

Factorization and Boundedness for Representations of Locally Compact Groups on Topological Vector Spaces

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Abstract. We (a) prove that continuous morphisms from locally compact groups to locally exponential (possibly infinite-dimensional) Lie groups factor through Lie quotients, recovering a result of Shtern’s on factoring norm-continuous representations on Banach spaces; (b) characterize the maximal almost-periodicity of the identity component $\mathbb{G}_0 \leq \mathbb{G}$ of a locally compact group in terms of sufficiently discriminating families of continuous functions on \mathbb{G} valued in Hausdorff spaces generalizing an analogous result by Kadison-Singer; (c) apply that characterization to recover the von Neumann kernel of \mathbb{G}_0 as the joint kernel of all appropriately bounded and continuous \mathbb{G} -representations on topological vector spaces extending Kallman’s parallel statement for unitary representations, and (d) provide large classes of complete locally convex topological vector spaces (e.g. arbitrary products of Fréchet spaces) with the property that compact-group representations thereon whose vectors all have finite-dimensional orbits decompose as finite sums of isotypic components. This last result specializes to one of Hofmann-Morris on representations on products of real lines.

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1. Introduction

The several themes the paper touches on all ultimately emerge from and point back to the study of norm-continuous (or uniformly continuous) representations of locally compact groups. The precise setting differs among sources: some [23, 24, 47] focus on unitary representations while others [14, 30, 46] discuss representations on Banach spaces. Similarly, the framework is occasionally limited to particular classes of groups: Lie (finite dimensional), or connected, or both. The common pattern is the constrained nature and structure of such representations.

As a case in point, consider the fact [46, Corollary 2] that a norm-continuous representation of a compact group \mathbb{G} on a Banach space E decomposes as a sum of finitely many *isotypic components* [17, Definition 4.21]:

$$E = \bigoplus_{i=1}^n E_{\alpha_i}, \quad \alpha_i \in \text{Irr}(\mathbb{G}) := \text{classes of irreducible } \mathbb{G}\text{-representations,}$$

where E_α , $\alpha \in \text{Irr}(\mathbb{G})$ is the set of vectors generating finite-dimensional subrepresentations that are direct sums of copies of α .

The unitary case also follows from [24, main result stated on p.257], while [10, Theorem 2.10] contains a number of other characterizations (again in the Banach setup). In [46], the key result delivering this finiteness is the factorization [46, Theorem 1] of every norm-continuous representation, of any locally compact group whatsoever, through a Lie-group quotient. Theorem 2.1 below records the observation that this goes through rather generally:

Theorem 1.1. *A continuous morphism $\mathbb{G} \rightarrow \mathbb{U}$ from a locally compact group to a topological group factors through a Lie quotient provided all compact subgroups of \mathbb{U} are finite-dimensional Lie.*

In particular, this applies to locally exponential Lie groups \mathbb{U} .

The phrase *Lie group* is construed as in [36, Definition II.1.1]: topological groups locally modeled on (possibly infinite-dimensional) *locally convex* [28, §18.1] topological vector spaces. Recall [36, Definition IV.1.1] that the *locally exponential* Lie groups are those locally diffeomorphic to their Lie algebra via an exponential function, and that they do indeed [36, Theorem IV.3.16] satisfy the compact-subgroup condition. [46, Theorem 1] then follows by further specializing to $\mathbb{U} := GL(E)$ for a Banach space E .

In the course of investigating norm-continuous unitary representations of a locally compact *connected* group \mathbb{G} , [24, Lemmas 1 and 2] (in a slight rephrasing) note that the joint kernel of those representations is the smallest closed normal subgroup $\mathbb{H} \trianglelefteq \mathbb{G}$ for which

$$\mathbb{G}/\mathbb{H} \cong \mathbb{R}^d \times \mathbb{L}, \quad d \in \mathbb{Z}_{\geq 0}, \quad \mathbb{L} \text{ compact connected.} \quad (1)$$

In other words, \mathbb{H} is precisely the *von Neumann kernel* [43, §0] of \mathbb{G} : the intersection of the kernels of finite-dimensional unitary \mathbb{G} -representations. Those results in turn rely heavily on the characterization [23, Lemma 1 and Corollary 4] of groups of the form (1) in terms of certain families of conjugacy-invariant functions thereon. This too generalizes (Theorem 2.4) so as to allow extending the discussion to representations on Banach spaces and beyond:

Theorem 1.2. *The identity component $\mathbb{G}_0 \leq \mathbb{G}$ of a locally compact group is of the form (1) if and only if there is a family of continuous functions $\mathbb{G} \xrightarrow{\varphi} X_\varphi$ into Hausdorff topological spaces such that*

- *for every $1 \neq g \in \mathbb{G}$ there is some φ with $\varphi(g) \neq \varphi(1)$;*
- *and for every φ the set-valued map*

$$\tilde{\varphi} : \mathbb{G} \ni g \mapsto \varphi(\mathbb{G}_0\text{-conjugacy class of } g) \in 2^{X_\varphi}$$

is upper hemicontinuous [2, Definition 17.2] at $1 \in \mathbb{G}$:

$$\forall \text{ neighborhood } U \ni \varphi(1), \exists \text{ neighborhood } V \ni 1 \in \mathbb{G} \text{ with } g \in V \implies \tilde{\varphi}(g) \subseteq U.$$

There are thus (Theorem 2.9 and Corollary 2.13) representation-theoretic characterizations of locally compact groups with identity components of the form (1) via broader classes of topological vector spaces (than Banach spaces, say)

Theorem 1.3. *Let \mathbb{G} be a locally compact group.*

- (1) *The identity component of \mathbb{G} is of the form (1) if and only if \mathbb{G} has a separating family of representations $\mathbb{G} \xrightarrow{\varphi} GL(E_\varphi)$ on topological vector spaces such that*
 - *φ is continuous for the topology of uniform convergence on members of a covering bornology [6, §III.1.1, Definition 1] $\mathfrak{M} \subseteq 2^{E_\varphi}$;*
 - *the family $\varphi(\mathbb{G}_0) \subset GL(E_\varphi)$ is equicontinuous;*
 - *and $\varphi(\mathbb{G}_0)M \in \mathfrak{M}$ for all $M \in \mathfrak{M}$.*
- (2) *Consequently, the von Neumann kernel of $\mathbb{G}_0 \leq \mathbb{G}$ is precisely the common kernel of all representations φ as in (1).*

The conditions of Theorem 1.3(1) specialize precisely to norm-continuity and norm-boundedness for representations on Banach spaces or, more generally, on *quasi-Banach* spaces [37, §A.1].

Section 3 varies the theme of isotypic-component finiteness, taking [17, Exercise E4.8] as a starting point. That result says that compact group representations on *weakly complete* topological vector spaces E (i.e. [17, Definition A7.8] products of real lines equipped with their product topology) decompose as finite sums of isotypic components provided every orbit $\mathbb{G}v$, $v \in E$ has finite-dimensional span. In the language of Definition 3.7, a *locally \mathbb{G} -finite* representation of a compact group \mathbb{G} on a weakly complete TVS is *globally \mathbb{G} -finite*.

It is, on the one hand, easy to recover the analogous result (Proposition 3.9) for *Fréchet spaces* E instead ([48, §10]: locally convex complete metrizable TVSs). On the other hand, weakly complete spaces will be Fréchet only when the family of real-line factors is at most countable. It is natural, then, to try to find a common generalization of the two results: a class of topological vector spaces that naturally houses both weakly complete and Fréchet spaces, for which local \mathbb{G} -finiteness entails its global counterpart. Theorem 3.15 does this in terms of realizing a TVS as a cofiltered limit of well-behaved quotients; we record here the more straightforwardly-stated consequence (Corollary 3.16):

Theorem 1.4. *A locally \mathbb{G} -finite representation of a compact group \mathbb{G} on a product of Fréchet spaces is a finite direct sum of isotypic components.*

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2. Factorization through well-behaved groups

We make fairly free use of terminology pertinent to topological vector spaces (TVSs for short), for which the reader can consult (say) [6, 28, 29, 34, 41, 44, 48] and countless others. Where the occasion demands them, the text provides specific references. Topological groups (so TVSs in particular) are assumed Hausdorff by default.

The same goes for material on *uniform spaces*, as covered, say, in [22, Chapter 7 and onward], [28, §5], [7, Chapter II], etc.

In particular, we take for granted notions such as

- the topology underlying a uniform structure [7, §II.1.2];
- the *left* and *right uniformities* [7, §III.3.1, Definition 1] on a topological group;
- the *product* uniformity [7, §II.2.6, Definition 4] on a Cartesian product of topological spaces;
- and the *completion* [7, §II.3.7] of a Hausdorff uniform space.

There are references to *(co)limits* [1, §11], mostly in categories of TVSS. Of particular importance are

- *cofiltered limits* [4, Tag 04AY], i.e. limits of diagrams of the form

$$E_i \xrightarrow{\pi_{ji}} E_j, \quad i \geq j \text{ in a directed (or filtered) poset } (I, \leq),$$

meaning that any two elements are dominated by another;

- and dually, *filtered colimits*, i.e. colimits of diagrams

$$E_i \xrightarrow{\iota_{ji}} E_j, \quad i \leq j \in \text{filtered } (I, \leq).$$

Other terms (respectively) are *projective* [28, §19.7] and *inductive* [28, §19.2] *limits*.

Although phrased in terms of representations on Banach spaces E and hence morphisms into Banach Lie groups $\mathbb{U} := GL(E)$, the core of the proof of [46, Theorem 1] can be reworked to deliver substantially more.

Theorem 2.1. (1) *Let \mathbb{U} be a (Hausdorff) topological group whose compact subgroups are finite-dimensional Lie. Any topological-group morphism $\mathbb{G} \xrightarrow{\varphi} \mathbb{U}$ from a locally compact group factors through a finite-dimensional Lie-group quotient of \mathbb{G} .*

(2) *In particular, the conclusion of (1) holds for locally exponential [36, Definition IV.1.1] Lie groups \mathbb{U} .*

Proof. Part (2) follows immediately from (1), since the latter's hypothesis is met [36, Theorem IV.3.16]: (even locally) compact subgroups of locally exponential Lie groups are finite-dimensional Lie. Settling, (1), then, is enough.

