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Extending Structures of Rota-Baxter Lie Algebras

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Abstract. We first introduce the notion of an extending datum of a Rota-Baxter Lie algebra through a vector space. We then construct a unified product for the Rota-Baxter Lie algebra with a vector space as a main ingredient in our approach. Finally, we solve the extending structures problem of Rota-Baxter Lie algebras, which generalizes and unifies two problems in the study of Rota-Baxter Lie algebras: the extension problem studied by Mishra-Das-Hazra and the factorization problem investigated by Lang-Sheng.

Mathematics Subject Classification: 17B38, 17B05, 17B60.

Key Words: Rota-Baxter Lie algebras, extending structures, crossed products, factorization problems.

1. Introduction

The aim of this paper is to solve the extending structure problem for Rota-Baxter Lie algebras which asks for the classification of all Rota-Baxter Lie algebraic structures on a given vector space by the approach of a general unified product.

1.1. Rota-Baxter (Lie) algebras

Let $\mathbf{k}$ be a field and $\lambda \in \mathbf{k}$. A *Rota-Baxter algebra* of weight λ is a pair (R, P), where R is an associative algebra over $\mathbf{k}$, and $P : R \to R$ is a linear operator which satisfies the *Rota-Baxter identity*

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy) \text{ for } x, y \in R. \tag{1}$$

Then P is called a *Rota-Baxter operator* of weight λ. The notion of a Rota-Baxter algebra was probably introduced in the first time by G. Baxter [11] in 1960 based on the study of Spitzer's identity in fluctuation theory. It is a natural algebraic interpretation of the formula of partial integration in analysis, as in the case of a differential algebra could be treated as an algebraic abstraction of differential equations. More precisely, let R be the $\mathbb{R}$-algebra of continuous functions on $\mathbb{R}$. Define $P : R \to R$ to be the integration

$$P(f)(x) = \int_0^x f(t)dt.$$

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Then the integration by parts formula

$$\int_0^x P(f)'(t)P(g)(t)dt = P(f)(x)P(g)(x) - \int_0^x P(f)(t)P(g)'(t)dt$$

is precisely Eq. (1) with $\lambda = 0$.

After 60 years developments, the study of Rota-Baxter algebras has become a vibrant research field and plays an important role in mathematics and mathematical physics, such as multiple zeta values [21], pre-Lie algebras [1, 6], Hopf algebras [16, 30, 40, 42, 43], (antisymmetric) infinitesimal bialgebras [5, 38], dendriform algebras [14, 15, 44], operads [34], $\mathcal{O}$-operators [10], quantum field theory [13], classical (associative) Yang-Baxter equations [7, 9, 10, 17]. For more details about Rota-Baxter algebras, see the monograph [18].

Completely independent of the above developments, the Rota-Baxter operator in the context of Lie algebras has its own motivations and developing history. In fact, it is an operator form of the classical Yang-Baxter equation (CYBE), named after the physicists C. N. Yang [39] and R. Baxter [12]. For a given Lie algebra $\mathfrak{g}$, the *classical Yang-Baxter equation* (CYBE) was defined by the following tensor form

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$. In [36], Semonov-Tian-Shansky showed that if there exists a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$, then a skew-symmetric solution r of the CYBE can be equivalently expressed as a linear operator $R : \mathfrak{g} \to \mathfrak{g}$ satisfying the following identity

$$[R(x), R(y)] = R([R(x), y]) + R([x, R(y)]), \quad \forall x, y \in \mathfrak{g}, \tag{2}$$

which is precisely the Rota-Baxter relation (of weight zero) in Eq. (1) in the context of Lie algebras. In this way, Eq. (2) is called an *operator form* of the CYBE. We would like to emphasize that this result was generalized by Kupershmidt [25] by the concept of an $\mathcal{O}$-operator, which was also called a relative Rota-Baxter operator.

Quite recently the discovery of some new algebraic and geometric structures in mathematics and physics [19] has led to a renewed interest in the study of Rota-Baxter Lie algebras and their connections with deformations and homotopy theory [27, 28, 37], cf. [37] for recent surveys. One of the main motivation comes from the emergence of Rota-Baxter Lie groups. In [19], Lang-Sheng-Guo showed that the differentiation of a Rota-Baxter Lie group is a Rota-Baxter Lie algebra of weight 1, generalizing the fact that the differentiation of a Lie group is a Lie algebra. Let G be a Lie group. A *Rota-Baxter Lie group* is a Lie group G together with a smooth map $\mathcal{B} : G \to G$ on G satisfying

$$\mathcal{B}(g_1)\mathcal{B}(g_2) = \mathcal{B}(g_1 \mathrm{Ad}_{\mathcal{B}(g_1)} g_2), \quad \forall g_1, g_2 \in G,$$

where $\mathrm{Ad}_g, g \in G$ is an adjoint action. Then the following formula

$$[u, v]_{\mathfrak{g}} = \frac{d^2}{dtds} |_{t,s=0} \exp^{tu} \exp^{sv} \exp^{-tu}, \quad \forall u, v \in \mathfrak{g}$$

captures the relationship between the Lie bracket $[,]_{\mathfrak{g}}$ and the Lie group multiplication. Let e be an identity of G and $\mathfrak{g} = T_e G$ the Lie algebra of G. Then the tangent map $B = \mathcal{B}_{\star e} : \mathfrak{g} \to \mathfrak{g}$ of $\mathcal{B}$ at the identity e is a Rota-Baxter operator of weight 1 and so $(\mathfrak{g}, [,]_{\mathfrak{g}}, B)$ is a Rota-Baxter Lie algebra of weight 1.

Viewing Rota-Baxter Lie algebras in the framework of bialgebras, Lang and Sheng [26] introduced the notion of a quadratic Rota-Baxter Lie algebra, which has a one-one correspondence to the factorizable Lie bialgebras. This leads to the introduction of a Rota-Baxter Lie bialgebra [26]. The study of Rota-Baxter Lie bialgebras was partly motivated by the theory of Manin triple approach. Except for the construction of a one-to-one correspondence between Manin triples of Rota-Baxter Lie algebras and Rota-Baxter Lie bialgebras [26], Bai-Guo-Liu-Ma [8] independently established a general bialgebra structure on Rota-Baxter Lie algebras following the approach of Manin triples to Lie bialgebras and antisymmetric infinitesimal bialgebras. A cohomology theory of a relative Rota-Baxter Lie algebra with coefficients in a representation was studied in [24].

1.2. Extending structures problem

The extending structures (ES) problem arose in the study of group theory by Agore and Militaru [2] around 2014. It can be regarded as a uniform of two well-known problems in group theory – the *extension problem* of Hölder [22] and the *factorization problem* of Ore [33], which served as baby models for the ES-problem approach [2]. Later, Agore and Militaru argued in favor of the study of ES-problems on several occasions. For example, the extending structures problem for Lie algebras and unital associative algebras are solved in [3] and [4] respectively. Apparently, the ES-problem becomes more popular when some new motivations were found, coming on one hand from interesting applications to conformal algebras [23] and on the other from close relation to the theory of 3-Lie algebras [41] and superalgebras [45].

Considering the fact that the Rota-Baxter Lie algebra is a generalization of the Lie algebra, it is almost a natural question to consider the following extending structures problem.

Problem 1.1. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra. Let E be a vector space containing $\mathfrak{g}$ as a subspace. Up to isomorphism of Rota-Baxter Lie algebras which stabilizes $(\mathfrak{g}, P_{\mathfrak{g}})$, describe and classify all the Rota-Baxter Lie algebraic structures $([-,-], P_E)$ on E such that $(\mathfrak{g}, P_{\mathfrak{g}})$ is a Rota-Baxter Lie subalgebra of $(E, [-,-], P_E)$.

In order to solve the above problem, we follow the steps of Agore and Militaru [3] on the construction of the Lie algebra case, we give the concept of an extending datum of a Rota-Baxter Lie algebra through a vector space and a Rota-Baxter Lie extending structure. Generalizing the results of Agore and Militaru [3] of the Lie algebra case, the description part and the classification part of the extending structure problem of Rota-Baxter Lie algebras are answered by a unified product of Rota-Baxter Lie algebras and a relative cohomology group. In particular, the non-abelian extension of Rota-Baxter Lie algebras in [32] and the factorization problem of Rota-Baxter Lie algebras in [26] appear to be special cases of the Rota-Baxter Lie extending structures problem.

This paper is organized as follows. In Section 2, we give the concept of an extending datum of Rota-Baxter Lie algebras with vector spaces and a unified product from an extending datum. Then the description part of the extending structure problem of Rota-Baxter Lie algebras is answered in Theorem 2.11. The classification part of the extending structure problem of Rota-Baxter Lie algebras is answered in Theorem

2.15 using a relative cohomology group. In Section 3, we show that the crossed products and bicrossed products of two Rota-Baxter Lie algebras are special cases of the unified product. In Section 4, we consider the flag extending structure of Rota-Baxter Lie algebras and give an answer to the calculation of the classifying object of the extending structure of the flag extending structure of Rota-Baxter Lie algebras.

Notation. Throughout this paper, let $\mathbf{k}$ be a field unless the contrary is specified, which will be the base ring of all modules, algebras, coalgebras, bialgebras, tensor products, as well as linear maps.

2. Unified products for Rota-Baxter Lie algebras

In this section, we first introduce an extending datum of a Rota-Baxter Lie algebra. We then give a construction of the unified product of a Rota-Baxter Lie algebra with a vector space. We prove that any Rota-Baxter Lie algebra on a vector space E containing a given Rota-Baxter Lie algebra as a Rota-Baxter Lie subalgebra is isomorphic to a unified product. Finally, the answer to the classification part of the extending structure problem is given.

Definition 2.1. Let λ be a fixed element of $\mathbf{k}$. A *Rota-Baxter Lie algebra of weight* λ is a Lie algebra $\mathfrak{g}$ together with a linear map $P_{\mathfrak{g}}$ such that

$$[P(g), P(h)] = P([P(g), h] + [g, P(h)] + \lambda[g, h]), \ \forall g, h \in \mathfrak{g}.$$

We denote it by $(\mathfrak{g}, [\cdot,\cdot], P_{\mathfrak{g}})$ or $(\mathfrak{g}, P_{\mathfrak{g}})$. ■

Throughout the rest of this paper, we say $(\mathfrak{g}, P_{\mathfrak{g}})$ is a Rota-Baxter Lie algebra to mean that $(\mathfrak{g}, P_{\mathfrak{g}})$ is a Rota-Baxter Lie algebra of weight λ.

