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The Isometry Group of the 4-Dimensional Oscillator Group

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Abstract. For each left invariant Riemannian metric on the oscillator group, we determine the Ricci transformation and the full isometry group.

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1. Introduction

The main purpose of this paper is to determine the Ricci transformation and the group of isometries of any left invariant Riemannian metric on the four-dimensional oscillator group. The classification of left invariant Riemannian metrics, modulo automorphisms, on Lie groups constitutes a fundamental problem in both differential and Riemannian geometry. For simply connected unimodular Lie groups, definitive classifications have been established by Ha and Lee [13] in dimension three and by Van Thuong [25] in dimension four. Boucetta and Chakkar [9] further contribute to this tapestry by classifying Lorentzian metrics in this setting. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, the classification of left invariant Riemannian metrics on G is equivalent to the classification of inner products on $\mathfrak{g}$. If two inner products on $\mathfrak{g}$ are equivalent, then they induce isometric left invariant Riemannian metrics on G. It was proved by Alekseevsky in [2] that the converse is true only when $\mathfrak{g}$ is of type (R). An interesting problem which follows the classification of these metrics, is the description of their isometry groups, which play an essential role in both mathematical and physical theories. The group of isometries of Riemannian and Lorentzian Lie groups, in particular the oscillator group, has a great application in physics; This comes from the fact that an element of the group of isometries preserves several notions which are important in physics like geodesics (the image of a geodesic by an isometry is a geodesic), curvatures (a plane and its image by the differential of an isometry have the same curvatures), Ricci transformation (the differential of an isometry fixing the identity element of G commutes with the Ricci transformation of (G, g)), and more. One can find a brief description of the role of isometries in cosmological models in the following thesis [16]. The group of isometries of a left invariant Riemannian metric g on G, is the group consisting of all diffeomorphisms of G preserving the metric g. Let us denote this group by $\mathrm{Isom}(G, g)$, then by [20] $\mathrm{Isom}(G, g)$ is a Lie group under the compact-open topology and acts on G

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transitively. The isotropy subgroup at the identity element e of G is the stabilizer of e with respect to this action and it is denoted by $\mathrm{Isom}(G,g)_e$. Hence the group $\mathrm{Isom}(G,g)$ decomposes as

$$\mathrm{Isom}(G,g) = L(G)\cdot \mathrm{Isom}(G,g)_e \cong G\cdot \mathrm{Isom}(G,g)_e$$

where $L(G)$ is the subgroup of left translations which is identified with G. In fact, any element $\theta \in \mathrm{Isom}(G,g)$ such that $\theta(e) = p \in G$ decomposes as

$$\theta = L_p \circ (L_{p^{-1}} \circ \theta) \quad \text{where } L_{p^{-1}} \circ \theta \in \mathrm{Isom}(G,g)_e.$$

The product $L(G)\cdot \mathrm{Isom}(G,g)_e$ is not in general a semidirect product, it will be a semidirect product only when $L(G)$ is normal in $\mathrm{Isom}(G,g)$. There are two interesting cases where $L(G)$ is normal in $\mathrm{Isom}(G,g)$, in fact if G is nilpotent, then

$$\mathrm{Isom}(G,g) = L(G) \rtimes \mathrm{Isom}(G,g)_e \quad [26,\ 27].$$

In this case, the group of isometries fixing the identity element e is exactly the group of isometric automorphisms of G (see Ha and Lee in [14]). The same result holds also when G is unimodular solvable of type (R) (see [14]). This result is a consequence of two results given by Gordon and Wilson [12] and Helgason [15]. The isometry groups of three-dimensional Lie groups are well understood, see the following papers [4, 5, 8, 14].

It is well known that the harmonic oscillator is a nonrelativistic system whose Schrödinger equations can be completely solved. The four-dimensional oscillator algebra $\mathfrak{osc}$ is the Lie algebra generated by H, P, Q and E with the bracket relations

$$[H,P] = -Q \qquad [H,Q] = P \qquad [P,Q] = E.$$

The Jacobi identity of $\mathfrak{osc}$ holds in the following model

$$2H = \frac{-\partial^2}{\partial x^2} + x^2 \qquad P = \frac{\partial}{\partial x} \qquad Q = x \qquad E = 1$$

acting on functions of x [23]. This is exactly the harmonic oscillator problem, and this is the reason why the Lie algebra $\mathfrak{osc}$ is called the oscillator algebra. Its associated connected Lie group Osc is also called the oscillator group. The general oscillator Lie algebras $\mathfrak{g}_m(\boldsymbol{\lambda}) = \mathfrak{g}_m(\lambda_1, \ldots, \lambda_m)$ are the real $(2m+2)$-dimensional solvable Lie algebras generated by $\{P, X_1, \ldots, X_m, Y_1, \ldots, Y_m, Q\}$ with the bracket relations

$$[X_i, Y_i] = P \qquad [X_i, Q] = -\lambda_i Y_i \qquad [Y_i, Q] = \lambda_i X_i$$

where λ_i are positive real numbers. The oscillator Lie groups $\mathrm{G}_m(\boldsymbol{\lambda})$ are the connected Lie groups with Lie algebras $\mathfrak{g}_m(\boldsymbol{\lambda}) = \mathfrak{g}_m(\lambda_1, \ldots, \lambda_m)$. These groups are the only non commutative, simply connected Lie groups which admit a bi-invariant Lorentzian metric. Their Lorentzian geometry and their lattices were studied by Medina and Revoy in [18].

If G is a Lie group and if L is a lattice in G, i.e. a cocompact discrete subgroup of G, then the quotient G/L is a compact, homogeneous Lorentzian manifold. In particular, if G is an oscillator group, then from the bi-invariance of the Lorentzian metric on G, G/L is a locally-symmetric Lorentzian manifold called oscillator manifold. These manifolds are the only Lorentzian, homogeneous manifolds of finite volume, with a non-compact group of isometries [1, 28]. With respect to a family of left invariant Lorentzian metrics, Gadea and Oubina in [11] gave all homogeneous pseudo-Riemannian structures on the oscillator groups, they also determined all reductive decompositions and the isometry group of the 4-dimensional oscillator group. One can find an important study of reductive homogeneous Lorentzian manifold in the following paper [3].

The study of Riemannian and Lorentzian Lie groups is a very rich subject, one can find other interesting and motivating results about the curvatures and the isometry groups of these Lie groups in the following references [7, 12, 22, 24].

In the 4-dimensional oscillator group studies, Kremlev and Nikonorov in [17] showed that there are three possibilities in the signature of the Ricci operator in this group denoted $A_{4,10}$ which are

$$(-,-,-,+) \quad (-,-,0,+) \quad (-,-,+,+)$$

In [25], Thuong showed that there are two Riemannian metrics, up to automorphism, on $\widetilde{Nil \rtimes S^1}$, where $\widetilde{Nil \rtimes S^1}$ refers to the four-dimensional oscillator group.

The main results of this paper are

1. We determine the Ricci transformation of four Riemannian metrics in two non-isometric clases on the oscillator group Osc.
2. The corresponding scalar curvatures are given.
3. We give their associated isometric automorphism groups.
4. We describe their associated isometry groups.

2. Ricci transformation of a Riemannian Lie group

In this section, we revisit the fundamental concepts of Riemann curvature, sectional curvature, and the Ricci transformation (also known as Ricci curvature or Ricci operator) within the context of a Riemannian Lie group. The Ricci transformation plays a significant role in characterizing the isometry group of a Riemannian Lie group; specifically, an isometry that fixes the identity element of G commutes with the Ricci transformation of (G, g). Therefore, understanding the commutant of the Ricci transformation simplifies the task of identifying the group of isometries. Consequently, our objective is to compute the Ricci transformation of left invariant metrics on the oscillator group to delineate the group of isometries associated with these metrics on this Lie group.

Let g be a left invariant Riemannian metric on a connected Lie group G and let ∇ be the Levi-Civita connection associated to the Riemannian Lie group (G, g). Then the *Riemann curvature tensor* R of G is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad \forall X, Y, Z \in \mathcal{X}(G)$$

where $\mathcal{X}(G)$ is the set of all vector fields of G. For any $p \in G$ and for any couple of linearly independent tangent vectors $X, Y \in T_pG$, the number

$$K(X,Y) = \frac{g(R(X,Y)X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

is called the *sectional curvature* associated with X and Y.

If X and Y are orthonormal, then

$$K(X,Y) = g(R(X,Y)X,Y).$$

If a basis $B = \{e_1, ..., e_n\}$ is an orthonormal basis of T_pG, then for any $x \in T_pG$, the *Ricci quadratic form* $r(x)$ is defined by

$$r(x) = -\sum_{i=1}^{n} K(x,e_i) = -\sum_{i=1}^{n} g(R(x,e_i)x,e_i).$$

If u is a unit vector, then $r(u)$ is called *Ricci curvature in the direction* u. The *Ricci transformation* $\hat{r}$ is the self adjoint endomorphism of T_pG defined by

$$\hat{r}(x) = \sum_{i=1}^{n} R(x,e_i)e_i.$$

This is related to the quadratic form r by the identity

$$r(x) = g(\hat{r}(x),x).$$

The eigenvalues of $\hat{r}$ are called *principal Ricci curvatures.* If we choose an orthonormal basis $\{e_1, ..., e_n\}$ consisting of eigenvectors, then the numbers $r(e_i)$ can be identified with the principal Ricci curvatures, hence they diagonalizes the Ricci transformation $\hat{r}$ and the trace of $\hat{r}$ given by

$$\rho = r(e_1) + r(e_2) + ... + r(e_n)$$

is called the *scalar curvature of* $\mathfrak{g}$. The collection of signs $\{\mathrm{sgn}(r(e_1)), ..., \mathrm{sgn}(r(e_n))\}$ is identified with the signature of the quadratic form r. (see [19]).

In general, it is not easy to calculate these curvatures using the geometrical approach. But in [19], Milnor gave the algebraic way to find these curvatures using the structure constants attached to the metric Lie algebra of the Riemannian Lie group (G,g).