As noted in the proof of the already-cited [46, Theorem 1], it is enough handle the restriction of φ to a judiciously-chosen open subgroup $\mathbb{H} \leq \mathbb{G}$, for we would then know that φ factors through a quotient $\mathbb{G} \twoheadrightarrow \overline{\mathbb{G}}$ having a finite-dimensional Lie quotient of \mathbb{H} as an open subgroup, so that $\overline{\mathbb{G}}$ is itself Lie.

We can choose [32, Lemma 2.3.1] the aforementioned open $\mathbb{H} \leq \mathbb{G}$ *almost connected* [17, preceding Theorem 10.89], i.e. with $\mathbb{H}/\mathbb{H}_0 = \mathbb{H}/\mathbb{G}_0$ compact. But then \mathbb{H} has [32, §4.6, Theorem] a compact normal subgroup $\mathbb{K} \trianglelefteq \mathbb{H}$ with \mathbb{H}/\mathbb{K} Lie (finite-dimensional). The (automatically compact) image $\varphi(\mathbb{K}) \leq \mathbb{U}$ is finite-dimensional Lie by assumption, so φ factors through the (finite-dimensional Lie [20, Theorem 7]) extension of the Lie group \mathbb{H}/\mathbb{K} by the Lie group $\varphi(\mathbb{K}) \cong \mathbb{K}/\ker \varphi|_{\mathbb{K}}$. ■

We record an alternative argument Theorem 2.1(2) assuming the Lie algebra $Lie(\mathbb{G})$ admits a continuous norm. That assumption, while weakening the result, still holds [36, Corollary IV.1.10] for *Banach* Lie groups [35, Definition IV.I], so the result still recovers (and generalizes) [46, Theorem 1] upon specializing \mathbb{U} to $GL(E)$ for Banach spaces E .

Proof of Theorem 2.1(2) when $Lie(\mathbb{U})$ admits a continuous norm. The assumptions on \mathbb{U} ensure [36, Problem II.5] that it contains *no small subgroups* [32, 2.20.1]: there are neighborhoods $V \ni 1 \in \mathbb{U}$ containing no non-trivial subgroups. But then

- every subgroup of the preimage $\varphi^{-1}V \subseteq \mathbb{G}$ will be annihilated by φ ;
- which goes for a normal compact subgroup $\mathbb{K} \trianglelefteq \mathbb{G}'$ of an open subgroup $\mathbb{G}' \leq \mathbb{G}$ with \mathbb{G}'/\mathbb{K} Lie (and there are such [32, §4.0]);
- which means that φ factors through the finite-dimensional Lie group obtained as the quotient of \mathbb{G} by the normal subgroup generated by \mathbb{K} . ■

Remark 2.2. (1) Theorem 2.1(2) generalizes [36, Theorem IV.3.16], which says that (automatically closed) locally compact subgroups of locally-exponential Lie groups are finite-dimensional Lie. The (first) proof given above does of course appeal to [36, Theorem IV.3.16], but in its weaker form pertaining to *compact* groups.

(2) Naturally, the assumed continuous norm on $Lie(\mathbb{U})$ in the second proof of Theorem 2.1(2) need not induce the topology on that space. In other words, $Lie(\mathbb{U})$ need not be *normable* [48, §11]. The space $H(\Omega)$ [48, §10, Example II] of holomorphic functions on an open subset $\Omega \subseteq \mathbb{C}^n$, with the topology of uniform convergence on compact sets, is [48, Proposition 34.4] a *Montel space* in the sense of [48, Definition 34.2]: Hausdorff, *locally convex* ([28, §18.1], [6, §II.4.1]), *barreled* ([29, §21.2], [48, Definition 33.1]) and such that every closed *bounded* [48, Definition 14.1] subset is compact.

Now, on the one hand normable Montel spaces are finite-dimensional [48, paragraph following Definition 34.2], so that $H(\Omega)$ cannot be normable if Ω is non-empty. On the other hand, holomorphic functions on *connected* Ω vanishing on a compact subset $K \subset \Omega$ with non-empty interior must vanish globally, so $\|\cdot\| := \sup_K |\cdot|$ is a continuous norm on $H(\Omega)$.

(3) [46, Footnote 1] expresses some concerns with the claim, in the course of the proof of [14, Corollary 2], that the image of a uniformly-continuous representation $\mathbb{G} \xrightarrow{\varphi} GL(E)$ of a (connected) locally compact group \mathbb{G} on a Banach space E is a (connected) locally compact group with no small subgroups.

That claim seems to me correct, in the charitable interpretation of equipping the image with the intrinsic topology (i.e. the *quotient topology* [7, §I.2.4, Example 1]) inherited from \mathbb{G}) rather than the subspace topology inherited from $GL(E)$ (the contrast between these two topologies is the flag raised by [46, Footnote 1]):

- $\mathbb{G}/\ker \varphi$ is locally compact in the quotient topology, the kernel being a closed subgroup;
- and it also satisfies the no-small-subgroups condition because its image in the *coarser* subspace topology on

$$\varphi(\mathbb{G}/\ker \varphi) \subseteq GL(E)$$

does: plainly, the no-small-subgroup requirement survives a *strengthening* of the topology.

For these reasons, the proof of [14, Corollary 2] appears to me to be correct. ■

The importance in [24] of working with *unitary* representations (as opposed to, say, representations on Banach spaces) and *connected* locally compact groups can be discerned by tracing through one of its main sources, [23]: [24, Lemmas 1 and 2] appeal to [23, Theorem 1 and Corollary 4] respectively, both of which point back to [23, Lemma 1]. The latter is a structure result stating that connected locally compact groups admitting families of continuous functions which

- are constant along conjugacy classes;
- and separate non-trivial conjugacy classes from $\{1\}$

are products of compact (connected) groups and Euclidean groups $(\mathbb{R}^d, +)$. In turn, the result appeals to the fact [21, Theorem 3] that said products are precisely the connected locally compact groups having fundamental systems of conjugacy-invariant compact 1-neighborhoods.

Per [23, Corollary 4], that structure result applies to connected locally compact groups with enough norm-continuous unitary representations $\mathbb{G} \xrightarrow{\varphi} \mathbb{U}(H)$ by applying the aforementioned [23, Lemma 1] to the functions of the form $\|\varphi(\bullet) - 1\|$. The unitary character of the representation is crucial here in ensuring the invariance of the functions on conjugacy classes. A relaxation of that unitary constraint thus entails a weakening of the conjugacy invariance of the functions in [23, Lemma 1]; this is what motivates Theorem 2.4 below.

Recall [2, Definition 17.2] that a set-valued map $X \xrightarrow{\psi} 2^Y$ for topological spaces X and Y (sometimes [2, Definition 17.1] referred to as a *correspondence* $X \rightsquigarrow Y$) is *upper hemicontinuous at* $x_0 \in X$ if the *upper inverse images* (or *preimages*) [2, §17.1, following Definition 17.1]

$$\psi_{-1}(U) := \{x \in X \mid \psi(x) \subseteq U\}, \quad \text{neighborhoods } U \supseteq \psi(x_0)$$

are neighborhoods of x_0 .

Remark 2.3. The (-1) -subscript notation is borrowed from [39, §1], where upper inverse images are termed *small preimages* (and *hemicontinuity* is *semi*continuity).

Theorem 2.4. *Let \mathbb{G} be a locally compact group with identity component \mathbb{G}_0 . The following conditions are equivalent.*

- (a) $1 \in \mathbb{G}$ has a fundamental system of \mathbb{G}_0 -conjugacy-invariant compact neighborhoods.
- (b) $\mathbb{G}_0 \cong \mathbb{R}^d \times \mathbb{L}$ for some compact connected \mathbb{L} and $d \in \mathbb{Z}_{\geq 0}$.
- (c) There is an open $\mathbb{H} \leq \mathbb{G}$ expressible as a cofiltered limit $\mathbb{H} \cong \varprojlim_i \mathbb{H}_i$ of (finite-dimensional Lie) quotients, with each \mathbb{H}_i containing an open subgroup of the form $\mathbb{R}^d \times \mathbb{L}_i$ for compact (finite-dimensional Lie) \mathbb{L}_i .
- (d) There is a family of real-valued functions $\mathbb{G} \xrightarrow{\varphi \in \mathcal{F}} \mathbb{R}$, separating $g \neq 1$ from 1 (i.e. distinguishing element non-triviality) and constant along \mathbb{G}_0 -conjugacy classes.
- (e) Same as (d), but replacing the conjugacy-class constancy with the formally weaker requirement that

$$\tilde{\varphi} := \mathbb{G} \ni g \mapsto \varphi(\mathbb{G}_0\text{-conjugacy class of } g) \in 2^{\mathbb{R}}, \quad \varphi \in \mathcal{F}$$

be upper hemicontinuous at $1 \in \mathbb{G}$.

(f) *Analogue of (d), but with $\varphi \in \mathcal{F}$ taking values in varying Hausdorff spaces X_φ in place of \mathbb{R} .*

(g) *Analogue of (e), but with $\varphi \in \mathcal{F}$ taking values in Hausdorff spaces X_φ .*

Proof. (a) \implies (b): As noted in [23, footnote 8], this is a consequence of [21, Theorem 3].

(b) \implies (c): Specifically, we prove the formally stronger, Lie branch of (c).

By [32, Lemma 2.3.1] again, \mathbb{G} has an open almost connected subgroup $\mathbb{H} \leq \mathbb{G}$. This will be the desired \mathbb{H} : the claim, in particular, is that *any* almost connected \mathbb{H} will do.

