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Lie Groups Endowed with a Left Invariant Riemannian Metric and a Flat Metric Connection with Skew-Symmetric Torsion

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Abstract. We give a complete description of Lie groups endowed with a left invariant Riemannian metric and a left invariant flat metric connection with skew-symmetric torsion.

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1. Introduction

Let (M, g) be a Riemannian manifold and ∇ a metric connection on M, i.e., $\nabla(g) = 0$. The torsion T of ∇ defines a $(3, 0)$ tensor field ω by

$$\omega(X, Y, Z) = g(T_X Y, Z), \quad X, Y, Z \in \Gamma(TM)$$

and when $\omega \in \Omega^3(M)$, ∇ is called a metric connection with skew-symmetric torsion. Flat metric connections with skew-symmetric torsion on a complete Riemannian manifold were completely classified by É. Cartan and J.A Shouten [3]. They proved the following theorem.

Theorem 1.1. *Let (M, g, ∇) be a simply connected complete Riemannian manifold endowed with a flat metric connection with skew-symmetric torsion. Then M is isometric to a Riemannian product with factors in one of the following classes:*

(i) *Euclidean spaces.*

(ii) *Simply connected, compact semi-simple Lie groups equipped with a bi-invariant metric.*

(iii) *The sphere S^7 with a Riemannian metric of constant sectional curvature.*

The Levi-Civita connection of a Euclidean space is obviously flat and has a skew-symmetric torsion actually trivial. Let G be a compact semi-simple Lie group endowed with a bi-invariant metric $\langle\ ,\ \rangle$. Then the connections ∇^+ and ∇^- given by

$$\nabla^+_X Y = [X, Y] \quad \text{and} \quad \nabla^-_X Y = 0 \tag{1}$$

for any left invariant vector fields X, Y are bi-invariant flat, metric and have skew-symmetric torsion. Moreover, the inversion $x \mapsto x^{-1}$ sends ∇^+ to ∇^-. The sphere S^7 has a family of flat metric connections with skew-symmetric torsion [1].

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This theorem has been investigated later by many authors leading to different proofs [1, 4, 10]. Actually, the existence of a flat metric connection with skew-symmetric torsion on a simply connected Riemannian manifold is equivalent to the existence of a global orthonormal frame of Killing vector fields [4] and this situation were studied even in the pseudo-Riemannian case [12, 13].

In this paper, we consider a connected Lie group G endowed with a left invariant metric $\langle\ ,\ \rangle$ and a left invariant flat metric connection with skew-symmetric torsion ∇. If G is compact semi-simple, one might naturally expect that $\langle\ ,\ \rangle$ must be bi-invariant and ∇ is either ∇^+ or ∇^-. Surprisingly, this is not always the case (see Subsection 2.1 and Theorem 1.4). Nonetheless, we establish an affirmative result when G is simple and we give a complete description of Lie groups having a left invariant metric and a left invariant flat metric connection with skew-symmetric torsion. We call such a Lie group *Riemannian Lie group with a FMSS-connection.* More precisely, we will prove the following two results.

Theorem 1.2. *Let $(K,\langle\ ,\ \rangle,\nabla)$ be a simply connected Riemannian Lie group with a FMSS-connection. Then there exists $(G,\langle\ ,\ \rangle_1,\nabla^1)$ and $(H,\langle\ ,\ \rangle_2,\nabla^2)$ two simply connected Riemannian Lie groups with a FMSS-connection, a right action α of G on H, a left action β of H on G such that:*

(1) *∇^1 is the Levi-Civita connection of $\langle\ ,\ \rangle_1$ and hence $\langle\ ,\ \rangle_1$ is flat,*
(2) *H is compact semi-simple,*
(3) *$(K,\langle\ ,\ \rangle,\nabla)$ isomorphic to the bi-crossed product of $(G,\langle\ ,\ \rangle_1,\nabla^1)$ and $(H,\langle\ ,\ \rangle_2,\nabla^2)$ by the mean of α and β.*

Corollary 1.3. *Let $(K,\langle\ ,\ \rangle,\nabla)$ be a solvable non abelian connected Riemannian Lie group with a FMSS-connection. Then ∇ is the Levi-Civita connection of $\langle\ ,\ \rangle$ and hence $\langle\ ,\ \rangle$ is flat. Moreover, K is 2-solvable.*

For the details of the notion of bi-crossed product of Lie groups and the associated matched pairs of Lie algebras one can see [6, 7], we will recall the essential on these notions in Section 2 and we will also precise the notion of bi-crossed product of two Riemannian Lie groups with a FMSS-connection (see Proposition 2.1). Simply connected Lie groups endowed with a flat left invariant metric were described in [8, Theorem 1.5] so to complete our study, we need to describe compact semi-simple Riemannian Lie groups with a FMSS-connection. This is the subject of the following theorem.