3. The 4-dimensional oscillator group

The matrix realization of the four-dimensional oscillator Lie algebra $\mathfrak{osc} \backsimeq \mathfrak{g}_1(\lambda)$ given by Biggs and Remsing [6] is

$$\mathfrak{osc} = \left\{ \begin{bmatrix} 0 & -y & z & -2x \\ 0 & 0 & \theta & z \\ 0 & -\theta & 0 & y \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(x,y,z,\theta) \Big/ x,y,z,\theta \in \mathbb{R} \right\}.$$

Its associated connected Lie group Osc is defined by:

$$\mathrm{Osc} =$$

$$\left\{ \begin{bmatrix} 1 & -y\cos(\theta) - z\sin(\theta) & z\cos(\theta) - y\sin(\theta) & -2x \\ 0 & \cos(\theta) & \sin(\theta) & z \\ 0 & -\sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 0 & 1 \end{bmatrix} = m(x,y,z,\theta) \Big/ x,y,z,\theta \in \mathbb{R} \right\}.$$

Consider the standard basis B of $\mathfrak{osc}$ such that

$$B=\Big\{E_1 = M(1,0,0,0),\ E_2 = M(0,1,0,0),\ E_3 = M(0,0,1,0),\ E_4 = M(0,0,0,1)\Big\}.$$

The only non-zero commutators of elements of B are

$$[E_2, E_3] = E_1 \qquad [E_2, E_4] = -E_3 \qquad [E_3, E_4] = E_2$$

Proposition 3.1. (1) *The center of $\mathfrak{osc}$ is $Z(\mathfrak{osc}) = \langle E_1\rangle$.*

(2) *The Lie algebra $\mathfrak{osc}$ is solvable.*

(3) *The Lie algebra $\mathfrak{osc}$ is not nilpotent.*

Proof. (1) The only element x in B verifying $[x, E_i] = 0$ for all $i = 1, ..., 4$ is E_1. Hence $Z(\mathfrak{osc}) = \langle E_1\rangle$.

(2) The derived serie of $\mathfrak{osc}$ is given by

$$\mathscr{D}^1(\mathfrak{osc}) = \langle E_1, E_2, E_3\rangle \quad \mathscr{D}^2(\mathfrak{osc}) = \langle E_1\rangle \quad \mathscr{D}^3(\mathfrak{osc}) = 0$$

Hence the Lie algebra $\mathfrak{osc}$ is solvable.

(3) The central serie of $\mathfrak{osc}$ is given by

$$\mathscr{C}^1(\mathfrak{osc}) = \langle E_1, E_2, E_3\rangle \quad \mathscr{C}^2(\mathfrak{osc}) = \langle E_1, E_2, E_3\rangle$$

Hence $\mathscr{C}^i(\mathfrak{osc}) = \langle E_1, E_2, E_3\rangle\ \forall i \geq 1$, thus $\mathfrak{osc}$ is not nilpotent. ■

4. Metrics on Osc and their Ricci transformations

Any left invariant Riemannian metric on $\mathfrak{osc}$ is equivalent, up to automorphism, and up to an identification with the set of upper triangular matrices with positive diagonal entries, to one of the following metrics [25]:

$$\mu_1 = \begin{bmatrix} \alpha' & \beta' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu' \end{bmatrix} \qquad \alpha', \nu' > 0, \quad \beta' \geq 0.$$

$$\mu_2 = \begin{bmatrix} \alpha' & \beta' & \gamma' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu' & 0 \\ 0 & 0 & 0 & \nu' \end{bmatrix} \qquad \alpha', \nu' > 0, \quad 0 < \mu' < 1, \quad \beta', \gamma' \geq 0.$$

Let us assume in the first case that $\beta' = \gamma' = 0$. From the Proposition 1.4 in [25], the symmetric, positive definite matrices associated to $\mu_1 = \mathrm{diag}\{\alpha', 1, 1, \nu'\}$ and to $\mu_2 = \mathrm{diag}\{\alpha', 1, \mu', \nu'\}$ are respectively

$$\begin{bmatrix} \frac{1}{\alpha'^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{\nu'^2} \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{\alpha'^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu'^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\nu'^2} \end{bmatrix}.$$

Put $\alpha = \frac{1}{\alpha'^2}$, $\mu = \frac{1}{\mu'^2}$ and $\nu = \frac{1}{\nu'^2}$. Thus, these metrics are given by

$$\langle .,. \rangle_1 = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix} \qquad \langle .,. \rangle_2 = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix}.$$

4.1. Ricci transformation of $\mathfrak{osc}$ with respect to the metric $\langle .,. \rangle_1$

Put
$$e_1 = \frac{E_1}{\sqrt{\alpha}}, \quad e_2 = E_2, \quad e_3 = E_3, \quad e_4 = \frac{E_4}{\sqrt{\nu}}.$$
Then $\mathscr{B} = \{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{osc}$ with respect to $\langle .,. \rangle_1$.

Proposition 4.1. *The Levi-Civita connection of the metric Lie algebra* $(\mathfrak{osc}, \langle .,. \rangle_1)$ *is described by the following equalities*

- $\nabla_{e_i} e_i = 0 \ \forall i = 1, ..., 4$
- $\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \frac{-1}{2}\sqrt{\alpha} e_3$
- $\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = \frac{1}{2}\sqrt{\alpha} e_2$
- $\nabla_{e_1} e_4 = \nabla_{e_4} e_1 = 0$
- $\nabla_{e_2} e_3 = \frac{1}{2}\sqrt{\alpha} e_1$
- $\nabla_{e_3} e_2 = \frac{-1}{2}\sqrt{\alpha} e_1$
- $\nabla_{e_2} e_4 = 0$
- $\nabla_{e_4} e_2 = \frac{1}{\sqrt{\nu}} e_3$
- $\nabla_{e_3} e_4 = 0$
- $\nabla_{e_4} e_3 = \frac{-1}{\sqrt{\nu}} e_2$

Proof. All the non-zero structure constants of $(\mathfrak{osc}, \langle .,. \rangle_1)$ are

- $\alpha_{231} = \sqrt{\alpha}$
- $\alpha_{243} = \frac{-1}{\sqrt{\nu}}$
- $\alpha_{342} = \frac{1}{\sqrt{\nu}}$
- $\alpha_{321} = -\sqrt{\alpha}$
- $\alpha_{423} = \frac{1}{\sqrt{\nu}}$
- $\alpha_{432} = \frac{-1}{\sqrt{\nu}}$

Thus, one can obtain the result by a straightforward computations using the following formula
$$\nabla_{e_i} e_j = \sum_{k=1}^{4} \frac{1}{2} \left(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij} \right) e_k. \quad \text{(see Section 5 in [19])}$$
∎

Theorem 4.2. *The Ricci transformation* $\hat{r}$ *of* $(\mathfrak{osc}, \langle .,. \rangle_1)$ *is given by*

$$\hat{r} = \begin{bmatrix} \frac{\alpha}{2} & 0 & 0 & 0 \\ 0 & \frac{-\alpha}{2} & 0 & 0 \\ 0 & 0 & \frac{-\alpha}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proof. The Ricci transformation is given by the following formulas

$$\begin{aligned} \hat{r}(e_1) &= R(e_1, e_2)e_2 + R(e_1, e_3)e_3 + R(e_1, e_4)e_4 \\ &= -\nabla_{e_2}\nabla_{e_1} e_2 - \nabla_{e_3}\nabla_{e_1} e_3 = \frac{\alpha}{2} e_1, \end{aligned}$$

$$\begin{aligned} \hat{r}(e_2) &= R(e_2, e_1)e_1 + R(e_2, e_3)e_3 + R(e_2, e_4)e_4 \\ &= -\nabla_{e_1}\nabla_{e_2} e_1 - \nabla_{e_3}\nabla_{e_2} e_3 - \nabla_{[e_2,e_3]} e_3 - \nabla_{[e_2,e_4]} e_4 = -\sqrt{\alpha}\nabla_{e_1} e_3 = \frac{-\alpha}{2} e_2, \end{aligned}$$

$$\begin{aligned}\hat{r}(e_3) &= R(e_3,e_1)e_1 + R(e_3,e_2)e_2 + R(e_3,e_4)e_4\\ &= -\nabla_{e_1}\nabla_{e_3}e_1 - \nabla_{e_2}\nabla_{e_3}e_2 - \nabla_{[e_3,e_2]}e_2 - \nabla_{[e_3,e_4]}e_4 = \sqrt{\alpha}\nabla_{e_1}e_2 = \frac{-\alpha}{2}e_3,\\ \hat{r}(e_4) &= R(e_4,e_1)e_1 + R(e_4,e_2)e_2 + R(e_4,e_3)e_3\\ &= -\nabla_{e_2}\nabla_{e_4}e_2 - \nabla_{[e_4,e_2]}e_2 - \nabla_{e_3}\nabla_{e_4}e_3 - \nabla_{[e_4,e_3]}e_3\\ &= \frac{-1}{\sqrt{\nu}}\nabla_{e_2}e_3 - \frac{1}{\sqrt{\nu}}\nabla_{e_3}e_2 + \frac{1}{\sqrt{\nu}}\nabla_{e_3}e_2 + \frac{1}{\sqrt{\nu}}\nabla_{e_2}e_3 = 0.\end{aligned}$$

■

Corollary 4.3. *The principal Ricci curvatures of* $(\mathfrak{osc}, \langle .,. \rangle_1)$ *are*

$$r(e_1) = \frac{\alpha}{2} > 0 \qquad r(e_2) = \frac{-\alpha}{2} \qquad r(e_3) = \frac{-\alpha}{2} < 0 \qquad r(e_4) = 0.$$

Hence the signature of the Ricci transformation $\hat{r}$ *is* $(+,-,-,0)$.

Proposition 4.4. *The scalar curvature of* $(\mathfrak{osc}, \langle .,. \rangle_1)$ *is* $\rho = \frac{-\alpha}{2} < 0$.