The celebrated Lie-group-approximation theorem for almost connected locally compact groups [32, §4.6, Theorem] says that \mathbb{H} (being almost connected) is the cofiltered limit of its Lie quotients (i.e. is *pro-Lie* in any of the alternative senses of the term [16, Theorem 3.39 and Definitions A, B and C preceding it]). Note also that the identity component $\mathbb{G}_0 = \mathbb{H}_0 \cong \mathbb{R}^d \times \mathbb{L}$ *surjects* onto each identity component $\mathbb{H}_{i,0}$, for instance because images of profinite groups (such as \mathbb{H}/\mathbb{H}_0) are again profinite [17, Exercise E1.13].

For an index i large enough to ensure that the \mathbb{R}^d factor in $\mathbb{H}_0 \cong \mathbb{R}^d \times \mathbb{L}$ embeds into $\mathbb{H}_{i,0}$, that identity component will be both in \mathbb{H}_i (because the latter is Lie) and of the desired form $\mathbb{R}^d \times \mathbb{L}_i$.

(c) \implies (d): This time we hypothesize the formally weaker, non-Lie version of (c).

We can first declare all functions $\varphi \in \mathcal{F}$ constant and non-zero on all non-trivial cosets of the given open subgroup $\mathbb{H} \leq \mathbb{G}$, so that they are also obviously constant on the \mathbb{G}_0 -conjugacy classes outside \mathbb{H} .

Next, the restrictions $\varphi|_{\mathbb{H}}$ will be pullbacks along the surjections $\mathbb{H} \twoheadrightarrow \mathbb{H}_i$ of functions defined on \mathbb{H}_i , so it suffices to turn attention to an individual \mathbb{H}_i . We will make implicit, repeated use of the fact that the identity component $\mathbb{H}_0 = \mathbb{G}_0$ surjects onto $\mathbb{H}_{i,0}$.

By assumption, \mathbb{H}_i has an open subgroup of the form $\mathbb{R}^d \times \mathbb{L}_i$ with compact \mathbb{L}_i . The factor \mathbb{R}^d being central in

$$\mathbb{H}_{i,0} \cong \mathbb{R}^d \times \mathbb{L}_{i,0},$$

the conjugation action by \mathbb{G}_0 factors through that of the *compact* group $\mathbb{L}_{i,0}$. But then the quotient

$$\mathbb{H}_i / (\mathbb{G}_0\text{-conjugation}) = \mathbb{H}_i / (\mathbb{L}_{i,0}\text{-conjugation}) \tag{2}$$

(with its quotient topology) is locally compact Hausdorff [7, §III.4.1, Proposition 2 and its Corollary 1 and §III.4.2, Proposition 3], hence [52, Theorem 19.3] also *Tychonoff* [52, Definition 14.8]: points can be separated from closed sets by continuous functions. We can now take the restrictions $\varphi|_{\mathbb{H}}$ of $\varphi \in \mathcal{F}$ to be the continuous real-valued functions on (2) with varying i , vanishing at (the image of) 1, precomposed with the quotient maps $\mathbb{H} \twoheadrightarrow \mathbb{H}_i$.

(d) \implies (e), (f), (g) are all formal. One last implication will complete the picture.

(g) \implies (a): This is very much parallel to the proof of [23, Lemma 1].

Fix a compact neighborhood $K \ni 1$ in \mathbb{G} . Having chosen, for each point $p \in \partial K$ on the (compact) boundary of K , a function $\varphi_p \in \mathcal{F}$ and disjoint neighborhoods

$$V_{p,1} \ni \varphi_p(1), \quad V_{p,p} \ni \varphi_p(p) \quad \text{in } X_{\varphi_p}$$

(possible by the separation and Hausdorff assumptions), there is a *finite* cover of ∂K with neighborhoods

$$U_i := \varphi_{p_i}^{-1}(V_{p_i,p_i}) \ni p_i, \quad 1 \leq i \leq n.$$

The upper hemicontinuity assumption then ensures the existence of a neighborhood $U \ni 1$ in \mathbb{G} such that

$$p \in U \implies \varphi_{p_i}(\mathbb{G}_0\text{-conjugacy class of } p) \subseteq V_{p_i,1}, \quad \forall 1 \leq i \leq n.$$

Because $V_{p_i,1}$ and V_{p_i,p_i} are disjoint, this means that the \mathbb{G}_0 -conjugacy classes of $p \in U$ cannot intersect any of the U_i and hence ∂K . But said conjugacy classes are connected, meaning that the union of the \mathbb{G}_0 -conjugates of U is (a relatively compact neighborhood of 1) contained in the prescribed K .

This settles the mutual equivalence of all of the listed conditions. \blacksquare

One reason for the verbosity of (c) is that in general, when \mathbb{G} is not Lie, the mutually-equivalent conditions of Theorem 2.4 do not imply the existence of an open subgroup of the form $\mathbb{R}^d \times \mathbb{L}$ (even if \mathbb{G} is assumed almost connected).

Example 2.5. Consider the group

$$\mathbb{G} := \mathbb{G}_0 \rtimes (\mathbb{Z}/2)^{\aleph_0}, \quad \mathbb{G}_0 := \mathbb{R} \times \mathbb{T}, \quad \mathbb{T} := (\mathbb{R}/\mathbb{Z})^{\aleph_0},$$

with the action defined as follows:

- the generator σ_n of the n -indexed factor $\mathbb{Z}/2$ acts by inversion on the n -indexed circle factor $(\mathbb{R}/\mathbb{Z})_n$ and trivially on $(\mathbb{R}/\mathbb{Z})_m$, $m \neq n$;
- and the same σ_n operates on the factor $\mathbb{R} \cong \mathbb{R} \times \{0\} \subset \mathbb{G}_0$ by

$$\mathbb{R} \xrightarrow[\text{(id, obvious surjection onto } n^{\text{th}} \text{ component)}]{\sigma_n} \mathbb{R} \times (\mathbb{R}/\mathbb{Z})^{\aleph_0}.$$

The centralizer in \mathbb{G} of the factor $\mathbb{R} \subset \mathbb{G}_0$ is precisely \mathbb{G}_0 , so in particular has infinite index in \mathbb{G} .

Remark 2.6. Example 2.5 is very much in the spirit of [33, Part II], and can serve the same purpose: the group \mathbb{G} therein is *maximal almost periodic* (or *MAP* [51, §1]: it has sufficiently many finite-dimensional unitary representations), but its *left and right uniform structures* [22, §7.1, post Definition 7.3] do not coincide.

Indeed, condition (b) in Theorem 2.4 means precisely [50, §32] that \mathbb{G}_0 is MAP. The group \mathbb{G} of Example 2.5 being almost connected, it is MAP ([33, Theorem 2] or [18, Corollary 3.1]) along with its identity component \mathbb{G}_0 .

On the other hand, locally compact groups whose left and right uniformities coincide contain [33, §4, Proposition] normal open subgroups of the form $\mathbb{R}^d \times \mathbb{L}$ with \mathbb{L} compact; the group \mathbb{G} of Example 2.5 does not. \blacksquare

As the discussion preceding Theorem 2.4 suggests, all of this affords generalizations of various results on the structure of unitary uniformly continuous representations on Banach spaces by connected locally compact groups. Theorem 2.9, for instance, is meant to extend [23, Corollary 4], where the locally compact group is assumed connected and the representation unitary. Some prefatory remarks are in order.

Conventions 2.7. We frequently work with locally convex TVSs, which phrase we occasionally shorten to *LCTVS*. $\mathcal{L}(E, F)$ denotes the space of continuous linear maps $E \rightarrow F$, with $\mathcal{L}(E, E)$ abbreviated to $\mathcal{L}(E)$.

Following [29, §39.1] or [6, §III.3.1] or [8, §X.1] in the broader context of uniform spaces, we write $\mathcal{L}_{\mathfrak{M}}(E, F)$ for $\mathcal{L}(E, F)$ equipped with the topology of uniform convergence on each member of a family \mathfrak{M} of subsets of E (the *\mathfrak{M} -topology*, or *topology of \mathfrak{M} -convergence*).

We assume \mathfrak{M} to be a *covering bornology* [6, §III.1.1, Definition 1]: closed under taking subsets and finite unions, and with $\bigcup_{\mathfrak{M}} B = E$. The \mathfrak{M} subscripts apply also to the spaces $GL(E)$ of (topological and linear) automorphisms of E , as in $GL_{\mathfrak{M}}(E)$. The members of \mathfrak{M} are often (but not necessarily) *bounded* [28, §15.6]. $\mathcal{L}_{\mathfrak{M}}(E, F)$ is

- Hausdorff if F is [8, §X.1.2, Proposition 1] (recall that we are assuming \mathfrak{M} covers E);
- a TVS and locally convex if F is [6, §III.3.1, Proposition 1], provided linear continuous maps $E \rightarrow F$ send members of \mathfrak{M} to bounded sets. In particular, this is the case [48, Proposition 14.2] provided \mathfrak{M} itself is bounded (i.e. consists of bounded sets). ■

Definition 2.8. Let \mathbb{G} be a topological group and E a TVS (typically but not necessarily locally convex).

- (1) For a family \mathfrak{M} as in Conventions 2.7 an *\mathfrak{M} -representation* of \mathbb{G} on E (or an *\mathfrak{M} - \mathbb{G} -representation*) is a continuous morphism $\mathbb{G} \rightarrow GL_{\mathfrak{M}}(E)$.
- (2) Plain (unqualified) representations are *\mathfrak{M} -representations*

$$\mathbb{G} \longrightarrow GL_s(E) := GL_{\mathfrak{M}}(E), \quad \mathfrak{M} := \{\text{all finite subsets}\}. \tag{3}$$

The ‘s’ subscript stands for the *simple* [29, post §39.1(4)] (or pointwise) topology.

