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Duflo's Conjecture for the Restriction of Tempered Representations of GL(n) to a Mirabolic Subgroup

Gang Liu and Jun Yu

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Abstract. We prove Duflo's conjecture in the setting of restriction of tempered representations of $\mathrm{GL}(n,\mathbb{R})$ (or $\mathrm{GL}(n,\mathbb{C})$) to a mirabolic subgroup. The proof is mainly based on Sahi's results concerning the branching laws in this setting and precise knowledge of the moment map that we obtain in this paper.

Mathematics Subject Classification: 22E46.

Key Words: Duflo's conjecture, Kirillov's conjecture, orbit method, tempered representations, moment map.

1. Introduction

By works of Kirillov, Kostant, Duflo and others, we know that geometric methods have played an important role in the development of representation theory and harmonic analysis. On the other hand, one of the central parts in harmonic analysis is branching problems. Notably, in a series of seminal papers (see for example [14]), Kobayashi initiated the study of discrete decomposability and admissibility for representations when restricted to non-compact subgroups. Then we may ask, despite the analytic nature of branching problems, can we interpret the branching laws in a geometric manner? Or, can branching laws be essentially determined by geometric data? The answer turns to be positive in some contexts. More precisely, in some contexts we can describe the branching laws in the framework of *orbit method* (or more generally, in the framework of *geometric quantization*). In this direction, the first success concerns compact groups in the thesis of Heckman ([13]). Later on, Guillemin-Sternberg formulated a *quantization commutes with reduction* conjecture (for a compact Lie group acting on a compact symplectic manifold), which was proved in special cases by themselves ([12]) and Vergne ([33], [34]), and established in full generality by Meinrenken ([21], [22]) and Tian-Zhang ([32]). Since then, this theory has been developed notably by Vergne ([35]), Ma-Zhang ([19]), Paradan-Vergne ([26]) and Paradan ([25]). Nevertheless, it should be pointed out that in all these works, the groups are always reductive (Actually, in all theses works, except that of Paradan, the groups are always compact). Encouraged by all these developments, Duflo formulated a general conjecture recently which aims at studying in a geometric manner the branching problem concerning a triple (G, H, π), where G is an almost algebraic group, H is an almost algebraic subgroup of G, and π is a *discrete series* of G.

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1.1. Duflo's conjecture

In this subsection, we state Duflo's conjecture. We start by outlining some essential ingredients in Duflo's orbit method. For more details and the general setting about Duflo's orbit method theory, we refer to Duflo's "CIME lectures" ([6]).

Let G be an almost algebraic real group with Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{g}^*$ the algebraic dual space of $\mathfrak{g}$. In the framework of Duflo's theory, two fundamental notions are *admissible* (in the sense of Duflo) and *well polarizable* (in the sense of Pukanszky) for G-coadjoint orbits in $\mathfrak{g}^*$. An admissible and well polarizable coadjoint orbit $\mathcal{O}$ is attached with at least one irreducible unitary representation of G. Moreover, two different such orbits are associated with non-equivalent representations. In general, the set of isomorphism classes of irreducible unitary G-representations associated to admissible and well polarizable G-coadjoint orbits is not the whole unitary dual $\hat{G}$. However, it is sufficient to describe the Plancherel formula of G (namely, the spectral decomposition of $L^2(G)$) by them ([7]). Especially, all *discrete series* representations of G (i.e., those appearing in the discrete spectrum of $L^2(G)$), and "almost" all *tempered* representations of G (i.e., those appearing in the support of the spectral decomposition of $L^2(G)$) are attached to admissible and *strongly regular* G-coadjoint orbits. Recall that an element $f \in \mathfrak{g}^*$ is called *strongly regular* if f is regular (i.e., the coadjoint orbit containing f is of maximal dimension) and its "reductive factor" $\mathfrak{s}(f) := \{X \in \mathfrak{g}(f) : \mathrm{ad}X \text{ is semisimple}\}$ is of maximal dimension among the reductive factors of all the regular elements in $\mathfrak{g}^*$ (note that f is regular implies that $\mathfrak{g}(f)$ is commutative). A coadjoint orbit $\mathcal{O}$ is called *strongly regular* if there exists an element $f \in \mathcal{O}$ (then every element in $\mathcal{O}$) which is strongly regular. It is known that each strongly regular coadjoint orbit is well polarizable. In particular, when G is reductive, all tempered representations with regular infinitesimal character are associated to admissible and strongly regular coadjoint orbits (and it is known that these representations are sufficient to describe the Plancherel formula of G).

Now let H be an almost algebraic subgroup of G with Lie algebra $\mathfrak{h}$. Let $\mathcal{O}$ be a G-coadjoint orbit in $\mathfrak{g}^*$. It is well known that equipped with the Kirillov-Kostant-Souriau symplectic form ω, $\mathcal{O}$ becomes a H-Hamiltonian space. The corresponding moment map is the natural projection $q : \mathcal{O} \to \mathfrak{h}^*$.

Let π be a discrete series of G, as stated above; it is attached to a strongly regular G-coadjoint orbit $\mathcal{O}_\pi$. Consider the restriction to H of π, $\pi|_H$. Then, Duflo's conjecture reads as follows:

(i) $\pi|_H$ is H-admissible (in the sense of Kobayashi, [14]) if and only if the moment map $q : \mathcal{O}_\pi \to \mathfrak{h}^*$ is *weakly proper*.

(ii) If $\pi|_H$ is H-admissible, then each irreducible H-representation σ which appears in $\pi|_H$ is attached to a *strongly regular* H-coadjoint orbit Ω (in the sense of Duflo) which is contained in $q(\mathcal{O}_\pi)$.

(iii) If $\pi|_H$ is H-admissible, then the multiplicity of each such σ can be expressed geometrically on the *reduced space* of Ω with respect to the moment map q.

Let us give some explanations for Duflo's conjecture. Firstly, the above notion of "H-admissible" due to Kobayashi means that $\pi|_H$ decomposes discretely and with finite multiplicities. Secondly, "weakly proper" in (i) means that the preimage (for q)

of each compact subset which is contained in $q(\mathcal{O}_\pi)\cap\Upsilon_{sr}$ is compact in $\mathcal{O}_\pi$. Here Υ_{sr} is the set of strongly regular elements in $\mathfrak{h}^*$. Moreover according to Duflo-Vargas's work ([9], [10]), each irreducible H-representation σ which appears in the spectrum of $\pi|_H$ is attached to a strongly regular H-coadjoint orbit (no matter whether $\pi|_H$ is H-admissible), but is not necessarily a discrete series. Furthermore, if $\pi|_H$ is H-admissible, then each H-irreducible representation appearing in $\pi|_H$ must be a discrete series ([9], [10]). So the statement in (ii) has a natural geometric meaning.

Concerning Duflo's conjecture, the following cases have been established:

(1) If G is compact, then Duflo's conjecture is a special case of the Spin^c version of *quantization commutes with reduction* principle (see [24]) .

(2) When G is reductive and H is a maximal compact subgroup, then Duflo's conjecture is a consequence of [23].

(3) More generally, if G and H are both reductive, then the assertions (ii) and (iii) of Duflo's conjecture are consequences of recent work of Paradan ([25]). Note that in this case, Dulfo-Vargas and Paradan ([9], [10], [25]) proved that $\pi|_H$ is H-admissible if and only if the moment map $q:\mathcal{O}_\pi\to\mathfrak{h}^*$ is *proper*. In order to prove the assertion (i) of Duflo's conjecture in this case, one needs to prove the equivalence between properness and weak properness of the moment map.

(4) For the case where G is a simple group of Hermitian type and H is a maximal solvable subgroup of G, Duflo's conjecture is established in [17].

(5) When $G=SU(2,1)$ and H is a minimal parabolic subgroup of G, Duflo's conjecture is established in [16].

(6) When $G=Spin(2n,1)$ and H is a minimal parabolic subgroup of G, Duflo's conjecture is established in [18].

Despite the works cited above, Duflo's conjecture is still not fully established. Nevertheless, it is natural to ask if Duflo's conjecture can be extended to other irreducible unitary representations which are not necessarily discrete series. For that purpose, tempered representations are reasonable candidates. Let us explain it in more detail.

Firstly, as we already mentioned above, the Plancherel formula of G can be described geometrically by admissible and strongly regular coadjoint orbits of G (especially almost all tempered representations are associated to admissible and strongly regular orbits). On the other hand, by recent work of Benoist-Kobayashi ([3]), if G is algebraic reductive and π is a tempered representation of G, then each irreducible sub-representation of $\pi|_H$ is tempered (where H is an algebraic subgroup of G). Thus in the case where π is a tempered representation with regular infinitesimal character of a reductive group G, and $\pi|_H$ is admissibly decomposable, Duflo's conjecture also makes sense. In this paper we prove this form of the generalization of Duflo's conjecture under the setting of Kirillov's conjecture.