Definition 2.2. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{g}', P_{\mathfrak{g}'})$ be two Rota-Baxter Lie algebras. A *morphism of Rota-Baxter Lie algebras* $f : (\mathfrak{g}, P_{\mathfrak{g}}) \to (\mathfrak{g}', P_{\mathfrak{g}'})$ is a morphism of Lie algebras $f : \mathfrak{g} \to \mathfrak{g}'$ such that $f \circ P_{\mathfrak{g}} = P_{\mathfrak{g}'} \circ f$. It is said to be an *isomorphism* if f is a linear isomorphism. ■

Let E be a vector space and $\mathfrak{g} \subset E$ a subspace of E. A subspace V of E is a *complement* of $\mathfrak{g}$ in E if $E = V + \mathfrak{g}$ and $V \cap \mathfrak{g} = 0$. Note that such a complement is unique up to isomorphism and its dimension is called the *codimension* of $\mathfrak{g}$ in E.

For later use, we recall the notion of a Rota-Baxter module over a Rota-Baxter Lie algebra given in [26].

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra. A *left* $\mathfrak{g}$*-module* is a vector space V together with a linear map $\triangleright : \mathfrak{g} \otimes V \to V$, called the left action of $\mathfrak{g}$ on V, such that $[g, h] \triangleright x = g \triangleright (h \triangleright x) - h \triangleright (g \triangleright x)$, for all $x \in V$ and $g, h \in \mathfrak{g}$.

The notion of right $\mathfrak{g}$-modules can be defined similarly. Note that any right $\mathfrak{g}$-module is a left $\mathfrak{g}$-module via $g \triangleright x := -x \triangleleft g$ and viceversa. Hence the category of right $\mathfrak{g}$-modules is isomorphic to the category of left $\mathfrak{g}$-modules.

Definition 2.4. [26, Definition 3.6] Let $(\mathfrak{g}, P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra. A *left* $(\mathfrak{g}, P_{\mathfrak{g}})$*-module* is a pair (V, T), where V is a vector space and $T : V \to V$ is a linear map, together with a linear map $\triangleright : \mathfrak{g} \otimes V \to V$ such that V is a left

$\mathfrak{g}$-module under the map $\triangleright$ and for $g \in \mathfrak{g}$ and $v \in V$,

$$P_{\mathfrak{g}}(g) \triangleright T(v) = T(P_{\mathfrak{g}}(g) \triangleright v + g \triangleright T(v) + \lambda g \triangleright v).$$

The notion of right $(\mathfrak{g}, P_{\mathfrak{g}})$-modules can be defined similarly.

Definition 2.5. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra and let (E, P_E) be a pair where E is a vector space and $P_E : E \to E$ is a linear map. Suppose $\mathfrak{g}$ is a subspace of E and V is a complement of $\mathfrak{g}$ in E. For a linear map $\phi : E \to E$ such that $\phi \circ P_E = P_E \circ \phi$, we consider the diagram:

$$\begin{array}{ccccc} (\mathfrak{g}, P_{\mathfrak{g}}) & \xrightarrow{i} & (E, P_E) & \xrightarrow{\pi} & V \\ {\scriptstyle id_{\mathfrak{g}}}\downarrow & & {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle id_V} \\ (\mathfrak{g}, P_{\mathfrak{g}}) & \xrightarrow{i} & (E, P_E) & \xrightarrow{\pi} & V \end{array}$$

where $\pi : E \to V$ is the canonical projection of $E = \mathfrak{g} + V$ on V and $i : \mathfrak{g} \to E$ is the inclusion map such that $i \circ P_{\mathfrak{g}} = P_E \circ i$. We say that $\phi : E \to E$ *stabilizes* $(\mathfrak{g}, P_{\mathfrak{g}})$ (resp. *co-stabilizes* V) if the left square (resp. the right square) of the above diagram is commutative.

Definition 2.6. Suppose that $(E, [\cdot,\cdot], P)$ and $(E, [\cdot,\cdot]', P')$ are two Rota-Baxter Lie algebras on E both containing $(\mathfrak{g}, P_{\mathfrak{g}})$ as a Rota-Baxter Lie subalgebra.

(1) $(E, [\cdot,\cdot], P)$ and $(E, [\cdot,\cdot]', P')$ are called *equivalent*, denoted $(E, [\cdot,\cdot], P) \equiv (E, [\cdot,\cdot]', P')$, if there exists a Rota-Baxter Lie algebra isomorphism

$$\phi : (E, [\cdot,\cdot], P) \to (E, [\cdot,\cdot]', P')$$

which stabilizes $(\mathfrak{g}, P_{\mathfrak{g}})$.

(2) $(E, [\cdot,\cdot], P)$ and $(E, [\cdot,\cdot]', P')$ are called *cohomologous*, denoted $(E, [\cdot,\cdot], P) \approx (E, [\cdot,\cdot]', P')$, if there exists a Rota-Baxter Lie algebra isomorphism

$$\phi : (E, [\cdot,\cdot], P) \to (E, [\cdot,\cdot]', P')$$

which stabilizes $(\mathfrak{g}, P_{\mathfrak{g}})$ and co-stabilizes V.

Remark 2.7. (1) Note that $\equiv$ and $\approx$ are equivalence relations on the set of all Rota-Baxter Lie algebras structures on E containing $(\mathfrak{g}, P_{\mathfrak{g}})$ as a Rota-Baxter Lie subalgebra and we denote by $Extd(E, (\mathfrak{g}, P_{\mathfrak{g}}))$ (resp. $Extd'(E, (\mathfrak{g}, P_{\mathfrak{g}}))$) the set of all equivalence classes via $\equiv$ (resp. $\approx$).

(2) By Definition 2.6, $Extd(E, (\mathfrak{g}, P_{\mathfrak{g}}))$ gives the set of all isomorphism classes of Rota-Baxter Lie algebras on E that stabilizes $(\mathfrak{g}, P_{\mathfrak{g}})$ and $Extd'(E, (\mathfrak{g}, P_{\mathfrak{g}}))$ gives the set of all isomorphism classes of Rota-Baxter Lie algebras on E from the point of extension theory.

(3) Since any two cohomologous Rota-Baxter Lie algebras structures on E are equivalent, there is a canonical projection

$$Extd'(E, (\mathfrak{g}, P_{\mathfrak{g}})) \to Extd(E, (\mathfrak{g}, P_{\mathfrak{g}})).$$

■

Generalizing the extending datum of Lie algebras through vector spaces [3], we give the following notion.

Definition 2.8. Let $(\mathfrak{g}, P_\mathfrak{g})$ be a Rota-Baxter Lie algebra and let V be a vector space.

(1) An *extending datum* of $(\mathfrak{g}, P_\mathfrak{g})$ through V is a system

$$\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$$

consisting of four bilinear maps

$$\triangleleft : V \times \mathfrak{g} \to V, \triangleright : V \times \mathfrak{g} \to \mathfrak{g}, \ f : V \times V \to \mathfrak{g}, \{-,-\} : V \times V \to V$$

and two linear maps $P_1 : V \to \mathfrak{g}$, $P_2 : V \to V$.

(2) Let $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ be an extending datum of $(\mathfrak{g}, P_\mathfrak{g})$ through V. Denote by $\mathfrak{g}\natural V$ the vector space $\mathfrak{g} \times V$ with a bilinear map $[-,-] : (\mathfrak{g} \times V) \times (\mathfrak{g} \times V) \to \mathfrak{g} \times V$ defined by

$$[(g,x),(h,y)] := ([g,h] + x \triangleright h - y \triangleright g + f(x,y), \{x,y\} + x \triangleleft h - y \triangleleft g) \quad (3)$$

and a linear map $\mathcal{P} : \mathfrak{g} \times V \to \mathfrak{g} \times V$ defined by

$$\mathcal{P}((g,x)) := (P_\mathfrak{g}(g) + P_1(x), P_2(x)), \quad (4)$$

for all $g, h \in \mathfrak{g}$ and $x, y \in V$. The object $\mathfrak{g}\natural V$ is called a *unified product* of $(\mathfrak{g}, P_\mathfrak{g})$ and V if it is a Rota-Baxter Lie algebra with the maps given by Eqs (3) and (4).

(3) The extending datum $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ is called a *Rota-Baxter Lie extending structure* of $(\mathfrak{g}, P_\mathfrak{g})$ through V if $\mathfrak{g}\natural V$ is a unified product of $(\mathfrak{g}, P_\mathfrak{g})$ and V.

(4) Denote by $\mathcal{L}((\mathfrak{g}, P_\mathfrak{g}), V)$ the set of all Rota-Baxter Lie extending structures of $(\mathfrak{g}, P_\mathfrak{g})$ through V. ■

Now we provide an equivalent characterization of the unified product $\mathfrak{g}\natural V$ of $(\mathfrak{g}, P_\mathfrak{g})$ and V as follows.