Representations are thus *separately continuous* [9, §VIII.2.1, Definition 1 (i)], in the sense that the induced action

$$\mathbb{G} \times E \rightarrow E \tag{4}$$

is continuous in each variable when the other is fixed (as customary: [17, Definition 2.1 and Remark 2.2] or [12, Definition 13.1.1] or [40, §2]). We thus also need

- (3) *jointly continuous* (or just *joint*) *\mathfrak{M} -representations* are [9, §VIII.2.1, Definition 1 (ii)] those for which (4) is continuous for the product topology on the domain.

Theorem 2.9. *For a locally compact group \mathbb{G} the following conditions are equivalent.*

- (a) $\mathbb{G}_0 \cong \mathbb{R}^d \times \mathbb{K}$ for some compact connected \mathbb{K} and $d \in \mathbb{Z}_{\geq 0}$.
- (b) \mathbb{G} has a separating family of unitary, norm-continuous representations, in the sense that

$$\mathbb{G} \ni g \neq 1 \implies \varphi(g) \neq 1 \text{ for some } \varphi \in \mathcal{F}.$$

- (c) \mathbb{G} has a separating family of norm-continuous representations $\mathbb{G} \xrightarrow{\varphi \in \mathcal{F}} GL(E)$ on Banach spaces $E = E_\varphi$, all bounded in the sense that $\sup_{g \in \mathbb{G}} \|\rho(g)\| < \infty$ for all $\varphi \in \mathcal{F}$.
- (d) \mathbb{G} has a separating family of \mathfrak{M} -representations $\mathbb{G} \xrightarrow{\varphi \in \mathcal{F}} GL_{\mathfrak{M}}(E)$ on topological vector spaces $E = E_\varphi$ such that

- for each $\varphi \in \mathcal{F}$, the family $\varphi(\mathbb{G}_0) \subset GL_{\mathfrak{M}}(E)$ is equicontinuous [28, §15.13] (equivalently [28, §15.13(1)], equicontinuous at $0 \in E$):

$$\forall \text{ neighborhood } U \ni 0, \quad \exists \text{ neighborhood } V \ni 0 \text{ with } \varphi(\mathbb{G}_0)V \subseteq U. \quad (5)$$

- and every $\varphi|_{\mathbb{G}_0}$ leaves \mathfrak{M} invariant:

$$\varphi(\mathbb{G}_0)M \in \mathfrak{M}, \quad \forall M \in \mathfrak{M}, \quad \forall \varphi \in \mathcal{F}. \quad (6)$$

Proof. (a) \implies (b): The hypothesis makes available all of the various mutually-equivalent conditions of Theorem 2.4, and in particular (c). Observe first that it will be enough to work with the open subgroup $\mathbb{H} \leq \mathbb{G}$ provided by that condition: assuming we have such representations $\mathbb{H} \xrightarrow{\varphi \in \mathcal{F}} U(E_\varphi)$ (unitary groups of the Hilbert spaces E_φ), we can *induce* them up to representations

$$\mathbb{G} \xrightarrow{\text{Ind}_{\mathbb{H}}^{\mathbb{G}} \varphi} GL(F_\varphi) \quad \text{for appropriate } F_\varphi.$$

The openness of \mathbb{H} will obviate many of the complications [5, §E.1] attendant to induction from arbitrary subgroups, so we can follow [25, §2.1]. Having fixed a unitary representation $\varphi \in \mathcal{F}$ on $E = E_\varphi$, set

$$F = F_\varphi := \left\{ \mathbb{G} \xrightarrow{f} E \mid f(gh) = \varphi(h^{-1})f(g), \forall g \in G, h \in H \text{ and } \sum_{x \in \mathbb{G}/\mathbb{H}} \|f(\tilde{x})\|^2 < \infty \right\},$$

where $\tilde{x} \in \mathbb{G}$ are arbitrary lifts of $x \in \mathbb{G}$. The lifts make no difference to $\|f(\tilde{x})\|$ because φ is unitary. The action of \mathbb{G} on F is (as expected)

$$\text{Ind}_{\mathbb{H}}^{\mathbb{G}} \varphi(g)f := f(g^{-1}\bullet),$$

and verifying that the desired properties transport over from the family $\{\varphi\}$ to its induced counterpart is a simple matter.

Having reduced the problem to the \mathbb{H} of Theorem 2.4(c), we can further pass on to the quotients \mathbb{H}_i (since we can regard \mathbb{H}_i -representations as \mathbb{H} -representations), and then reduce again to the open subgroups $\mathbb{R}^d \times \mathbb{L}_i \leq \mathbb{H}_i$.

All in all, it is enough to focus on groups of the form $\mathbb{R}^d \times \mathbb{K}$, for which the claim is obvious: simply take the φ to be external tensor products of irreducible unitary \mathbb{R}^d - and \mathbb{K} -representations, both of which will be finite-dimensional (and hence

norm-continuous). Or: this is the easy direction of the characterization [50, §32] of connected locally compact maximally almost periodic groups, and goes through verbatim in the disconnected case as well.

(b) \implies (c) is immediate.

(c) \implies (d): The family of (c) satisfies the conditions in (d). Indeed, norm-continuity means continuity for the *strong* [29, post §39.1(4)] topology

$$GL_b(E) := GL_{\mathfrak{M}}(E), \quad \mathfrak{M} := \{\text{all bounded subsets}\}.$$

For Banach (or indeed, complete metrizable [28, §15.13(2)]) spaces E equicontinuity is precisely uniform boundedness, which then also implies (6).

(d) \implies (a): It will be most convenient to check condition (g) of Theorem 2.4. Indeed, simply take the requisite φ to be the representations provided by the hypothesis, i.e.

$$\mathbb{G} \xrightarrow{\varphi} \mathcal{L}_{\mathfrak{M}}(E_{\varphi}), \quad \varphi \in \mathcal{F}.$$

The separation requirement is built into the hypothesis, so the one (mildly) sticking point is upper hemicontinuity. Verifying it means showing that for fixed $\varphi \in \mathcal{F}$ operating on $E := E_{\varphi}$

$$\mathbb{G} \ni g \text{ close to } 1 \implies \varphi(Ad_{\mathbb{G}_0}(g)) \subset \text{small neighborhood } U \ni 1 \in \mathcal{L}_{\mathfrak{M}}(E),$$

where

$$Ad_{\mathbb{G}_0}(g) := \{sgs^{-1} \mid s \in \mathbb{G}_0\}$$

is the \mathbb{G}_0 -conjugacy class of g . Being contained in U , in turn, means that all operators in $\varphi(Ad_{\mathbb{G}_0}(g))$ are uniformly close to the identity on a fixed bounded subset $M \in \mathfrak{M}$. We write ‘ \sim ’ for ‘is close to’. The conclusion follows from the two constraints imposed on the individual φ :

- the set $\varphi(\mathbb{G}_0)M$ still belongs to \mathfrak{M} by (6);
- so that $g \in \mathbb{G}$ close to 1 are uniformly close to the identity on that set:

$$\varphi(g)\varphi(s^{-1}) \sim \varphi(s^{-1}) \text{ uniformly on } M \text{ over } g \sim 1;$$
- whence $\varphi(s)\varphi(g)\varphi(s^{-1}) \sim \varphi(s)\varphi(s^{-1}) = 1$ uniformly on M over $g \sim 1$ by the equicontinuity assumption. ■

Remark 2.10. The two conditions (5) and (6) of Theorem 2.9(d) are not exactly unrelated. For one thing, assuming

$$\mathfrak{M} := \{\text{all bounded sets}\}, \tag{7}$$

the equicontinuity requirement (5) plainly implies the boundedness condition (6): having fixed bounded $B \subseteq E$ and a neighborhood $U \ni 0 \in E$,

$$\begin{aligned} U &\supseteq \varphi(\mathbb{G}_0)V \quad \text{for some origin neighborhood } V \text{ by (5)} \\ &\supseteq \lambda\varphi(\mathbb{G}_0)B \quad \text{for small scalars } \lambda \text{ because } B \subseteq \frac{1}{\lambda}V \text{ is bounded,} \end{aligned}$$

and hence $\varphi(\mathbb{G}_0)B$ is bounded.

If in addition (to (7)) E is *locally bounded* (i.e. has a bounded origin neighborhood; equivalently [28, §15.10]: E is *quasi-normable* or *p-normable* for some $0 < p \leq 1$)

then (5) and (6) are easily proven equivalent. In particular, this applies to normable spaces (= locally bounded + locally convex [28, §15.10(4)]). Nevertheless, in the generality of Theorem 2.9(d) the two conditions are independent: The Examples 2.11 and 2.12 show that each can in principle occur without the other. ■

Example 2.11. Assuming a compact group \mathbb{G} acts on E *jointly* continuously via $\mathbb{G} \xrightarrow{\varphi} GL_{\mathfrak{M}}(E)$, the family $\varphi(\mathbb{G})$ is easily seen to be equicontinuous [6, §VIII.2.1, Remark 2)]. On the other hand, if, say, \mathfrak{M} is the family of finite sets and \mathbb{G}_0 has at least one infinite orbit, then of course \mathfrak{M} will not be preserved by the action in the sense of (6).

Example 2.12. Assuming (6) even for the family \mathfrak{M} of *all* bounded sets, (5) need not follow. The subsequent construction will do double duty in Example 3.2. We work over the complex numbers, so as to fall within the scope of [38]. Specifically, we adapt [38, Example 2.11] to the present purposes.

Take $\mathbb{G} := \mathbb{S}^1$, say, and let E be the *pro- C^* -algebra* [38, Definition 1.1] of functions on \mathbb{G} defined by analogy to the aforementioned example (where the base space is $[0, 1]$ instead of the circle):

$$E := \mathcal{C}_{\mathbb{C}}(\mathbb{G}) := \{\text{continuous functions } \mathbb{G} \rightarrow \mathbb{C}\},$$

with the topology of uniform convergence on every countable compact subset $F \subset \mathbb{G}$ with finitely many cluster points (call such $F \subset \mathbb{G}$ *distinguished*, per [38, Definition 2.5]). The topology on E is complete and locally convex (by general theory [28, §19.8(1) and §19.10(2)]: E is a cofiltered limit of C^* -algebras $\mathcal{C}_{\mathbb{C}}(F)$). \mathbb{G} acts on E by translation on the common domain \mathbb{G} of the functions.