1.2. Works on Kirillov's conjecture

For each $n\geq 1$, let $G_n(k)=\mathrm{GL}(n,k)$ ($k=\mathbb{R}$ or $\mathbb{C}$), and $P_n(k)$ be the mirabolic subgroup consisting of matrices whose last row is $(0,0,\dots,0,1)$. Kirillov's conjecture asserts that the restriction to $P_n(k)$ of any irreducible unitary representation of $G_n(k)$ is still irreducible. Bernstein first proved Kirillov's conjecture over non-archimedean local fields ([5]). In [29] Sahi proved Kirillov's conjecture when $k = \mathbb{C}$, and when $k = \mathbb{R}$ and π is a tempered representation. Then in [2] Baruch proved Kirillov's conjecture over archimedean local fields in general. Moreover, Sahi, et al determined $\pi|P_n(k)$ for any irreducible unitary representation π of $G_n(k)$. Let us recall some results of Sahi and his collaborators.

Note that we have $P_n(k) \cong k^{n-1} \rtimes G_{n-1}(k)$, where k^{n-1} is the unipotent radical of $P_n(k)$. It is clear that $P_n(k)$ has two orbits in $(k^{n-1})^*$: $\{0\}$ and $(k^{n-1})^* - \{0\}$. Let $\xi \in (k^{n-1})^* - \{0\}$ be defined by $\xi(x_1, ..., x_{n-1}) = x_{n-1}$. Then $\mathrm{Stab}_{G_{n-1}(k)}(\xi) \cong P_{n-1}(k)$. By Mackey's method ([20]) one deduces that each irreducible unitary representation ρ of $P_n(k)$ is obtained in one of the following two ways:

(i) by trivially extending an irreducible unitary representation π' of $G_{n-1}(k)$ to $P_n(k)$. In this case, we write $\rho = E\pi'$;

(ii) then, by extending an irreducible unitary representation ρ' of $P_{n-1}(k)$ to $k^{n-1} \rtimes P_{n-1}(k)$ by the character ξ and then inducing to $P_n(k)$. That is, $\rho = \mathrm{Ind}_{k^{n-1} \rtimes P_{n-1}(k)}^{P_n(k)}(\rho' \otimes \xi)$. In this case, we write $\rho = I\rho'$.

Then, E (or I) is a functor from the additive category of unitary representations of $G_{n-1}(k)$ (or $P_{n-1}(k)$) to the additive category of unitary representations of $P_n(k)$ and

$$\widehat{P_n(k)} = E(\widehat{G_{n-1}(k)}) \bigsqcup I(\widehat{P_{n-1}(k)}).$$

Here $\widehat{S}$ denotes the unitary dual of a Lie group S. Taking induction, one shows: each irreducible unitary representation ρ of $P_n(k)$ is of the form $\rho = I^{j-1}E\sigma$ for a unique integer j $(1 \leq j \leq n)$ and a unique $\sigma \in \widehat{G_{n-j}(k)}$. It is known that

(a) $P_n(k)$ admits a unique tempered representation $\tau \in \widehat{P_n(k)}$.

(b) $\tau = I^{n-1}E\mathbf{1}$, where $\mathbf{1}$ is the trivial representation of $G_0(k)$ (a trivial group).

(c) According to a result of Sahi ([29],Theorem 3.1), for each tempered representation π of $G_n(k)$, $\pi|_{P_n(k)} = \tau = I^{n-1}E\mathbf{1}$.

A unitary representation π of $G_n(k)$ is said to be *adducible* if $\pi|_{P_n(k)} = I^{j-1}E\sigma$ for some integer j $(1 \leq j \leq n)$ and some unitary representation σ of $G_{n-j}(k)$. The integer j and the representation σ are uniquely determined by π. The integer j is called the *depth* of π, denoted by $\mathrm{depth}(\pi)$. We write $\sigma = A\pi$ and call it the *adduced representation* of π. Kirillov's conjecture (now a theorem) implies that any irreducible unitary representation of $\mathrm{G}_n(k)$ is adducible.

Let π_i $(1 \leq i \leq s)$ be an irreducible unitary representation of $\mathrm{GL}_{n_i}(k)$, where $\sum_{1 \leq i \leq s} n_i = n$. Write $\pi_1 \times \cdots \times \pi_s$ for the representation of $\mathrm{GL}_n(k)$ obtained by unitary parabolic induction from the representation $\pi_1 \otimes \cdots \otimes \pi_s$ of a standard parabolic subgroup of $\mathrm{GL}_n(k)$ with Levi subgroup $\mathrm{GL}_{n_1}(k) \times \cdots \times \mathrm{GL}_{n_s}(k)$.

Theorem 1.1 ([29], Theorem 2.1). *Let π_i be an irreducible unitary representation of $G_{n_i}(k)$, and $n = \sum_{1 \leq i \leq s} n_i$. Then the depth of $\pi_1 \times \cdots \times \pi_s$ is equal to $\sum_{1 \leq i \leq s} \mathrm{depth}(\pi_i)$, and we have*

$$A(\pi_1 \times \cdots \times \pi_s) = A\pi_1 \times \cdots \times A\pi_s.$$

In [36] Vogan classified the unitary dual of $\mathrm{GL}_n(\mathbb{C})$ (and $\mathrm{GL}_n(\mathbb{R})$). Vogan proved that: any irreducible unitary representation π of $\mathrm{GL}_n(\mathbb{C})$ can be written in a unique way as $\pi = \pi_1 \times \cdots \times \pi_s$ with each π_i a unique character or a Stein complementary series. Any irreducible unitary representation π of $\mathrm{GL}_n(\mathbb{R})$ can be written in a unique way as $\pi = \pi_1 \times \cdots \times \pi_s$ with each π_i in the following list:

(i) unitary character;

(ii) Stein complementary series;

(iii) Speh representation;

(iv) Speh complementary representation.

Recall that discrete series of $\mathrm{GL}(2,\mathbb{R})$ (which are irreducible unitary representations π such that $\pi|_{\mathrm{SL}(2,\mathbb{R})}$ is a discrete series, which are also called relative discrete series by some authors) are examples of Speh representations. By Vogan's classification, it suffices to treat the restriction to $P_n(k)$ of each building block representation. Apparently, a unitary character χ of $\mathrm{G}_n(k)$ has depth 1 and $A\chi = \chi|_{G_{n-1}(k)}$. In [29] it is shown that discrete series of $\mathrm{GL}(2,\mathbb{R})$ has depth 2, so the restriction to $P_n(k)$ of all tempered representations of $G_n(k)$ are treated. Moreover, Stein complementary series are treated in [30] in general, especially these representations have depth 2 and the adduced representations are still Stein complementary series. Speh representations are treated in [31] in general, especially these representations have depth 2 and the adduced representations are still Speh representations. Speh complementary series are treated in [1] in general, in particular these representations have depth 4 and the adduced representations are still Speh complementary series.

1.3. Main theorem of this paper

In [6] Duflo reduced the classification of the unitary dual of a general real algebraic group to the classification of the unitary dual of reductive groups([6], [7]). Duflo's approach is based on Mackey's method of unitary induced representations, and is carried out in an inductive manner. For the mirabolic group $P_n(k)$, it is very simple, which is just Sahi's construction by two functors E and I as recalled in Subsection 1.2.

As in [7], any tempered representation of $G_n(k)$ with regular infinitesimal character is attached to a regular semisimple (which is equivalent to strongly regular for reductive groups) coadjoint orbit. For the group $P_n(k)$, as show in Proposition 3.7 there is only one strongly regular coadjoint orbit, denoted by Ω, which is an open and dense orbit. Thus, $P_n(k)$ has a unique tempered representation up to isomorphism, which is actually a discrete series. As in the previous subsection, we denote the unique tempered representation of $P_n(k)$ by τ. Then, τ is attached to Ω, and $\tau = I^{n-1}E\mathbf{1}$.

Let π be a tempered representation of G with regular infinitesimal character. Then π is attached to a regular semisimple coadjoint orbit $\mathcal{O}_\pi$ (in the sense of Duflo). Let $q : \mathcal{O}_\pi \to \mathfrak{p}_n(k)^*$ be the moment map of the $P_n(k)$-Hamiltonian space $\mathcal{O}_\pi$. The following theorem is our main result in this paper (see Theorem 4.4 and Theorem 5.4), which could be regarded as a generalization of Duflo's conjecture to tempered representations in this setting.

Theorem 1.2. *There are only finitely many $P_n(k)$ orbits in $q(\mathcal{O}_\pi)$, and the unique open coadjoint orbit Ω is contained in $q(\mathcal{O}_\pi)$. Moreover,*

(1) $\pi|_{P_n(k)}$ *is isomorphic to* τ.

(2) *the moment map* $q : \mathcal{O}_\pi \to \mathfrak{p}_n(k)^*$ *is* weakly proper, *but* not proper.

(3) *the* reduced space *of* Ω *with respect to the moment map* q *is a singleton.*

Especially, in the spirit of the orbit method, the geometric picture described in the above theorem corresponds well to the classic results (a), (b) and (c) in the previous subsection.