Proof. We have $\rho = \sum_{j=1}^{4} r(e_j) = \frac{-\alpha}{2} < 0.$ ■

Remark 4.5. Put $\alpha' = \lambda > 0$ i.e. $\alpha = \frac{1}{\lambda^2}$. The Ricci curvatures $r(e_i), i = 1,2,3$ are exactly the Ricci curvatures of the Heisenberg three-dimensional Lie algebra $\mathfrak{heis}_3$ with respect to the scalar product

$$\langle .,. \rangle_\lambda = \mathrm{diag}\{\lambda, \lambda, 1\} \quad \text{(see [13])}.$$

4.2. Ricci transformation of $\mathfrak{osc}$ with respect to the metric $\langle .,. \rangle_2$

Put $\qquad e_1 = \frac{E_1}{\sqrt{\alpha}} \qquad e_2 = E_2 \qquad e_3 = \frac{E_3}{\sqrt{\mu}} \qquad e_4 = \frac{E_4}{\sqrt{\nu}}.$

Then $\mathscr{B} = \{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of $\mathfrak{osc}$ with respect to $\langle .,. \rangle_2$.

Proposition 4.6. *The associated Levi-Civita connection to the metric Lie algebra* $(\mathfrak{osc}, \langle .,. \rangle_2)$ *is given by the following formulas*

- $\nabla_{e_i}e_i = 0 \quad \forall i = 1, ..., 4$
- $\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}e_3$
- $\nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}e_2$
- $\nabla_{e_1}e_4 = \nabla_{e_4}e_1 = 0$
- $\nabla_{e_2}e_3 = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}e_1 + \frac{1}{2}\frac{\mu-1}{\sqrt{\mu\nu}}e_4$
- $\nabla_{e_3}e_2 = \frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}e_1 + \frac{1}{2}\frac{\mu-1}{\sqrt{\mu\nu}}e_4$
- $\nabla_{e_2}e_4 = \frac{1}{2}\frac{1-\mu}{\sqrt{\mu\nu}}e_3$
- $\nabla_{e_4}e_2 = \frac{1}{2}\frac{1+\mu}{\sqrt{\mu\nu}}e_3$
- $\nabla_{e_3}e_4 = \frac{1}{2}\frac{1-\mu}{\sqrt{\mu\nu}}e_2$
- $\nabla_{e_4}e_3 = \frac{-1}{2}\frac{1+\mu}{\sqrt{\mu\nu}}e_2$

Proof. All the non-zero structure constants of our metric Lie algebra are

- $\alpha_{231} = \sqrt{\frac{\alpha}{\mu}}$
- $\alpha_{243} = -\sqrt{\frac{\mu}{\nu}}$
- $\alpha_{342} = \frac{1}{\sqrt{\mu\nu}}$
- $\alpha_{321} = -\sqrt{\frac{\alpha}{\mu}}$
- $\alpha_{423} = \sqrt{\frac{\mu}{\nu}}$
- $\alpha_{432} = \frac{-1}{\sqrt{\mu\nu}}$

We use the same formula used in the proof of the Proposition 4.1. ■

Theorem 4.7. *The Ricci transformation* $\hat{r}$ *of* $(\mathfrak{osc}, \langle .,. \rangle_2)$ *is given by*

$$\hat{r} = \begin{bmatrix} \frac{\alpha}{2\mu} & 0 & 0 & 0 \\ 0 & \frac{-\mu^2-\alpha\nu+1}{2\mu\nu} & 0 & 0 \\ 0 & 0 & \frac{\mu^2-\alpha\nu-1}{2\mu\nu} & 0 \\ 0 & 0 & 0 & \frac{-(\mu-1)^2}{2\mu\nu} \end{bmatrix}.$$

Proof. By definition of $\hat{r}$, one gets that

$$\hat{r}(e_1) = \sum_{i=1}^{4} R(e_1, e_i)e_i = R(e_1, e_2)e_2 + R(e_1, e_3)e_3 + R(e_1, e_4)e_4.$$

We can easily see that

$$R(e_1, e_2)e_2 = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}\nabla_{e_2}e_3, \qquad R(e_1, e_3)e_3 = \frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}\nabla_{e_3}e_2, \qquad R(e_1, e_4)e_4 = 0.$$

Hence we get the following result

$$\hat{r}(e_1) = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}\Big(\nabla_{e_2}e_3 - \nabla_{e_3}e_2\Big) = \frac{\alpha}{2\mu}\, e_1.$$

Similarly, the element $\hat{r}(e_2)$ is defined by

$$\hat{r}(e_2) = \sum_{i=1}^{4} R(e_2, e_i)e_i = R(e_2, e_1)e_1 + R(e_2, e_3)e_3 + R(e_2, e_4)e_4.$$

One can easily get that

$$R(e_2, e_1)e_1 = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}\nabla_{e_1}e_3, \qquad R(e_2, e_4)e_4 = \frac{1}{2}\,\frac{\mu-1}{\sqrt{\mu\nu}}\nabla_{e_4}e_3 + \sqrt{\frac{\mu}{\nu}}\nabla_{e_3}e_4$$

$$R(e_2, e_3)e_3 = \frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}\nabla_{e_3}e_1 - s\,\frac{1}{2}\,\frac{\mu-1}{\sqrt{\mu\nu}}\nabla_{e_3}e_4 - \sqrt{\frac{\alpha}{\mu}}\nabla_{e_1}e_3.$$

Therefore, the sum of these elements gives the following result

$$\hat{r}(e_2) = R(e_2, e_1)e_1 + R(e_2, e_3)e_3 + R(e_2, e_4)e_4 = \frac{-\mu^2 - \alpha\nu + 1}{2\mu\nu}\, e_2.$$

Similarly, we obtain that

$$\hat{r}(e_3) = \frac{\mu^2 - \alpha\nu - 1}{2\mu\nu}\, e_3 \quad \text{and} \quad \hat{r}(e_4) = \frac{-(\mu-1)^2}{2\mu\nu}\, e_4. \qquad \blacksquare$$

Corollary 4.8. *The principal Ricci curvatures of* $(\mathfrak{osc}, \langle .,. \rangle_2)$ *are*

- $r(e_1) = \frac{\alpha}{2\mu} > 0$
- $r(e_2) = \frac{-\mu^2-\alpha\nu+1}{2\mu\nu} < 0$
- $r(e_3) = \frac{\mu^2-\alpha\nu-1}{2\mu\nu}$
- $r(e_4) = \frac{-(\mu-1)^2}{2\mu\nu} < 0$

Hence there are three possibilities in the signature of the Ricci transformation $\hat{r}$

(1) *If* $\mu^2 < \alpha\nu + 1$, *then* $\hat{r}$ *has signature* $(+,-,-,-)$.

(2) *If* $\mu^2 = \alpha\nu + 1$, *then* $\hat{r}$ *has signature* $(+,-,0,-)$.

(3) *If* $\mu^2 > \alpha\nu + 1$, *then* $\hat{r}$ *has signature* $(+,-,+,-)$.

These signatures are exactly the possible signatures in the oscillator algebra $A_{4,10}$ *described by Kremlev and Nikonorov in* [17].

Remark 4.9. Since $Z(\mathfrak{osc}) = \langle e_1 \rangle$, we see that $r(e_1) = \frac{\alpha}{2\mu} > 0$.

Proposition 4.10. *The scalar curvature of* $(\mathfrak{osc}, \langle .,. \rangle_2)$ *is strictly negative and is given by*

$$\rho = -\frac{(\mu-1)^2 + \alpha\nu}{2\mu\nu} < 0.$$

Proof. The scalar curvature is $\rho = \sum_{i=1}^{4} r(e_i)$. ■

Now assume that $\beta' > 0$ and $\gamma' = 0$. Then our metric is given by

$$\mu_3 = \begin{bmatrix} \alpha' & \beta' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu' & 0 \\ 0 & 0 & 0 & \nu' \end{bmatrix} \quad \alpha', \nu', \beta' > 0, \quad 0 < \mu' \leq 1.$$

Note that the metric μ_1 does not contains the metric μ_2.
In fact we have $\mathrm{Aut}(\widetilde{Osc})_{\langle .,. \rangle_1} \neq \mathrm{Aut}(\widetilde{Osc})_{\langle .,. \rangle_2}$ (see Theorem 5.3). But the metric μ_3 contains the following metric

$$\eta = \begin{bmatrix} \alpha' & \beta' & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \nu' \end{bmatrix} \quad \alpha', \nu', \beta' > 0.$$

Hence we only need to study the metric μ_3. From the Proposition 1.4 in [25], the symmetric, positive definite matrix associated to μ_3 is given by

$$\langle .,. \rangle_3 = \begin{bmatrix} \frac{1}{\alpha'^2} & \frac{-\beta'}{\alpha'^2} & 0 & 0 \\ \frac{-\beta'}{\alpha'^2} & 1 + (\frac{\beta'}{\alpha'})^2 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu'^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\nu'^2} \end{bmatrix}.$$

Put $\alpha = \frac{1}{\alpha'^2}$, $\beta = \beta'$, $\mu = \frac{1}{\mu'^2}$ and $\nu = \frac{1}{\nu'^2}$. Thus, this metric is given by

$$\langle .,. \rangle_3 = \begin{bmatrix} \alpha & -\alpha\beta & 0 & 0 \\ -\alpha\beta & 1 + \alpha\beta^2 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix}.$$

4.3. Ricci transformation of $\mathfrak{osc}$ with respect to the metric $\langle.,.\rangle_3$

The basis $B=\{E_1,E_2,E_3,E_4\}$ is not orthonormal with respect to $\langle.,.\rangle_3$. Using the Gram-Schmidt process, we obtain the following $\langle.,.\rangle_3$-orthonormal basis of $\mathfrak{osc}$

$$e_1=\frac{E_1}{\sqrt{\alpha}} \qquad e_2=E_2+\beta E_1 \qquad e_3=\frac{E_3}{\sqrt{\mu}} \qquad e_4=\frac{E_4}{\sqrt{\nu}}.$$