The bounded subsets $B \subseteq E$ are precisely those bounded in the usual sense:

$$\sup_{f \in B, x \in \mathbb{G}} |f(x)| < \infty.$$

Indeed, abstract boundedness means [48, Proposition 14.5] uniform boundedness on every distinguished $F \subseteq \mathbb{G}$, and a sequence $x_n \in \mathbb{G} = \mathbb{S}^1$ with

$$|f_n(x_n)| \xrightarrow{n} \infty, \quad f_n \in B$$

would entail unboundedness on the distinguished F consisting of a convergent subsequence of (x_n) together with its limit. It follows from this that (6) holds: naturally, translations of sets of functions uniformly bounded on \mathbb{G} are again such. (5) does not hold though: no matter how large a distinguished $F \subset \mathbb{G}$ is, there are [52, Theorems 15.8 and 17.10] $f \in E$ vanishing on F and taking the value 1 at some fixed $x \in \mathbb{G} \setminus F$. ■

Recall (e.g. [43, §0]) that the *von Neumann kernel* $\mathfrak{n}(\mathbb{G}) \trianglelefteq \mathbb{G}$ of a locally compact group \mathbb{G} is the intersection of the kernels of finite-dimensional unitary \mathbb{G} -representations; it is a closed characteristic subgroup, as observed in loc.cit. [24, Lemmas 1 and 2] can be rephrased as saying that for connected locally compact \mathbb{G} , $\mathfrak{n}(\mathbb{G})$ can also be recovered as the intersection of the kernels of all norm-continuous unitary representations (on perhaps infinite-dimensional Hilbert spaces). The following immediate consequence of Theorem 2.9 amplifies that result.

Corollary 2.13. For a locally compact group \mathbb{G} , the von Neumann kernel $\mathfrak{n}(\mathbb{G}_0)$ of its identity component is precisely the joint kernel of all

- (a) norm-continuous unitary \mathbb{G} -representations;
- (b) or norm-continuous, bounded representations on Banach spaces;
- (c) or representations $\mathbb{G} \xrightarrow{\varphi} GL_{\mathfrak{M}}(E)$ on topological vector spaces with $\varphi|_{\mathbb{G}_0}$ equicontinuous and \mathfrak{M} -preserving in the sense of (5) and (6) respectively.

3. Locally and globally finite representations

Recall [17, Exercise E4.8]: a representation of a compact group \mathbb{G} on a (real) *weakly complete* [17, Definition A7.8] TVS E (i.e. a topological product of real lines) for which the space

$$E_{fin} := \{v \in E \mid \dim \text{span } \mathbb{G}v < \infty\} \tag{8}$$

of \mathbb{G} -finite [17, Definition 3.1] vectors is all of E has finitely many *isotypic components* [17, Definition 4.21]. Taking a cue from that source, the present section’s aim is to analyze that automatic-finiteness phenomenon in more depth.

Remark 3.1. (1) As noted in the Conventions 2.7, actions $\mathbb{G} \times E \rightarrow E$ attached to representations of topological groups \mathbb{G} on TVSs E are assumed *separately* continuous in [17, Definition 2.1]. On the other hand, the hint provided for [17, Exercise E4.8] appeals to [17, Theorem 4.22], valid for *jointly* continuous representations (and in general *not* without that hypothesis: Example 3.2).

The reason why all is well is that being products of barreled spaces, weakly complete TVSs are again barreled [29, §27.1(5)] (indeed, Montel [28, §27.2(4)]), and hence separately continuous representations thereon of locally compact groups are automatically jointly continuous by [9, §VIII.2.1, Proposition 1] or [40, Proposition 13.2].

(2) The term *weakly complete* recalled in (1) above is liable to cause some confusion: it is obviously used differently in [28, §10.6(6)], where infinite-dimensional spaces are *never* weakly complete (in that text’s sense). The difference lies in this: the weakly complete spaces of [17, Lemma A7.7 and Definition A7.8] are cofiltered limits of finite-dimensional real or complex vector spaces with their *standard* topologies, whereas in [28, §10.4(8)] the topologies in question have closed cofinite subspaces as a basis of neighborhoods for the origin, so that the resulting finite-dimensional quotients are *discrete*.

In short: the difference between [17, Appendix 7] and [28, §10] is that between equipping the ground field (\mathbb{R} or \mathbb{C} in the cases of interest here) with the standard or the discrete topology respectively.

(3) In reference to (1) above, it might be worth pointing out to what extent joint continuity is necessary in the decomposition [17, Theorem 4.22], for compact \mathbb{G} , of a \mathbb{G} -representation as a completed direct sum of isotypic components. Unwinding that argument back to [17, Theorem 3.36(vi)], it becomes apparent that joint continuity is appealed to in arguing that the idempotent *averaging operator* [17, Definition 3.32]

$$E \ni v \mapsto \int_{\mathbb{G}} gv \, d\mu_{Haar}(g) \in E, \tag{9}$$

picking out the \mathbb{G} -invariant elements of E , is continuous. Example 3.2 shows that this is no longer the case absent *joint* continuity.

Example 3.2. Example 2.12 will do for this purpose as well: E consists of

$$\mathbb{G} := \mathbb{S}^1 \xrightarrow[\text{continuous}]{f} \mathbb{C}$$

with the topology of uniform convergence on every distinguished (meaning countable with finitely many cluster points) closed $F \subset \mathbb{G}$, and \mathbb{G} acts by translation. The separate continuity of the action is checked immediately, whereas *joint* continuity decidedly fails. This can be verified

- directly, by dualizing [38, Theorem 2.7] the action $\mathbb{G} \times E \rightarrow E$ to the translation action $\mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$ and observing that the latter does not preserve the family of distinguished compact sets $F \subset \mathbb{G}$ mentioned above;
- or, more to the point and downstream from [17, Theorem 3.36(vi)], by noting that the averaging operator (9) is not continuous because its kernel

$$\left\{ f \in \mathcal{C}_{\mathbb{C}}(\mathbb{G}) \mid \int_{\mathbb{G}} f \, d\mu_{\text{Haar}} = 0 \right\}$$

is not closed in the topology used here.

Indeed, said kernel is dense in E : for every continuous $\mathbb{G} \xrightarrow{f} \mathbb{C}$ and every distinguished $F \subset \mathbb{G}$ one can alter f outside F so as to obtain a function f' with vanishing integral but with $f'|_F \equiv f|_F$. ■

Remark 3.3. Naturally, by Remark 3.1(1), the space E of Example 3.2 is not barreled. This also follows directly from [19, Remark 4.1], giving a necessary condition for a commutative pro- C^* -algebra to be barreled. ■

We dwell (via Example 3.4) on the issue of joint versus separate continuity for actions for a moment, because it acts as a kind of motivating nexus for a number of other problems one is naturally led to consider in the present context: the (non-) extensibility of an action on a (locally convex) TVS to its *completion* [28, §15.3], the interplay between being barreled and normed (Remark 3.5(1)), the strict gradation of completeness properties [17, Table 3.1] pertinent to actions, and so on.

Example 3.4. This is again a separately but not jointly continuous action (or rather a family of such), where the main ingredient is a completeness deficiency. For locally compact \mathbb{G} set

$$E := \left\{ \mathbb{G} \xrightarrow[\text{continuous}]{f} \mathbb{R} \right\} \quad \text{with the topology of pointwise convergence,} \quad (10)$$

acted upon by left translation:

$$\mathbb{G} \times E \xrightarrow{\triangleright} E, \quad f \longmapsto f^{g\triangleright} \rightarrow f(g^{-1}\cdot).$$

This is a separately continuous action precisely because we are operating on *continuous* functions.

If the action were jointly continuous, though, it would extend (even jointly: Lemma 3.6) continuously to the completion

$$\widehat{E} = \mathbb{R}^{\mathbb{G}} \text{ with the product topology.}$$

Such an extension exists *precisely* in the trivial case when \mathbb{G} is discrete; otherwise $g \mapsto f(g^{-1}\cdot)$ is pointwise-discontinuous whenever $f \in E$ is itself discontinuous.

Remark 3.5. (1) As already recalled in Remark 3.1(1), separate continuity automatically entails its joint counterpart for actions on barreled spaces. This is also the case for actions (of locally compact groups) on *metrizable* LCTVSs (so in particular also for normable spaces), regardless of completeness. The point is not moot, as normed spaces can fail to be barreled: the space of eventually-vanishing sequences with the ℓ^2 norm of [28, §21.5, concluding paragraph] is one such, as pointed out in [28, §28.1, post (5)] (also with the supremum norm [26, example 16, p.40]).