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2. Coadjoint action, the dual map and the moment map

Let $n \geq 1$, and $k = \mathbb{R}$ or $\mathbb{C}$. Write $G_n(k) = \mathrm{GL}(n, k)$,

$$P_n(k) = \left\{ \begin{pmatrix} A & \alpha \\ 0_{1\times(n-1)} & 1 \end{pmatrix} : A \in \mathrm{GL}(n-1, k), \alpha \in k^{n-1} \right\}.$$

In the literature $P_n(k)$ is called "mirabolic subgroup". Write $\mathfrak{g}_n(k) = \mathfrak{gl}(n, k)$, which is the Lie algebra of $G_n(k)$. Write

$$\mathfrak{p}_n(k) = \left\{ \begin{pmatrix} A & \alpha \\ 0_{1\times(n-1)} & 0 \end{pmatrix} : A \in \mathfrak{gl}(n-1, k), \alpha \in k^{n-1} \right\},$$

which is the Lie algebra of $P_n(k)$. Write $\mathfrak{g}_n(k)^*$ (resp. $\mathfrak{p}_n(k)^*$) for the dual space of $\mathfrak{g}_n(k)$ (resp. of $\mathfrak{p}_n(k)$). Then, $G_n(k)$ acts on $\mathfrak{g}_n(k)^*$ through

$$(g \cdot f)\xi = f(g^{-1} \cdot \xi), \ \forall g \in G_n(k), \forall f \in \mathfrak{g}_n(k)^*, \forall \xi \in \mathfrak{g}_n(k).$$

This is called the *coadjoint action*, and a $G_n(k)$-orbit in $\mathfrak{g}_n(k)^*$ is called a coadjoint orbit. Similarly, one defines the coadjoint action of $P_n(k)$ on $\mathfrak{p}_n(k)^*$ and corresponding coadjoint orbits. Write

$$(\xi, \eta) = \mathrm{tr}(\xi\eta), \ \forall \xi, \eta \in \mathfrak{g}_n(k).$$

This gives a $G_n(k)$ conjugation invariant nondegenerate bilinear form on $\mathfrak{g}_n(k)$. It also gives a $G_n(k)$ equivariant isomorphism

$$\mathrm{pr} : \mathfrak{g}_n(k) \to \mathfrak{g}_n(k)^*, \quad \xi \mapsto f \quad \text{defined by} \quad f(\eta) = \mathrm{tr}(\xi\eta), \ \forall \eta \in \mathfrak{g}_n(k).$$

Through pr, the above bilinear form on $\mathfrak{g}_n(k)$ induces a nondegenerate bilinear form on $\mathfrak{g}_n(k)^*$ defined by

$$(f, f') = (\mathrm{pr}^{-1}(f), \mathrm{pr}^{-1}(f')), \forall f, f' \in \mathfrak{g}_n(k)^*.$$

Write $\qquad \overline{\mathfrak{p}}_n(k) = \left\{ \begin{pmatrix} A & 0_{(n-1)\times 1} \\ \alpha^t & 0 \end{pmatrix} : A \in \mathfrak{gl}(n-1,k), \alpha \in k^{n-1} \right\}.$

Define $\mathrm{pr}' : \mathfrak{g}_n(k) \to \mathfrak{p}_n(k)^*$ by $(\mathrm{pr}'(\xi))(\eta) = \mathrm{tr}(\xi\eta),\ \forall \eta \in \mathfrak{p}_n(k)$.
It is easy to show that

$$\ker(\mathrm{pr}') = \left\{ \begin{pmatrix} 0_{(n-1)\times(n-1)} & \alpha \\ 0_{1\times(n-1)} & t \end{pmatrix} : \alpha \in k^{n-1}, t \in k. \right\}$$

It is clear that we have $\mathfrak{g}_n(k) = \ker(\mathrm{pr}') \oplus \overline{\mathfrak{p}}_n(k)$ as a linear space. In consequence $\mathrm{pr}'|_{\overline{\mathfrak{p}}_n(k)} : \overline{\mathfrak{p}}_n(k) \to \mathfrak{p}_n(k)^*$ is a linear isomorphism. In this way, any element in $f \in \mathfrak{p}_n(k)^*$ could be represented by

$$f = \mathrm{pr}'(\xi) = \mathrm{pr}(\xi)|_{\mathfrak{p}_n(k)}$$

for a unique $\xi \in \overline{\mathfrak{p}}_n(k)$. The *restriction map* $q : \mathfrak{g}_n(k)^* \to \mathfrak{p}_n(k)^*$ is defined by

$$f \mapsto f' = f|_{\mathfrak{p}_n(k)}.$$

For any $\xi \in \mathfrak{g}_n(k)$, we have $q(\mathrm{pr}(\xi)) = \mathrm{pr}'(\xi)$.

3. Classification of P-coadjoint orbits

Recall that for any Lie group G, the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ is determined by

$$(\mathrm{ad}(X)f)(Y) = -f([X,Y]),\ \forall X, Y \in \mathfrak{g}, \forall f \in \mathfrak{g}^*.$$

Write $\qquad N_n(k) = \left\{ \begin{pmatrix} I_{n-1} & \alpha \\ 0_{1\times(n-1)} & 1 \end{pmatrix} : \alpha \in k^{n-1} \right\}$

and $\qquad \mathfrak{n}_n(k) = \left\{ \begin{pmatrix} 0_{n-1} & \alpha \\ 0_{1\times(n-1)} & 0 \end{pmatrix} : \alpha \in k^{n-1} \right\}.$

Then, $N_n(k)$ is the unipotent radical of $P_n(k)$, and $\mathfrak{n}_n(k)$ is its Lie algebra (it is the nilradical of $\mathfrak{p}_n(k)$). Write

$$L_n(k) = \left\{ \begin{pmatrix} A & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 1 \end{pmatrix} : A \in G_{n-1}(k) \right\}$$

and $\qquad \mathfrak{l}_n(k) = \left\{ \begin{pmatrix} A & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 0 \end{pmatrix} : A \in \mathfrak{g}_{n-1}(k) \right\}.$

Then, $L_n(k)$ (resp. $\mathfrak{l}_n(k)$) is a Levi subgroup (resp. Levi subalgebra) of $P_n(k)$ (resp. of $\mathfrak{l}_n(k)$). Note that there is an exact sequence of $P_n(k)$ modules,

$$0 \to \mathfrak{n}_n(k) \to \mathfrak{p}_n(k) \to \mathfrak{l}_n(k) \to 0.$$

Dually, there is an exact sequence of $P_n(k)$ modules,

$$0 \to \mathfrak{l}_n(k)^* \to \mathfrak{p}_n(k)^* \to \mathfrak{n}_n(k)^* \to 0.$$

Lemma 3.1. *For any $n \geq 1$, there is an identification*

$$\mathfrak{p}_n(k)^*/P_n(k) = \mathfrak{l}_n(k)^*/L_n(k) \bigsqcup \mathfrak{p}_{n-1}(k)^*/P_{n-1}(k).$$

Proof. From the exact sequence $0 \to \mathfrak{l}_n(k)^* \to \mathfrak{p}_n(k)^* \to \mathfrak{n}_n(k)^* \to 0$, we get

$$\mathfrak{p}_n(k)^*/P_n(k) = \mathfrak{l}_n(k)^*/P_n(k) \bigsqcup (\mathfrak{p}_n(k)^* - \mathfrak{l}_n(k)^*)/P_n(k).$$

As N_n acts trivially on $\mathfrak{l}_n(k)^*$, we get

$$\mathfrak{l}_n(k)^*/P_n(k) = \mathfrak{l}_n(k)^*/L_n(k).$$

Choose an element $\overline{h} \in \mathfrak{p}_n(k)^*$ such that $0 \neq h = \overline{h}|_{\mathfrak{n}_n(k)^*} \in \mathfrak{n}_n(k)^*$ and $\overline{h}|_{\mathfrak{l}_n(k)} = 0$. Due to the fact that $L_n(k)$ acts transitively on $\mathfrak{n}_n(k)^* - \{0\}$, any $P_n(k)$ orbit in $\mathfrak{p}_n(k)^* - \mathfrak{l}_n(k)^*$ intersects with $\mathfrak{l}_n(k)^* + \overline{h}$. From this we get

$$(\mathfrak{p}_n(k)^* - \mathfrak{l}_n(k)^*)/P_n(k) \cong (\mathfrak{l}_n(k)^* + \overline{h})/P_n(k)^h,$$

where $P_n(k)^h = \mathrm{Stab}_{P_n(k)}(h)$. Further, write $L_n(k)^h = \mathrm{Stab}_{L_n(k)}(h)$, and write $\mathfrak{p}_n(k)^h$ (resp. $\mathfrak{l}_n(k)^h$) for the Lie algebra of $P_n(k)^h$ (resp. of $L_n(k)^h$). As $N_n(k)$ is abelian, we have $N_n(k) \subset P_n(k)^h$. Thus,

$$P_n(k)^h = N_n(k) \rtimes L_n(k)^h.$$

Since $N_n(k)$ acts trivially on $\mathfrak{l}_n(k)^*$, its action on $\mathfrak{l}_n(k)^* + \overline{h}$ is through translations.