Theorem 1.4. *Let $(G,\langle\ ,\ \rangle,\nabla)$ be a connected compact semi-simple Riemannian Lie group with a FMSS-connection. Denote by $\mathfrak{g}=T_eG$ the Lie algebra of G. Then the torsion of ∇ is parallel with respect to both ∇ and the Levi-Civita connection of $\langle\ ,\ \rangle$ and there exists $a,b:\mathfrak{g}\longrightarrow\mathfrak{g}$ such that a (resp. b) is an homomorphism (resp. anti-homomorphism) of Lie algebras, a bi-invariant metric $\langle\ ,\ \rangle_0$ on G, $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ where $\mathfrak{g}_1=\ker a$, $\mathfrak{g}_2=\ker b$,*

$$\langle X,Y\rangle=\begin{cases}\langle a(X),a(Y)\rangle_0 & \textit{if}\quad X,Y\in\mathfrak{g}_2,\\ \langle b(X),b(Y)\rangle_0 & \textit{if}\quad X,Y\in\mathfrak{g}_1,\\ \langle a(X),b(Y)\rangle_0 & \textit{if}\quad X\in\mathfrak{g}_2,Y\in\mathfrak{g}_1,\end{cases}$$

and $$\nabla_X Y = \begin{cases} 0 \quad \text{if} \quad X \in \mathfrak{g}_1, \\ [X,Y] \quad \text{if} \quad X, Y \in \mathfrak{g}_2 \\ (\phi)^{-1}[a(X), b(Y)] \quad \text{if} \quad X \in \mathfrak{g}_2, Y \in \mathfrak{g}_1, \end{cases}$$

where $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$, $X_1 + X_2 \mapsto b(X_1) + a(X_2)$. *Moreover, the torsion* T *defines a Lie bracket on* $\mathfrak{g}$ *and* $\langle\ ,\ \rangle$ *is bi-invariant with respect to this Lie bracket.*

Corollary 1.5. *Let* $(G, \langle\ ,\ \rangle, \nabla)$ *be a compact connected simple Riemannian Lie group with a FMSS-connection. Then* $\langle\ ,\ \rangle$ *is bi-invariant and* ∇ *is equal either to* ∇^+ *or* ∇^-.

It is a well-known fact that a Riemannian manifold endowed with a flat metric connection with skew-symmetric torsion is locally symmetric and has nonnegative Ricci curvature (see [1]). Hence Theorem 1.4 provides examples of left invariant metrics not bi-invariant which are locally symmetric and have positive Ricci curvature.

The paper is organized as follows. In Section 2, we give an example of a Riemannian Lie group with a FMSS-connection where the metric and the connection are not bi-invariant, we make the notion of bi-crossed product appearing in Theorem 1.2 more precise and we prove this theorem. In Section 3, we prove Theorem 1.4 by using two lemmas (Lemmas 3.1–3.2) which are interesting in their own.

Notation. For any Lie group G and for any $x \in G$, we denote by L_x and R_x, respectively, the left and the right multiplication by x.

2. Bi-crossed product of Riemannian Lie groups with a FMSS-connection and the proof of Theorem 1.2

In this section, we give an example of a Riemannian Lie group with a FMSS-connection such that both the metric and the connection are not bi-invariant, we define the bi-crossed product of two Riemannian Lie groups with a FMSS-connection and we prove Theorem 1.2.

2.1. Example

On any compact semi-simple Lie group endowed with a bi-invariant metric, the connections ∇^+ and ∇^- given by (1) are bi-invariant, flat and metric with skew-symmetric torsion.

We give now an example of a Lie group endowed with a left invariant metric which is not bi-invariant and which admits a left invariant (not bi-invariant) flat metric connection with skew-symmetric torsion.

Let $\mathfrak{h}$ be a semi-simple Lie algebra endowed with a bi-invariant scalar product $\langle\ ,\ \rangle_0$, i.e., for any $X, Y, Z \in \mathfrak{h}$,

$$\langle [X,Y], Z\rangle_0 + \langle [X,Z], Y\rangle_0 = 0.$$

Consider the Lie algebra $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ product of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ where $\mathfrak{h}_i$ is a copy of $\mathfrak{h}$. Let $F : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_2$ be an isomorphism of Lie algebras.

Let G be a Lie group with $\mathfrak{g}$ as the Lie algebra of left invariant vector fields. We define on G a left invariant metric $\langle\ ,\ \rangle$ and a left invariant connection ∇ by

$$\langle X,Y\rangle=\begin{cases}\langle X,Y\rangle_0 & \text{if}\quad X,Y\in\mathfrak{h}_2,\\ \langle X,Y\rangle_0+\langle F(X),F(Y)\rangle_0 & \text{if}\quad X,Y\in\mathfrak{h}_1,\\ -\langle X,F(Y)\rangle_0 & \text{if}\quad X\in\mathfrak{h}_2, Y\in\mathfrak{h}_1,\end{cases}$$

and

$$\nabla_XY=\begin{cases}0 & \text{if}\quad X\in\mathfrak{h}_1,\\ [F(Y),X] & \text{if}\quad X\in\mathfrak{h}_2, Y\in\mathfrak{h}_1,\\ [X,Y] & \text{if}\quad X,Y\in\mathfrak{h}_2.\end{cases}$$

One can check that $\nabla(\langle\ ,\ \rangle)=0$, the curvature of ∇ vanishes and its torsion is skew-symmetric. Note that $\langle\ ,\ \rangle$ is not a bi-invariant metric since, for any $X,Y\in\mathfrak{h}_2$ and $Z\in\mathfrak{h}_1$,

$$\langle[X,Y],Z\rangle+\langle[X,Z],Y\rangle=-\langle F(Z),[X,Y]\rangle_0.$$

Moreover, ∇ is not bi-invariant since, for $X\in\mathfrak{h}_2$, $Y\in\mathfrak{h}_1$, $Z\in\mathfrak{h}_1$,

$$[Z,\nabla_XY]-\nabla_{[Z,X]}Y-\nabla_X[Z,Y]=-[[F(Z),F(Y)],X].$$

2.2. Bi-crossed product of two Riemannian Lie groups with an FMSS-connection

Let us recall the notions of matched pair of Lie algebras and matched pair of Lie groups and the associated bi-crossed product (see [6, 7] for details).