The non-zero brackets in the basis $\mathscr{B}=\{e_1,e_2,e_3,e_4\}$ are

$$[e_2,e_3]=\sqrt{\frac{\alpha}{\mu}}e_1 \qquad [e_2,e_4]=-\sqrt{\frac{\mu}{\nu}}e_3 \qquad [e_3,e_4]=\frac{1}{\sqrt{\mu\nu}}e_2-\beta\sqrt{\frac{\alpha}{\mu\nu}}e_1.$$

Hence all the non-zero structure constants are

- $\alpha_{231}=\sqrt{\frac{\alpha}{\mu}}$
- $\alpha_{321}=-\sqrt{\frac{\alpha}{\mu}}$
- $\alpha_{342}=\frac{1}{\sqrt{\mu\nu}}$
- $\alpha_{432}=\frac{-1}{\sqrt{\mu\nu}}$
- $\alpha_{243}=-\sqrt{\frac{\mu}{\nu}}$
- $\alpha_{423}=\sqrt{\frac{\mu}{\nu}}$
- $\alpha_{341}=-\beta\sqrt{\frac{\alpha}{\mu\nu}}$
- $\alpha_{431}=\beta\sqrt{\frac{\alpha}{\mu\nu}}$

Proposition 4.11. *The Levi-Civita connection of the metric Lie algebra $(\mathfrak{osc},\langle.,.\rangle_3)$ is described by the following equalities*

- $\nabla_{e_i}e_i=0 \quad \forall i=1,...,4$
- $\nabla_{e_1}e_2=\nabla_{e_2}e_1=\frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}e_3$
- $\nabla_{e_1}e_3=\nabla_{e_3}e_1=\frac{1}{2}\sqrt{\frac{\alpha}{\mu}}e_2+\frac{\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_4$
- $\nabla_{e_1}e_4=\nabla_{e_4}e_1=\frac{-\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_3$
- $\nabla_{e_2}e_3=\frac{1}{2}\sqrt{\frac{\alpha}{\mu}}e_1+\frac{1}{2}\frac{\mu-1}{\sqrt{\mu\nu}}e_4$
- $\nabla_{e_3}e_2=\frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}e_1+\frac{1}{2}\frac{\mu-1}{\sqrt{\mu\nu}}e_4$
- $\nabla_{e_2}e_4=\frac{1}{2}\frac{1-\mu}{\sqrt{\mu\nu}}e_3$
- $\nabla_{e_4}e_2=\frac{1}{2}\frac{1+\mu}{\sqrt{\mu\nu}}e_3$
- $\nabla_{e_3}e_4=\frac{-\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_1+\frac{1}{2}\frac{1-\mu}{\sqrt{\mu\nu}}e_2$
- $\nabla_{e_4}e_3=\frac{\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_1-\frac{1}{2}\frac{1+\mu}{\sqrt{\mu\nu}}e_2$

Proof. It suffices to use the same formula as we used in the proposition 4.1. ■

Theorem 4.12. *The Ricci transformation $\hat{r}$ of $(\mathfrak{osc},\langle.,.\rangle_3)$ is given by*

$$\hat{r}=\begin{bmatrix} \frac{\alpha(\nu+\beta^2)}{2\mu\nu} & \frac{-\beta\sqrt{\alpha}}{2\mu\nu} & 0 & 0 \\ \frac{-\beta\sqrt{\alpha}}{2\mu\nu} & \frac{-\mu^2-\alpha\nu+1}{2\mu\nu} & 0 & \frac{-\alpha\beta}{2\mu\sqrt{\nu}} \\ 0 & 0 & \frac{\mu^2-\alpha(\nu+\beta^2)-1}{2\mu\nu} & 0 \\ 0 & \frac{-\alpha\beta}{2\mu\sqrt{\nu}} & 0 & -\frac{\alpha\beta^2+(\mu-1)^2}{2\mu\nu} \end{bmatrix}.$$

Proof. The elements $\hat{r}(e_i)$ are described by the following

$$\hat{r}(e_1)=\sum_{i=1}^{4}R(e_1,e_i)e_i=R(e_1,e_2)e_2+R(e_1,e_3)e_3+R(e_1,e_4)e_4.$$

One can verify that

$$\begin{aligned} R(e_1,e_2)e_2&=\frac{\alpha}{4\mu}e_1+\frac{\sqrt{\alpha\nu}(\mu-1)}{4\mu\nu}e_4\\ R(e_1,e_3)e_3&=\frac{\alpha(\nu+\beta^2)}{4\mu\nu}e_1-\frac{\beta\sqrt{\alpha}(1-\mu)}{4\mu\nu}e_2\frac{\sqrt{\alpha\nu}(\mu-1)}{4\mu\nu}e_4\\ R(e_1,e_4)e_4&=\frac{\alpha\beta^2}{4\mu\nu}e_1-\frac{\beta\sqrt{\alpha}(1+\mu)}{4\mu\nu}e_2. \end{aligned}$$

Hence summing these quantities we obtain that $\hat{r}(e_1) = \frac{\alpha(\nu+\beta^2)}{4\mu\nu}e_1 - \frac{\beta\sqrt{\alpha}}{2\mu\nu}e_2$. Similarly, $\hat{r}(e_2)$ is defined by $\hat{r}(e_2) = R(e_2,e_1)e_1 + R(e_2,e_3)e_3 + R(e_2,e_4)e_4$. We obtain

$$\begin{aligned} R(e_2,e_1)e_1 &= \frac{\alpha}{4\mu}e_2 + \frac{\alpha\beta}{4\mu\sqrt{\nu}}e_4 \\ R(e_2,e_3)e_3 &= \frac{\beta\sqrt{\alpha}(\mu-1)}{4\mu\nu}e_1 + \frac{(1-\mu)^2-3\alpha\nu}{4\mu\nu}e_2 - \frac{3\alpha\beta}{4\mu\sqrt{\nu}}e_4 \\ R(e_2,e_4)e_4 &= -\frac{\beta\sqrt{\alpha}(\mu+1)}{4\mu\nu}e_1 + \frac{1+2\mu-3\mu^2}{4\mu\nu}e_2. \end{aligned}$$

Therefore, the sum of these quantities gives

$$\hat{r}(e_2) = \frac{-\beta\sqrt{\alpha}}{2\mu\nu}e_1 + \frac{-\mu^2-\alpha\nu+1}{2\mu\nu}e_2 - \frac{\alpha\beta}{2\mu\sqrt{\nu}}e_4.$$

The element $\hat{r}(e_3)$ is given by $\hat{r}(e_3) = R(e_3,e_1)e_1 + R(e_3,e_2)e_2 + R(e_3,e_4)e_4$. We get

$$\begin{aligned} R(e_3,e_1)e_1 &= \frac{\alpha(\nu+\beta^2)}{4\mu\nu}e_3 \\ R(e_3,e_2)e_2 &= \frac{(1-\mu)^2-3\alpha\nu}{4\mu\nu}e_3 \\ R(e_3,e_4)e_4 &= \frac{-3\alpha\beta^2+\mu^2+2\mu-3}{4\mu\nu}e_3. \end{aligned}$$

Hence the sum of these elements gives $\hat{r}(e_3) = \dfrac{\mu^2-\alpha(\nu+\beta^2)-1}{2\mu\nu}e_3$.

For $\hat{r}(e_4)$, one can verify that

$$\begin{aligned} R(e_4,e_1)e_1 &= \frac{\alpha\beta}{4\mu\sqrt{\nu}}e_2 + \frac{\alpha\beta^2}{4\mu\nu}e_4 \\ R(e_4,e_2)e_2 &= \frac{\sqrt{\alpha\nu}(\mu-1)}{4\mu\nu}e_1 + \frac{-3\mu^2+2\mu+1}{4\mu\nu}e_4 \\ R(e_4,e_3)e_3 &= -\frac{\sqrt{\alpha\nu}(\mu-1)}{4\mu\nu}e_1 - \frac{3\alpha\beta}{4\mu\sqrt{\nu}}e_2 + \frac{-3\alpha\beta^2+\mu^2+2\mu-3}{4\mu\nu}e_4. \end{aligned}$$

Summing these elements we get that $\hat{r}(e_4) = \frac{-\alpha\beta}{2\mu\sqrt{\nu}}e_2 - \frac{\alpha\beta^2+(\mu-1)^2}{2\mu\nu}e_4$. ■

Corollary 4.13. *The scalar curvature of* $(\mathfrak{osc}, \langle .,. \rangle_3)$ *is strictly negative and is given by*

$$\rho = -\frac{(\mu-1)^2+\alpha(\nu+\beta^2)}{2\mu\nu} < 0.$$

Proof. The scalar curvature ρ is the trace of $\hat{r}$, thus

$$\rho = \mathrm{Trace}(\hat{r}) = -\frac{(\mu-1)^2+\alpha(\nu+\beta^2)}{2\mu\nu} < 0.$$ ■

Now, assume that $\beta', \gamma' > 0$. Then our metric is

$$\mu_4 = \begin{bmatrix} \alpha' & \beta' & \gamma' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu' & 0 \\ 0 & 0 & 0 & \nu' \end{bmatrix} \quad \alpha', \nu', \beta', \gamma' > 0, \quad 0 < \mu' < 1.$$

The symmetric positive definite matrix associated to this matrix is

$$\langle .,.\rangle_4 = \begin{bmatrix} \frac{1}{\alpha'^2} & \frac{-\beta'}{\alpha'^2} & \frac{-\gamma'}{\alpha'^2\mu'} & 0 \\ \frac{-\beta'}{\alpha'^2} & 1+(\frac{\beta'}{\alpha'})^2 & \frac{\beta'\gamma'}{\alpha'^2\mu'} & 0 \\ \frac{-\gamma'}{\alpha'^2\mu'} & \frac{\beta'\gamma'}{\alpha'^2\mu'} & \frac{1}{\mu'^2}+\frac{\gamma'^2}{\mu'^2\alpha'^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\nu'^2} \end{bmatrix}.$$