Actually, $$\exp(X) \cdot (g + \overline{h}) = (g + \mathrm{ad}(X)\overline{h}) + \overline{h}$$

for any $X \in \mathfrak{n}_n(k)$ and any $g \in \mathfrak{l}_n(k)^*$. Now we claim that

$$\mathfrak{l}_n(k)^*/\,\mathrm{ad}(\mathfrak{n}_n(k))\overline{h} = (\mathfrak{l}_n(k)^h)^*. \tag{1}$$

One direct consequence of this equality is that

$$(\mathfrak{l}_n(k)^* + h)/P_n(k)^h \cong \mathfrak{l}_n(k)^h/L_n(k)^h.$$

It is easy to check directly that $L_n(k)^h \cong P_{n-1}(k)$. Then it follows that

$$(\mathfrak{p}_n(k)^* - \mathfrak{l}_n(k)^*)/P_n(k) \cong \mathfrak{p}_{n-1}(k)^*/P_{n-1}(k).$$

Therefore, $$\mathfrak{p}_n(k)^*/P_n(k) = \mathfrak{l}_n(k)^*/L_n(k) \bigsqcup \mathfrak{p}_{n-1}(k)^*/P_{n-1}(k).$$

Now we show the claim for $\mathfrak{l}_n(k)^*/\,\mathrm{ad}(\mathfrak{n}_n(k))\overline{h} = (\mathfrak{l}_n(k)^h)^*$.
In fact, since the pairing between $\mathfrak{l}_n(k)$ and $\mathfrak{l}_n(k)^*$ is nondegenerate. It is equivalent to show

$$\{\xi \in \mathfrak{l}_n(k) : (\mathrm{ad}(\eta)\overline{h})(\xi) = 0, \forall \eta \in \mathfrak{n}_n(k)\} = \mathfrak{l}_n(k)^h. \tag{2}$$

This follows from

$$(\mathrm{ad}(\eta)\overline{h})(\xi) = -\overline{h}([\eta, \xi]) = \overline{h}([\xi, \eta]) = -(\mathrm{ad}(\xi)\overline{h})(\eta).$$ ■

Lemma 3.2. *Assume* $\overline{h}|_{\mathfrak{l}_n(k)^*} = 0$ *and* $0 \neq h = \overline{h}|_{\mathfrak{n}_n(k)^*} \in \mathfrak{n}_n(k)^*$. *Then,*

(1) $\mathrm{ad}(\mathfrak{l}_n(k)^h)\overline{h} = 0$.

(2) *the map* $\mathfrak{n}_n(k) \to \mathfrak{l}_n(k)^*, \eta \mapsto (\mathrm{ad}\,\eta)\overline{h}$ *is injective.*

Proof. For assertion (1), let $\xi \in \mathfrak{l}_n(k)^h$. For any $\eta \in \mathfrak{n}_n(k)$, $(\mathrm{ad}(\xi)\overline{h})(\eta) = 0$ since $\mathrm{ad}(\xi)\overline{h} \in \mathfrak{l}_n(k)^*$. For any $\eta \in \mathfrak{l}_n(k)$,

$$(\mathrm{ad}(\xi)\overline{h})(\eta) = -\overline{h}([\xi, \eta]) = 0$$

since $\overline{h}|_{\mathfrak{l}_n(k)^*} = 0$. Thus, $\mathrm{ad}(\xi)\overline{h} = 0$. Therefore, $\mathrm{ad}(\mathfrak{l}_n(k)^h)\overline{h} = 0$.
For assertion (2), we may assume that $\overline{h} = \mathrm{pr}'(\xi)$, where $\xi' = \begin{pmatrix} 0_{n-1} & 0_{(n-1)\times 1} \\ \beta'^t & 0 \end{pmatrix}$, $\beta'^t = (0, \dots, 0, 1)$. Then, a direct calculation of $\mathrm{ad}(\eta)\overline{h} = \mathrm{pr}'([\eta, \xi'])$ $(\eta \in \mathfrak{n}_n(k))$ shows the assertion. ■

Now assume $\overline{h}|_{\mathfrak{l}_n(k)^*} = 0$ and $0 \neq h = \overline{h}|_{\mathfrak{n}_n(k)} \in \mathfrak{n}_n(k)^*$. Choose a complement of $(\mathrm{ad}(\mathfrak{n}_n(k))\overline{h}$ in $\mathfrak{l}_n(k)^*$, denoted by V_h.

Lemma 3.3. *Assume $\overline{h}|_{\mathfrak{l}_n(k)^*} = 0$ and $0 \neq h = \overline{h}|_{\mathfrak{n}_n(k)^*} \in \mathfrak{n}_n(k)^*$. Then, we have the following assertions:*

(1) *each $P_n(k)$ orbit intersecting with $\mathfrak{l}_n(k)^* + \overline{h}$ has a representative of the form $f = g + \overline{h}$ where $g \in V_h$.*

(2) *two elements $g_1 + \overline{h}$ and $g_2 + \overline{h}$ ($g_1, g_2 \in V_h$) are in one same $P_n(k)$ orbit if and only if $[g_1]$ and $[g_2]$ are in one same $L_n(k)^h$ orbit, where*

$$[g_i] = g_i + \mathrm{ad}(\mathfrak{n}_n(k))\overline{h} \in \mathfrak{l}_n(k)^* / \mathrm{ad}(\mathfrak{n}_n(k))\overline{h} = (\mathfrak{l}_n(k)^{\overline{h}})^*$$

($i = 1, 2$) are considered as elements in $(\mathfrak{l}_n(k)^h)^*$.

(3) *assume $g \in V_h$. Then, the map*

$$\mathrm{Stab}_{P_n(k)}(g + \overline{h}) \to L_n(k),\ nl \mapsto l$$

($n \in N_n(k)$, $l \in L_n(k)$) gives an isomorphism

$$\mathrm{Stab}_{P_n(k)}(g + \overline{h}) \cong \mathrm{Stab}_{L_n(k)^h}([g]).$$

Proof. The assertion (1) follows from Equation (1).
For assertion (2), we show the necessarity first. Assume $g_2 + \overline{h} = x \cdot (g_1 + \overline{h})$ for some $x \in P_n(k)$. Then, $x \in P_n(k)^h = N_n(k) \rtimes L_n(k)^h$. Write $x = nl = \exp(\xi)l$ for some $\xi \in \mathfrak{n}_n(k)$ and $l \in L_n(k)^h$. Then,

$$\exp(-\xi) \cdot (g_2 + \overline{h}) = l \cdot (g_1 + \overline{h}).$$

By Lemma 3.2, we have $l \cdot \overline{h} = 0$. As $\mathfrak{n}_n(k)$ is an abelian ideal of $\mathfrak{p}_n(k)$, we have

$$\exp(-\xi) \cdot (g_2 + \overline{h}) = (g_2 - \mathrm{ad}(\xi)\overline{h}) + \overline{h}.$$

Thus, $g_2 - \mathrm{ad}(\xi)\overline{h} = l \cdot g_1$. This just means, $[g_1]$ and $[g_2]$ are in one $L_n(k)^h$ orbit. The sufficiency could be shown with similar facts used in showing the necessarity.
For assertion (3), write $n = \exp(\xi)$ $(\xi \in \mathfrak{n}_n(k))$. By the above proof for (2), we see that $nl \in \mathrm{Stab}_{P_n(k)}(g + \overline{h})$ if and only if $l \in L_n(k)^h$ and

$$g - \mathrm{ad}(\xi)\overline{h} = l \cdot g.$$

The last is just the condition for $l \in \mathrm{Stab}_{L_n(k)^h}([g])$. Thus, the map

$$\mathrm{Stab}_{P_n(k)}(g + \overline{h}) \to L_n(k),\ nl \mapsto l$$

gives a surjection $\mathrm{Stab}_{P_n(k)}(g + \overline{h}) \to \mathrm{Stab}_{L_n(k)^h}([g])$.

On the other hand, suppose $l = 1$. Then, $\mathrm{ad}(\xi)\overline{h} = 0$. By Lemma 3.2(2), this implies that $\xi = 0$. Thus, the above surjection $\mathrm{Stab}_{P_n(k)}(g+\overline{h}) \to \mathrm{Stab}_{L_n(k)^h}([g])$ is an isomorphism. ■

We have some remarks regarding Lemma 3.3.

(1) $(\mathrm{ad}(\mathfrak{n}_n(k))\overline{h}$ is stable under the action of $L_n(k)^h$, but $L_n(k)^h$ is not a reductive subgroup. Actually one can verify that in general, it is impossible to take V_h stable under $L_n(k)^h$.

(2) we have $\mathrm{Stab}_{L_n(k)^h}(g) \subset \mathrm{Stab}_{P_n(k)}(g+\overline{h})$ and $\mathrm{Stab}_{L_n(k)^h}(g) \subset \mathrm{Stab}_{L_n(k)^h}([g])$, but neither is an equality in general.