Theorem 2.9. *Let $(\mathfrak{g}, P_\mathfrak{g})$ be a Rota-Baxter Lie algebra. Let V be a vector space and $\Omega((\mathfrak{g}, P_\mathfrak{g}), V)$ an extending datum of $(\mathfrak{g}, P_\mathfrak{g})$ through V. Then the following statements are equivalent:*

(1) *$\mathfrak{g}\natural V$ is a unified product of $(\mathfrak{g}, P_\mathfrak{g})$ and V;*

(2) *For any $g, h \in \mathfrak{g}$ and $x, y, z \in V$,*

(a) $f(x,x) = 0$, $\{x,x\} = 0$*;*

(b) $(V, \triangleleft)$ *is a right $\mathfrak{g}$-module;*

(c) $x \triangleright [g,h] = [x \triangleright g, h] + [g, x \triangleright h] + (x \triangleleft g) \triangleright h - (x \triangleleft h) \triangleright g$*;*

(d) $\{x,y\} \triangleleft g = \{x, y \triangleleft g\} + \{x \triangleleft g, y\} + x \triangleleft (y \triangleright g) - y \triangleleft (x \triangleright g)$*;*

(e) $\{x,y\} \triangleright g = x \triangleright (y \triangleright g) - y \triangleright (x \triangleright g) + [g, f(x,y)] + f(x, y \triangleleft g) + f(x \triangleleft g, y)$*;*

(f) $f(x,\{y,z\}) + f(y,\{z,x\}) + f(z,\{x,y\}) + x \triangleright f(y,z) + y \triangleright f(z,x) + z \triangleright f(x,y) = 0$*;*

(g) $\{x,\{y,z\}\} + \{y,\{z,x\}\} + \{z,\{x,y\}\} + x \triangleleft f(y,z) + y \triangleleft f(z,x) + z \triangleleft f(x,y) = 0$*;*

(h) $[P_{\mathfrak{g}}(g), P_1(y)] - P_2(y) \triangleright P_{\mathfrak{g}}(g) + P_{\mathfrak{g}}(y \triangleright P_{\mathfrak{g}}(g)) + P_1(y \triangleleft P_{\mathfrak{g}}(g)) - P_{\mathfrak{g}}([\mathfrak{g}, P_1(y)]) + P_{\mathfrak{g}}(P_2(y) \triangleright g) + P_1(P_2(y) \triangleleft g) + \lambda P_{\mathfrak{g}}(y \triangleright g) + \lambda P_1(y \triangleleft g) = 0$;

(i) $P_2(y) \triangleleft P_{\mathfrak{g}}(g) - P_2(y \triangleleft P_{\mathfrak{g}}(g)) - P_2(P_2(y) \triangleleft g) - \lambda P_2(y \triangleleft g) = 0$;

(j) $[P_1(x), P_1(y)] + P_2(x) \triangleright P_1(y) - P_2(y) \triangleright P_1(x) + f(P_2(x), P_2(y)) + P_{\mathfrak{g}}(y \triangleright P_1(x)) - P_{\mathfrak{g}}(f(P_2(x), y)) - P_1(\{P_2(x), y\}) + P_1(y \triangleleft P_1(x)) - P_{\mathfrak{g}}(x \triangleright P_1(y)) - P_{\mathfrak{g}}(f(x, P_2(y))) - P_1(\{x, P_2(y)\}) - P_1(x \triangleleft P_1(y)) - \lambda P_{\mathfrak{g}}(f(x,y)) - \lambda P_1(\{x,y\}) = 0$;

(k) $\{P_2(x), P_2(y)\} + P_2(x) \triangleleft P_1(y) - P_2(y) \triangleleft P_1(x) - P_2(\{P_2(x), y\}) + P_2(y \triangleleft P_1(x)) - P_2(\{x, P_2(y)\}) - P_2(x \triangleleft P_1(y)) - \lambda P_2(\{x,y\}) = 0$.

Proof. By Theorem 3.2 of [3], $\mathfrak{g}\natural V$ is a Lie algebra under the product given by (3) if and only if Eqs. (2a)–(2g) hold. Next we show that the operator $\mathcal{P}$ given by Eq. (4) is a Rota-Baxter operator if and only if Eqs. (2h)–(2k) hold. Since $(g,x) = (g,0)+(0,x)$ in $\mathfrak{g}\natural V$ and $\mathcal{P}$ given by Eq. (4) is a linear map, we only need to consider the action of $\mathcal{P}$ on all generators of $\mathfrak{g}\natural V$, i.e. the set $\{(g,0)|g \in \mathfrak{g}\} \cup \{(0,x)|x \in V\}$. As $[-,-]$ is antisymmetric, there are three steps to consider. First, $\mathcal{P}$ is a Rota-Baxter operator for any $(g,0),(h,0)$:

$$
\begin{aligned}
&[\mathcal{P}((g,0)), \mathcal{P}((h,0))] - \mathcal{P}\Big([\mathcal{P}(g,0),(h,0)] + [(g,0),\mathcal{P}(h,0)] + \lambda[(g,0),(h,0)]\Big)\\
&= [(P_{\mathfrak{g}}(g),0),(P_{\mathfrak{g}}(h),0)] - \mathcal{P}\big([(P_{\mathfrak{g}}(g),0),(h,0)] + [(g,0),(P_{\mathfrak{g}}(h),0)] + \lambda[(g,0),(h,0)]\big)\\
&= ([P_{\mathfrak{g}}(g),P_{\mathfrak{g}}(h)],0) - \Big(P_{\mathfrak{g}}([P_{\mathfrak{g}}(g),h] + [g,P_{\mathfrak{g}}(h)] + \lambda[g,h]),0\Big) \quad \text{(by Eqs. (3)–(4))}\\
&= 0 \quad (\text{as } P_{\mathfrak{g}} \text{ being a Rota-Baxter operator of } \mathfrak{g}).
\end{aligned}
$$

Next we prove that $\mathcal{P}$ is a Rota-Baxter operator for any $(g,0),(0,y)$ if and only if Eqs. (2h)–(2i) hold. Indeed,

$$
\begin{aligned}
&[\mathcal{P}((g,0)), \mathcal{P}((0,y))] - \mathcal{P}\Big([\mathcal{P}(g,0),(0,y)] + [(g,0),\mathcal{P}(0,y)] + \lambda[(g,0),(0,y)]\Big)\\
&= [(P_{\mathfrak{g}}(g),0),(P_1(y),P_2(y))] - \mathcal{P}\Big([(P_{\mathfrak{g}}(g),0),(0,y)] + [(g,0),(P_1(y),P_2(y))]\\
&\quad + \lambda[(g,0),(0,y)]\Big) \quad \text{(by Eq. (4))}\\
&= \Big([P_{\mathfrak{g}}(g),P_1(y)] - P_2(y) \triangleright P_{\mathfrak{g}}(y), -P_2(y) \triangleleft P_{\mathfrak{g}}(g)\Big) - \mathcal{P}\Big(\big(-y \triangleright P_{\mathfrak{g}}(g), -y \triangleleft P_{\mathfrak{g}}(g)\big)\\
&\quad + \big([g,P_1(y)] - P_2(y) \triangleright g, -P_2(y) \triangleleft g\big) + \lambda\big(-y \triangleright g, -y \triangleleft g\big)\Big) \quad \text{(by Eq. (3))}\\
&= \Big([P_{\mathfrak{g}}(g),P_1(y)] - P_2(y) \triangleright P_{\mathfrak{g}}(y), -P_2(y) \triangleleft P_{\mathfrak{g}}(g)\Big)\\
&\quad - \Big(P_{\mathfrak{g}}(-y \triangleright P_{\mathfrak{g}}(g)) + P_1(-y \triangleleft P_{\mathfrak{g}}(g)), P_2(-y \triangleleft P_{\mathfrak{g}}(g))\Big)\\
&\quad - \Big(P_{\mathfrak{g}}([g,P_1(y)]) - P_{\mathfrak{g}}(P_2(y) \triangleright g) + P_1(-P_2(y) \triangleleft g), P_2(-P_2(y) \triangleleft g)\Big)\\
&\quad - \lambda\Big(P_{\mathfrak{g}}(-y \triangleright g) + P_1(-y \triangleleft g), P_2(-y \triangleleft g)\Big) \quad \text{(by Eq. (4))}\\
&= \Big([P_{\mathfrak{g}}(g),P_1(y)] - P_2(y) \triangleright P_{\mathfrak{g}}(g) + P_{\mathfrak{g}}(y \triangleright P_{\mathfrak{g}}(g)) + P_1(y \triangleleft P_{\mathfrak{g}}(g)) - P_{\mathfrak{g}}([g,P_1(y)])\\
&\quad + P_{\mathfrak{g}}(P_2(y) \triangleright g) + P_1(P_2(y) \triangleleft g) + \lambda P_{\mathfrak{g}}(y \triangleright g) + \lambda P_1(y \triangleleft g),\\
&\quad P_2(y) \triangleleft P_{\mathfrak{g}}(g) - P_2(y \triangleleft P_{\mathfrak{g}}(g)) - P_2(P_2(y) \triangleleft g) - \lambda P_2(y \triangleleft g)\Big).
\end{aligned}
$$

Hence $\mathcal{P}$ is a Rota-Baxter operator for $(g,0),(0,y)$ if and only if Eqs. (2h)–(2i) hold. Finally, we prove that $\mathcal{P}$ is a Rota-Baxter operator for $(0,x),(0,y)$ if and only if Eqs. (2j)–(2k) hold. Indeed,