Let G and H be two Lie groups, and let $\alpha: H\times G\longrightarrow H$, $(h,g)\mapsto\alpha_g(h)$, and $\beta: H\times G\longrightarrow G$, $(h,g)\mapsto\beta_h(g)$, be two smooth maps satisfying

1. $\alpha_e=\mathrm{Id}_H$ and $\beta_e=\mathrm{Id}_G$,
2. $\beta_{h_1h_2}=\beta_{h_1}\circ\beta_{h_2}$,
3. $\alpha_{g_1g_2}=\alpha_{g_2}\circ\alpha_{g_1}$,
4. $\alpha_g(h_1h_2)=\alpha_{\beta_{h_2}(g)}(h_1)\alpha_g(h_2)$,
5. $\beta_h(g_1g_2)=\beta_h(g_1)\beta_{\alpha_{g_1}(h)}(g_2)$.

(G,H,α,β) is called a *matched pair of Lie groups*. The product

$$(g_1,h_1)(g_2,h_2)=(g_1\beta_{h_1}(g_2),\alpha_{g_2}(h_1)h_2)$$

defines a Lie group structure on $G\times H$. We denote by $G\bowtie H$ the obtained Lie group called the *bi-crossed product* of (G,H,α,β). Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively.

For any $g\in G$ and $h\in H$, $\alpha_g(e)=e$ and $\beta_h(e)=e$ and hence $T_e\alpha: G\longrightarrow \mathrm{GL}(\mathfrak{h})$, $g\mapsto T_e\alpha_g$ and $T_e\beta: H\longrightarrow\mathrm{GL}(\mathfrak{g})$ are, respectively, an anti-homomorphism and a homomorphism of Lie groups. Differentiating these maps at the identity gives rise to an anti-representation $\widetilde{\alpha}:\mathfrak{g}\longrightarrow\mathrm{gl}(\mathfrak{h})$ and a representation $\widetilde{\beta}:\mathfrak{h}\longrightarrow\mathrm{gl}(\mathfrak{g})$. The Lie algebra of $G\bowtie H$ is equal to $\mathfrak{g}\oplus\mathfrak{h}$ with the Lie bracket given by

$$[a,b]=[a,b]_{\mathfrak{g}},\ [u,v]=[u,v]_{\mathfrak{h}}\quad\text{and}\quad [u,a]=\widetilde{\beta}(u)(a)+\widetilde{\alpha}(a)(u), \tag{2}$$

for any $a,b\in\mathfrak{g}$ and $u,v\in\mathfrak{h}$. Moreover,

$$\begin{cases}\widetilde{\alpha}(a)([u,v])=[u,\widetilde{\alpha}(a)(v)]+[\widetilde{\alpha}(a)(u),v]+\widetilde{\alpha}(\widetilde{\beta}(v)(a))(u)-\widetilde{\alpha}(\widetilde{\beta}(u)(a))(v),\\ \widetilde{\beta}(u)([a,b])=[a,\widetilde{\beta}(u)(b)]+[\widetilde{\beta}(u)(a),b]-\widetilde{\beta}(\widetilde{\alpha}(b)(u))(a)+\widetilde{\beta}(\widetilde{\alpha}(a)(u))(b).\end{cases} \tag{3}$$

Conversely, a matched pair of Lie algebras is a couple of Lie algebras $(\mathfrak{g},\mathfrak{h})$, an anti-representation $\widetilde{\alpha}:\mathfrak{g}\longrightarrow \mathrm{gl}(\mathfrak{h})$ and a representation $\widetilde{\beta}:\mathfrak{h}\longrightarrow \mathrm{gl}(\mathfrak{g})$ satisfying (3). Then $\mathfrak{g}\oplus\mathfrak{h}$ endowed with the bracket (2) is a Lie algebra. Let G and H be, respectively, the simply connected Lie groups associated to $\mathfrak{g}$ and $\mathfrak{h}$. If G is compact, Majid showed [7, Theorem 4.2] that $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be integrated to a right action α of G on H and a left action β of H on G such that (G,H,α,β) is a matched pair of Lie groups and $G\bowtie H$ is the simply connected Lie group associated to the Lie algebra $\mathfrak{g}\oplus\mathfrak{h}$.

The following proposition introduces the notion of bi-crossed product of two Riemannian Lie groups with a FMSS-connection.