To verify the claim, let $\mathbb{G} \times E \xrightarrow{\triangleright} E$ be a separately continuous action of locally compact \mathbb{G} on metrizable locally convex E , whose topology must [28, §18.2(2)] be induced by countably many *seminorms* [28, §14.1]

$$p_0 \leq p_1 \leq \dots \leq p_n \leq \dots, \quad n \in \mathbb{Z}_{\geq 0}.$$

Each individual operator $g \triangleright$ is continuous by assumption, so that for fixed $n \in \mathbb{Z}_{\geq 0}$ we have

$$p_n(g \triangleright \cdot) \leq Mp_m \quad \text{for some } M, m \in \mathbb{Z}_{\geq 0}.$$

In other words, we have

$$\mathbb{G} = \bigcup_{M, m \in \mathbb{Z}_{\geq 0}} {}_n\mathbb{G}_{\leq M|m}, \quad {}_n\mathbb{G}_{\leq M|m} := \{g \in \mathbb{G} \mid p_n(g \triangleright \cdot) \leq Mp_m\}.$$

By separate continuity each

$${}_n\mathbb{G}_{\leq M|m} = \bigcap_{p_m(v) \leq 1} \{g \in \mathbb{G} \mid p_n(g \triangleright v) \leq M\} \text{ is closed,}$$

so for every fixed n some ${}_n\mathbb{G}_{\leq M|m}$ must contain a non-empty open subset of \mathbb{G} (locally compact Hausdorff spaces are *Baire* [45, Proposition 20.18]). Next, cover a fixed but arbitrary compact origin neighborhood $K \ni 1 \in \mathbb{G}$ with finitely many right translates of that ${}_n\mathbb{G}_{\leq M|m}$:

$$K \subseteq \bigcup_{i=1}^s ({}_n\mathbb{G}_{\leq M|m}) \cdot g_i$$

The continuity of the individual g_i ensures that for sufficiently large m' they all map the *unit semi-ball* [48, Definition 7.6]

$$B_{p_{m'} \leq 1} := \{v \in E \mid p_{m'}(v) \leq 1\}$$

into the unit semi-ball $B_{p_m \leq 1}$ (which, by its very definition, ${}_n\mathbb{G}_{\leq M|m}$ then maps into $B_{p_n \leq M}$). But then

$$({}_n\mathbb{G}_{\leq M|m}) \cdot g_i \subseteq {}_n\mathbb{G}_{\leq M|m'}, \quad \forall 1 \leq i \leq s,$$

and hence

$$\forall n, \quad \exists M, m' \text{ with } K \subseteq {}_n\mathbb{G}_{\leq M|m'}.$$

This means precisely that $\{g \triangleright\}_{g \in K}$ is equicontinuous; equivalently [9, §VIII.2.1, Remark (2) following Definition 1], \triangleright is (jointly) continuous.

(2) Several kinds of action pathology interface in Example 3.4. Consider the following conditions on a separately continuous action on the LCTVS E :

- joint continuity;
- extensibility to a jointly continuous action on the completion \widehat{E} ;
- extensibility to a separately continuous action on \widehat{E} .

When \widehat{E} is barreled (as was the case in Example 3.4, where it was even weakly complete; cf. Remark 3.1(1)) the three are equivalent (by Lemma 3.6 and automatic joint continuity for barreled spaces); they thus stand or fall simultaneously.

(3) In (10) we could have considered functions defined on any homogeneous space \mathbb{G}/\mathbb{H} for closed $\mathbb{H} \leq \mathbb{G}$, to the same effect: non-extensibility of the action as soon as (the locally compact Hausdorff space) \mathbb{G}/\mathbb{H} is not discrete.

It is well-known that non-discrete homogeneous spaces of locally compact groups are uncountable (e.g. [11, Remark 2.A.18]); Example 3.4 gives an intriguing (albeit roundabout) verification of this: by item (1) above $\mathbb{R}^{\mathbb{G}/\mathbb{H}}$ cannot be metrizable in its product topology, which it would be (precisely [28, §18.3(6)]) if \mathbb{G}/\mathbb{H} were countable. ■

The following (presumably well-known) result is included here for ease of access and reference.

Lemma 3.6. *Let*
$$\mathbb{G} \times X \xrightarrow{\triangleright} X \tag{11}$$

be a continuous action of a locally compact group on a Hausdorff uniform space, with each $X \xrightarrow{g \triangleright} X$ uniformly continuous.

The canonical extension $\mathbb{G} \times \widehat{X} \rightarrow \widehat{X}$ of \triangleright to the completion of X is again continuous.

Proof. The functoriality [7, §II.3.7, Theorem 3] of the construction $X \mapsto \widehat{X}$ does indeed provide a canonical extension as in the statement: every (uniformly continuous)

$$X \xrightarrow{g \triangleright} X \text{ extends uniquely to } \widehat{X} \xrightarrow{g \triangleright} \widehat{X} \text{ (also uniformly continuous),}$$

and the associativity and unitality of the resulting action follow from said functoriality. The issue is the joint continuity of the extension $\mathbb{G} \times \widehat{X} \rightarrow \widehat{X}$. First recast [31, §VII.8, equation (2)] the initial action (11) as a map

$$X \longrightarrow \text{CONT}(\mathbb{G}, X) \tag{12}$$

uniformly continuous for the uniformity of compact convergence [8, §X.1.3, example III] on the codomain. Said compact-convergence uniformity is complete [8, §X.1.6, Corollary 3 to Theorem 2], so (12) fits into

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \widehat{X} & \xrightarrow{\quad} & \text{CONT}(\mathbb{G}, \widehat{X}). \\ & \searrow & & \nearrow & \\ & & \text{CONT}(\mathbb{G}, X) & & \end{array}$$

The upper right-hand map is the counterpart (via [31, §VII.8, (2)] again) of the extended action $\mathbb{G} \times \widehat{X} \rightarrow \widehat{X}$. ■

[17, Exercise E4.8] suggests the natural problem of extending it to actions on broader classes of locally convex TVSSs. It will first be convenient to introduce some handy terminology.

- Definition 3.7.**
- (1) A representation φ of a locally compact group on a TVS E is *locally \mathbb{G} -finite* if the space E_{fin} of (8) coincides with E .
 - (2) For compact \mathbb{G} , φ is (*globally*) *\mathbb{G} -finite* if it is locally \mathbb{G} -finite and has finitely many isotypic components.
 - (3) E is *\mathbb{G} -globalizing* (for compact \mathbb{G}) if every jointly continuous \mathbb{G} -representation on E is \mathbb{G} -finite as soon as it is locally so.
 - (4) E is *globalizing* if it is \mathbb{G} -globalizing for arbitrary compact \mathbb{G} . ■

In this language, [17, Exercise E4.8] says that weakly complete TVSSs are globalizing. It is not difficult to show (Proposition 3.9) that *Fréchet* or (F)-spaces (i.e. [28, §18.2] complete metrizable locally convex) also are. Among products of real (or complex) lines, only those with $\leq \aleph_0$ factors are metrizable [28, §18.3(6)]; weakly complete spaces, then, mostly are not. On the other hand, arbitrary complete (even barreled) LCTVSSs are certainly not all globalizing, by Example 3.8. *Some* constraint on the carrier space is thus necessary for globalization to go through.

Example 3.8. Fix an arbitrary (possibly infinite) family of irreducible (hence finite-dimensional) representations $\mathbb{G} \xrightarrow{\varphi} GL(V_\varphi)$ of a compact group, and equip the direct sum $V := \bigoplus_\varphi V_\varphi$ with its *direct-sum locally convex topology* [28, §18.5] and the obvious direct-sum \mathbb{G} -action.

V is complete and locally convex [28, §18.5(3)]. It is also barreled, as locally convex direct sums of barreled spaces always are ([28, §27.1(3)], [44, §II.7.2, Corollary 1]); indeed, it is even bornological by [28, §28.4(1)], a stronger property [28, §28.1(2)] (given completeness) than being barreled.

The action plainly being separately continuous (since every vector has non-zero components in only finitely many V_φ), it is also jointly so by [9, §VIII.2.1, Proposition 1] again. Every vector is \mathbb{G} -finite by construction; if on the other hand the family $\{\varphi\}$ contained infinitely many isomorphism types to begin with, then the resulting representation on V of course has infinitely many isotypic components.

Proposition 3.9. *Fréchet spaces are globalizing in the sense of Definition 3.7(4).*

Proof. Naturally, only the local-to-global implication is interesting. Therefore let $\mathbb{G} \xrightarrow{\varphi} GL_s(E)$ (notation as in (3)) be a representation on an (F)-space, automatically jointly continuous [9, §VIII.2.1, Proposition 1] because (F)-spaces are barreled, see [28, §21.5(3)].

The topology of E is by assumption induced by a complete metric d . Were there infinitely many isotypic components $E_n \leq E$, $n \in \mathbb{Z}_{\geq 0}$, we could pick

$$v_n \in E_n, \quad 0 < \|v_n\| := d(0, v_n) < \frac{1}{2^n}.$$

The sequence of partial sums $v_0 + \dots + v_n$ is Cauchy and hence convergent to $v := \sum_n v_n$, and the (continuous [17, Theorem 4.22]) projections $E \twoheadrightarrow E_n$ all fail to annihilate the limit v . This contradicts the local \mathbb{G} -finiteness, and we are done. ■

Another simple observation:

Proposition 3.10. *Let \mathbb{G} be a compact group and $\mathbb{G} \xrightarrow{\varphi} GL_s(E)$ a jointly continuous representation on a complete LCTVS expressible \mathbb{G} -equivariantly as a cofiltered limit*

$$E \cong \varprojlim_i E_i, \quad E_i := E/H_i \text{ complete, } H_i \leq E \text{ closed } \mathbb{G}\text{-invariant.} \quad (13)$$

If the induced \mathbb{G} -representations on E_i are all \mathbb{G} -finite, then

$$E \cong \prod_{\alpha \in \text{Irr } \mathbb{G}} E_\alpha \quad (\text{product of its isotypic components}),$$

and $E_\alpha \neq 0 \iff E_{i,\alpha} \neq 0$ for some i .

Proof. The finiteness assumption ensures that every quotient $E_i = E/H_i$ is a product

$$E_i \cong \prod_{\alpha \in \text{Irr } \mathbb{G}} E_{i,\alpha},$$

and the conclusion follows immediately from the fact that in any category limits commute [31, §IX.2]:

$$E \cong \varprojlim_i \prod_{\alpha} E_{i,\alpha} \cong \prod_{\alpha} \varprojlim_i E_{i,\alpha} \cong \prod_{\alpha} E_\alpha. \quad \blacksquare \quad (14)$$

Remark 3.11. (1) The quotients $E_i = E/H_i$ in the statement are not necessarily equipped with the quotient topology: that is the *finest* possibility, but the result allows coarser options.