The following lemma is easy to show.

Lemma 3.4. *If $h = 0$, then $f \in \mathfrak{l}_n(k)^* \subset \mathfrak{p}_n(k)^*$. In this case,*

$$\mathrm{Stab}_{P_n(k)}(f) = N_n(k) \rtimes \mathrm{Stab}_{L_n(k)}(f).$$

Theorem 3.5. *A $P_n(k)$ coadjoint orbit $P_n(k)\cdot f$ is characterized by an integer j $(0 \le j \le n-1)$ and an $L_{n-j}(k)$ coadjoint orbit $L_{n-j}(k)\cdot g$. Moreover, we have $\mathrm{Stab}_{P_n(k)} f \cong N_{n-j}(k) \rtimes \mathrm{Stab}_{L_{n-j}(k)}(g)$.*

Proof. Applying Lemma 3.1 inductively, we get

$$\mathfrak{p}_n(k)^*/P_n(k) = (\bigsqcup_{0\le j\le n-2} \mathfrak{l}_{n-j}(k)^*/L_{n-j}(k)) \bigsqcup \mathfrak{p}_1(k)^*/P_1(k).$$

As both $P_1(k)$ and $L_1(k)$ are the trivial group, we get

$$\mathfrak{p}_1(k)^*/P_1(k) = \mathfrak{l}_1(k)^*/L_1(k) = \text{singleton}.$$

Thus,
$$\mathfrak{p}_n(k)^*/P_n(k) = \bigsqcup_{0\le j\le n-1} \mathfrak{l}_{n-j}(k)^*/L_{n-j}(k). \tag{3}$$

For any j, $L_{n-j}(k) \cong \mathrm{G}_{n-1-j}(k)$. Thus, we get the first statement of the theorem. If a $P_n(k)$ coadjoint orbit $P_n(k)\cdot f$ is characterized by an integer j $(0 \le j \le n-1)$ and an $L_{n-j}(k)$ coadjoint orbit $L_{n-j}\cdot g$, then

$$\mathrm{Stab}_{P_n(k)} f \cong N_{n-j}(k) \rtimes \mathrm{Stab}_{L_{n-j}(k)}(g)$$

by Lemma 3.3(3) and Lemma 3.4. ■

With Theorem 3.5, we not only classified P coadjoint orbits, but also calculated their *stabilizers*. Note that $N_{n-j}(k) \cong k^{n-1-j}$.

Definition 3.6. In Theorem 3.5, we call $j+1$ the *depth* of a $P_n(k)$-coadjoint orbit $P_n(k)\cdot f$. ■

By Theorem 3.5 we have $\mathrm{Stab}_{P_n(k)}(f) \cong N_{n-j}(k) \rtimes \mathrm{Stab}_{L_{n-j}(k)}(g)$.

We call f a *semisimple element* (and call $P_n(k)\cdot f$ a *semisimple orbit*) if g is a semisimple element in $\mathfrak{l}_{n-j}(k)^*$. The latter means that $\mathrm{Stab}_{L_{n-j}(k)}(g)$ is an a reductive (algebraic) subgroup of $L_{n-j}(k)\cong \mathrm{GL}(n-1-j,k)$. In this case we call the $(n-1-j)$ eigenvalues of $\mathrm{pr}^{-1}(g)\in\mathfrak{gl}(n-1-j,k)$ the eigenvalues of f.

When $j=n-1$, we have $\mathrm{Stab}_{P_n(k)}(f)=\mathrm{Stab}_{L_{n-j}(k)}(g)=\{1\}$.

This gives an open orbit in $\mathfrak{p}_n(k)^*$. When $0\le j\le n-2$, we have

$$\dim \mathrm{Stab}_{P_n(k)}(f)=(n-j-1)+\dim \mathrm{Stab}_{L_{n-j}(k)}(g)\ge 2(n-1-j).$$

Thus, $P_n(k)\cdot f$ is an orbit of codimension$\ge 2(n-1-j)\ge 2$. Moreover, by Lemma 3.1 and the proof of Theorem 3.5, we see that the complement of the above open orbit in $\mathfrak{p}_n(k)^*$ is a closed subset of codimension one. This shows the following.

Proposition 3.7. *There is a unique open $P_n(k)$-coadjoint orbit in $\mathfrak{p}_n(k)^*$, which we denote by Ω. It is dense and its complement is a codimension one closed subset. For any $f\in\Omega$, we have $\mathrm{Stab}_{P_n(k)}(f)=1$.*
In particular , Ω is the only strongly regular coadjoint orbit of $P_n(k)$ in $\mathfrak{p}_n(k)^*$. Consequently, $P_n(k)$ has one and only one discrete series representation, which we denote by τ. Moreover τ is attached to Ω, and $L^2(P_n(k))\cong\tau\otimes\tau$.

We give some more precise information about the unique open $P_n(k)$-orbit in $\mathfrak{p}_n(k)^*$ below.

Example 3.8. Let

$$\xi_n=\begin{pmatrix} 0 & 0 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 1 & 0\end{pmatrix},$$

and $f_n=\mathrm{pr}'(\xi_n)\in\mathfrak{p}_n(k)^*$. Write h for the projection of f_n to $\mathfrak{n}_n(k)^*$. By calculation one sees that $L_n(k)^h=P_{n-1}(k)$, and the element

$$f_n+\mathrm{ad}(\mathfrak{n}_n(k))\overline{h}\in(\mathfrak{l}_n(k)^h)^*$$

is equal to $\mathrm{pr}'(\xi_{n-1})\in\mathfrak{p}_{n-1}(k)^*=(\mathfrak{l}_n(k)^h)^*$. Taking induction, one sees that f_n represents the unique open $P_n(k)$ orbit in $\mathfrak{p}_n(k)^*$.

Lemma 3.9. *Let $k=\mathbb{C}$. Let $a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}\in\mathbb{C}$ be such that $a_i\ne a_j$ ($\forall i,j$, $1\le i<j\le n-1$) and $b_j\ne 0$ ($\forall j$, $1\le j\le n-1$). Set*

$$\xi_n'=\begin{pmatrix} a_1 & 0 & \dots & 0 & 0\\ 0 & a_2 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & a_{n-1} & 0\\ b_1 & b_2 & \dots & b_{n-1} & 0\end{pmatrix}$$

and $f_n'=\mathrm{pr}'(\xi_n)\in\mathfrak{p}_n(k)^*$. Then f_n' represents the unique open $P_n(k)$ orbit in $\mathfrak{p}_n(k)^*$.

Proof. Suppose $g=\begin{pmatrix} B & \beta\\ 0_{1\times(n-1)} & 1\end{pmatrix}\in\mathrm{Stab}_{P_n}(f_n')$, where $B\in\mathrm{GL}(n-1,\mathbb{C})$ and $\beta\in\mathbb{C}^{n-1}$. Write $A=\mathrm{diag}\{a_1,\dots,a_{n-1}\}$ and $\alpha=(b_1,\dots,b_{n-1})^t$.

Then, $$\xi_n' = \begin{pmatrix} A & 0_{(n-1)\times 1} \\ \alpha^t & 0 \end{pmatrix}.$$

As $g \cdot f_n' = g\,\mathrm{pr}'(\xi_n') = \mathrm{pr}'(g\xi_n' g^{-1})$, we see that

$$g \in \mathrm{Stab}_{P_n}(f_n) \Leftrightarrow g\xi_n' g^{-1} - \xi_n' \in \mathfrak{n}_n',$$

where
$$\begin{aligned}\mathfrak{n}_n' &= \{\eta \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr}(\eta\theta) = 0, \forall \theta \in \mathfrak{p}_n\} \\ &= \Big\{ \begin{pmatrix} 0_{n-1} & \alpha \\ 0_{1\times(n-1)} & t \end{pmatrix} : \alpha \in \mathbb{C}^{n-1}, t \in \mathbb{C} \Big\}.\end{aligned}$$

By calculation

$$g\xi_n' g^{-1} = \begin{pmatrix} BAB^{-1} + \beta\alpha^t B^{-1} & -BAB^{-1}\beta - \beta\alpha^t B^{-1}\beta \\ \alpha^t B^{-1} & -\alpha^t B^{-1}\beta \end{pmatrix}.$$

Thus, $$g \in \mathrm{Stab}_{P_n}(f_n) \Leftrightarrow BAB^{-1} + \beta\alpha^t B^{-1} = A \text{ and } \alpha^t B^{-1} = \alpha^t.$$

This is equivalent to: $AB - BA = \beta\alpha^t$ and $\alpha^t B = \alpha^t$.

Write $B = (x_{i,j})_{(n-1)\times(n-1)}$ and $\beta^t = (y_j)_{1\le j\le(n-1)}$.