Proposition 2.1. *Let $(G,\langle\ ,\ \rangle_1,\nabla^1)$ and $(H,\langle\ ,\ \rangle_2,\nabla^2)$ be two simply connected Riemannian Lie groups with a FMSS-connection and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras, respectively. Suppose that there exists a anti-representation $\widetilde{\alpha}:\mathfrak{g}\longrightarrow \mathrm{so}(\mathfrak{h},\langle\ ,\ \rangle_2)$ and a representation $\widetilde{\beta}:\mathfrak{h}\longrightarrow \mathrm{so}(\mathfrak{g},\langle\ ,\ \rangle_1)$ such that, for any $a,b\in\mathfrak{g}$ and for any $u,v\in\mathfrak{h}$,*

$$\begin{cases}\widetilde{\alpha}(a)(T^2_uv)=T^2_u\widetilde{\alpha}(a)(v)-T^2_v\widetilde{\alpha}(a)(u),\\ \widetilde{\beta}(u)(T^1_ab)=T^1_a\widetilde{\beta}(u)(b)-T^1_b\widetilde{\beta}(u)(a),\\ \widetilde{\alpha}(a)(u\bullet_2 v)=\widetilde{\alpha}(a)(u)\bullet_2 v+u\bullet_2(\widetilde{\alpha}(a)(v))-\widetilde{\alpha}(\widetilde{\beta}(u)(a))(v),\\ \widetilde{\beta}(u)(a\bullet_1 b)=\widetilde{\beta}(u)(a)\bullet_1 b+a\bullet_1\widetilde{\beta}(u)(b)+\widetilde{\beta}(\widetilde{\alpha}(a)(u))(b),\end{cases}\tag{4}$$

where T^i is the torsion of ∇^i and $\bullet_i$ is the restriction of ∇^i to the Lie algebra. Then $(\mathfrak{g},\mathfrak{h},\widetilde{\alpha},\widetilde{\beta})$ is a matched pair of Lie algebras and if $(\mathfrak{g},\mathfrak{h},\widetilde{\alpha},\widetilde{\beta})$ can be integrated to a matched pair of Lie groups (G,H,α,β) then the bi-crossed product $G\bowtie H$ carries a structure of Riemannian Lie group with a FMSS-connection.

Proof. We endow $\mathfrak{g}\oplus\mathfrak{h}$ with the bracket (2) and the scalar product $\langle\ ,\ \rangle=\langle\ ,\ \rangle_1+\langle\ ,\ \rangle_2$. Since, for any $a,b\in\mathfrak{g}$ and $u,v\in\mathfrak{h}$,

$$[a,b]_{\mathfrak{g}}=a\bullet_1 b-b\bullet_1 a-T^1_ab\quad\text{and}\quad[u,v]_{\mathfrak{h}}=u\bullet_2 v-v\bullet_2 u-T^2_uv$$

the system (4) implies (3) and hence $(\mathfrak{g},\mathfrak{h},\widetilde{\alpha},\widetilde{\beta})$ is a matched pair of Lie algebras. We define now on $\mathfrak{g}\oplus\mathfrak{h}$ the algebra product $\bullet$ by

$$X\bullet Y=\begin{cases}X\bullet_1 Y & \text{if}\quad X,Y\in\mathfrak{g},\\ X\bullet_2 Y & \text{if}\quad X,Y\in\mathfrak{h},\\ -\widetilde{\alpha}(X)(Y) & \text{if}\quad X\in\mathfrak{g},Y\in\mathfrak{h},\\ \widetilde{\beta}(X)(Y) & \text{if}\quad X\in\mathfrak{h},Y\in\mathfrak{g}.\end{cases}$$

If $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be integrated, $G\bowtie H$ is a Lie group whose Lie algebra is $\mathfrak{g}\oplus\mathfrak{h}$ and $\langle\ ,\ \rangle$ and $\bullet$ define on $G\bowtie H$ a left invariant metric also denoted by $\langle\ ,\ \rangle$ and a left invariant connection ∇. Let us show that $(G\bowtie H,\langle\ ,\ \rangle,\nabla)$ is a Riemannian Lie group with a FMSS-connection. Indeed, if we denote by L_X the endomorphism $Y\mapsto X\bullet Y$, it is obvious that L_X is skew-symmetric with respect to $\langle\ ,\ \rangle$ and the torsion of $\bullet$ is given by

$$T_XY=\begin{cases}T^1_XY & \text{if}\quad X,Y\in\mathfrak{g},\\ T^2_XY & \text{if}\quad X,Y\in\mathfrak{h},\\ -T_YX=0 & \text{if}\quad X\in\mathfrak{g},Y\in\mathfrak{h},\end{cases}$$

and one can see that T is skew-symmetric with respect to $\langle\ ,\ \rangle$.

Denote by R the curvature of ∇. Since ∇^1 and ∇^2 are flat, we get easily that $R(X,Y)Z=0$ whenever $X,Y,Z\in\mathfrak{g}$ or $X,Y,Z\in\mathfrak{h}$.

For $a,b\in\mathfrak{g}$ and $u,v\in\mathfrak{h}$, a direct computation gives:

$$\begin{cases} R(a,b)u=-\widetilde{\alpha}([a,b])(u)-\widetilde{\alpha}(a)\circ\widetilde{\alpha}(b)(u)+\widetilde{\alpha}(b)\circ\widetilde{\alpha}(a)(u),\\ R(u,v)a=\widetilde{\beta}([u,v])(a)-\widetilde{\beta}(u)\circ\widetilde{\beta}(v)(a)+\widetilde{\beta}(v)\circ\widetilde{\beta}(u)(a),\\ R(a,u)v=-\widetilde{\alpha}(a)(u)\bullet_2 v+\widetilde{\alpha}(\widetilde{\beta}(u)(a))(v)+\widetilde{\alpha}(a)(u\bullet_2 v)-u\bullet_2(\widetilde{\alpha}(a)(v)),\\ R(a,u)b=-\widetilde{\beta}(\widetilde{\alpha}(a)(u))(b)-\widetilde{\beta}(u)(a)\bullet_1 b-a\bullet_1\widetilde{\beta}(u)(b)+\widetilde{\beta}(u)(a\bullet_1 b).\end{cases}\tag{5}$$