$$
\begin{aligned}
&[\mathcal{P}((0,x)),\mathcal{P}((0,y))]-\mathcal{P}\Big([\mathcal{P}((0,x)),(0,y)]+[(0,x),\mathcal{P}((0,y))]+\lambda[(0,x),(0,y)]\Big)\\
&=\big[\big(P_1(x),P_2(x)\big),\big(P_1(y),P_2(y)\big)\big]-\mathcal{P}\Big(\,\big[\big(P_1(x),P_2(x)\big),(0,y)\big]\\
&\quad+\big[(0,x),\big(P_1(y),P_2(y)\big)\big]+\lambda[(0,x),(0,y)]\Big)\\
&=\Big([P_1(x),P_1(y)]+P_2(x)\triangleright P_1(y)-P_2(y)\triangleright P_1(x)+f\big(P_2(x),P_2(y)\big),\{P_2(x),P_2(y)\}\\
&\quad+P_2(x)\triangleleft P_1(y)-P_2(y)\triangleleft P_1(x)\Big)-\mathcal{P}\Big(\big(-y\triangleright P_1(x)+f(P_2(x),y),\\
&\quad\{P_2(x),y\}-y\triangleleft P_1(x)\big)+\big(x\triangleright P_1(y)+f(x,P_2(y)),\{x,P_2(y)\}+x\triangleleft P_1(y)\big)\\
&\quad+\lambda\big(f(x,y),\{x,y\}\big)\Big)\\
&=\Big([P_1(x),P_1(y)]+P_2(x)\triangleright P_1(y)-P_2(y)\triangleright P_1(x)+f(P_2(x),P_2(y)),\{P_2(x),P_2(y)\}\\
&\quad+P_2(x)\triangleleft P_1(y)-P_2(y)\triangleleft P_1(x)\Big)-\Big(P_{\mathfrak{g}}(-y\triangleright P_1(x))+P_{\mathfrak{g}}(f(P_2(x),y))\\
&\quad+P_1(\{P_2(x),y\})-P_1(y\triangleleft P_1(x)),P_2(\{P_2(x),y\})-P_2(y\triangleleft P_1(x))\Big)\\
&\quad-\Big(P_{\mathfrak{g}}(x\triangleright P_1(y))+P_{\mathfrak{g}}(f(x,P_2(y)))+P_1(\{x,P_2(y)\})+P_1(x\triangleleft P_1(y)),P_2(\{x,P_2(y)\})\\
&\quad+P_2(x\triangleleft P_1(y))\Big)-\lambda\Big(P_{\mathfrak{g}}(f(x,y))+P_1(\{x,y\}),P_2(\{x,y\})\Big)\\
&=\Big([P_1(x),P_1(y)]+P_2(x)\triangleright P_1(y)-P_2(y)\triangleright P_1(x)+f(P_2(x),P_2(y))+P_{\mathfrak{g}}(y\triangleright P_1(x))\\
&\quad-P_{\mathfrak{g}}(f(P_2(x),y))-P_1(\{P_2(x),y\})+P_1(y\triangleleft P_1(x))-P_{\mathfrak{g}}(x\triangleright P_1(y))\\
&\quad-P_{\mathfrak{g}}(f(x,P_2(y)))-P_1(\{x,P_2(y)\})-P_1(x\triangleleft P_1(y))-\lambda P_{\mathfrak{g}}(f(x,y))\\
&\quad-\lambda P_1(\{x,y\}),\{P_2(x),P_2(y)\}+P_2(x)\triangleleft P_1(y)-P_2(y)\triangleleft P_1(x)-P_2(\{P_2(x),y\})\\
&\quad+P_2(y\triangleleft P_1(x))-P_2(\{x,P_2(y)\})-P_2(x\triangleleft P_1(y))-\lambda P_2(\{x,y\})\Big).
\end{aligned}
$$

Hence $\mathcal{P}$ is a Rota-Baxter operator for $(0,x),(0,y)$ if and only if Eqs. (2j) and (2k) hold. This completes the proof. ■

Remark 2.10. Note that Eqs. (2b) and (2k) show that (V,P_2) is a right $(\mathfrak{g},P_{\mathfrak{g}})$-module under the action $\triangleleft$. ■

Let $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)=(\triangleleft,\triangleright,f,\{-,-\},P_1,P_2)$ be a Rota-Baxter Lie extending structure and $\mathfrak{g}\natural V$ the associated unified product. Then there is a canonical inclusion

$$i_{\mathfrak{g}}:\mathfrak{g}\to\mathfrak{g}\natural V,\quad i_{\mathfrak{g}}(g)=(g,0),\quad \text{for any}\quad g\in\mathfrak{g}.$$

Note that $i_{\mathfrak{g}}\circ P_{\mathfrak{g}}=\mathcal{P}\circ i_{\mathfrak{g}}$. Then $i_{\mathfrak{g}}$ is an injective morphism of Rota-Baxter Lie algebras. Hence $(\mathfrak{g},P_{\mathfrak{g}})$ can be viewed as a Rota-Baxter Lie subalgebra of $\mathfrak{g}\natural V$ through the identification $\mathfrak{g}\cong i_{\mathfrak{g}}(\mathfrak{g})\cong\mathfrak{g}\times\{0\}$.

Next we will prove that any Rota-Baxter Lie algebra structure on a vector space E containing $(\mathfrak{g},P_{\mathfrak{g}})$ as a Rota-Baxter Lie subalgebra is isomorphic to a unified product of $(\mathfrak{g},P_{\mathfrak{g}})$ through a vector space.

Now we arrive at our first main result in this section.

Theorem 2.11. *Let $(\mathfrak{g}, P_\mathfrak{g})$ be a Rota-Baxter Lie algebra. Let E be a vector space containing $\mathfrak{g}$ as a subspace and $(E, [-,-], P_E)$ a Rota-Baxter Lie algebra such that $(\mathfrak{g}, P_\mathfrak{g})$ is a Rota-Baxter Lie subalgebra of $(E, [-,-], P_E)$. Then there exists a Rota-Baxter Lie extending structure*

$$\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$$

of $(\mathfrak{g}, P_\mathfrak{g})$ through a subspace V of E and an isomorphism of Rota-Baxter Lie algebras

$$(E, P_E) \cong (\mathfrak{g}\natural V, \mathcal{P})$$

that stabilizes $(\mathfrak{g}, P_\mathfrak{g})$ and co-stabilizes V.

Proof. Since $\mathbf{k}$ is a field, we have a linear map $p : E \to \mathfrak{g}$ such that $p(g) = g$, for all $g \in \mathfrak{g}$. Define $V := \ker(p)$. Then V is a complement of $\mathfrak{g}$ in E, $p \circ p = p$ and $p \circ (id_E - p) = 0$. Now define the extending datum of $(\mathfrak{g}, P_\mathfrak{g})$ through V as follows:

$$\begin{aligned}
\triangleright : V \times \mathfrak{g} \to \mathfrak{g}, &\quad x \triangleright g := p([x, g]),\\
\triangleleft : V \times \mathfrak{g} \to V, &\quad x \triangleleft g := [x, g] - p([x, g]),\\
f : V \times V \to \mathfrak{g}, &\quad f(x, y) := p([x, y]),\\
\{,\} : V \times V \to V, &\quad \{x, y\} := [x, y] - p([x, y]),\\
P_1 : V \to \mathfrak{g}, &\quad P_1(x) = p(P_E(x)),\\
P_2 : V \to V, &\quad P_2(x) = P_E(x) - p(P_E(x)),
\end{aligned}$$

for any $g \in \mathfrak{g}$ and $x, y \in V$. Note that by definition of the map p, for any $g \in \mathfrak{g}$ and $x, y \in V$, $x \triangleleft g \in V$, $\{x, y\} \in V$ and $P_2(x) \in V$. Hence the above maps are all well-defined linear maps or bilinear maps. Now we prove that $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ is a Rota-Baxter Lie extending structure of $(\mathfrak{g}, P_\mathfrak{g})$ through V and

$$\phi : \mathfrak{g}\natural V \to E, \quad \phi(g, x) := g + x$$

is an isomorphism of Rota-Baxter Lie algebras that stabilizes $\mathfrak{g}$ and co-stabilizes V. Analogue to the proof of Theorem 3.4 in [3],

$$\phi : \mathfrak{g} \times V \to E, \quad \phi(g, x) := g + x$$

is a linear isomorphism between the vector space $\mathfrak{g} \times V$ and the Lie algebra E with the inverse given by

$$\phi^{-1}(y) := (p(y), y - p(y)), \quad \text{for all } \ y \in E.$$

As ϕ is a linear isomorphism, there exists a unique Rota-Baxter Lie algebra structure on $\mathfrak{g} \times V$ with the Rota-Baxter operator and the product on $\mathfrak{g} \times V$ given by

$$\mathcal{P}((g, x)) := \phi^{-1}(P_E(\phi(g, x))), \quad [(g, x), (h, y)] := \phi^{-1}([\phi(g, x), \phi(h, y)]), \qquad (5)$$

for all $g, h \in \mathfrak{g}$ and $x, y \in V$, such that ϕ is an isomorphism of Rota-Baxter Lie algebras. In fact, by the proof of Theorem 3.4 in [3], the product in Eq. (5) coincides with the one defined by Eq. (3). Hence, it remains to prove that the Rota-Baxter operator in Eq. (5) coincides with the one defined by Eq. (4).

Indeed, for any $g \in \mathfrak{g}$ and $x \in V$, we have

$$
\begin{aligned}
&\mathcal{P}((g,x)) = \phi^{-1}(P_E(\phi(g,x))) = \phi^{-1}(P_E(g+x))\\
&= \phi^{-1}(P_E(g) + P_E(x)) = \phi^{-1}(P_{\mathfrak{g}}(g) + P_E(x))\\
&\qquad\qquad \text{(by } (\mathfrak{g}, P_{\mathfrak{g}}) \text{ being a Rota-Baxter Lie subalgebra of } (E, P_E))\\
&= \phi^{-1}(P_{\mathfrak{g}}(g) + p(P_E(x)) + P_E(x) - p(P_E(x)))\\
&= (P_{\mathfrak{g}}(g) + p(P_E(x)), P_E(x) - p(P_E(x))) \quad \text{(by } p \circ p = p \text{ and } p \circ (id_E - p) = 0)\\
&= (P_{\mathfrak{g}}(x) + P_1(x), P_2(x)).
\end{aligned}
$$

Finally, we can check the following diagram is commutative

$$
\begin{array}{ccccc}
(\mathfrak{g}, P_{\mathfrak{g}}) & \xrightarrow{i_{\mathfrak{g}}} & (\mathfrak{g}\natural V, \mathcal{P}) & \xrightarrow{q} & V\\
{\scriptstyle id_{\mathfrak{g}}}\downarrow & & {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle id_V}\\
(\mathfrak{g}, P_{\mathfrak{g}}) & \xrightarrow{i_{\mathfrak{g}}} & (E, P_E) & \xrightarrow{\pi} & V
\end{array}
$$

where $\pi : E \to V$ is the projection of $E = \mathfrak{g} + V$ on the vector space V and $q : \mathfrak{g}\natural V \to V, q(g,x) := x$ is the canonical projection. Hence ϕ stabilizes $(\mathfrak{g}, P_{\mathfrak{g}})$ and V. This completes the proof. ■

Remark 2.12. By Theorem 2.11, any Rota-Baxter Lie algebra structure on a vector space E containing $(\mathfrak{g}, P_{\mathfrak{g}})$ as a Rota-Baxter Lie subalgebra is isomorphic to a unified product of $(\mathfrak{g}, P_{\mathfrak{g}})$ through a given complement V of $\mathfrak{g}$ in E. Hence the classification of all Rota-Baxter Lie algebra structures on E that contains $(\mathfrak{g}, P_{\mathfrak{g}})$ as a Rota-Baxter Lie subalgebras is equivalent to the classification of all unified products $\mathfrak{g}\natural V$, associated to all Rota-Baxter Lie extending structures $\Omega((\mathfrak{g}, P_{\mathfrak{g}}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$, for a given complement V of $\mathfrak{g}$ in E. ■

To classify all unified products $\mathfrak{g}\natural V$, we need the following lemma.