Put $\alpha = \frac{1}{\alpha'^2}$, $\beta = \beta'$, $\gamma = \gamma'$, $\mu = \frac{1}{\mu'^2}$ and $\nu = \frac{1}{\nu'^2}$. Thus, this metric is given by

$$\langle .,.\rangle_4 = \begin{bmatrix} \alpha & -\alpha\beta & -\alpha\gamma\sqrt{\mu} & 0 \\ -\alpha\beta & 1+\alpha\beta^2 & \alpha\beta\gamma\sqrt{\mu} & 0 \\ -\alpha\gamma\sqrt{\mu} & \alpha\beta\gamma\sqrt{\mu} & \mu+\alpha\gamma^2\mu & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix}.$$

4.4. Ricci transformation of $\mathfrak{osc}$ with respect to the metric $\langle .,.\rangle_4$

Using the Gram-Schmidt process, we obtain the following orthonormal basis of $\mathfrak{osc}$ with respect to the inner product $\langle .,.\rangle_4$

$$e_1 = \frac{E_1}{\sqrt{\alpha}} \qquad e_2 = E_2 + \beta E_1 \qquad e_3 = \frac{E_3}{\sqrt{\mu}} + \gamma E_1 \qquad e_4 = \frac{E_4}{\sqrt{\nu}}.$$

The non-zero brackets in the basis $\mathscr{B} = \{e_1, e_2, e_3, e_4\}$ are

$$[e_2,e_3] = \sqrt{\frac{\alpha}{\mu}}e_1 \qquad [e_2,e_4] = -\sqrt{\frac{\mu}{\nu}}e_3 + \gamma\sqrt{\frac{\alpha\mu}{\nu}}e_1 \qquad [e_3,e_4] = -\beta\sqrt{\frac{\alpha}{\mu\nu}}e_1 + \frac{1}{\sqrt{\mu\nu}}e_2.$$

Hence all the non-zero structure constants are

- $\alpha_{231} = \sqrt{\frac{\alpha}{\mu}}$
- $\alpha_{423} = \sqrt{\frac{\mu}{\nu}}$
- $\alpha_{321} = -\sqrt{\frac{\alpha}{\mu}}$
- $\alpha_{341} = -\beta\sqrt{\frac{\alpha}{\mu\nu}}$
- $\alpha_{342} = \frac{1}{\sqrt{\mu\nu}}$
- $\alpha_{431} = \beta\sqrt{\frac{\alpha}{\mu\nu}}$
- $\alpha_{432} = \frac{-1}{\sqrt{\mu\nu}}$
- $\alpha_{241} = \gamma\sqrt{\frac{\alpha\mu}{\nu}}$
- $\alpha_{243} = -\sqrt{\frac{\mu}{\nu}}$
- $\alpha_{421} = -\gamma\sqrt{\frac{\alpha\mu}{\nu}}$

Proposition 4.14. *The associated Levi-Civita connection to the metric Lie algebra $(\mathfrak{osc}, \langle .,.\rangle_4)$ is described by the following formulas*

- $\nabla_{e_i}e_i = 0 \quad \forall i = 1,...,4$
- $\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}e_3 - \frac{\gamma}{2}\sqrt{\frac{\alpha\mu}{\nu}}e_4$
- $\nabla_{e_1}e_3 = \nabla_{e_3}e_1 = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}e_2 + \frac{\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_4$
- $\nabla_{e_1}e_4 = \nabla_{e_4}e_1 = \frac{\gamma}{2}\sqrt{\frac{\alpha\mu}{\nu}}e_2 - \frac{\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_3$
- $\nabla_{e_2}e_3 = \frac{1}{2}\sqrt{\frac{\alpha}{\mu}}e_1 + \frac{1}{2}\frac{\mu-1}{\sqrt{\mu\nu}}e_4$
- $\nabla_{e_3}e_2 = \frac{-1}{2}\sqrt{\frac{\alpha}{\mu}}e_1 + \frac{1}{2}\frac{\mu-1}{\sqrt{\mu\nu}}e_4$
- $\nabla_{e_2}e_4 = \frac{\gamma}{2}\sqrt{\frac{\alpha\mu}{\nu}}e_1 + \frac{1}{2}\frac{1-\mu}{\sqrt{\mu\nu}}e_3$
- $\nabla_{e_4}e_2 = \frac{-\gamma}{2}\sqrt{\frac{\alpha\mu}{\nu}}e_1 + \frac{1}{2}\frac{1+\mu}{\sqrt{\mu\nu}}e_3$
- $\nabla_{e_3}e_4 = \frac{-\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_1 + \frac{1}{2}\frac{1-\mu}{\sqrt{\mu\nu}}e_2$
- $\nabla_{e_4}e_3 = \frac{\beta}{2}\sqrt{\frac{\alpha}{\mu\nu}}e_1 - \frac{1}{2}\frac{1+\mu}{\sqrt{\mu\nu}}e_2$

Theorem 4.15. *The Ricci transformation* $\hat{r}$ *of* $(\mathfrak{osc}, \langle .,. \rangle_4)$ *is given by*

$$\hat{r} = \begin{bmatrix} \frac{\alpha(\nu+\beta^2+\mu^2\gamma^2)}{2\mu\nu} & \frac{-\beta\sqrt{\alpha}}{2\mu\nu} & \frac{-\gamma\mu\sqrt{\alpha}}{2\nu} & 0 \\ \frac{-\beta\sqrt{\alpha}}{2\mu\nu} & -\frac{\alpha(\nu+\mu^2\gamma^2)+\mu^2-1}{2\mu\nu} & \frac{\alpha\beta\gamma}{2\nu} & \frac{-\alpha\beta}{2\mu\sqrt{\nu}} \\ \frac{-\gamma\mu\sqrt{\alpha}}{2\nu} & \frac{\alpha\beta\gamma}{2\nu} & -\frac{\alpha(\nu+\beta^2)+1-\mu^2}{2\mu\nu} & -\frac{\alpha\gamma}{2\sqrt{\nu}} \\ 0 & \frac{-\alpha\beta}{2\mu\sqrt{\nu}} & -\frac{\alpha\gamma}{2\sqrt{\nu}} & -\frac{\alpha(\beta^2+\mu^2\gamma^2)+(\mu-1)^2}{2\mu\nu} \end{bmatrix}.$$

Proof. The proof is similar to the one of the precedent case, after heavy and careful calculations we get that

$$\begin{aligned} R(e_1,e_2)e_2 &= \frac{\alpha(\nu+\mu^2\gamma^2)}{4\mu\nu}e_1 + \frac{\gamma\sqrt{\alpha}(1-\mu)}{4\mu\nu}e_3 + \frac{\sqrt{\alpha}(\mu-1)}{4\mu\sqrt{\nu}}e_4 \\ R(e_1,e_3)e_3 &= \frac{\alpha(\nu+\beta^2)}{4\mu\nu}e_1 - \frac{\beta\sqrt{\alpha}(1-\mu)}{4\mu\nu}e_2 - \frac{\sqrt{\alpha\nu}(\mu-1)}{4\mu\nu}e_4 \\ R(e_1,e_4)e_4 &= \frac{\alpha(\beta^2+\mu^2\gamma^2)}{4\mu\nu}e_1 - \frac{\beta\sqrt{\alpha}(1+\mu)}{4\mu\nu}e_2 - \frac{\gamma\sqrt{\alpha}(1+\mu)}{4\mu\nu}e_3. \end{aligned}$$

Hence summing these quantities we obtain that

$$\hat{r}(e_1) = \frac{\alpha(\nu+\beta^2+\mu^2\gamma^2)}{2\mu\nu}e_1 - \frac{\beta\sqrt{\alpha}}{2\mu\nu}e_2 - \frac{\gamma\mu\sqrt{\alpha}}{2\nu}e_3.$$

Similarly, we get

$$\begin{aligned} R(e_2,e_1)e_1 &= \frac{\alpha(\nu+\mu^2\gamma^2)}{4\mu\nu}e_2 - \frac{\alpha\beta\gamma}{4\nu}e_3 + \frac{\alpha\beta}{4\mu\sqrt{\nu}}e_4 \\ R(e_2,e_3)e_3 &= \frac{\beta\sqrt{\alpha}(\mu-1)}{4\mu\nu}e_1 + \frac{(1-\mu)^2-3\alpha\nu}{4\mu\nu}e_2 - \frac{3\alpha\beta}{4\mu\sqrt{\nu}}e_4 \\ R(e_2,e_4)e_4 &= -\frac{\beta\sqrt{\alpha}(1+\mu)}{4\mu\nu}e_1 + \frac{-3\gamma^2\alpha\mu^2-3\mu^2+2\mu+1}{4\mu\nu}e_2 + \frac{3\alpha\beta\gamma}{4\nu}e_3. \end{aligned}$$

Therefore, the sum of these quantities gives

$$\hat{r}(e_2) = \frac{-\beta\sqrt{\alpha}}{2\mu\nu}e_1 - \frac{\alpha(\nu+\mu^2\gamma^2)+\mu^2-1}{2\mu\nu}e_2 + \frac{\alpha\beta\gamma}{2\nu}e_3 - \frac{\alpha\beta}{2\mu\sqrt{\nu}}e_4.$$

For the element $\hat{r}(e_3)$, we obtain

$$\begin{aligned} R(e_3,e_1)e_1 &= -\frac{\alpha\beta\gamma}{4\nu}e_2 + \frac{\alpha(\nu+\beta^2)}{4\mu\nu}e_3 + \frac{\alpha\gamma}{4\sqrt{\nu}}e_4 \\ R(e_3,e_2)e_2 &= -\frac{\gamma\sqrt{\alpha}(\mu-1)}{4\nu}e_1 + \frac{(1-\mu)^2-3\alpha\nu}{4\mu\nu}e_3 - \frac{3\alpha\gamma}{4\sqrt{\nu}}e_4 \\ R(e_3,e_4)e_4 &= -\frac{\gamma\sqrt{\alpha}(\mu+1)}{4\nu}e_1 + \frac{3\alpha\beta\gamma}{4\nu}e_2 + \frac{-3\alpha\beta^2+\mu^2+2\mu-3}{4\mu\nu}e_3. \end{aligned}$$

