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On $(\alpha, 1, 0)$-Derivations of Anti-Commutative Algebras

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Abstract. The aim of this paper is to investigate $(\alpha, 1, 0)$-derivations of anti-commutative algebras. We show that when the base field is of characteristic zero, the dimensions of the spaces of $(\alpha, 1, 0)$-derivations yield an infinite one-parameter family of invariant functions under algebra isomorphism. Furthermore, we demonstrate that this infinite family reduces to only three distinct functions when we restrict our focus to the class of Lie algebras. This reduction addresses an open problem regarding the behavior of these invariants in the context of Lie algebras. Additionally, we establish sharp bounds for these invariant functions.

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Key Words: Anti-commutative algebras, Lie algebras, invariants of algebra, extended derivations of algebras, isomorphism problem.

1. Introduction

The concept of (α, β, γ)-*derivations* of Lie algebras, an extension of δ-*derivations* (originally introduced and studied by Filippov [2]), was first proposed by Hrivnák and Novotný [5] to define new invariants of (Lie) algebras. If two algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, then the corresponding spaces of (α, β, γ)-derivations, $\mathcal{D}(\alpha, \beta, \gamma)(\mathfrak{A})$ and $\mathcal{D}(\alpha, \beta, \gamma)(\mathfrak{B})$, are isomorphic vector spaces, making $\operatorname{Dim} \mathcal{D}(\alpha, \beta, \gamma)(\square)$ an *invariant* of algebras.

The study of derivations of algebras and their generalizations has played a key role in uncovering structural results for various classes of algebras (see [4, Section 6] and references therein). For instance, (α, β, γ)-derivations have been shown to provide sufficient invariants to distinguish three-dimensional complex Lie algebras up to isomorphism.

It is worth noting that (α, β, γ)-*derivations* are a special case of *extended derivations* (see [1]), which have proven useful in the study of *algebra degenerations*, an active research topic with applications in mathematical physics, differential geometry, harmonic analysis, and algebra.

Let $C^2(V; V)$ denote the space of antisymmetric bilinear maps on an n-dimensional vector space V over a field $\mathbb{K}$. An open problem posed by Hrivnák and Novotný [5, § 4], specifically in their fifth concluding remark, is to determine the behavior of the one-parameter family of invariant functions $\{\phi_{n,t}\}_{t \in \mathbb{K}}$, where

$$\phi_{n,t} : C^2(V; V) \to \{0, 1, \dots, n^2\}$$

is defined by $\phi_{n,t}(\mu) = \operatorname{Dim} \mathcal{D}(t, 1, 0)(\mathfrak{A})$, with $\mathfrak{A} = (V, \mu)$. Is this family infinite?

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Moreover, is the restriction $\widehat{\phi}_{n,t}$ of $\phi_{n,t}$ to $\mathcal{L}(V)$, the *variety of Lie algebra laws on* V, finite?

In this paper, we answer both questions affirmatively and establish sharp bounds for $\widehat{\phi}_{n,t}$. The proofs provided are elementary in nature, relying on well-known techniques.

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Conventions. For simplicity, we assume throughout this paper that all $\mathbb{K}$-algebras are finite dimensional and that the base field $\mathbb{K}$ is of characteristic zero.

Definition 1.1. [5, §2] Let $\mathfrak{A} = (V, \mu)$ be an anti-commutative $\mathbb{K}$-algebra and let $\alpha, \beta, \gamma \in \mathbb{K}$ be three constants. An (α, β, γ)*-derivation* of $\mathfrak{A}$ is a linear transformation D from V to itself, such that

$$\alpha D\mu(X, Y) = \beta\mu(DX, Y) + \gamma\mu(X, DY)$$

for all $X, Y \in V$. The set all (α, β, γ)-derivations of $\mathfrak{A}$ is denoted by $\mathcal{D}(\alpha, \beta, \gamma)(\mathfrak{A})$.

Notation. Let U, V, $\widetilde{V}$ and W be vector spaces over $\mathbb{K}$. Let us write by $L(V;\widetilde{V})$ the set of all linear transformations from V to $\widetilde{V}$, and let $\Theta_{V;\widetilde{V}} \in L(V;\widetilde{V})$ denote *the zero map from* V *to* $\widetilde{V}$, which is defined by $\Theta_{V;\widetilde{V}}(X) = 0_{\widetilde{V}}$ for all $X \in V$ (or simply $\Theta_{V;\widetilde{V}}(X) = 0$ when unambiguous).

If W is a vector subspace of V, we denote by $\iota_{W;V}$ (or simply ι_W when no confusion can arise) the *inclusion map* from W to V. Recall that any linear transformation $T \in L(\widetilde{V}; V)$ whose image is contained in W induces a unique linear map $\overline{T}$ in $L(\widetilde{V}; W)$ such that $T = \iota_{W;V} \circ \overline{T}$ (*restriction of the codomain of a linear map*).

If U and W are *complementary subspaces* of V; i.e. V is an *internal direct sum* of U and W, we write $\pi_U^{U\oplus W}$ and $\pi_W^{U\oplus W}$ for the *projection maps* from V onto U and W, respectively (or simply π_U and π_W when the decomposition $V = U \oplus W$ is clear from the context). Let $Q \in L(U, \widetilde{V})$ and $T \in L(W, \widetilde{V})$ be arbitrary linear maps. Let us denote by $Q \oplus T$ the unique linear map in $L(V;\widetilde{V})$ such that $(Q \oplus T) \circ \iota_{U;V} = Q$ and $(Q \oplus T) \circ \iota_{W;V} = T$ (*characteristic property of the direct sum*).