Lemma 2.13. *Let*
$\Omega((\mathfrak{g}, P_{\mathfrak{g}}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$, $\Omega'((\mathfrak{g}, P_{\mathfrak{g}}), V) = (\triangleleft', \triangleright', f', \{-,-\}', P_1', P_2')$ *be two Rota-Baxter Lie extending structures of* $(\mathfrak{g}, P_{\mathfrak{g}})$ *through* V *and let* $\mathfrak{g}\natural V, \mathfrak{g}\natural' V$ *be their corresponding associated unified products. Then there exists a bijection between the set of all morphisms of Rota-Baxter Lie algebras* $\psi : \mathfrak{g}\natural V \to \mathfrak{g}\natural' V$ *which stabilizes* $(\mathfrak{g}, P_{\mathfrak{g}})$*, and the set of pairs* (r, v)*, where* $r : V \to \mathfrak{g}$*,* $v : V \to V$ *are two linear maps satisfying the following conditions: for any* $g \in \mathfrak{g}$*,* $x, y \in V$*,*

(1) $v(x) \triangleleft' g = v(x \triangleleft g)$;

(2) $r(x \triangleleft g) = [r(x), g] - x \triangleright g + v(x) \triangleright' g$;

(3) $v(\{x,y\}) = \{v(x), v(y)\}' + v(x) \triangleleft' r(y) - v(y) \triangleleft' r(x)$;

(4) $r(\{x,y\}) = [r(x), r(y)] + v(x) \triangleright' r(y) - v(y) \triangleright' r(x) + f'(v(x), v(y)) - f(x,y)$;

(5) $P_1(x) = P_{\mathfrak{g}}(r(x)) + P_1'(v(x)) - r(P_2(x))$;

(6) $v(P_2(x)) = P_2'(v(x))$.

Under the above bijection, the morphism of Rota-Baxter Lie algebras

$$\psi=\psi_{(r,v)}:\mathfrak{g}\natural V\to\mathfrak{g}\natural' V$$

corresponding to (r,v) *is given by*

$$\psi(g,x)=(g+r(x),v(x)),$$

for any $g\in\mathfrak{g}$ *and* $x\in V$. *Moreover,* $\psi=\psi_{(r,v)}$ *is an isomorphism of Rota-Baxter Lie algebras if and only if* $v:V\to V$ *is a bijection and* $\psi=\psi_{(r,v)}$ *co-stabilizes* V *if and only if* $v=id_V$.

Proof. By the proof of Lemma 3.5 in [3], a linear map $\psi:\mathfrak{g}\natural V\to\mathfrak{g}\natural' V$ which stabilizes $\mathfrak{g}$ is uniquely determined by two linear maps $r:V\to\mathfrak{g}$ and $v:V\to V$ such that $\psi(g,x)=(g+r(x),v(x))$ for all $g\in\mathfrak{g}$ and $x\in V$. Moreover, ψ is a morphism of Lie algebras if and only if the conditions (1)–(4) hold. Denote by $\mathcal{P}$ and $\mathcal{P}'$ the Rota-Baxter operators associated to $\mathfrak{g}\natural V$ and $\mathfrak{g}\natural' V$. Next, we show that ψ is compatible with the Rota-Baxter operators, i.e. $\psi\circ\mathcal{P}=\mathcal{P}'\circ\psi$, if and only if the conditions (5)–(6) hold. For $g,h\in\mathfrak{g}$, we have

$$\psi\circ\mathcal{P}((g,0))=\psi((P_{\mathfrak{g}}(g),0))=(P_{\mathfrak{g}}(g),0)=\mathcal{P}'((g,0))=\mathcal{P}'\circ\psi((g,0)).$$

ψ is compatible with the Rota-Baxter operators under this case. For $x\in V$, we get

$$\begin{aligned}&\psi\circ\mathcal{P}((0,x))-\mathcal{P}'\circ\psi((0,x))=\psi((P_1(x),P_2(x)))-\mathcal{P}'((r(x),v(x)))\\&=(P_1(x)+r(P_2(x)),v(P_2(x)))-(P_{\mathfrak{g}}(r(x))+P_1'(v(x)),P_2'(v(x))).\end{aligned}$$

Hence $\psi\circ\mathcal{P}((0,x))=\mathcal{P}'\circ\psi((0,x))$ if and only if the conditions (5)–(6) hold. Moreover, by the proof of Lemma 3.5 in [3], $\psi=\psi_{(r,v)}$ is an isomorphism if and only if $v:V\to V$ is a bijection and $\psi=\psi_{(r,v)}$ co-stabilizes V if and only if $v=id_V$. ■

By Lemma 2.13, we give the following definition.

Definition 2.14. Let $(\mathfrak{g},P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra and let V be a vector space. Two Rota-Baxter Lie extending structures of $(\mathfrak{g},P_{\mathfrak{g}})$ through V, $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)=(\triangleleft,\triangleright,f,\{-,-\},P_1,P_2)$ and $\Omega'((\mathfrak{g},P_{\mathfrak{g}}),V)=(\triangleleft',\triangleright',f',\{-.-\}',P_1',P_2')$ are called *equivalent*, denoted by

$$\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)\equiv\Omega'((\mathfrak{g},P_{\mathfrak{g}}),V),$$

if there exists a pair (r,v) of linear maps, where $r:V\to\mathfrak{g}$ and $v\in Aut_{\mathbf{k}}(V)$ such that

$$(\triangleleft',\triangleright',f',\{-,-\}',P_1',P_2')$$

is obtained from $(\triangleleft,\triangleright,f,\{-,-\},P_1,P_2)$ using (r,v) via

$$\begin{aligned}x\triangleleft' g&=v(v^{-1}(x)\triangleleft g),\\x\triangleright' g&=r(v^{-1}(x)\triangleleft g)+v^{-1}(x)\triangleright g+[g,r(v^{-1}(x))],\\f'(x,y)&=f(v^{-1}(x),v^{-1}(y))+r(\{v^{-1}(x),v^{-1}(y)\})+[r(v^{-1}(x)),r(v^{-1}(y))]\\&\quad-r(v^{-1}(x)\triangleleft r(v^{-1}(y)))-v^{-1}(x)\triangleright r(v^{-1}(y))+r(v^{-1}(y)\triangleleft r(v^{-1}(x)))\\&\quad+v^{-1}(y)\triangleright r(v^{-1}(x)),\\\{x,y\}'&=v(\{v^{-1}(x),v^{-1}(y)\})-v(v^{-1}(x)\triangleleft r(v^{-1}(y)))+v(v^{-1}(y)\triangleleft r(v^{-1}(x))),\\P_1'(x)&=P_1(v^{-1}(x))-P_{\mathfrak{g}}(r(v^{-1}(x)))+r(P_2(v^{-1}(x))),\\P_2'(x)&=v(P_2(v^{-1}(x))).\end{aligned}$$

Recall that $Extd(E,(\mathfrak{g},P_{\mathfrak{g}}))$ is the set of all equivalence classes of Rota-Baxter Lie algebra structures on E which stabilizes $(\mathfrak{g},P_{\mathfrak{g}})$. Now we give a classification of $Extd(E,(\mathfrak{g},P_{\mathfrak{g}}))$ as our second main results.

Theorem 2.15. *Let $(\mathfrak{g},P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra, E a vector space that contains $\mathfrak{g}$ as a subspace and V a complement of $\mathfrak{g}$ in E. Then:*

(1) *$\equiv$ is an equivalence relation on the set $\mathcal{L}((\mathfrak{g},P_{\mathfrak{g}}),V)$ of all Rota-Baxter Lie extending structures of $(\mathfrak{g},P_{\mathfrak{g}})$ through V.*
Denote by $H^2_{(\mathfrak{g},P_{\mathfrak{g}})}(V,(\mathfrak{g},P_{\mathfrak{g}})):=\mathcal{L}((\mathfrak{g},P_{\mathfrak{g}}),V)/\equiv$, the quotient set.

(2) *The map*
$$\Phi: H^2_{(\mathfrak{g},P_{\mathfrak{g}})}(V,(\mathfrak{g},P_{\mathfrak{g}}))\to Extd(E,(\mathfrak{g},P_{\mathfrak{g}})),$$
$$\overline{(\triangleleft,\triangleright,f,\{-,-\},P_1,P_2)}\mapsto(\mathfrak{g}\natural V,[-,-],\mathcal{P})$$
is a bijection, where $\overline{(\triangleleft,\triangleright,f,\{-,-\},P_1,P_2)}$ is the equivalence class of $(\triangleleft,\triangleright,f,\ \{-,-\},P_1,P_2)$ via $\equiv$.

Proof. (1) By Theorems 2.9, 2.11 and Lemma 2.13, $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)\equiv\Omega'((\mathfrak{g},P_{\mathfrak{g}}),V)$ in the sense of Definition 2.14 if and only if there exists an isomorphism of Rota-Baxter Lie algebras $\psi:\mathfrak{g}\natural V\to\mathfrak{g}\natural' V$ which stabilizes $(\mathfrak{g},P_{\mathfrak{g}})$. Therefore, $\equiv$ is an equivalence relation on the set $\mathcal{L}((\mathfrak{g},P_{\mathfrak{g}}),V)$ of all Rota-Baxter Lie extending structures.