(2) The last isomorphism of (14) relies implicitly on the fact that for compact \mathbb{G} , the functor

$$E \mapsto E_\alpha, \quad \alpha \in \text{Irr}(\mathbb{G})$$

picking out the α -isotypic component is *continuous* [31, §V.4] (i.e. preserves arbitrary limits) on any of the various categories of TVSs equipped with \mathbb{G} actions. This is a composite of several simple observations:

- E_α can be recovered as the space

$$E_\alpha = (V_\alpha^* \otimes E)^\mathbb{G} \otimes V_\alpha, \quad V_\alpha := \text{carrier space of } \alpha$$

of invariants in the *tensor product* [40, §2] of the \mathbb{G} -representations α and E ;

- tensoring with the finite-dimensional V_α is continuous (because [1, Proposition 13.4] it is easily seen to preserve products and *equalizers* [1, Definition 7.51]);
- and taking \mathbb{G} -invariants means forming a limit (equalizer of the maps induced by the individual $g \in \mathbb{G}$) so commutes with limits [31, §IX.2]. \blacksquare

In particular:

Corollary 3.12. *If $\mathbb{G} \xrightarrow{\varphi} GL_s(E)$ as in Proposition 3.10 (assuming (13)) is locally \mathbb{G} -finite then it is \mathbb{G} -finite.*

Proof. It is in any case the product of its isotypic components by Proposition 3.10, so strictly larger than the *sum* of those isotypic components if infinitely many of them are non-zero. ■

Remark 3.13. A cofiltered limit (15) is what, by analogy to the dual colimit situation [28, §19.4], might be called *strict*: for indices $i \leq j$ the map $E_j \rightarrow E_i$ is onto, E_i is equipped with the resulting quotient topology. The terminology is also familiar in somewhat different contexts, e.g. the *strict pro-objects* of (or rather dual to) [3, §8.12.1].

Note, though, that the requirement that the structure maps

$$E \cong \varprojlim_i E_i \longrightarrow E_j$$

of the limit (15) be onto is an additional requirement: this need not be so for arbitrary strict cofiltered limits, as Example 3.14 shows. ■

Example 3.14. There are trivial cofiltered limits of non-trivial (complete) LCTVSSs, strict in the sense of Remark 3.13. One can simply adapt the analogous set-theoretic version [49] of this remark: given a cofiltered system

$$X_j \twoheadrightarrow X_i, \quad i \leq j$$

of set surjections with empty limit in SET, consider the resulting cofiltered system

$$\mathbb{R}^{\oplus X_j} \twoheadrightarrow \mathbb{R}^{\oplus X_i}, \quad i \leq j$$

of quotient maps, with the direct sums equipped with their locally convex direct-sum topology [28, §18.5]. The limiting LCTVS will then be trivial. ■

In light of the remarks preceding Example 3.8, it seems apposite to search for a common generalization of [17, Exercise E4.8] and Proposition 3.9 by extending those results to a natural class of LCTVSSs broad enough to house both Fréchet and weakly complete spaces. The following result is one way to achieve that goal.

Theorem 3.15. *Suppose a complete LCTVS E is expressible as a cofiltered limit*

$$E \cong \varprojlim_i E_i, \quad E_i := E/H_i \text{ complete} \tag{15}$$

for a family of closed subspaces $H_i \leq E$, filtered when ordered by reverse inclusion. If for each i the countable power $E_i^{\aleph_0}$ is globalizing, so is E .

Before moving on to the proof, we record one consequence that motivated the investigation to begin with: Corollary 3.16 generalizes both Proposition 3.9 (obviously) and [17, Exercise E4.8], by specializing to products of copies of \mathbb{R} or \mathbb{C} , which are of course Fréchet.

Corollary 3.16. *Arbitrary products of Fréchet spaces are globalizing.*

Proof. Indeed, countable products of Fréchet spaces are again Fréchet [28, §18.3(6)] and hence globalizing by Proposition 3.9, and arbitrary products $E := \prod_i E_i$ are expressible as cofiltered limits

$$E \cong \varprojlim_{|\mathcal{F}| < \infty} \left(E_{\mathcal{F}} := \prod_{i \in \mathcal{F}} E_i \right)$$

meeting the requirements of Theorem 3.15. ■

It will be worthwhile to pick up a few more auxiliary observations along the way. Recall [34, §4.9] that a(n automatically closed [34, Theorem 4.9.3]) subspace $E_0 \leq E$ of a TVS is *complemented* if $E \cong E_0 \oplus E_1$ linearly and topologically.

Lemma 3.17. *The property of being (\mathbb{G} -)globalizing passes over to complemented subspaces.*

Proof. A \mathbb{G} -action on a complemented subspace $E_0 \leq E$ extends over to E by making \mathbb{G} act trivially on a complement $E_1 \leq E$ with $E \cong E_0 \oplus E_1$, (local) \mathbb{G} -finiteness transports over between the original action and the extension, etc. ■

In particular, the hypothesis of Theorem 3.15 now implies that *finite* powers of each E_i are also globalizing, since they are summands in countable powers; we take this for granted.

Proof of Theorem 3.15. One difficulty is that a jointly continuous representation $\mathbb{G} \xrightarrow{\varrho} GL_s(E)$, fixed for the duration of the proof, need not be compatible with the limit realization (15). Part of the proof will supply that deficiency.

(I) **Reduction to separable \mathbb{G} .** That is, we argue here that it is enough to assume \mathbb{G} *separable* in the usual [52, Problem 5F] sense that it has a dense countable subset.

Indeed, irreducible representations being uniquely determined by their characters [17, Corollary 4.5], countably infinitely many inequivalent irreducible \mathbb{G} -representations will always return infinitely many mutually inequivalent irreducible summands if restricted to an appropriate countably-generated (hence separable) closed subgroup $\mathbb{H} \leq \mathbb{G}$. The other relevant properties such as (local) finiteness are also preserved upon restricting from \mathbb{G} to \mathbb{H} , so we may work with the latter throughout.

We henceforth assume \mathbb{G} separable.

(II) **Reduction to \mathbb{G} -equivariant limits (15).** The goal here is to show that the conclusion holds generally if it holds under the assumption that the $H_i \leq E$ of (15) are \mathbb{G} -invariant. We do this by substituting

$$[\mathbb{G}]H_i := \bigcap_{g \in \mathbb{G}} gH_i \quad \text{for } H_i \quad (16)$$

respectively. Observe first that in (16) we can always intersect over g ranging over any dense subset $D \subseteq \mathbb{G}$. Indeed, H_i is [28, §18.1(3)] the joint kernel of those continuous *seminorms* [28, §14.1] p_s on E which vanish on H_i , so that

$$\bigcap_{g \in \mathbb{G}} gH_i = \bigcap_{g \in \mathbb{G}; s} \ker p_s(g-) = \bigcap_{g \in D; s} \ker p_s(g-)$$

by the continuity of the action: for fixed $v \in E$ and index s we have

$$p_s(gv) = 0, \forall g \in \mathbb{G} \quad \Leftrightarrow \quad p_s(gv) = 0, \forall g \in \text{dense subset } D \subseteq \mathbb{G}.$$

Our separability assumption then allows us to take D countable; assume also that $1 \in D$ for convenience. Consider the embedding

$$E/[\mathbb{G}]H_i \hookrightarrow \prod_{g \in D} E/gH_i, \quad (17)$$

equipping its domain with the subspace topology.

The image is the graph of the TVS morphism

$$E/H_i \xrightarrow{(1 \neq g \in D)} \prod_{1 \neq g \in D} E/gH_i,$$

where

$$E/H_i \xrightarrow[\cong]{g} E/gH_i$$

denotes the isomorphism induced by the action of $g \in G$. Said image is thus complemented (so also complete, because closed) in the product

$$\prod_{g \in D} E/gH_i \cong E/H_i \times \prod_{1 \neq g \in D} E/gH_i,$$

assumed globalizing. This makes $E/[\mathbb{G}]H_i$ (with the subspace topology (17)) globalizing, and we are done upon observing that the limit realization (15) also implies the analogous

$$E \cong \varprojlim_i E/[\mathbb{G}]H_i.$$

(III) **Conclusion, assuming (15) is \mathbb{G} -equivariant.** Simply apply Corollary 3.12. ■

Remark 3.18. The proof of Theorem 3.15 required some care in distinguishing the quotient topology on the domain of (17) from the subspace topology induced by that embedding. This is not a moot point, as the two need not coincide. The problem (of whether or not they do) is moreover linked to the issue of sums of closed subspaces failing to be closed. Consider

$$\text{closed } E_0, E_1 \leq \text{complete metrizable TVS } E.$$

Quotienting everything in sight by $E_0 \cap E_1$ (which will not affect complete metrizable-ability [28, pre §15.11(4)]), we may assume that E_0 and E_1 intersect trivially. Paraphrased, [27, Theorem 1] (stated there for Banach spaces but going through in the complete metrizable case because the key ingredient is the *closed graph theorem* [28, §12.12(3)]) says that

$$E_0 + E_1 \leq E \iff E/E_0 \times E/E_1$$

is a homeomorphism onto its image if and only if $E_0 + E_1 \leq E$ is closed. That it isn't always is well known (see for instance [15, §15 and p.110] for two different approaches to constructing examples of such pathologies). ■

Consider, in closing, yet another way in which the isotypic-component machinery of [17, Chapters 3 and 4] can go awry, this time when local convexity is absent.

Example 3.19. Set $\mathbb{G} := \mathbb{S}^1$ and, for fixed $0 < p < 1$, consider the space $E := L^p(\mathbb{G}, \mu_{Haar})$ (complex-valued functions) of [28, §15.9]. This is the preminent example of a *quasi-Banach* space [37, §A.1] (equivalently [42, Theorem 3.2.1], complete locally bounded TVS): complete with respect to the topology induced by

$$\|f\| := \|f\|_p^p := \int_{\mathbb{G}} |f|^p \, d\mu_{Haar}$$

This is a p -norm in the sense of [28, §15.10], i.e. a function satisfying the usual axioms of a norm except for homogeneity, which is instead replaced by p -homogeneity: $\|\lambda f\| = |\lambda|^p \|f\|$ for scalars λ .