From $AB - BA = \beta\alpha^t$, we get

$$(a_i - a_j)x_{i,j} = b_j y_i, \forall i, j, 1 \le i, j \le n-1.$$

Let $i = j$, from $b_j \neq 0$ we get $y_j = 0$. Thus, $\beta = 0$. Let $i \neq j$, from $y_i = 0$ we get $x_{i,j} = 0$. Thus, B is a diagonal matrix. From $\alpha^t B = \alpha^t$ and all entries of α are not equal to 0, we get $B = I_{n-1}$. Hence, $\mathrm{Stab}_{P_n}(f_n') = 1$. Therefore, f_n' represents the unique open $P_n(k)$ orbit in $\mathfrak{p}_n(k)^*$. ■

4. Moment map in the $\mathrm{GL}(n,\mathbb{C})$ case

Let $G = \mathrm{GL}(n,\mathbb{C})$ and

$$P = P_n(\mathbb{C}) = \Big\{ \begin{pmatrix} A & \alpha \\ 0_{1\times(n-1)} & 1 \end{pmatrix} : A \in \mathrm{GL}(n-1,\mathbb{C}), \alpha \in \mathbb{C}^{n-1} \Big\}.$$

Let T be the maximal torus of G consisting of diagonal matrices in G. Let $\vec{a} = (a_1, ..., a_n) \in \mathbb{C}^n$ with $a_i \neq a_j$ $(1 \le i < j \le n)$. Write

$$\xi = \xi_{\vec{a}} = \mathrm{diag}\{a_1, ..., a_n\} \in \mathfrak{t} \subset \mathfrak{g} \quad \text{and} \quad f = f_{\vec{a}} = \mathrm{pr}(\xi) \in \mathfrak{g}^*.$$

Then, f is a regular semisimple element in $\mathfrak{g}^*$, and any regular semisimple orbit $\mathcal{O} \subset \mathfrak{g}^*$ is of the form $\mathcal{O} = \mathcal{O}_f = G \cdot f$ with $f = f_{\vec{a}}$ as above.
From now on, we fix $\vec{a}$, ξ and f in this section.

Lemma 4.1. *There are $2^n - 1$ P orbits in $\mathcal{O}_f$. One of them is Zariski open and dense, and the union of the rest is a codimension one Zariski closed subset.*

Proof. With the assumption, $\mathrm{Stab}_G(f) = T$. Thus, $P\backslash\mathcal{O}_f = P\backslash G/T$.
It is clear that the map $PgT \mapsto Tg^{-1}P$ gives a bijection $P\backslash G/T \cong T\backslash G/P$.
Consider the transitive G-action on $\mathbb{C}^n - \{0\}$:

$$g \cdot (x_1, ..., x_n)^t = (g^{-1})^t (x_1, \ldots, x_n)^t, \quad \forall g \in G, \ \forall (x_1, \ldots, x_n) \in \mathbb{C}^n - \{0\}.$$

Let $v_0 = (0, ..., 0, 1)^t$. Then, $\mathrm{Stab}_G(v_0) = P$.

Thus, $G/P = \mathbb{C}^n - \{0\}$ and $T\backslash G/P = T\backslash(\mathbb{C}^n - \{0\})$.

For any $I = \{i_1, ..., i_k\}$ where $1 \le i_1 < i_2 < ... < i_k \le n$ and $1 \le k \le n$, let $v_I = (x_1, \ldots, x_n)^t \in \mathbb{C}^n - \{0\}$ be defined by $x_i = 1$ if $i \in \{i_1, \ldots, i_k\}$, and $x_i = 0$ if $i \notin \{i_1, \ldots, i_k\}$. Note that T acts on $\mathbb{C}^n - \{0\}$ through

$$\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} \cdot (x_1, ..., x_n)^t = (\lambda_1^{-1} x_1, ..., \lambda_n^{-1} x_n)^t.$$

It is easy to see that $\{v_I : \emptyset \neq I \subset \{1, ..., n\}\}$ represent all different T orbits in $\mathbb{C}^n - \{0\}$. This shows the lemma. ∎

Write $I_0 = \{1, 2, ..., n\}$. For any $\emptyset \neq I \subset I_0$, introduce a matrix $g_I \in G$. In the case of $n \in I$, define g_I by

$$g_I = \begin{pmatrix} I_{n-1} & 0_{(n-1)\times 1} \\ \beta^t & 1 \end{pmatrix},$$

where $\beta^t = (x_1, \ldots, x_{n-1})$ with $x_i = 1$ if $i \in I$, and $x_i = 1$ if $i \notin I$. In the case of $n \notin I$, let $k = \max\{1 \le i \le n-1 : i \in I\}$, and define g_I by

$$g_I = \begin{pmatrix} I_{k-1} & 0_{(k-1)\times 1} & 0_{(k-1)\times(n-1-k)} & 0_{(k-1)\times 1} \\ 0_{1\times(k-1)} & 0 & 0_{1\times(n-1-k)} & 1 \\ 0_{(n-1-k)\times(k-1)} & 0_{(n-1-k)\times 1} & I_{n-1-k} & 0_{(n-1-k)\times 1} \\ \beta'^t & 1 & 0_{1\times(n-1-k)} & 0 \end{pmatrix},$$

where $\beta'^t = (x_1, \ldots, x_{k-1})$ with

$$x_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

Proposition 4.2. *$\{g_I \cdot f : \emptyset \neq I \subset I_0\}$ represent all different P orbits in $\mathcal{O}_f$. Among them, $Pg_{I_0} \cdot f$ is Zariski open and dense.*

Proof. One can show that $g_I^{-1} \cdot v_0 = v_I$ for any I. By the proof of Lemma 4.1, $\{g_I \cdot f : \emptyset \neq I \subset I_0\}$ represent all different P orbits in $\mathcal{O}_f$.

It is clear that $Tg_{I_0}^{-1} \cdot v_0$ is a Zarisk open and dense subset of $\mathbb{C}^n - \{0\}$. Thus, $Pg_{I_0} \cdot f$ is Zarisk open and dense in $\mathfrak{p}_n(k)^*$. ∎

Lemma 4.3. *For any $\emptyset \neq I \subset I_0$, the element $q(g_I \cdot f)$ is semisimple, its depth is equal to $\#I$ and its eigenvalues are $\{a_i : i \in I_0 - I\}$.*

Proof. We have

$$q(g_I \cdot f) = q(g_I \cdot \mathrm{pr}(\xi)) = q(\mathrm{pr}(g_I \cdot \xi)) = \mathrm{pr}'(g_I \cdot \xi).$$

In the case of $n \in I$, by calculation we have $\mathrm{pr}'(g_I \cdot \xi) = \mathrm{pr}'(\xi_I)$, where

$$\xi_I = \begin{pmatrix} \mathrm{diag}\{a_1, \ldots, a_{n-1}\} & 0_{(n-1)\times 1} \\ \beta'^t & 0 \end{pmatrix})$$

with $\beta'^t = (y_1, \ldots, y_{n-1})$, where

$$y_i = \begin{cases} a_i - a_n & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

Separating $I_0 = \{1, 2, \ldots, n\}$ into the disjoint union of two subsets: I and $I_0 - I$, we see that ξ_I is a block diagonal matrix. The one with rows and columns indexed by I is of the form in Lemma 3.9; the one with rows and columns indexed by $I_0 - I$ is a diagonal matrix with eigenvalues $\{a_i : i \in I_0 - I\}$.

We know that there is a unique open P orbit in $\mathfrak{p}^*$, and any matrix of the form in Example 3.8 or Lemma 3.9 is in this orbit. From this, substituting ξ_I by a P conjugate matrix ξ'_I, we could make ξ'_I still a block diagonal matrix with two blocks indexed by I and $I_0 - I$ respectively, with the part indexed by the set I of the form in Example 3.8 (with degree of $\#I$, instead of n), and the part indexed by $I_0 - I$ a diagonal matrix with eigenvalues $\{a_i : i \in I_0 - I\}$. Applying the reduction in Theorem 3.5 $\#I - 1$ times, we arrive at a diagonal matrix in $\mathfrak{l}_{n+1-\#I}(\mathbb{C})$ with eigenvalues $\{a_i : i \in I_0 - I\}$. That just means, $g_I \cdot f$ is semisimple, with depth $\#I$, and eigenvalues $\{a_i : i \in I_0 - I\}$.