(2) By Lemma 2.13, the map Φ is a well-defined injection. By Theorem 2.11, Φ is a surjection. Hence Φ is a bijection. ■

Remark 2.16. Let $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)=(\triangleleft,\triangleright,f,\{-,-\},P_1,P_2)$ and $\Omega'((\mathfrak{g},P_{\mathfrak{g}}),V)=(\triangleleft',\triangleright',f',\{-.-\}',P_1',P_2')$ be two Rota-Baxter Lie extending structures of $(\mathfrak{g},P_{\mathfrak{g}})$ through V. They are called *cohomologous*, denoted by $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)\approx\Omega'((\mathfrak{g},P_{\mathfrak{g}}),V)$, if $\triangleleft'=\triangleleft$, $P_2'=P_2$ and there exists a linear map $r:V\to\mathfrak{g}$ such that

$$\begin{aligned}
x\triangleright' g&=x\triangleright g+r(x\triangleleft g)-[r(x),g],\\
f'(x,y)&=f(x,y)+r(\{x,y\})+[r(x),r(y)]+y\triangleright r(x)-x\triangleright r(y)+r(y\triangleleft r(x))\\
&\quad-r(x\triangleleft r(y)),\\
\{x,y\}'&=\{x,y\}-x\triangleleft r(y)+y\triangleleft r(x),\\
P_1'(x)&=P_1(x)-P_{\mathfrak{g}}(r(x))+r(P_2(x)),
\end{aligned}$$

for all $g\in\mathfrak{g}$ and $x,y\in V$.

Similar to the proof of Theorem 2.15, we conclude that $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V)\approx\Omega'((\mathfrak{g},P_{\mathfrak{g}}),V)$ if and only if there exists an isomorphism of Rota-Baxter Lie alebras $\phi:\mathfrak{g}\natural V\to\mathfrak{g}\natural' V$ which stabilizes $(\mathfrak{g},P_{\mathfrak{g}})$ and co-stabilizes V. Then $\approx$ is also an equivalent relation on the set $\mathcal{L}((\mathfrak{g},P_{\mathfrak{g}}),V)$ of all Rota-Baxter Lie extending structures of $(\mathfrak{g},P_{\mathfrak{g}})$ through V. Define
$$H^2(V,(\mathfrak{g},P_{\mathfrak{g}})):=\mathcal{L}((\mathfrak{g},P_{\mathfrak{g}}),V)/\approx,$$
then the map
$$H^2(V,(\mathfrak{g},P_{\mathfrak{g}}))\to Extd'(E,\mathfrak{g}),\quad \overline{(\triangleleft,\triangleright,f,\{-,-\})}\mapsto(\mathfrak{g}\natural V,[-,-],\mathcal{P})$$
is a bijection between $H^2(V,(\mathfrak{g},P_{\mathfrak{g}}))$ and the isomorphism classes of all Rota-Baxter Lie algebras structures on E which stabilizes $(\mathfrak{g},P_{\mathfrak{g}})$ and co-stabilizes V.

3. Special cases of unified products

In the section, we first introduce the notion of crossed products and bicrossed products of Rota-Baxter Lie algebras. Then we prove that both products are two special cases of the unified products of Rota-Baxter Lie algebras.

Throughout the rest of this paper, we shall omit the map if it is one of the maps $\triangleleft, \triangleright, f, \{-,-\}, P_1$ or P_2 of an extending datum $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ and it is trivial.

3.1. Crossed products and the extension problem

In this subsection, we introduce the concept of a crossed product of Rota-Baxter Lie algebras and show that the extension problem studied by Mishra-Das-Hazra [32] is a special case of the Rota-Baxter Lie extending structures problem.

Let $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\})$ be an extending datum of $(\mathfrak{g}, P_\mathfrak{g})$ through V such that $\triangleleft$ is a trivial map, i.e. $x \triangleleft g = 0$, for all $x \in V$ and $g \in \mathfrak{g}$. It follows from Theorem 2.9 that $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleright, f, \{-,-\})$ is a Rota-Baxter Lie extending structure of $(\mathfrak{g}, P_\mathfrak{g})$ through V if and only if $(V, \{-,-\}, P_2)$ is a Lie algebra that satisfy the following relations:

$$
\begin{aligned}
&f(x,x) = 0, \quad x \triangleright [g,h] = [x \triangleright g, h] + [g, x \triangleright h],\\
&\{x,y\} \triangleright g = x \triangleright (y \triangleright g) - y \triangleright (x \triangleright g) + [g, f(x,y)],\\
&f(x, \{y,z\} + f(y, \{z,x\}) + f(z, \{x,y\}) + x \triangleright f(y,z) + y \triangleright f(z,x) + z \triangleright f(x,y) = 0,\\
&[P_\mathfrak{g}(g), P_1(y)] - P_2(y) \triangleright P_\mathfrak{g}(g) + P_\mathfrak{g}(y \triangleright P_\mathfrak{g}(g)) - P_\mathfrak{g}([g, P_1(y)]) + P_\mathfrak{g}(P_2(y) \triangleright g)\\
&\quad + \lambda P_\mathfrak{g}(y \triangleright g) = 0,\\
&[P_1(x), P_1(y)] + P_2(x) \triangleright P_1(y) - P_2(y) \triangleright P_1(x) + f(P_2(x), P_2(y)) + P_\mathfrak{g}(y \triangleright P_1(x))\\
&\quad - P_\mathfrak{g}(f(P_2(x), y)) - P_1(\{P_2(x), y\}) - P_\mathfrak{g}(x \triangleright P_1(y)) - P_\mathfrak{g}(f(x, P_2(y)))\\
&\quad - P_1(\{x, P_2(y)\}) - \lambda P_\mathfrak{g}(f(x,y)) - \lambda P_1(\{x,y\}) = 0.
\end{aligned}
$$

for any $g, h \in \mathfrak{g}$ and $x, y, z \in V$.

In this case, the associated unified product $\mathfrak{g}\natural V$ is the *crossed product* of the Rota-Baxter Lie algebras $(\mathfrak{g}, P_\mathfrak{g})$ and (V, P_2). A system $((\mathfrak{g}, P_\mathfrak{g}), (V, P_2), \triangleright, f, P_1)$ consisting of two Rota-Baxter Lie algebras $(\mathfrak{g}, P_\mathfrak{g}), (V, P_2)$, two bilinear maps $\triangleright : V \times \mathfrak{g} \to \mathfrak{g}$, $f : V \times V \to \mathfrak{g}$ and a linear map $P_1 : V \to \mathfrak{g}$ satisfying the above conditions is called a *crossed system of Rota-Baxter Lie algebras.* The Rota-Baxter Lie algebra $(\mathfrak{g}\natural V, [-,-], \mathcal{P})$ associated to the crossed system $((\mathfrak{g}, P_\mathfrak{g}), (V, P_2), \triangleright, f, P_1)$ is defined as follows: for any $g, h \in \mathfrak{g}$ and $x, y \in V$,

$$
\begin{aligned}
&[(g,x),(h,y)] := ([g,h] + x \triangleright h - y \triangleright g + f(x,y),\\
&\{x,y\}),\ \mathcal{P}((g,x)) := (P_\mathfrak{g}(g) + P_1(x), P_2(x)).
\end{aligned}
$$

Note that the crossed system provides the answer to the following restricted version of the extending structures problem:

Problem 3.1. *Let $(\mathfrak{g}, P_\mathfrak{g})$ be a Rota-Baxter Lie algebra, E a vector space containing $\mathfrak{g}$ as a subspace. Describe and classify all Rota-Baxter Lie algebra structures on E such that $(\mathfrak{g}, P_\mathfrak{g})$ is a Rota-Baxter ideal of E.*

Let $((\mathfrak{g}, P_\mathfrak{g}), (V, P_2), \triangleright, f, P_1)$ be a crossed system of two Rota-Baxter Lie algebras.

Since $[(g,0),(h,y)] = ([g,h]-y\triangleright g,0)$ and $\mathcal{P}((g,0)) = (P_{\mathfrak{g}}(g),0)$, we have that $\mathfrak{g}\cong\mathfrak{g}\times\{0\}$ is a Rota-Baxter ideal of the Rota-Baxter Lie algebra $\mathfrak{g}\natural V$. Next we will prove that any Rota-Baxter Lie algebra structure on a vector space E containing $(\mathfrak{g},P_{\mathfrak{g}})$ as a Rota-Baxter ideal can be described by a crossed product of Rota-Baxter Lie algebras.

Corollary 3.2. *Let $(\mathfrak{g},P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra and let E be a vector space containing $\mathfrak{g}$ as a subspace. Then any Rota-Baxter Lie algebra structure on E that contains $(\mathfrak{g},P_{\mathfrak{g}})$ as a Rota-Baxter ideal is isomorphic to a Rota-Baxter Lie algebra associated to a crossed system $((\mathfrak{g},P_{\mathfrak{g}}),(V,P_2),\triangleright,f,P_1)$.*

Proof. Suppose (E,P_E) is a Rota-Baxter Lie algebra such that $(\mathfrak{g},P_{\mathfrak{g}})$ is a Rota-Baxter ideal in (E,P_E). Then $(\mathfrak{g},P_{\mathfrak{g}})$ is also a Rota-Baxter Lie subalgebra of (E,P_E). Hence by Theorem 2.11, there is a Rota-Baxter Lie extending structure $\Omega((\mathfrak{g},P_{\mathfrak{g}}),V) = (\triangleleft,\triangleright,f,\{-,-\})$ of $(\mathfrak{g},P_{\mathfrak{g}})$ through a subspace V of E and an isomorphism of Rota-Baxter Lie algebras $E\cong\mathfrak{g}\natural V$. As $(\mathfrak{g},P_{\mathfrak{g}})$ is a Rota-Baxter ideal of (E,P_E), for any $x\in V$ and $g\in\mathfrak{g}$, we have that $[x,g]\in\mathfrak{g}$ and hence $p([x,g]) = [x,g]$, where p is the linear map in the proof of Theorem 2.11. Thus, $x\triangleleft_p g = [x,g]-p([x,g]) = 0$ and (E,P_E) is isomorphic to the Rota-Baxter Lie algebra associated to the crossed system $((\mathfrak{g},P_{\mathfrak{g}}),(V,P_2),\triangleright,f,P_1)$, where $V=\ker(p)$. ■

