{gl}_q = \mathfrak{gl}_3$, and the exponent of w encodes a weight of $\mathfrak{gl}_n = \mathfrak{gl}_1$. Using (13), we obtain

$$\mathbf{b}(0) = \frac{\prod_{1\le i<j\le 3}\left(1-\frac{x_j}{x_i}\right)\prod_{1\le i<j\le 3}\left(1-\frac{y_j}{y_i}\right)\prod_{1\le i,j\le 3}\left(1-\frac{y_j}{x_i}\right)}{\prod_{1\le i\le 3}\left(1-\frac{w}{x_i}\right)\prod_{1\le i\le 3}\left(1-\frac{y_i}{w}\right)}.$$

Upon expanding $\mathbf{b}(0)$ to a sufficiently high order, we program Mathematica to retain only those terms in which the exponent vectors for (x_1, x_2, x_3) and (y_1, y_2, y_3) are both weakly decreasing, corresponding to dominant weights for $\mathfrak{gl}_3$. (Since $n = 1$ in this example, there is no need to do the same for the lone variable w.) In the remaining sum, we then find the coefficient of $e^{[\![\lambda,0,0]\!]}$, namely, of w^0 – that is, we collect all terms in which the power of w is 0. This coefficient is the following sum of terms in the variables x_i and y_i; in light of the remark before this example, we write the terms in descending order with respect to total degree in the y_i:

$$\frac{y_1^2y_2^2y_3^2}{x_1^2x_2^2x_3^2} - \frac{y_1^2y_2y_3}{x_1x_2x_3^2} + \frac{y_1^2y_2}{x_1x_2x_3} + \frac{y_1y_2y_3}{x_2x_3^2} - \frac{y_1y_2}{x_2x_3} + 1$$

$$= e^{(-2,-2,-2,0,2,2,2)} - e^{(-1,-1,-2,0,2,1,1)} + e^{(-1,-1,-1,0,2,1,0)} + e^{(0,-1,-2,0,1,1,1)} - e^{(0,-1,-1,0,1,1,0)} + e^{(0,0,0,0,0,0,0)}$$

$$= e^{[\![0,(2,2,2),(2,2,2)]\!]} - e^{[\![0,(2,1,1),(2,1,1)]\!]} + e^{[\![0,(1,1,1),(2,1,0)]\!]} + e^{[\![0,(2,1,0),(1,1,1)]\!]} - e^{[\![0,(1,1,0),(1,1,0)]\!]} + e^{[\![0,(0,0,0),(0,0,0)]\!]}.$$

This yields precisely the BGG resolution produced above by Maple.

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