So $R=0$ if and only if $\widetilde{\alpha}$ is an anti-representation, $\widetilde{\beta}$ is a representation and the last two equations of (4) hold. ■

2.3. Proof of Theorem 1.2

Proof. Let $(K,\langle\ ,\ \rangle,\nabla)$ be a simply connected Riemannian Lie group endowed with a left invariant flat metric with skew-symmetric connection and T the torsion of ∇. We denote by $\mathfrak{k}$ the Lie algebra of K. For any $X,Y,Z\in\mathfrak{k}$, we have the following formulas established in [1],

$$\begin{cases}\nabla_XY=\nabla^0_XY+\frac{1}{2}T_XY,\quad \mathrm{ric}^0(X,Y)=-\frac{1}{4}\mathrm{tr}(T_X\circ T_Y)\geq 0,\\ 3\nabla_X(T)(Y,Z)=T_XT_YZ+T_YT_ZY+T_ZT_XY,\end{cases}\tag{6}$$

where ∇^0 is the Levi-Civita connection of $\langle\ ,\ \rangle$, ric^0 its Ricci curvature and $T_X:\mathfrak{k}\longrightarrow\mathfrak{k}$, $Y\mapsto T_XY$.

Since $\langle\ ,\ \rangle$ and ∇ are left invariant, they induce a scalar product on $\mathfrak{k}$ also denoted by $\langle\ ,\ \rangle$ and an algebra product $\bullet:\mathfrak{k}\times\mathfrak{k}\longrightarrow\mathfrak{k}$, $(X,Y)\mapsto X\bullet Y=\nabla_XY$. For any $X\in\mathfrak{k}$, both T_X and L_X are skew-symmetric where $\mathrm{L}_XY=X\bullet Y$. Moreover, the vanishing of the curvature of ∇ is equivalent to

$$\mathrm{L}_{[X,Y]}=[\mathrm{L}_X,\mathrm{L}_Y].$$

Consider $\mathfrak{g}=\{X\in\mathfrak{k},T_X=0\}$ and $\mathfrak{h}=\mathfrak{g}^\perp$ its orthogonal with respect to $\langle\ ,\ \rangle$. Since T is skew-symmetric then $\mathfrak{h}=\mathrm{span}\{T_XY,X,Y\in\mathfrak{k}\}$.

For $Y\in\mathfrak{g}$, by using the third formula in (6), we get $T_{X\bullet Y}Z=0$ for any $X,Z\in\mathfrak{k}$ and hence $\mathfrak{g}$ is a left ideal of $(\mathfrak{k},\bullet)$ and we denote by $\bullet_1$ the restriction of $\bullet$ to $\mathfrak{g}$. Since for any $X\in\mathfrak{k}$, L_X is skew-symmetric, we deduce that $\mathfrak{h}$ is also a left ideal of $(\mathfrak{k},\bullet)$ and we denote by $\bullet_2$ the restriction of $\bullet$ to $\mathfrak{h}$. For any $a,b\in\mathfrak{g}$ and $u,v\in\mathfrak{h}$, we have

$$[a,b]=a\bullet_1 b-b\bullet_1 a,\ [u,v]=u\bullet_2 v-v\bullet_2 u-T_uv\quad\text{and}\quad [u,a]=u\bullet a-a\bullet u.$$

This shows that $\mathfrak{g}$ and $\mathfrak{h}$ are subalgebras, and that the linear maps $\widetilde{\alpha}:\mathfrak{g}\longrightarrow\mathrm{so}(\mathfrak{h})$, $a\mapsto-(\mathrm{L}_a)_{|\mathfrak{h}}$ and $\widetilde{\beta}:\mathfrak{h}\longrightarrow\mathrm{so}(\mathfrak{g})$, $u\mapsto(\mathrm{L}_u)_{|\mathfrak{g}}$ are, respectively, an anti-representation and a representation of Lie algebras, and for any $a\in\mathfrak{g}$ and $u\in\mathfrak{h}$

$$[u,a]=\widetilde{\beta}(u)(a)+\widetilde{\alpha}(a)(u).$$

Thus $(\mathfrak{g},\mathfrak{h},\widetilde{\alpha},\widetilde{\beta})$ is a matched pair of Lie algebras and for any $X,Y\in\mathfrak{k}$,

$$X\bullet Y=\begin{cases}X\bullet_1 Y & \text{if}\quad X,Y\in\mathfrak{g},\\ X\bullet_2 Y & \text{if}\quad X,Y\in\mathfrak{h},\\ -\widetilde{\alpha}(X)(Y) & \text{if}\quad X\in\mathfrak{g},Y\in\mathfrak{h},\\ \widetilde{\beta}(X)(Y) & \text{if}\quad X\in\mathfrak{h},Y\in\mathfrak{g}.\end{cases}$$

The curvature of $\bullet$ is given by (5) so the vanishing of the curvature and (3) imply (4). Thus we can apply Proposition 2.1.