Hence the sum of these elements shows that

$$\hat{r}(e_3) = \frac{-\gamma\mu\sqrt{\alpha}}{2\nu}e_1 + \frac{\alpha\beta\gamma}{2\nu}e_2 - \frac{\alpha(\nu+\beta^2)+1-\mu^2}{2\mu\nu}e_3 - \frac{\alpha\gamma}{2\sqrt{\nu}}e_4.$$

Finally, the element $\hat{r}(e_4)$ is given by

$$\begin{aligned} R(e_4,e_1)e_1 &= \frac{\alpha\beta}{4\mu\sqrt{\nu}}e_2 + \frac{\alpha\gamma}{4\sqrt{\nu}}e_3 + \frac{\alpha(\beta^2+\mu^2\gamma^2)}{4\mu\nu}e_4 \\ R(e_4,e_2)e_2 &= \frac{\sqrt{\alpha}(\mu-1)}{4\mu\sqrt{\nu}}e_1 - \frac{3\alpha\gamma}{4\sqrt{\nu}}e_3 + \frac{-3\gamma^2\alpha\mu^2-3\mu^2+2\mu+1}{4\mu\nu}e_4 \\ R(e_4,e_3)e_3 &= -\frac{\sqrt{\alpha}(\mu-1)}{4\mu\sqrt{\nu}}e_1 - \frac{3\alpha\beta}{4\mu\sqrt{\nu}}e_2 + \frac{-3\alpha\beta^2+\mu^2+2\mu-3}{4\mu\nu}e_4. \end{aligned}$$

Hence summing these elements we get the following result

$$\hat{r}(e_4) = \frac{-\alpha\beta}{2\mu\sqrt{\nu}}e_2 - \frac{\alpha\gamma}{2\sqrt{\nu}}e_3 - \frac{\alpha(\beta^2+\mu^2\gamma^2)+(\mu-1)^2}{2\mu\nu}e_4. \qquad \blacksquare$$

Corollary 4.16. *The scalar curvature of $(\mathfrak{osc}, \langle .,. \rangle_4)$ is strictly negative and is given by*

$$\rho = -\frac{(\mu-1)^2+\alpha(\nu+\beta^2+\mu^2\gamma^2)}{2\mu\nu} < 0.$$

Proof. The scalar curvature ρ is the trace of $\hat{r}$, hence

$$\rho = \mathrm{Trace}(\hat{r}) = -\frac{(\mu-1)^2+\alpha(\nu+\beta^2+\mu^2\gamma^2)}{2\mu\nu} < 0. \qquad \blacksquare$$

5. The isometry group of the oscillator group Osc

Let us give some well known and useful facts about the isometry group of a left invariant Riemannian metric on a Lie group.

Let $\mathfrak{g}$ be a Lie algebra and let G be its associated simply connected Lie group. The group of isometries of G with respect to the left invariant Riemannian metric g is defined by

$$\mathrm{Isom}(G,g) = \{\theta \in \mathrm{Diffeo}(G)/\theta^* g = g\}$$

where $\mathrm{Diffeo}(G)$ is the diffeomorphism group of G and $\theta^* g$ is the pullback of g by θ (see [14]). The connected component of the identity of $\mathrm{Isom}(G,g)$ will be denoted by $\mathrm{Isom}_0(G,g)$. The group $\mathrm{Isom}(G,g)$ acts transitively on G, the isotropy subgroup of $\mathrm{Isom}(G,g)$ at the identity element e of G is defined by

$$\mathrm{Isom}(G,g)_e = \{\theta \in \mathrm{Isom}(G,g)/\theta(e) = e\}$$

its connected component of the identity will be denoted by $\mathrm{Isom}_0(G,g)_e$.

Since G is simply connected, we have $\mathrm{Aut}(G) \simeq \mathrm{Aut}(\mathfrak{g})$ and we have an action of this group on the set $\mathscr{L}$ of all left invariant Riemannian metrics on G given by

$$\mathrm{Aut}(\mathfrak{g}) \times \mathscr{L} \longrightarrow \mathscr{L}, \quad (\theta, g) \longmapsto \theta^* g = g_\theta$$

where $g_\theta(u,v) = g(\theta^{-1}u, \theta^{-1}v)\ \forall u,v \in \mathfrak{g}$. See [14] for more detail.

We denote the isotropy subgroup of $\mathrm{Aut}(\mathfrak{g})$ at g by $\mathrm{Aut}(\mathfrak{g})_g = \{\theta \in \mathrm{Aut}(\mathfrak{g})/\theta^* g = g\}$.

Definition 5.1. An automorphism of $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$ satisfying $\theta^* g = g$ is called an isometric automorphism of G.

Let θ be an automotphism of G, the pullback θ^*g of g by θ is defined by

$$\theta^*g(u,v) = g\left(\theta_*^{-1}(u), \theta_*^{-1}(v)\right) \quad \forall u, v \in \mathfrak{g}$$

where θ_* is the differential of θ at the identity element e of G. This equation is equivalent to the following matrix equation $[g] = [\theta_*]^t[\theta^*g][\theta_*]$. Hence, in term of matrix calculation, the group of isometric automorphisms of G is given by [14]

$$\mathrm{Aut}(G)_g = \left\{\theta \in \mathrm{Aut}(G) \ / \ [g] = [\theta_*]^t[g][\theta_*]\right\}.$$

Put $L(G)$ (resp. $R(G)$) for the group of all left translations (resp. right translations) on G. Let G be a compact, connected, simple Lie group and let g be a left invariant Riemannian metric on G.

The following is a well known result $\mathrm{Isom}_0(G,g) \subset L(G)R(G)$ [21].

Since $\mathrm{Aut}(G)_g \subset \mathrm{Isom}(G,g)_e$, we get the following important result

$$L(G) \rtimes \mathrm{Aut}(G)_g \subset L(G) \cdot \mathrm{Isom}(G,g)_e = \mathrm{Isom}(G,g) \text{ [14]}.$$

We have [14] $$L(G) \trianglelefteq \mathrm{Isom}(G,g) \Leftrightarrow \mathrm{Isom}(G,g)_e = \mathrm{Aut}(G)_g. \tag{1}$$

$$L(G) \trianglelefteq \mathrm{Isom}_0(G,g) \Leftrightarrow \mathrm{Isom}_0(G,g)_e = \mathrm{Aut}_0(G)_g. \tag{2}$$

Where $L(G) \trianglelefteq \mathrm{Isom}(G,g)$ means that $L(G)$ is normal in $\mathrm{Isom}(G,g)$.

Lemma 5.2. [14] *Let θ be a diffeomorphism of G which fixes the identity element e of G, and let g be a left invariant Riemannian metric on G. The following statements are equivalent*

(1) *θ is an isometry of g.*

(2) *For each $p \in G$, the differential of $L_{\theta(p)^{-1}}\theta L_p$ at e is an orthogonal transformation on the inner product space (T_eG, g).*

We denote by $\widetilde{Osc}$ the simply connected Lie group associated to the Lie algebra $\mathfrak{osc}$. To find $\mathrm{Isom}(\widetilde{Osc}, \langle .,. \rangle_i)$ $i = 1, ..., 4$, it is necessary to describe $\mathrm{Aut}(\widetilde{Osc})_{\langle .,. \rangle_i}$. The automorphism group of the 4-dimensional oscillator algebra is given by

$$\mathrm{Aut}(\mathfrak{osc}) =$$

$$\left\{ \begin{bmatrix} \sigma(x^2+y^2) & wy - \sigma tx & -wx - \sigma ty & u \\ 0 & x & y & t \\ 0 & -\sigma y & \sigma x & w \\ 0 & 0 & 0 & \sigma \end{bmatrix} \Big/ x, y, u, t, w \in \mathbb{R}, x^2 + y^2 \neq 0 \right\}, \tag{3}$$

where $\sigma = \pm 1$ (see [6, 10, 25]).

From here on, to simplify notations, we put $G = \widetilde{Osc}$ and $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{osc}$.

Theorem 5.3. *The group of isometric automorphisms of G is given by*

$$\mathrm{Aut}(G)_{\langle .,. \rangle_1} = \mathrm{diag}\left\{\varepsilon, \mathrm{O}(2), \varepsilon\right\}, \qquad \mathrm{Aut}(G)_{\langle .,. \rangle_2} \cong (\mathbb{Z}_2)^2$$

$$\mathrm{Aut}(G)_{\langle .,. \rangle_3} \cong \mathbb{Z}_2 \qquad \mathrm{Aut}(G)_{\langle .,. \rangle_4} = \{I_4\}.$$

where $$\varepsilon = \det \mathrm{O}(2) = \begin{cases} 1 & \text{in } \mathrm{SO}(2) \\ -1 & \text{in } \mathrm{O}(2) - \mathrm{SO}(2) \end{cases}$$

Proof. Let $A \in \mathrm{Aut}(G)$, then by (3) A is of the form

$$A = \begin{bmatrix} \sigma(x^2+y^2) & wy-\sigma tx & -wx-\sigma ty & u \\ 0 & x & y & t \\ 0 & -\sigma y & \sigma x & w \\ 0 & 0 & 0 & \sigma \end{bmatrix}$$

where $x^2+y^2 \neq 0$ and $\sigma = \pm 1$. Therefore

$$A \in \mathrm{Aut}(G)_{\langle .,. \rangle_1} \;\Leftrightarrow\; A^t\langle .,. \rangle_1 A = \langle .,. \rangle_1 \;\Leftrightarrow\; A \in \mathrm{diag}\{\varepsilon, \mathrm{O}(2), \varepsilon\}.$$