In the case of $n \in I$, let $k = \max\{i : i \in I\}$. By calculation we have

$$g_I \cdot f = \mathrm{pr}'(g_I \cdot \xi) = \mathrm{pr}'(\xi_I),$$

where
$$\xi_I = \begin{pmatrix} \mathrm{diag}\{a_1, \ldots, a_{k-1}, a_n, a_{k+1}, \ldots, a_{n-1}\} & 0_{(n-1)\times 1} \\ \beta'^t & 0 \end{pmatrix}$$

with $\beta'^t = (y_1, \ldots, y_{k-1}, \underbrace{0, \ldots, 0}_{n-k})$, where $y_i = \begin{cases} a_i - a_k & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$

Write
$$I' = (I - \{k\}) \cup \{n\}.$$

Separating $I_0 = \{1, 2, \ldots, n\}$ into the disjoint union of two subsets: I' and $I_0 - I'$, we see that ξ_I is a block diagonal matrix. The one with rows and columns indexed by I' is of the form in Lemma 3.9; the one with rows and columns indexed by $I_0 - I'$ is a diagonal matrix with eigenvalues $\{a_i : i \in I_0 - I\}$. The proof of the rest is the same as in the case of $n \in I$. ■

Theorem 4.4. *The set $p(\mathcal{O}_f)$ consists of $2^n - 1$ semisimple P orbits, with the unique open P coadjoint orbit in $\mathfrak{p}^*$ among them. Moreover,*

(1) *the moment map $q : \mathcal{O}_f \to \mathfrak{p}^*$ is* weakly proper*, but* not proper over image *(so in particular* not proper*).*

(2) *the* reduced space *of the unique open P orbit in $\mathfrak{p}^*$ (with respect to the moment map q) is a singleton.*

Proof. By Lemma 4.1, $\mathcal{O}_f$ is the union of $2^n - 1$ P orbits. In Lemma 4.3, we described the image of the moment map $q : \mathfrak{g}^* \to \mathfrak{p}^*$ for each of these P orbits. Particularly, we see that: each of them is semisimple, and different P orbits in $\mathcal{O}_f$ are mapped to different P orbits in $\mathfrak{p}^*$ (due to $a_i \neq a_j$), and the unique open P orbit in $\mathfrak{p}^*$ is among them (for $I = I_0$). This shows that the moment map *is weakly proper.* Now, for any $\emptyset \neq I \subset I_0$, write $j = \#I - 1$. Then,

$$\mathrm{Stab}_P(g_I \cdot f) = P \cap g_I T g_I^{-1} = g_I (T \cap g_I^{-1} P g_I) g_I^{-1},$$

and
$$T \cap g_I P g_I^{-1} = \mathrm{Stab}_T(g_I^{-1} \cdot v_0),$$

which is a torus isomorphic to $(\mathbb{C}^\times)^{n-1-j}$.

From the description of $p(g_I \cdot f)$ in Lemma 4.3, by Lemma 3.3 and Lemma 3.4 we have

$$\mathrm{Stab}_P(p(g_I \cdot f)) \cong \mathbb{C}^{n-1-j} \rtimes (\mathbb{C}^\times)^{n-1-j}.$$

From this, we see that the reduced space of $p(g_I \cdot f)$ is isomorphic to

$$\mathrm{Stab}_P(p(g_I \cdot f))/\,\mathrm{Stab}_P(g_I \cdot f) \cong \mathbb{C}^{n-1-j}.$$

Hence, p *is non-proper*. Moreover, for a compact subset $K \subset p(\mathcal{O}_f)$, $p^{-1}(K)$ is compact if and only if K is contained in the unique open orbit in $\mathfrak{p}^*$. Then p *is not proper over image*.

The second statement also follows from the description of the moment map. ■

5. Moment map in the $\mathbf{GL}(n, \mathbb{R})$ case

Now let $G = \mathrm{GL}(n, \mathbb{R})$, and

$$P = P_n(\mathbb{R}) = \left\{ \begin{pmatrix} A & \alpha \\ 0_{1\times(n-1)} & 1 \end{pmatrix} : A \in \mathrm{GL}(n-1, \mathbb{R}), \alpha \in \mathbb{R}^{n-1} \right\}.$$

Let $0 \le k \le \lfloor \frac{n}{2} \rfloor$. Let $z_1, \ldots, z_n$ be n distinct complex numbers with $z_{2j-1} = a_j + \mathbf{i}b_j$ $(1 \le j \le k)$, $z_{2j} = a_j - \mathbf{i}b_j$ $(1 \le j \le k)$, and $z_{2k+j} = a_{2k+j}$ $(1 \le j \le n-2k)$, where $a_1, \ldots, a_k, a_{2k+1}, \ldots, a_n, b_1, \ldots, b_k \in \mathbb{R}$ and $b_1 \cdots b_k \neq 0$.

Write
$$\vec{z} = (z_1, ..., z_n) \in \mathbb{C}^n.$$

Set
$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Write
$$\xi = \xi_{\vec{z}} = \mathrm{diag}\{a_1 I_2 + b_1 J, \ldots, a_k I_2 + b_k J, a_{2k+1}, \ldots, a_n\} \in \mathfrak{g},$$

and
$$f = f_{\vec{z}} = \mathrm{pr}(\xi) \in \mathfrak{g}^*.$$

Write
$$T_k = \left\{ \begin{array}{l} \mathrm{diag}\{\lambda_1 I_2 + \mu_1 J, \ldots, \lambda_k I_2 + \mu_k J, \lambda_{2k+1}, \ldots, \lambda_n\} : \\ \quad \lambda_1, \ldots, \lambda_k, \lambda_{2k+1}, \ldots, \lambda_n, \mu_1, \ldots, \mu_k \in \mathbb{R}, \\ \quad (\lambda_1^2 + \mu_1^2) \cdots (\lambda_k^2 + \mu_k^2)\lambda_{2k+1} \cdots \lambda_n \neq 0 \end{array} \right\},$$

which is a maximal torus in G. It is clear that $\mathrm{Stab}_G(f) = T_k$. Thus, f is a regular semisimple element in $\mathfrak{g}^*$. On the other hand, any regular semisimple orbit $\mathcal{O} \subset \mathfrak{g}^*$ is of the form $\mathcal{O} = G \cdot f_{\vec{z}}$.

From now on, we fix $\vec{z}$, ξ and f in this subsection.

Lemma 5.1. *There are $2^{n-k}-1$ P orbits in $\mathcal{O}_f$. One of them is open and dense, and the union of the rest is a codimension one (if $k < \frac{n}{2}$) or two (if $k = \frac{n}{2}$) closed subset.*

Proof. Let G act on $\mathbb{R}^n - \{0\}$ through

$$g \cdot (x_1, \ldots, x_n)^t = (g^{-1})^t (x_1, \ldots, x_n)^t, \quad \forall g \in G, \ \forall (x_1, \ldots, x_n)^t \in \mathbb{R}^n - \{0\}.$$

Write
$$v_0 = (0, \ldots, 0, 1)^t \in \mathbb{R}^n - \{0\}.$$

Then,
$$\mathrm{Stab}_G(v_0) = P.$$

Hence, $\mathbb{R}^n - \{0\} = G/P$. Similar as in the proof of Lemma 4.3, we have identifications

$$P\backslash\mathcal{O}_f = P\backslash G/T_k \cong T_k\backslash G/P = T_k\backslash(\mathbb{R}^n - \{0\}).$$

Write $I_0^{(1)} = \{1, \dots, k\}$, $I_0^{(2)} = \{2k+1, \dots, n\}$. For any $I_1 \subset I_0^{(1)}$ and $I_2 \subset I_0^{(2)}$ with $\#I_1 + \#I_2 > 0$, let $v_{I_1,I_2} = (x_1, \dots, x_n) \in \mathbb{R}^n - \{0\}$ be given by

$$(x_{2i-1}, x_{2i}) = \begin{cases} (0,1) & \text{if } i \in I_1, \\ (0,0) & \text{if } i \in I_0^{(1)} - I_1, \end{cases}$$

and

$$x_j = \begin{cases} 1 & \text{if } j \in I_2, \\ 0 & \text{if } j \in I_0^{(2)} - I_2. \end{cases}$$

Note that T_k acts on $\mathbb{R}^n - \{0\}$ through

$$\begin{aligned} &\operatorname{diag}\{\lambda_1 I_2 + \mu_1 J, \dots, \lambda_k I_2 + \mu_k J, \lambda_{2k+1}, \dots, \lambda_n\} \cdot (x_1, ..., x_n)^t \\ &= ((\lambda_1^2 + \mu_1^2)^{-1}(x_1\lambda_1 + y_1\mu_1), (\lambda_1^2 + \mu_1^2)^{-1}(-x_1\mu_1 + y_1\lambda_1), \dots, \\ &\quad (\lambda_k^2 + \mu_k^2)^{-1}(x_k\lambda_k + y_k\mu_k), (\lambda_k^2+\mu_k^2)^{-1}(-x_k\mu_k+y_k\lambda_k), \lambda_{2k+1}^{-1}x_{2k+1}, \dots, \lambda_n^{-1}x_n)^t. \end{aligned}$$

It is easy to see that $\{v_{I_1,I_2} : \emptyset \neq I \subset \{1, ..., n\}\}$ represent all different T_k orbits in $\mathbb{R}^n - \{0\}$. This shows the lemma. ■

For (I_1, I_2) as in the proof of the above lemma, we define a matrix $g_{I_1,I_2} \in G$ as follows. If $n \in I_2$, define

$$g_{I_1,I_2} = \begin{pmatrix} I_{n-1} & 0_{(n-1)\times 1} \\ \beta^t & 1 \end{pmatrix},$$

where $\beta^t = (x_1, \dots, x_{n-1})$ with

$$(x_{2i-1}, x_{2i}) = \begin{cases} (0,1) & \text{if } i \in I_1, \\ (0,0) & \text{if } i \in I_0^{(1)} - I_1, \end{cases}$$

and

$$x_j = \begin{cases} 1 & \text{if } j \in I_2 - \{n\}, \\ 0 & \text{if } j \in I_0^{(2)} - I_2. \end{cases}$$