Let $(G,\langle\ ,\ \rangle_1,\nabla^1)$ be the simply connected Riemannian Lie group with FSSM-connection associated to $(\mathfrak{g},\langle\ ,\ \rangle_{|\mathfrak{g}},\bullet_1)$ and $(H,\langle\ ,\ \rangle_2,\nabla^2)$ be the simply connected Riemannian Lie group with FSSM-connection associated to $(\mathfrak{h},\langle\ ,\ \rangle_{|\mathfrak{h}},\bullet_2)$. Since the torsion of ∇^1 vanishes then ∇^1 is the Levi-Civita connection of $\langle\ ,\ \rangle_1$ and hence $\langle\ ,\ \rangle_1$ is flat. According to (6) and since for any $u\in\mathfrak{h}\setminus\{0\}$, $T_u\neq 0$, the Ricci curvature of $\langle\ ,\ \rangle_2$ is positive and hence H is compact. According to [7, Theorem 4.2], $\widetilde{\alpha}$ and $\widetilde{\beta}$ can be integrated to a right action α of G on H and a left action β of H on G and K is isomorphic to the bi-crossed product of $G\bowtie H$ and we can conclude by using Proposition 2.1. ■

Corollary 1.3 is an immediate consequence of the fact that when K is solvable $H=\{e\}$ and G is 2-solvable (see [8, Theorem 1.5])

3. Proof of Theorem 1.4

In this section, we give a proof of Theorem 1.4. It is based on two lemmas.

Lemma 3.1. *Let (G,g) be a connected and simply connected Lie group endowed with a left invariant metric and let $\phi:G\longrightarrow S^n\times M$ be an isometry where $S^n\times M$ is a Riemannian product. Then $n=3$.*

Proof. Put $(s,a)=\phi(e)$. We consider the embedding $i:S^n\longrightarrow G$ given by $i(x)=\phi^{-1}(x,a)$. Let us show that $i(S^n)$ is a closed subgroup of G.

Since g is left invariant then it is complete and hence the metric on $S^n\times M$ is complete. The inclusion $I^0(S^n)\times I^0(M)\longrightarrow I^0(S^n\times M)$ is, according to [5, Theorem 3.5, p. 240], an isomorphism where $I^0(\bullet)$ is the connected component of the identity in the group of isometries. On the other hand, G is connected and the metric is left invariant so, for any $b\in G$, $\phi\circ L_b\circ\phi^{-1}\in I^0(S^n\times M)$ and hence

$$\phi\circ L_b\circ\phi^{-1}(x,y)=(\mu_b(x),\nu_b(y)) \tag{7}$$

where $\mu_b\in I^0(S^n)$ and $\nu_b\in I^0(M)$. We have $i(s)=e$ and for any $x\in S^n$,

$$\phi\circ L_{i(x)}\circ\phi^{-1}(s,a)=\phi(i(x))=(x,a)=(\mu_{i(x)}(s),\nu_{i(x)}(a)).$$

Thus
$$\mu_{i(x)}(s)=x\quad\text{and}\quad\nu_{i(x)}(a)=a. \tag{8}$$

Moreover, $\mu_{i(x)}$ is an isometry of S^n so there exists a unique $x^{-1}\in S^n$ such that $\mu_{i(x)}(x^{-1})=s$.

Now, for any $x, y \in S^n$, we have

$$\begin{aligned} i(x).i(y) &= \phi^{-1}(x,a).\phi^{-1}(y,a) = L_{i(x)}\phi^{-1}(y,a) \\ &\overset{(7)}{=} \phi^{-1}(\mu_{i(x)}(y), \nu_{i(x)}(a)) \overset{(8)}{=} \phi^{-1}(\mu_{i(x)}(y), a) = i(\mu_{i(x)}(y)). \end{aligned}$$

This shows that $i(S^n)$ is stable by the group product and $i(x^{-1}) = i(x)^{-1}$. Since S^3 is the only simply connected sphere which has a structure of Lie group we can conclude. ∎

Lemma 3.2. *Let G be a simply connected compact semi-simple Lie group endowed with a bi-invariant metric h_1 and a left invariant metric h_2, and let*

$$\Phi : (G, h_2) \longrightarrow (G, h_1)$$

be an isometry with $\Phi(e) = e$. Then there exist $A, B : G \longrightarrow G$ where A (resp. B) is an homomorphism (resp. anti-homomorphism) of Lie groups such that for any $x \in G$, $\Phi(x) = A(x)B(x)$. In particular, if G is simple $A = 0$ or $B = 0$ and hence h_2 is bi-invariant.

Proof. We have $I^0(G, h_1) = L(G)R(G)$ (see [9, Corollary 4.3]) and hence the map $\psi : G \times G \longrightarrow \mathrm{Iso}^\circ(G, h_2)$ given by $(g, h) \mapsto \Phi^{-1} L_g R_h \Phi$ is an homomorphism[1] of Lie groups and $\ker \psi = \{(g, g^{-1}),\ g \in Z(G)\}$ where $Z(G)$ is the center of G.