Thus we find that $\mathrm{Aut}(G)_{\langle .,. \rangle_1} = \mathrm{diag}\{\varepsilon, \mathrm{O}(2), \varepsilon\}$. For the metric $\langle .,. \rangle_2$, one gets

$$A \in \mathrm{Aut}(G)_{\langle .,. \rangle_2} \Leftrightarrow A^t\langle .,. \rangle_2 A = \langle .,. \rangle_2$$

$$\Leftrightarrow A \in \left\{ I_4, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}.$$

This is exactly the Klein 4-group, thus $\mathrm{Aut}(G)_{\langle .,. \rangle_2} \cong (\mathbb{Z}_2)^2$.
For the third metric $\langle .,. \rangle_3$, we have $A \in \mathrm{Aut}(G)_{\langle .,. \rangle_3} \Leftrightarrow A^t\langle .,. \rangle_3 A = \langle .,. \rangle_3$. This matrix equation implies that $\sigma^2(x^2+y^2)^2 = 1$, i.e. $x^2+y^2 = 1$. Then the last component in the first line of the matrix $A^t\langle .,. \rangle_3 A$ implies that $u = t\beta$. If we replace that in the last component of the matrix $A^t\langle .,. \rangle_3 A$, we get $t^2 + w^2\mu + \nu = \nu$, hence $t = w = u = 0$. Now put

$$A = \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & x & y & 0 \\ 0 & -\sigma y & \sigma x & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix},$$

therefore one can easily see that

$$A \in \mathrm{Aut}(G)_{\langle .,. \rangle_3} \;\Leftrightarrow\; A^t\langle .,. \rangle_3 A = \langle .,. \rangle_3 \;\Leftrightarrow\; A \in \left\{ I_4, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\}.$$

Thus $\mathrm{Aut}(G)_{\langle .,. \rangle_3} \cong \mathbb{Z}_2$.
Finally, one gets that $A \in \mathrm{Aut}(G)_{\langle .,. \rangle_4} \Leftrightarrow A = I_4$. The justification is similar to that of the previous case, the last component in the first line of the matrix $A^t\langle .,. \rangle_4 A$ implies that $u = t\beta + \gamma\sqrt{\mu}w$. If we replace that in the last component of the matrix $A^t\langle .,. \rangle_4 A$, we obtain $t^2 + w^2\mu + \nu = \nu$, hence $t = w = u = 0$. Now put

$$A = \begin{bmatrix} \sigma & 0 & 0 & 0 \\ 0 & x & y & 0 \\ 0 & -\sigma y & \sigma x & 0 \\ 0 & 0 & 0 & \sigma \end{bmatrix},$$

and assume that $\sigma = 1$.

If $A^t\langle.,.\rangle_4 A = \langle.,.\rangle_4$, then the first and the second components of the first line of the two matrices $A^t\langle.,.\rangle_4 A$ and $\langle.,.\rangle_4$ give the following system

$$\begin{cases} -\beta x + \gamma\sqrt{\mu}y = -\beta \\ -\gamma\sqrt{\mu}x - \beta y = -\gamma\sqrt{\mu} \end{cases}$$

One can calculate x in the first equation and replace it in the second equation, this shows that $y = 0$, hence $x = 1$. Assume that $\sigma = -1$, by the same remark in the previous case we obtain the following system

$$\begin{cases} \beta x + \gamma\sqrt{\mu}y = -\beta \\ -\gamma\sqrt{\mu}x + \beta y = -\gamma\sqrt{\mu} \end{cases}$$

The solution of this system is the following

$$\begin{cases} x = \frac{\gamma^2\mu - \beta^2}{\beta^2+\gamma^2\mu^2} \\ y = \frac{-2\beta\gamma\sqrt{\mu}}{\beta^2+\gamma^2\mu^2} \end{cases}$$

The second component in the second line of the two matrices $A^t\langle.,.\rangle_4 A$ and $\langle.,.\rangle_4$ forces the following equality

$$(1+\alpha\beta^2)x^2 + 2\alpha\beta\gamma\sqrt{\mu}xy + (\mu\alpha\gamma^2 + \mu)y^2 = 1 + \alpha\beta^2.$$

We can easily verify that this equality is verified only when $\mu = 1$. But we have $\mu > 1$, hence there is no automorphism preserving the metric $\langle.,.\rangle_4$ when $\sigma = -1$. Consequently $\mathrm{Aut}(G)_{\langle.,.\rangle_4} = \{I_4\}$. ■

Proposition 5.4. *The Lie group G is unimodular.*

Proof. A simple verification shows that $\mathrm{Tr}(\mathrm{ad}\, E_i) = 0$, $\forall i = 1, ..., 4$. ■

Proposition 5.5. *The Lie group G is not of type (R).*

Proof. The characteristic polynomial of $\mathrm{ad}\, E_4$ is $P = X^2(X^2+1)$. We see that P has two complex eigenvalues. ■

Since our Lie group G is not of type (R), we cannot claim directly that

$$\mathrm{Isom}(G) = G \rtimes \mathrm{Aut}(G)_{\langle.,.\rangle_i} \quad i = 1, ..., 4.$$

Recall that we have in hand the following transitive action

$$\mathrm{Isom}(G, g) \times G \longrightarrow G, \quad (\theta, x) \longmapsto \theta(x)$$

Consider the isotropy representation given by

$$\mathrm{Is}_e : \mathrm{Isom}(G, g)_e \longrightarrow GL(\mathfrak{g}) \quad \theta \longmapsto \theta_* = d_e\theta$$

By Lemma 5.2, if $\theta \in \mathrm{Isom}(G, g)_e$, then $\theta_* \in \mathrm{O}(\mathfrak{g}, g)$. Consider the vector subspace of $\mathrm{GL}(\mathfrak{g})$ defined by

$$\mathcal{C}(\hat{r}) = \left\{\psi \in \mathrm{GL}(\mathfrak{g}) / \psi \circ \hat{r} = \hat{r} \circ \psi\right\}.$$

If $\theta \in \mathrm{Isom}(G,g)_e$, then $\theta_* \in \mathcal{C}(\hat{r})$ (see Corollary 2.10 [14]). Hence the group $\mathrm{Isom}(G,g)_e$ can be injected in the intersection $\mathrm{O}(\mathfrak{g},g) \cap \mathcal{C}(\hat{r})$.
The following lemma is important

Lemma 5.6. (Corollary 2.8 in [14]) *If* $\mathrm{Isom}(G,g)_e$ *is a finite group, then* $\mathrm{Isom}(G,g)_e = \mathrm{Aut}(G)_g$.

5.1. The isometry group of G with respect to the metric $\langle .,. \rangle_1$

Theorem 5.7. *The connected component of the group of isometries of the four-dimensional oscillator group with respect to the Riemannian metric* $\langle .,. \rangle_1$ *is given by*

$$\mathrm{Isom}_0\left(G, \langle .,. \rangle_1\right) \cong G \rtimes \mathrm{SO}(2).$$

Proof. The Ricci transformation $\hat{r}$ of $(G, \langle .,. \rangle_1)$ is equal to

$$\hat{r} = \begin{bmatrix} \frac{\alpha}{2} & 0 & 0 & 0 \\ 0 & \frac{-\alpha}{2} & 0 & 0 \\ 0 & 0 & \frac{-\alpha}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\mathcal{C}(\hat{r})$ is given by

$$\mathcal{C}(\hat{r}) = \left\{ \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & a & c & 0 \\ 0 & b & d & 0 \\ 0 & 0 & 0 & y \end{bmatrix} \Big/ a,b,c,d,x,y \in \mathbb{R} \right\}.$$

Thus, we obtain that $\mathrm{O}\left(\mathfrak{g}, \langle .,. \rangle_1\right) \cap \mathcal{C}(\hat{r}) = \mathrm{diag}\left\{\pm 1, \mathrm{O}(2), \pm 1\right\}$. We have

$$\mathrm{diag}\left\{\varepsilon, \mathrm{O}(2), \varepsilon\right\} = \mathrm{Aut}(G)_{\langle .,. \rangle_1} \subset \mathrm{Isom}\left(G, \langle .,. \rangle_1\right)_e \subset \mathrm{diag}\left\{\pm 1, \mathrm{O}(2), \pm 1\right\}.$$

Hence

$$\mathrm{Aut}_0(G)_{\langle .,. \rangle_1} = \mathrm{Isom}_0\left(G, \langle .,. \rangle_1\right)_e = \mathrm{diag}\left\{1, \mathrm{SO}(2), 1\right\} \cong \mathrm{SO}(2).$$

By (2), $L(G)$ is normal in $\mathrm{Isom}_0\left(G, \langle .,. \rangle_1\right)$. Therefore

$$\mathrm{Isom}_0\left(G, \langle .,. \rangle_1\right) \cong G \rtimes \mathrm{SO}(2). \qquad \blacksquare$$

5.2. The isometry group of G with respect to the metric $\langle .,. \rangle_2$

Now, let us study the isometry group of $(G, \langle .,. \rangle_2)$. We describe the following cases associated to the Ricci transformation $\hat{r}$ of $(G, \langle .,. \rangle_2)$.