\mathbb{G} acts by rotations $f \xrightarrow{g^v} f(g^{-1}\cdot)$ jointly continuously, and

$$E_{fin} = \text{span} \{z \mapsto z^n \mid n \in \mathbb{Z}\} \subset E$$

is of course dense and algebraically a direct sum of its isotypic components $E_n := \text{span}\{z^n\}$, $n \in \mathbb{Z}$. There are, however, no continuous spectral projections $E \rightarrow E_n$ as in [17, Theorem 4.22], for none of the spaces E_n are complemented in E , see [28, §15.9(10)].

Remark 3.20. The problem in Example 3.19 stems from the fact that one would like to define said spectral projections by integration (as in (9) for instance), and vector-valued integration is problematic in the absence of convexity. Recall first [42, Theorem 3.5.1] that the *Riemann integrability* [42, §3.5] of all E -valued continuous functions on a compact interval *implies* the local convexity of the complete metrizable TVS E . Sufficiently regular functions (e.g. analytic [42, Theorem 3.5.2] or more generally, sufficiently continuously differentiable [42, Theorem 3.5.4]) are integrable, but in the present case most

$$\mathbb{S}^1 \ni g \mapsto gv \in E, \quad v \in E$$

are *not* continuously differentiable: simply adapt the well-known example [13, Chapter 3, Example 8] of a nowhere differentiable continuous function on $[0, 1]$, by extending it to a \mathbb{Z} -periodic function on the real line and hence a piecewise-continuous function on $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$.

References

- [1] J. Adámek, H. Herrlich, G. E. Strecker: *Abstract and concrete categories: the joy of cats*, Repr. Theory Appl. Categ. 2006/17 (2006) 1–507.
- [2] C. D. Aliprantis, K. C. Border: *Infinite Dimensional Analysis. A Hitchhiker’s Guide*, 3rd ed., Springer, Berlin (2006).
- [3] M. Artin, A. Grothendieck, J. L. Verdier, N. Bourbaki, P. Deligne, B. Saint-Donat (eds): *Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964. Théorie des Topos et Cohomologie Étale des Schémas. (SGA 4). Tome 1: Théorie des Topos. Exposés I à IV. 2e éd., Lecture Notes in Mathematics 269*, Springer, Cham (1972).
- [4] The Stacks Project Authors: *Stacks project*, available at <https://stacks.math.columbia.edu/>.
- [5] B. Bekka, P. de la Harpe, A. Valette: *Kazhdan’s property (T)*, New Mathematical Monographs 11, Cambridge University Press, Cambridge (2008).
- [6] N. Bourbaki: *Elements of Mathematics. Topological Vector spaces. Chap. 1–5*, soft-cover printing of the 1st English edition of 1987, Springer, Berlin (2003).
- [7] N. Bourbaki: *Elements of Mathematics. General Topology. Part 1*, Actualites Scientifiques et Industrielles, Hermann, Paris (1966).
- [8] N. Bourbaki: *Elements of Mathematics. General Topology. Part 2*, Actualites Scientifiques et Industrielles, Hermann, Paris (1966).
- [9] N. Bourbaki: *Elements of Mathematics. Integration II. Chapters 7–9*, Springer, Berlin (2004).
- [10] A. Chirvasitu: *(Quantum) discreteness, spectrum compactness and uniform continuity*, arXiv: 2310.15139v1 (2023).

- [11] Y. Cornuier, P. de la Harpe: *Metric Geometry of Locally Compact Groups*, EMS Tracts in Mathematics 25, European Mathematical Society, Zürich (2016).
- [12] J. Dixmier: *C^* -Algebras*, North-Holland Mathematical Library 15, North-Holland Publishing Company, Amsterdam (1977).
- [13] B. R. Gelbaum, J. M. H. Olmsted: *Counterexamples in Analysis*, corrected reprint of the second edition edition 1965, Dover Publications, Mineola (2003).
- [14] D. Gurarie: *Banach uniformly continuous representations of Lie groups and algebras*, J. Funct. Analysis 36 (1980) 401–407.
- [15] P. R. Halmos: *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, 2nd ed., Chelsea Publishing Company, New York (1957).
- [16] K. H. Hofmann, S. A. Morris: *The Lie Theory of Connected Pro-Lie Groups. A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups*, EMS Tracts in Mathematics 2, European Mathematical Society, Zürich (2007).
- [17] K. H. Hofmann, S. A. Morris: *The Structure of Compact Groups – a Primer for the Student – a Handbook for the Expert*, 4th ed., De Gruyter Studies in Mathematics 25, De Gruyter, Berlin (2020).
- [18] T. J. Huang: *A characterization of maximally almost periodic groups*, Proc. Amer. Math. Soc. 75/1 (1979) 59–62.
- [19] A. Inoue: *Locally C^* -algebra*, Mem. Fac. Sci. Kyushu Univ. Ser. A 25 (1971) 197–235.
- [20] K. Iwasawa: *On some types of topological groups*, Ann. Math. (2) 50 (1949) 507–558.
- [21] K. Iwasawa: *Topological groups with invariant compact neighborhoods of the identity*, Ann. Math. (2) 54 (1951) 345–348.
- [22] I. James: *Topologies and Uniformities*, exp. and rev. version of the first edition 1987, Springer Undergraduate Mathematics Series, Springer, London (1999).
- [23] R. V. Kadison, I. M. Singer: *Some remarks on representations of connected groups*, Proc. Natl. Acad. Sci. USA 38 (1952) 419–423.
- [24] R. R. Kallman: *A characterization of uniformly continuous unitary representations of connected locally compact groups*, Michigan Math. J. 16 (1969) 257–263.
- [25] E. Kaniuth, K. F. Taylor: *Induced Representations of Locally Compact Groups*, Cambridge Tracts in Mathematics 197, Cambridge University Press, Cambridge (2013).
- [26] S. M. Khaleelulla: *Counterexamples in Topological Vector Spaces*, Lecture Notes in Mathematics 936, Springer, Cham (1982).
- [27] H. Kober: *A theorem on Banach spaces*, Compos. Math. 7 (1939) 135–140.
- [28] G. Köthe: *Topological Vector Spaces I*, Grundlehren der mathematischen Wissenschaften 159, Springer, New York (1969).
- [29] G. Köthe: *Topological Vector Spaces II*, Grundlehren der mathematischen Wissenschaften 237, Springer, New York (1979).
- [30] D. Luminet, A. Valette: *Faithful uniformly continuous representations of Lie groups*, J. London Math. Soc., II. Ser. 49/1 (1994) 100–108.
- [31] S. Mac Lane: *Categories for the Working Mathematician*, 2nd ed., Graduate Texts in Mathematics 5, Springer, New York (1998).
- [32] D. Montgomery, L. Zippin: *Topological Transformation Groups*, reprint of the 1955 original, Krieger Publishing Co., Huntington (1974).

- [33] S. Murakami: *Remarks on the structure of maximally almost periodic groups*, Osaka Math. J. 2 (1950) 119–129.
- [34] L. Narici, E. Beckenstein: *Topological Vector Spaces*, 2nd ed., Pure and Applied Mathematics 296, CRC Press, Boca Raton (2011).
- [35] K.-H. Neeb: *Infinite-dimensional groups and their representations*, in: *Lie Theory*, Progress in Mathematics 228, Birkhäuser, Boston (2004) 213–328.
- [36] K.-H. Neeb: *Towards a Lie theory of locally convex groups*, Japanese J. Math. 1/2 (2006) 291–468.
- [37] M. Pavlović: *Function Classes on the Unit Disc. An Introduction*, 2nd rev. and ext. ed., De Gruyter Studies in Mathematics 52, De Gruyter, Berlin (2019).
- [38] N. C. Phillips: *Inverse limits of C^* -algebras*, J. Operator Theory 19/1 (1988) 159–195.
- [39] D. Repovš, P. V. Semenov: *Continuous selections of multivalued mappings*, in: *Recent Progress in General Topology III*, Prague Symposium 2001, Atlantis Press, Amsterdam (2014) 711–749.
- [40] A. Robert: *Introduction to the Representation Theory of Compact and Locally Compact Groups*, London Mathematical Society Lecture Notes Series 80, Cambridge University Press, Cambridge (1983).
- [41] A. P. Robertson, W. Robertson: *Topological Vector Spaces*, 2nd ed., Cambridge Tracts in Mathematics 53, Cambridge University Press, Cambridge (1980).
- [42] S. Rolewicz: *Metric Linear Spaces*, 2nd ed., Mathematics and its Applications 20, Kluwer Academic Publishers Group, Dordrecht (1985).
- [43] S. Rothman: *The von Neumann kernel and minimally almost-periodic groups*, Trans. Am. Math. Soc. 259 (1980) 401–421.
- [44] H. H. Schaefer, M. P. Wolff: *Topological Vector Spaces*, 2nd ed., Graduate Texts in Mathematics 3, Springer, New York (1999).
- [45] E. Schechter: *Handbook of Analysis and its Foundations*, Academic Press, San Diego (1997).
- [46] A. I. Shtern: *Norm continuous representations of locally compact groups*, Russ. J. Math. Physics 15/4 (2008) 552–553.
- [47] I. M. Singer: *Uniformly continuous representations of Lie groups*, Ann. Math. (2) 56 (1952) 242–247.
- [48] F. Trèves: *Topological Vector Spaces, Distributions and Kernels*, reprint of the 1967 original, Dover, Mineola (2006).
- [49] W. C. Waterhouse: *An empty inverse limit*, Proc. Amer. Math. Soc. 36 (1973) 618.
- [50] A. Weil: *L'Intégration dans les Groupes Topologiques et ses Applications*, Actualités Scientifiques et Industrielles 869, Hermann, Paris (1940).
- [51] T. W. Wilcox: *On the structure of maximally almost periodic groups*, Bull. Amer. Math. Soc. 73 (1967) 732–734.
- [52] S. Willard: *General Topology*, reprint of the 1970 original, Dover, Mineola (2004).

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