Since $Z(G)$ is discrete then ψ a covering. So there exists and open neighborhood U of e in G such that $\psi_{|U \times U}$ is a diffeomorphism from $U \times U$ to $\psi(U \times U)$. Let $U_0 \subset U$ be an open neighborhood of U such that $U_0.U_0 \in U$ and $V_0 \subset V$ such that $L(V_0) = \{L_g, g \in V_0\} \subset \psi(U_0 \times U_0)$ and $L(V) \subset \psi(U \times U)$. For any $g \in V$, there exists two unique $A(g), B(g) \in U$ such that

$$L_g = \Phi^{-1} \circ L_{A(g)} \circ R_{B(g)} \circ \Phi. \tag{9}$$

For any $g, h \in V_0$,

$$\begin{aligned} L_{gh} = L_g \circ L_h &= \Phi^{-1} \circ L_{A(g)} \circ R_{B(g)} \circ \Phi \circ \Phi^{-1} \circ L_{A(h)} \circ R_{B(h)} \circ \Phi \\ &= \Phi^{-1} \circ L_{A(g)} \circ R_{B(g)} \circ L_{A(h)} \circ R_{B(h)} \circ \Phi \\ &= \Phi^{-1} \circ L_{A(g)A(h)} \circ R_{B(h)B(g)} \circ \Phi \\ &= \psi(A(g)A(h), B(h)B(g)) \end{aligned}$$

Since $A(g), A(h), B(g), B(h) \in U_0$ then $A(g)A(h), B(h)B(g) \in U$ and hence $gh \in V$ and

$$A(gh) = A(g)A(h) \quad \text{and} \quad B(gh) = B(h)B(g).$$

We conclude that $T_e A_{|V_0} : \mathfrak{g} \longrightarrow \mathfrak{g}$ (resp. $T_e B_{|V_0} : \mathfrak{g} \longrightarrow \mathfrak{g}$) is a homomorphism (resp. an anti-homomorphism) of Lie algebras. Since G is simply connected, $A_{|V_0}$ and $B_{|V_0}$ can be extended to $A, B : G \longrightarrow G$. Moreover, V_0 generates G thus the relation (9) holds for any $g \in G$ and hence, for any $g, h \in G$, $\Phi(gh) = A(g)\Phi(h)B(g)$ and $\Phi(g) = A(g)B(g)$. Finally, $\ker A$ and $\ker B$ are normal subgroups and if G is simple then either $A = 0$ or $B = 0$. ∎

[1] Note that $G \times G$ is endowed with the product $(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_2 b_1)$.

Proof of Theorem 1.4 Let (G, g, ∇) be a connected compact semi-simple Riemannian Lie group with FMSS-connection. Note first that if the theorem is true for the universal covering of G it is true for G so we can suppose that G is simply connected. According to Theorem 1.1 and Lemma 3.1, there exists an isometry $\Phi : (G, g, \nabla) \longrightarrow (H, h, \nabla^+)$ where (H, h, ∇^+) is a connected compact semi-simple simply connected Lie group endowed with a bi-invariant metric and the connection given by (1). Connected compact semi-simple simply connected Lie groups which are homeomorphic are actually isomorphic as Lie groups (see [2]) so we can suppose that $H = G$. We can suppose also that $\Phi(e) = e$ and we can apply Lemma 3.2. Thus there exists $A, B : G \longrightarrow G$ where A (resp. B) is an homomorphism (resp. anti-homomorphism) of Lie groups such that for any $x \in G$, $\Phi(x) = A(x)B(x)$. Put $G_1 = \ker A$, $G_2 = \ker B$, $a = T_eA$ and $b = T_eB$, $\phi = T_e\Phi = a + b$, $\langle\ ,\ \rangle = g(e)$ and $\langle\ ,\ \rangle_0 = h(e)$.

We have $G_1 = \ker A$ and $G_2 = \ker B$ are normal subgroups of G, $G_1 \cap G_2 = \{e\}$ and $\dim G_1 + \dim G_2 = \dim G$. Moreover, $\mathfrak{g}_1 = T_eG_1 = \ker a$ and $\mathfrak{g}_2 = T_eG_2 = \ker b$ are ideals of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\langle \mathfrak{g}_1, \mathfrak{g}_2\rangle_0 = 0$. For any $X \in \mathfrak{g}$ and $x \in G$, we have

$$\begin{aligned} T_x\Phi(T_eL_gX) &= \frac{d}{dt}_{t=0} A(x\exp(tX))B(x\exp(tX)) \\ &= \frac{d}{dt}_{t=0} A(x)A(\exp(tX))B(\exp(tX))B(x) \\ &= T_eR_{B(x)}L_{A(x)}(a(X) + b(X)). \end{aligned}$$

Since h is bi-invariant and g is left invariant, Φ is an isometry if and only if, for any $X, Y \in \mathfrak{g}$,

$$\langle a(X) + b(X), a(Y) + b(Y)\rangle_0 = \langle X, Y\rangle.$$

This is equivalent to

$$\langle X, Y\rangle = \begin{cases} \langle a(X), a(Y)\rangle_0 & \text{if} \quad X, Y \in \mathfrak{g}_2, \\ \langle b(X), b(Y)\rangle_0 & \text{if} \quad X, Y \in \mathfrak{g}_1, \\ \langle a(X), b(Y)\rangle_0 & \text{if} \quad X \in \mathfrak{g}_2, Y \in \mathfrak{g}_1. \end{cases}$$