Definition 3.3. [32, Definition 3.1] Let $(\mathfrak{g},P_{\mathfrak{g}})$ and $(\mathfrak{h},P_{\mathfrak{h}})$ be two Rota-Baxter Lie algebras. A *non-abelian extension* of $(\mathfrak{g},P_{\mathfrak{g}})$ by $(\mathfrak{h},P_{\mathfrak{h}})$ is a Rota-Baxter Lie algebra $(\mathfrak{k},P_{\mathfrak{k}})$ equipped with a short exact sequence of Rota-Baxter Lie algebras

$$0\to\mathfrak{h}\to\mathfrak{k}\to\mathfrak{g}\to 0.$$

■

The equivalent classes of non-abelian extension of $(\mathfrak{g},P_{\mathfrak{g}})$ by $(\mathfrak{h},P_{\mathfrak{h}})$ is characterized in [32]. Actually, the restricted version of the extending structures problem is in face an equivalent reformulation of the non-abelian extension problem. Indeed, the Rota-Baxter Lie algebra $\mathfrak{g}\natural V$ associated to a crossed system $((\mathfrak{g},P_{\mathfrak{g}}),(V,P_2),\triangleright,f,P_1)$ is an extension of (V,P_2) by $(\mathfrak{g},P_{\mathfrak{g}})$ via the following sequence

$$0\to\mathfrak{g}\xrightarrow{i}\mathfrak{g}\natural V\xrightarrow{\pi}V\to 0,$$

where $i:\mathfrak{g}\to\mathfrak{g}\natural V$, $i(g)=(g,0)$ and $\pi:\mathfrak{g}\natural V\to V$, $\pi(g,x)=x$. Conversely, let $(\mathfrak{k},P_{\mathfrak{k}})$ be an extension of (V,P_2) by $(\mathfrak{g},P_{\mathfrak{g}})$. Then there is an exact sequence of Rota-Baxter Lie algebras

$$0\to\mathfrak{g}\xrightarrow{i}\mathfrak{k}\xrightarrow{\pi}V\to 0.$$

As $\mathfrak{g}\cong Im(i)=Ker(pi)$, $(\mathfrak{g},P_{\mathfrak{g}})$ can be viewed as a Rota-Baxter ideal of $(\mathfrak{k},P_{\mathfrak{k}})$. By Corollary 3.2, there exists a crossed system $((\mathfrak{g},P_{\mathfrak{g}}),(V,P_2),\triangleright,f)$ of Rota-Baxter Lie algebras such that $\mathfrak{g}\natural V\cong\mathfrak{k}$ as Rota-Baxter Lie algebras.

3.2. Matched pairs and the factorization problem

In this subsection, we first recall the concept of a matched pair of Lie algebras [29, 31] and then show that how the factorization problem studied by Lang and Sheng [26] is a special case of Theorem 2.15.

Definition 3.4. *A matched pair of Lie algebras* consists of two Lie algebras $(\mathfrak{g}, [-,-])$ and $(V, \{-,-\})$, which satisfy that

(1) $\mathfrak{g}$ is a left V-module under $\triangleright : V \otimes \mathfrak{g} \to \mathfrak{g}$,

(2) V is a right $\mathfrak{g}$-module under $\triangleleft : V \otimes \mathfrak{g} \to V$ and

(3) for all $g, h \in \mathfrak{g}$ and $u, v \in V$ we have,

$$u \triangleright [g,h] = [u \triangleright g, h] + [g, u \triangleright h] + (u \triangleleft g) \triangleright h - (u \triangleleft h) \triangleright g,$$
$$\{u, v\} \triangleleft g = \{u, v \triangleleft g\} + \{u \triangleleft g, v\} + u \triangleleft (v \triangleright g) - v \triangleleft (u \triangleright g). \qquad \blacksquare$$

The notion of representations of Rota-Baxter associative algebras was introduced in [20] and further studied in [35]. The concept of representations of Rota-Baxter Lie algebras of weight 0 was given in [24] and the general notion of representations of Rota-Baxter Lie algebras was given in [26] in the study of matched pairs of Rota-Baxter Lie algebras.

Definition 3.5. [26, Definition 3.12] A *matched pair of Rota-Baxter Lie algebras* consists of two Rota-Baxter Lie algebras $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{h}, P_{\mathfrak{h}})$, which satisfy that $(\mathfrak{g}, P_{\mathfrak{g}})$ is a left $(\mathfrak{h}, P_{\mathfrak{h}})$-module under $\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}$, $(\mathfrak{h}, P_{\mathfrak{h}})$ is a right $(\mathfrak{g}, P_{\mathfrak{g}})$-module under $\triangleleft : V \otimes \mathfrak{g} \to V$ and $(\mathfrak{g}, \mathfrak{h}, \triangleleft, \triangleright)$ is a matched pair of Lie algebras. $\blacksquare$

Let $\Omega((\mathfrak{g}, P_{\mathfrak{g}}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ be an extending datum of $(\mathfrak{g}, P_{\mathfrak{g}})$ through V such that f and P_1 are the trivial maps, i.e. $f(x,y) = 0$ and $P_1(x) = 0$, for all $x, y \in V$. Then by Theorem 2.9, $\Omega((\mathfrak{g}, P_{\mathfrak{g}}), V) = (\triangleleft, \triangleright, \{-,-\}, P_2)$ is a Rota-Baxter Lie extending structure of $(\mathfrak{g}, P_{\mathfrak{g}})$ through V if and only $(V, \{-,-\}, P_2)$ is a Rota-Baxter Lie algebra and $((\mathfrak{g}, P_{\mathfrak{g}}), (V, P_2), \triangleleft, \triangleright)$ is a matched pair of Rota-Baxter Lie algebras. In this case, we denote the associated unified product $\mathfrak{g} \natural V$ by $\mathfrak{g} \bowtie V$ and call it the *bicrossed product* of the matched pair $((\mathfrak{g}, P_{\mathfrak{g}}), (V, P_2), \triangleleft, \triangleright)$ of Rota-Baxter Lie algebras.

The bicrossed product of two Rota-Baxter Lie algebras is the construction which provides the answer for the following factorization problem.

Problem 3.6. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{h}, P_{\mathfrak{h}})$ be two Rota-Baxter Lie algebras. Describe and classify all Rota-Baxter Lie algebras $(\mathfrak{k}, P_{\mathfrak{k}})$ that factorize through $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{h}, P_{\mathfrak{h}})$, i.e. $(\mathfrak{k}, P_{\mathfrak{k}})$ contains $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{h}, P_{\mathfrak{h}})$ as Rota-Baxter Lie subalgebras such that $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ and $\mathfrak{g} \cap \mathfrak{h} = \{0\}$.

By Proposition 3.14 in [26], a Rota-Baxter Lie algebra $(\mathfrak{k}, P_{\mathfrak{k}})$ factorizes through $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{h}, P_{\mathfrak{h}})$ if and only if there is a matched pair of Rota-Baxter Lie algebras $((\mathfrak{g}, P_{\mathfrak{g}}), (\mathfrak{h}, P_{\mathfrak{h}}), \triangleleft, \triangleright)$ such that $\mathfrak{g} \bowtie \mathfrak{h} \cong (\mathfrak{k}, P_{\mathfrak{k}})$. Hence the factorization problem can be restated as:

Problem 3.7. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ and $(\mathfrak{h}, P_{\mathfrak{h}})$ be two Rota-Baxter Lie algebras. Describe the set of all matched pairs $((\mathfrak{g}, P_{\mathfrak{g}}), (\mathfrak{h}, P_{\mathfrak{h}}), \triangleleft, \triangleright)$ and classify up to an isomorphism all bicrossed product $\mathfrak{g} \bowtie \mathfrak{h}$.

As a bicrossed product of a matched pair $((\mathfrak{g}, P_{\mathfrak{g}}), (V, P_2), \triangleleft, \triangleright)$ of Rota-Baxter Lie algebras is a special case of the unified product of an extending datum $\Omega((\mathfrak{g}, P_{\mathfrak{g}}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ of $(\mathfrak{g}, P_{\mathfrak{g}})$ through V such that f and P_1 are the trivial maps, Problem 3.7 can be characterized by Theorem 2.15 as a special case.

4. Flag extending structures

In this section, we apply our main theorem to a flag extending structure of Rota-Baxter Lie algebras as a special case.

Given a vector space E containing a Rota-Baxter Lie algebra $(\mathfrak{g}, P_{\mathfrak{g}})$ as a subspace, there is a complement V of $\mathfrak{g}$ in E. Although Theorem 2.15 gives an classification of the extending structure problem of Rota-Baxter Lie algebras, there is a computational problem.

Problem 4.1. How to compute the object $H^2_{(\mathfrak{g},P_{\mathfrak{g}})}(V, (\mathfrak{g}, P_{\mathfrak{g}}))$ explicitly and how to obtain all Rota-Baxter Lie algebra structures on E which contains $(\mathfrak{g}, P_{\mathfrak{g}})$ as a Rota-Baxter Lie subalgebra?

Now we provide a way of answering the problem for a special case.

Definition 4.2. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra and let E be a vector space containing $\mathfrak{g}$ as a subspace. A Rota-Baxter Lie algebra on E such that $(\mathfrak{g}, P_{\mathfrak{g}})$ is a Rota-Baxter Lie subalgebra is called a *flag extending structure* of $(\mathfrak{g}, P_{\mathfrak{g}})$ to E if there is a finite chain of Rota-Baxter Lie subalgebras of E

$$(\mathfrak{g}, P_{\mathfrak{g}}) = (E_0, P_0) \subset (E_1, P_1) \subset \cdots \subset (E_m, P_m) = (E, P_E)$$

such that E_i has codimension 1 in E_{i+1}, for all $i = 0, \ldots, m-1$. ■

All flag extending structures of $(\mathfrak{g}, P_{\mathfrak{g}})$ to E can be described recursively: the first step is to describe and classify all unified products $\mathfrak{g}\natural V_1$, for a 1-dimensional vector space V_1; then replace $(\mathfrak{g}, P_{\mathfrak{g}})$ with $\mathfrak{g}\natural V_1$, describe and classify all unified products of $(\mathfrak{g}\natural V_1)\natural V_2$, where V_1 and V_2 are vector spaces of dimension 1. After m steps, we obtain all flag extending structure of $(\mathfrak{g}, P_{\mathfrak{g}})$ to E.