If $I_2 \neq \emptyset$ and $n \in I_0^{(2)} - I_2$, set $k' = \max\{1 \le i \le n-1 : i \in I_2\}$ and define

$$g_{I_1,I_2} = \begin{pmatrix} I_{k'-1} & 0_{(k'-1)\times 1} & 0_{(k'-1)\times(n-1-k')} & 0_{(k'-1)\times 1} \\ 0_{1\times(k'-1)} & 0 & 0_{1\times(n-1-k')} & 1 \\ 0_{(n-1-k')\times(k'-1)} & 0_{(n-1-k')\times 1} & I_{n-1-k'} & 0_{(n-1-k')\times 1} \\ \beta'^t & 1 & 0_{1\times(n-1-k')} & 0 \end{pmatrix},$$

where $\beta'^t = (x_1, \dots, x_{k'-1})$ with

$$(x_{2i-1}, x_{2i}) = \begin{cases} (0,1) & \text{if } i \in I_1, \\ (0,0) & \text{if } i \in I_0^{(1)} - I_1, \end{cases}$$

and

$$x_j = \begin{cases} 1 & \text{if } j \in I_2, \\ 0 & \text{if } j \in I_0^{(2)} - I_2. \end{cases}$$

If $I_2 = \emptyset$ and $\frac{n}{2} \in I_1$, define

$$g_{I_1,I_2} = \begin{pmatrix} I_{n-1} & 0_{(n-1)\times 1} \\ \beta^t & 1 \end{pmatrix},$$

where $\beta^t = (x_1, \ldots, x_{n-1})$ with

$$(x_{2i-1}, x_{2i}) = \begin{cases} (0,1) & \text{if } i \in I_1 - \{\frac{n}{2}\}, \\ (0,0) & \text{if } i \in I_0^{(1)} - I_1, \end{cases}$$

and $x_{n-1} = 0$. If $I_2 = \emptyset$ and $\frac{n}{2} \notin I_1$, set $k' = \max\{1 \leq i < \frac{n}{2} : i \in I_1\}$ and define

$$g_{I_1,I_2} = \begin{pmatrix} I_{2k'-1} & 0_{(2k'-1)\times 1} & 0_{(2k'-1)\times(n-1-2k')} & 0_{(2k'-1)\times 1} \\ 0_{1\times(2k'-1)} & 0 & 0_{1\times(n-1-2k')} & 1 \\ 0_{(n-1-2k')\times(2k'-1)} & 0_{(n-1-2k')\times 1} & I_{n-1-2k'} & 0_{(n-1-2k')\times 1} \\ \beta'^t & 1 & 0_{1\times(n-1-2k')} & 0 \end{pmatrix},$$

where $\beta'^t = (x_1, \ldots, x_{2k'-1})$ with

$$(x_{2i-1}, x_{2i}) = \begin{cases} (0,1) & \text{if } i \in I_1 - \{\frac{k'}{2}\}, \\ (0,0) & \text{if } i \in I_0^{(1)} - I_1, \end{cases}$$

and $x_{2k'-1} = 0$.

Proposition 5.2. *$\{g_{I_1,I_2} \cdot f : (\emptyset, \emptyset) \neq (I_1, I_2) \subset (I_0^{(1)}, I_0^{(2)})\}$ represent all different P-orbits in $\mathcal{O}_f$. Among them, $Pg_{I_0^{(1)}, I_0^{(2)}} \cdot f$ is open and dense.*

Proof. One can show that $g_{I_1,I_2}^{-1} \cdot v_0 = v_{I_1,I_2}$ for any (I_1, I_2). By the proof of Lemma 4.1, this indicates that

$$\{g_{I_1,I_2} \cdot f : (\emptyset, \emptyset) \neq (I_1, I_2) \subset (I_0^{(1)}, I_0^{(2)})\}$$

represent all different P-orbits in $\mathcal{O}_f$.

It is clear that $Tg^{-1}_{I_0^{(1)}, I_0^{(2)}} \cdot v_0$ is an open and dense subset in $\mathbb{R}^n - \{0\}$.

Thus, $Pg_{I_0^{(1)}, I_0^{(2)}} \cdot f$ is an open and dense P-orbit. ∎

Lemma 5.3. *For any $(\emptyset, \emptyset) \neq (I_1, I_2) \subset (I_0^{(1)}, I_0^{(2)})$, the element $p(g_{I_1,I_2} \cdot f)$ is semisimple, its depth is equal to $2\#I_1 + \#I_2$, and its eigenvalues are*

$$\{z_{2i-1}, z_{2i}, z_j : i \in I_0^{(1)} - I_1, j \in I_0^{(2)} - I_2\}.$$

Proof. Regard $\mathfrak{p}_n(\mathbb{R})$ as a real form of $\mathfrak{p}_n(\mathbb{C})$. Then, $\mathfrak{p}_n(\mathbb{R})^*$ is naturally contained in $\mathfrak{p}_n(\mathbb{C})^*$ as a real form of it. By the proof of Theorem 3.5, we see that this imbedding does not change the depth and eigenvalues. It is convenient to do conjugation regarding $P_n(\mathbb{C})$ to see the depth and eigenvalues of $p(g_{I_1,I_2} \cdot f)$, and hence shows the lemma. ∎

The following theorem can be shown similarly as Theorem 4.4.

Theorem 5.4. *The set $p(\mathcal{O}_f)$ consists of $2^{n-k}-1$ semisimple P-orbits, with the unique open P-orbit in $\mathfrak{p}^*$ among them. Moreover,*

(1) *the moment map $q : \mathcal{O}_f \to \mathfrak{p}^*$ is* weakly proper*, but* not proper over image *(so especially* not proper*).*

(2) *the* reduced space *of the unique open P-orbit in $\mathfrak{p}^*$ with respect to the moment map q is a singleton.*

Let us give more details on the *non-properness* of the moment map q in Theorem 5.4. Actually, analogous to the $\mathrm{GL}(n,\mathbb{C})$ case, we write

$$j = 2\#I_1 + \#I_2 - 1 \in [0, n-1]$$

for any $(\emptyset,\emptyset) \neq (I_1, I_2) \subset (I_0^{(1)}, I_0^{(2)})$. Then, with Lemma 5.3 one can show that

$$\mathrm{Stab}_P(g_{I_1,I_2} \cdot f) \cong \mathrm{U}(1)^{k-\#I_1} \times (\mathbb{R}^\times)^{n-k-\#I_1-\#I_2}.$$

On the other hand, Lemma 3.3 and Lemma 3.4 indicate that

$$\mathrm{Stab}_P(p(g_{I_1,I_2} \cdot f)) \cong \mathbb{R}^{n-1-j} \rtimes (\mathrm{U}(1)^{k-\#I_1} \times (\mathbb{R}^\times)^{n-k-\#I_1-\#I_2}).$$

From this, we see that the reduced space of $p(g_{I_1,I_2} \cdot f)$ is isomorphic to

$$\mathrm{Stab}_P(p(g_{I_1,I_2} \cdot f)) / \mathrm{Stab}_P(g_{I_1,I_2} \cdot f) \cong \mathbb{R}^{n-1-j}.$$

Hence, *p is non-proper.* Moreover, for a compact subset $K \subset p(\mathcal{O}_f)$, $p^{-1}(K)$ is compact if and only if K is contained in the unique open orbit in $\mathfrak{p}^*$. Then *p is not proper over image.*

After we posed our paper on arXiv, the authors received an email from M. Raïs who informed us that in [27] it is already shown by another method that there is a unique open and dense $P_n(k)$-coadjoint orbit in $\mathfrak{p}_n(k)^*$. Moreover, according to a result due to Duflo-Richardson (see the appendix of [15]), $q(\mathcal{O}_f)$ contains a non-empty Zariski open subset of $\mathfrak{p}_n(k)^*$, which implies in our context that $q(\mathcal{O}_f)$ contains the unique open and dense coadjoint orbit of P. However, our description of moment map is much more precise, and this precise geometric description (combined with Sahi's description for the branching laws) enables us to prove Duflo's conjecture in the setting of this paper. Especially, based on our treatment of moment map, we prove that the moment map is weakly proper but neither proper nor proper over image, this point gives more evidence that Duflo's *weakly proper* notion should be the right criterion to characterize the H-admissibility for general algebraic groups.

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Gang Liu, Institut Elie Cartan de Lorraine, Université de Lorraine, 3 rue Augustin Fresnel, Metz, France; gang.liu@univ-lorraine.fr.

Jun Yu, Beijing International Center for Mathematical Research, Peking University, Beijing, P. R. China; junyu@math.pku.edu.cn.

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