Now, for any $X, Y \in \mathfrak{g}$, we have

$$\nabla_X Y = \nabla^g_X Y + \frac{1}{2} T_X Y,$$

where ∇^g is the Levi-Civita connection of g and T is the torsion of ∇. But Φ sends ∇ on ∇^+ and hence

$$T_X Y = \phi^{-1}(T^+_{\phi(X)}\phi(Y))$$

where T^+ is the torsion of ∇^+ which is given by $T^+_X Y = [X, Y]$. By using the fact that a (resp. b) an homomorphism (resp. anti-homomorphism) of Lie algebras, we get

$$T_X Y = \begin{cases} -[X, Y] & \text{if} \quad X, Y \in \mathfrak{g}_1, \\ [X, Y] & \text{if} \quad X, Y \in \mathfrak{g}_2, \\ -T_Y X = \phi^{-1}[b(X), a(Y)] & \text{if} \quad X \in \mathfrak{g}_1, Y \in \mathfrak{g}_2. \end{cases}$$

Let us compute ∇. Note that since h is bi-invariant, for any $X,Y,Z\in\mathfrak{g}$,

$$\langle [X,Y],Z\rangle_0+\langle [X,Z],Y\rangle_0=0.$$

For $X,Y,Z\in\mathfrak{g}_1$, we have

$$\begin{aligned}\langle\nabla_XY,Z\rangle &= -\frac{1}{2}\langle[b(X),b(Y)],b(Z)\rangle_0-\frac{1}{2}\langle[b(Z),b(Y)],b(X)\rangle_0\\ &\quad -\frac{1}{2}\langle[b(Z),b(Y)],b(X)\rangle_0+\frac{1}{2}\langle[b(X),b(Y)],b(Z)\rangle_0=0.\end{aligned}$$

For $X,Y\in\mathfrak{g}_1$, $Z\in\mathfrak{g}_2$ we have

$$\langle\nabla_XY,Z\rangle=-\frac{1}{2}\langle[b(X),b(Y)],a(Z)\rangle_0+\frac{1}{2}\langle[b(X),b(Y)],a(Z)\rangle_0=0.$$

For $X,Z\in\mathfrak{g}_1$, $Y\in\mathfrak{g}_2$ we have

$$\begin{aligned}\langle\nabla_XY,Z\rangle &= \frac{1}{2}\langle[Z,X],Y\rangle+\frac{1}{2}\langle\phi^{-1}[b(X),a(Y)],Z\rangle\\ &= -\frac{1}{2}\langle[b(Z),b(X)],a(Y)\rangle_0+\frac{1}{2}\langle[b(X),a(Y)],b(Z)\rangle_0=0.\end{aligned}$$

For $X\in\mathfrak{g}_1$, $Y,Z\in\mathfrak{g}_2$ we have

$$\begin{aligned}\langle\nabla_XY,Z\rangle &= \frac{1}{2}\langle[Z,Y],X\rangle+\frac{1}{2}\langle\phi^{-1}[b(X),a(Y)],Z\rangle\\ &= \frac{1}{2}\langle[a(Z),a(Y)],b(X)\rangle_0+\frac{1}{2}\langle[b(X),a(Y)],a(Z)\rangle_0=0.\end{aligned}$$

For $X,Y\in\mathfrak{g}_2$ and $Z\in\mathfrak{g}_1$, we have

$$\begin{aligned}\langle\nabla_XY,Z\rangle &= -\langle\overline{\nabla}_XZ,Y\rangle=-\langle\phi^{-1}[a(X),b(Z)],Y\rangle\\ &= -\langle[a(X),b(Z)],a(Y)\rangle_0=\langle[a(X),a(Y)],b(Z)\rangle_0=\langle[X,Y],Z\rangle.\end{aligned}$$

For $X,Y,Z\in\mathfrak{g}_2$, we have

$$\begin{aligned}\langle\nabla_XY,Z\rangle &= \frac{1}{2}\langle[X,Y],Z\rangle+\frac{1}{2}\langle[Z,X],Y\rangle+\frac{1}{2}\langle[Z,Y],X\rangle+\frac{1}{2}\langle[X,Y],Z\rangle\\ &= \langle[X,Y],Z\rangle.\end{aligned}$$

Finally,

$$\nabla_XY=\begin{cases}0 \quad \text{if} \quad X\in\mathfrak{g}_1,\\ [X,Y] \quad \text{if} \quad X,Y\in\mathfrak{g}_2\\ \phi^{-1}[a(X),b(Y)] \quad \text{if} \quad X\in\mathfrak{g}_2, Y\in\mathfrak{g}_1.\end{cases}$$

We have $\nabla(T)=0$ as a consequence of the fact that $(G,\langle\ ,\ \rangle,\nabla)$ is isometric to a compact semi-simple Lie group with a bi-invariant metric. Moreover, from (6), we deduce that T defines on $\mathfrak{g}$ a Lie bracket for which the metric is bi-invariant. ■

Remark 3.3. Let $(G,\langle\ ,\ \rangle_0)$ be a Lie group endowed with a bi-invariant pseudo-Riemannian metric and $a:\mathfrak{g}\longrightarrow\mathfrak{g}$, $b:\mathfrak{g}\longrightarrow\mathfrak{g}$ as in Theorem 1.4. Then G endowed with the metric $\langle\ ,\ \rangle$ and the connection ∇ given in Theorem 1.4 is a pseudo-Riemannian Lie group with a FMSS-connection. Since there are nilpotent and solvable Lie groups which carry bi-invariant pseudo-Riemannian metrics we can use this remark to build nilmanifolds or solvmanifolds with a pseudo-Riemannian metric and flat metric connection with skew-symmetric torsion.

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