Now we show that all unified products of $\mathfrak{g}\natural V$ with V a 1-dimensional vector space are parameterized by the space $ExDer((\mathfrak{g}, P_{\mathfrak{g}}))$ of all extended derivations of $(\mathfrak{g}, P_{\mathfrak{g}})$. Now we recall the notion of twisted derivation of a Lie algebra [3].

Definition 4.3. [3, Definition 5.2] Let $\mathfrak{g}$ be a Lie algebra. A *twisted derivation* of $\mathfrak{g}$ is a pair (ε, D), where $\varepsilon : \mathfrak{g} \to \mathbf{k}$ and $D : \mathfrak{g} \to \mathfrak{g}$ are two linear maps such that for all $g, h \in \mathfrak{g}$ we have

$$\varepsilon([g, h]) = 0, \tag{6}$$

$$D([g, h]) = [D(g), h] + [g, D(h)] + \varepsilon(g)D(h) - \varepsilon(h)D(g). \tag{7}$$

Definition 4.4. Let $(\mathfrak{g}, P_{\mathfrak{g}})$ be a Rota-Baxter Lie algebra. A *extended derivation* of $(\mathfrak{g}, P_{\mathfrak{g}})$ is a quadruple $(\varepsilon, D, g_0, k_0)$, where $\varepsilon : \mathfrak{g} \to \mathbf{k}$ and $D : \mathfrak{g} \to \mathfrak{g}$ are two linear maps, $g_0 \in \mathfrak{g}$ and $k_0 \in \mathbf{k}$ such that (ε, D) is a twisted derivation of $\mathfrak{g}$ and for all $g \in \mathfrak{g}$,

$$\begin{aligned} [P_{\mathfrak{g}}(g), g_0] \ &- k_0 D(P_{\mathfrak{g}}(g)) + P_{\mathfrak{g}}(D(P_{\mathfrak{g}}(g))) + \varepsilon(P_{\mathfrak{g}}(g))g_0 - P_{\mathfrak{g}}([g, g_0]) \\ &+ P_{\mathfrak{g}}(k_0 D(g)) + k_0\varepsilon(g)g_0 + \lambda P_{\mathfrak{g}}(D(g)) + \lambda\varepsilon(g)g_0 = 0, \end{aligned} \tag{8}$$

$$k_0^2\varepsilon(g) + \lambda k_0\varepsilon(g) = 0. \tag{9}$$

Denote the set of all extended derivations of $(\mathfrak{g}, P_\mathfrak{g})$ by $ExDer((\mathfrak{g}, P_\mathfrak{g}))$. We shall now prove that the set of all Rota-Baxter Lie extending structures $\mathcal{L}((\mathfrak{g}, P_\mathfrak{g}), V)$ of a Rota-Baxter Lie algebra $(\mathfrak{g}, P_\mathfrak{g})$ through a 1-dimensional vector space V is parameterized by $ExDer((\mathfrak{g}, P_\mathfrak{g}))$.

Proposition 4.5. *Let $(\mathfrak{g}, P_\mathfrak{g})$ be a Rota-Baxter Lie algebra and let V be a vector space of dimension 1 with a basis $\{x\}$. Then there exists a bijection between the set $\mathcal{L}((\mathfrak{g}, P_\mathfrak{g}), V)$ of all Rota-Baxter Lie extending structures of $(\mathfrak{g}, P_\mathfrak{g})$ through V and the space $ExDer((\mathfrak{g}, P_\mathfrak{g}))$ of all extended derivations of $(\mathfrak{g}, P_\mathfrak{g})$. Through the above bijection, the Rota-Baxter Lie extending structure $\Omega((\mathfrak{g}, P_\mathfrak{g}), V) = (\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)$ associated to $(\varepsilon, D, g_0, k_0) \in ExDer((\mathfrak{g}, P_\mathfrak{g}))$ is given by*

$$x \triangleleft g = \varepsilon(g)x,\ x \triangleright g = D(g),\ f = 0,\ \{-,-\} = 0, P_1(x) = g_0,\ P_2(x) = k_0 x,$$

for all $g \in \mathfrak{g}$ and $x \in V$. Denote the corresponding unified product $\mathfrak{g}\natural V$ by $\mathfrak{g}\natural_{(\varepsilon,D,g_0,k_0)}V$.

Proof. Note that the set $\mathcal{L}((\mathfrak{g}, P_\mathfrak{g}), V)$ of all Rota-Baxter Lie extending structures of $(\mathfrak{g}, P_\mathfrak{g})$ through V is equivalent to the set of all the maps $\{(\triangleleft, \triangleright, f, \{-,-\}, P_1, P_2)\}$, where $\triangleleft : V \times \mathfrak{g} \to \mathfrak{g}$, $\triangleright : V \times \mathfrak{g} \to V$, $f : V \times V \to \mathfrak{g}$, $\{-,-\} : V \times V \to V$, $P_1 : V \to \mathfrak{g}$ and $P_2 : V \to V$ satisfying the conditions (2a)–(2k) of Theorem 2.9. Since V has dimension 1, the condition (2a) is equivalent to that f and $\{-,-\}$ are trivial maps. Again by the dimension of V is 1, there is a bijection between the set of all bilinear maps $\triangleleft : V \times \mathfrak{g} \to V$ and the set of all linear maps $\varepsilon : \mathfrak{g} \to \mathbf{k}$. The bijection is given such that the action $\triangleleft : V \times \mathfrak{g} \to V$ associated to ε is given by the formula: $x \triangleleft g = \varepsilon(g)x$, for all $g \in \mathfrak{g}$. Similarly, any bilinear map $\triangleright : V \times \mathfrak{g} \to \mathfrak{g}$ is uniquely determined by a linear map $D : \mathfrak{g} \to \mathfrak{g}$ via the formula: $x \triangleright g = D(g)$, for all $g \in \mathfrak{g}$. Moreover, the linear map $P_1 : V \to \mathfrak{g}$ is uniquely determined by an element $g_0 \in \mathfrak{g}$ via the formula: $P_1(x) = g_0$ and the linear map $P_2 : V \to V$ is uniquely determined by an element $k_0 \in \mathbf{k}$ via the formula: $P_2(x) = k_0 x$.

Since $dim_\mathbf{k}(V) = 1$, $f = 0$, and $\{-,-\} = 0$, we find that the conditions (2d)–(2g), (2j)–(2k) hold automatically. Moreover, the condition (2b) is equivalent to Eq. (6), the condition (2c) is equivalent to Eq. (7), the condition (2h) is equivalent to Eq. (8) and the condition (2i) is equivalent to Eq. (9). This completes the proof. ■

Definition 4.6. Let $(\varepsilon, D, g_0, k_0)$ and $(\varepsilon', D', g_0', k_0')$ be two extended derivations of a Rota-Baxter Lie algebra $(\mathfrak{g}, P_\mathfrak{g})$. $(\varepsilon, D, g_0, k_0)$ and $(\varepsilon', D', g_0', k_0')$ are *equivalent*, denoted by $(\varepsilon, D, g_0, k_0) \equiv (\varepsilon', D', g_0', k_0')$, if $\varepsilon = \varepsilon', k_0 = k_0'$ and there exists an element $g_1 \in \mathfrak{g}$ and a nonzero element $k_1 \in \mathbf{k}$ such that for any $g \in g$,

$$D(g) = k_1 D'(g) + [g_1, g] - \varepsilon(g) g_1, \tag{10}$$

$$g_0 = P_\mathfrak{g}(g_1) + k_1 g_0' - k_0 g_1. \tag{11}$$

Theorem 4.7. *Let $(\mathfrak{g}, P_\mathfrak{g})$ be a Rota-Baxter Lie algebra of codimension 1 in a vector space E and let V be a complement of $\mathfrak{g}$ in E with basis $\{x\}$. Then $\equiv$ is an equivalent relation of the set $ExDer((\mathfrak{g}, P_\mathfrak{g}))$ of all extended derivations of $(\mathfrak{g}, P_\mathfrak{g})$ and $Extd(E, \mathfrak{g}) \cong H^2_{(\mathfrak{g},P_\mathfrak{g})}(V, (\mathfrak{g}, P_\mathfrak{g})) \cong ExDer((\mathfrak{g}, P_\mathfrak{g}))/\equiv$.*

Proof. By Lemma 2.13, there is an isomorphism of Rota-Baxter Lie algebras between $\mathfrak{g}\natural_{(\varepsilon,D,g_0,k_0)}V$ and $\mathfrak{g}\natural_{(\varepsilon',D',g_0',k_0')}$ which stabilizes $(\mathfrak{g},P_{\mathfrak{g}})$ if and only if there are linear maps $r:V\to\mathfrak{g}$ and $v:V\to V$ satisfying the conditions (1)–(6) and v is a bijection. Since $V=\mathbf{k}\{x\}$, the map $r:V\to\mathfrak{g}$ is uniquely determined by an element $g_1\in\mathfrak{g}$ such that $r(x)=g_1$ and the bijective map $v:V\to V$ is uniquely determined by a nonzero element $k_1\in\mathbf{k}$ such that $v(x)=k_1x$. Note that $f=f'=0$ and $\{-,-\}=\{-,-\}'=0$ in the corresponding Rota-Baxter Lie extending structures, the conditions (3)–(4) hold trivially. The condition (1) is equivalent to $k_1\varepsilon'(g)x=k_1\varepsilon(g)x$, for all $g\in\mathfrak{g}$. As $k_1\neq 0$, $\varepsilon=\varepsilon'$. The condition (2) is equivalent to Eq. (10) and the condition (5) is equivalent to Eq. (11). Moreover, the condition (6) is equivalent to $k_0k_1x=k_0'k_1x$ and hence $k_0=k_0'$ by $k_1\neq 0$. Thus $\mathfrak{g}\natural_{(\varepsilon,D,g_0,k_0)}V\cong\mathfrak{g}\natural_{(\varepsilon',D',g_0',k_0')}V$ as Rota-Baxter Lie algebras if and only if $(\varepsilon,D,g_0,k_0)\equiv(\varepsilon',D',g_0',k_0')$. This completes the proof. ■

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