1. If $\mu^2 > \alpha\nu + 1$, then the signature of $\hat{r}$ is $(+,-,+,-)$. We have

$$r(e_1) - r(e_3) = 0 \quad \Leftrightarrow \quad \mu^2 = 2\alpha\nu + 1$$
$$r(e_2) - r(e_4) = 0 \quad \Leftrightarrow \quad 2\mu + \alpha\nu = 2 \text{ impossible because } \mu > 1.$$

Hence the possible forms of the Ricci transformation in this case are

$$\hat{r} = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \quad \text{or} \quad \hat{r} = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_1 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}.$$

2. If $\mu^2 < \alpha\nu + 1$, then the signature of $\hat{r}$ is $(+,-,-,-)$. We have

$$\begin{aligned} r(e_2) - r(e_3) = 0 &\Leftrightarrow 2(1-\mu^2) = 0 \text{ impossible because } \mu > 1 \\ r(e_3) - r(e_4) = 0 &\Leftrightarrow 2\mu^2 = 2\mu + \alpha\nu \\ r(e_2) - r(e_4) = 0 &\Leftrightarrow 2 = 2\mu + \alpha\nu \text{ impossible.} \end{aligned}$$

Hence the possible forms of the Ricci transformation in this case are

$$\hat{r} = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \quad \text{or} \quad \hat{r} = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_3 \end{bmatrix}.$$

3. If $\mu^2 = \alpha\nu + 1$, then the signature of $\hat{r}$ is $(+,-,0,-)$. The possible form of $\hat{r}$ is

$$\hat{r} = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}.$$

In the two cases where the Ricci transformation $\hat{r}$ is given by

$$\begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix}$$

i.e all the eigenvalues of $\hat{r}$ are distincts, we obtain that

$$\mathrm{O}(\mathfrak{g}, \langle .,. \rangle_2) \cap \mathcal{C}(\hat{r}) = \operatorname{diag}\{\pm 1, \pm 1, \pm 1, \pm 1\} \cong (\mathbb{Z}_2)^4.$$

Since $\operatorname{Isom}(G, \langle .,. \rangle_2)_e \subset \mathrm{O}(\mathfrak{g}, \langle .,. \rangle_2) \cap \mathcal{C}(\hat{r})$, then $\operatorname{Isom}(G, \langle .,. \rangle_2)_e$ is a finite group. By Lemma 5.6, we obtain $\operatorname{Isom}(G, \langle .,. \rangle_2)_e = \operatorname{Aut}(G)_{\langle .,. \rangle_2} \cong (\mathbb{Z}_2)^2$.
The two cases where $\hat{r}$ is given by

$$\begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_1 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_3 \end{bmatrix}$$

are similar by a permutation, we consider $\hat{r} \backsimeq \operatorname{diag}\{r_2, r_1, r_1, r_4\}$. Then one can find that

$$\mathcal{C}(\hat{r}) = \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix} \Big/ a, b, c, d, e, f \in \mathbb{R} \right\}.$$

Hence $\mathrm{O}(\mathfrak{g}, \langle .,. \rangle_2) \cap \mathcal{C}(\hat{r}) = \operatorname{diag}\{\pm 1, \pm 1, \pm 1, \pm 1\} \cong (\mathbb{Z}_2)^4$.
Since $\operatorname{Isom}(G, \langle .,. \rangle_2)_e \subset \mathrm{O}(\mathfrak{g}, \langle .,. \rangle_2) \cap \mathcal{C}(\hat{r})$, then $\operatorname{Isom}(G, \langle .,. \rangle_2)_e$ is a finite group. By Lemma 5.6, we get that $\operatorname{Isom}(G, \langle .,. \rangle_2)_e = \operatorname{Aut}(G)_{\langle .,. \rangle_2} \cong (\mathbb{Z}_2)^2$.
Therefore we have proved the following theorem

Theorem 5.8. *The isometry group of the four-dimensional oscillator group with respect to the Riemannian metric* $\langle .,. \rangle_2$ *is given (up to permutations in* $\hat{r}$*) by*

$$\mathrm{Isom}(G, \langle .,. \rangle_2) \cong G \rtimes (\mathbb{Z}_2)^2 .$$

5.3. The isometry group of G with respect to the metric $\langle .,. \rangle_3$

Recall that the Ricci transformation $\hat{r}$ of $(G, \langle .,. \rangle_3)$ is equal to

$$\hat{r} = \begin{bmatrix} \frac{\alpha(\nu+\beta^2)}{2\mu\nu} & \frac{-\beta\sqrt{\alpha}}{2\mu\nu} & 0 & 0 \\ \frac{-\beta\sqrt{\alpha}}{2\mu\nu} & \frac{-\mu^2-\alpha\nu+1}{2\mu\nu} & 0 & \frac{-\alpha\beta}{2\mu\sqrt{\nu}} \\ 0 & 0 & \frac{\mu^2-\alpha(\nu+\beta^2)-1}{2\mu\nu} & 0 \\ 0 & \frac{-\alpha\beta}{2\mu\sqrt{\nu}} & 0 & -\frac{(\mu-1)^2+\alpha\beta^2}{2\mu\nu} \end{bmatrix}.$$

Since $\hat{r}$ is a self adjoint endomorphism of $\mathfrak{g}$, there exists a matrix $P \in \mathrm{O}(4)$ such that

$$\hat{r} = PDP^{-1} \quad \text{where} \quad D = \mathrm{diag}\{r_1, r_2, r_3, r_4\}.$$

To determine the intersection $\mathrm{O}(\mathfrak{g}, \langle .,. \rangle_3) \cap \mathcal{C}(\hat{r})$, we consider the following automorphism

$$\Phi : \mathrm{M}_4(\mathbb{R}) \longrightarrow \mathrm{M}_4(\mathbb{R}), \quad A \longmapsto P^{-1}AP$$

The image of $\mathcal{C}(\hat{r})$ by Φ is exactly $\mathcal{C}(D)$, in fact we have

$$\begin{aligned} A \in \mathcal{C}(\hat{r}) &\Leftrightarrow A\hat{r} = \hat{r}A \Leftrightarrow APDP^{-1} = PDP^{-1}A \\ &\Leftrightarrow P^{-1}APD = DP^{-1}AP \Leftrightarrow \Phi(A) \in \mathcal{C}(D). \end{aligned}$$

Thus, $\mathcal{C}(\hat{r})$ and $\mathcal{C}(D)$ are isomorphic. By Kremlev and Nikonorov [17], the possibilities in the signature of $\hat{r}$ are

$$(+,-,0,-) \qquad (+,-,+,-) \qquad (+,-,-,-)$$

We distinguish the following cases (up to permutations)

$$D = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{bmatrix} \qquad D = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_3 \end{bmatrix}$$

$$D = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & 0 \\ 0 & 0 & 0 & r_2 \end{bmatrix} \qquad D = \begin{bmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & r_1 & 0 \\ 0 & 0 & 0 & r_2 \end{bmatrix}$$

The vector subspaces $\mathcal{C}(D)$ associated to D are respectively

$$\left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \Big/ a_i \in \mathbb{R} \right\} \quad \left\{ \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & a_4 & 0 \\ 0 & a_3 & a_5 & 0 \\ 0 & 0 & 0 & a_6 \end{bmatrix} \Big/ a_i \in \mathbb{R} \right\}$$

$$\left\{ \begin{bmatrix} a_1 & a_3 & 0 & 0 \\ a_2 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & a_7 \\ 0 & 0 & a_6 & a_8 \end{bmatrix} \Big/ a_i \in \mathbb{R} \right\} \quad \left\{ \begin{bmatrix} a_1 & a_4 & a_7 & 0 \\ a_2 & a_5 & a_8 & 0 \\ a_3 & a_6 & a_9 & 0 \\ 0 & 0 & 0 & a_{10} \end{bmatrix} \Big/ a_i \in \mathbb{R} \right\}.$$

Then, in all cases associated to $\mathcal{C}(D)$ (in the third case we assume that $\mu \neq \nu$ in $\langle .,. \rangle_3$), the intersection $\mathrm{O}(\mathfrak{g}, \langle .,. \rangle_3) \cap \mathcal{C}(\hat{r})$ is the group consisting of the following elements

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

This group is isomorphic to the group $(\mathbb{Z}_2)^3$.

Since $\mathrm{Isom}\,(G, \langle .,. \rangle_3)_e \subset \mathrm{O}\,(\mathfrak{g}, \langle .,. \rangle_3) \cap \mathcal{C}(\hat{r})$, then $\mathrm{Isom}\,(G, \langle .,. \rangle_3)_e$ is a finite group.

By Lemma 5.6, we get that $\mathrm{Isom}\,(G, \langle .,. \rangle_3)_e = \mathrm{Aut}(G)_{\langle .,. \rangle_3} \cong \mathbb{Z}_2$.

Accordingly, we have stated the following theorem

Theorem 5.9. *The group of isometries of the four-dimensional oscillator group with respect to the Riemannian metric $\langle .,. \rangle_3$ is given (up to permutations in $\hat{r}$) by*

$$\mathrm{Isom}\,(G, \langle .,. \rangle_3) \cong G \rtimes \mathbb{Z}_2.$$

5.4. The isometry group of G with respect to the metric $\langle .,. \rangle_4$

Theorem 5.10. *The isometry group of the four-dimensional oscillator group G with respect to the Riemannian metric $\langle .,. \rangle_4$ is given (up to permutations in $\hat{r}$) by*

$$\mathrm{Isom}\,(G, \langle .,. \rangle_4) \cong G.$$

Proof. By [17], the possibilities in the signature of $\hat{r}$ are

$$(+,-,0,-) \qquad (+,-,+,-) \qquad (+,-,-,-)$$

Following the same steps in the determination of the isometry group of $(G, \langle .,. \rangle_3)$, one can find that the intersection $\mathrm{O}\,(\mathfrak{g}, \langle .,. \rangle_4) \cap \mathcal{C}(\hat{r})$ is consisting of the following elements

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \right\} \cong (\mathbb{Z}_2)^2.$$

Since $\mathrm{Isom}\,(G, \langle .,. \rangle_4)_e \subset \mathrm{O}\,(\mathfrak{g}, \langle .,. \rangle_4) \cap \mathcal{C}(\hat{r})$, then $\mathrm{Isom}\,(G, \langle .,. \rangle_4)_e$ is a finite group. By Lemma 5.6, we get that $\mathrm{Isom}\,(G, \langle .,. \rangle_4)_e = \mathrm{Aut}(G)_{\langle .,. \rangle_4} = \{I_4\}$.
Consequently $\mathrm{Isom}\,(G, \langle .,. \rangle_4) \cong G$. ∎

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