Notation. [Centralizer and derived algebra] Let $\mathfrak{A} = (V, \mu)$ be an anti-commutative algebra, and let $S \subseteq \mathfrak{A}$ be any subset. Let us denote by $\mathcal{C}_{\mathfrak{A}}(S)$ the *centralizer* of S in $\mathfrak{A}$, and by $\mathfrak{A}^{(2)}$, the *derived algebra* of $\mathfrak{A}$.

Remark 1.2. Centralizers are vector subspaces of $\mathfrak{A}$ and they need not be subalgebras on account of the lack of associativity or similar properties (such as *Jacobi identity*). If $t \neq 0$ and D is a $(t, 1, 0)$-derivation of $\mathfrak{A}$, then it is clear that D preserves centralizers; and in particular *the center* of $\mathfrak{A}$, $\mathcal{Z}(\mathfrak{A}) := \mathcal{C}_{\mathfrak{A}}(\mathfrak{A})$. ■

Remark 1.3. It is worth remarking that if D is a $(t,1,0)$-derivation of $\mathfrak{A} = (V,\mu)$, then D is also a $(t,0,1)$-derivation of $\mathfrak{A}$ (because of the anti-commutative property). Therefore, $\mathcal{D}(t,1,0)(\mathfrak{A}) \subseteq \mathcal{D}(0,1,-1)(\mathfrak{A})$. ∎

2. The family $\{\phi_{n,t}\}_{t\in\mathbb{K}}$ is infinite

To answer the first question, we present a one-parameter family of pairwise non-isomorphic anti-commutative algebras of dimension n, which can be distinguished using the family $\{\phi_{n,t}\}_{t\in\mathbb{K}}$. If t is not a root of unity and $t \neq 0,1$, and if $\mathfrak{A}$ is a centerless anti-commutative algebra over an algebraically closed field $\mathbb{K}$, then it is a simple matter to see that any $(t,1,0)$-derivation of $\mathfrak{A}$ is a nilpotent linear transformation. So, we start by assuming that the dimension of V is four and let $\{e_1,e_2,e_3,e_4\}$ be a basis of V. Consider the linear transformation $D: V \to V$ defined by $e_1 \mapsto e_2$, $e_3 \mapsto e_4$ and $e_2, e_4 \mapsto 0$. Let $t \in \mathbb{K}$ be arbitrary. We now ask: what are all four-dimensional anti-commutative algebras for which D is a $(t,1,0)$-derivation? Such algebras are of the form:

$$\begin{cases} \mu(e_1,e_3) = ae_1 + ce_2 + be_3 + de_4, \\ \mu(e_1,e_4) = tae_2 + tbe_4, \\ \mu(e_2,e_3) = tae_2 + tbe_4, \end{cases}$$

where $a,b,c,d \in \mathbb{K}$ are free variables. Note the dependence of the *structure constants* of these algebras on the parameter t.

Proposition 2.1. *Let $\mathfrak{A}_s = (V,\mu_s)$ be the one-parameter family of four-dimensional anti-commutative algebras given by*

$$\mu_s = \{\ \mu_s(e_1,e_3) = e_1,\ \mu_s(e_1,e_4) = se_2,\ \mu_s(e_2,e_3) = se_2,$$

with $s \in \mathbb{K}$. If $t \neq 0,1$, then the value of the invariant function $\phi_{4,t}$ at $\mathfrak{A}_s$ is

$$\phi_{4,t}(\mathfrak{A}_s) = \begin{cases} 6, & \textit{if } s{=}0 \\ 1, & \textit{if } s{=}t \\ 0, & \textit{otherwise} \end{cases}$$

and therefore $\phi_{4,t_1} = \phi_{4,t_2}$ if and only if $t_1 = t_2$ (with $t_1,t_2 \neq 0,1$).

Proof. Let $A: V \to V$ be a linear transformation defined by $Ae_j = \sum_{i=1}^{4} A_j^i e_i$, for $j = 1,\ldots,4$, where A_j^i are constants in $\mathbb{K}$. We want to determine the conditions that A must satisfy to be in $\mathcal{D}(t,1,0)(\mathfrak{A}_s)$. We consider the case $s \neq 0$, in which the derived algebra of $\mathfrak{A}_s$ is generated by $\{e_1,e_2\}$ and the centralizer of e_3 is generated by $\{e_3,e_4\}$. Suppose that $A \in \mathcal{D}(t,1,0)(\mathfrak{A}_s)$. Since $t \neq 0$, any $(t,1,0)$-derivation preserves the derived algebra and centralizers, we obtain $A_j^i = 0$ for each $i = 3,4$, $j = 1,2$, and also for each $i = 1,2$, $j = 3,4$. This implies that $A_2^1 = 0$ and $A_4^3 = 0$, because $\mu_s(Ae_2,e_4) = \mu_s(e_2,Ae_4) = tD\mu_s(e_2,e_4) = 0$; and so $Ae_2 \in \operatorname{span}\{e_1,e_2\} \cap \mathcal{C}_{\mathfrak{A}_s}(e_4)$ and $Ae_4 \in \operatorname{span}\{e_3,e_4\} \cap \mathcal{C}_{\mathfrak{A}_s}(e_2)$.

Next we study the equality between

$$tA\mu_s(e_1,e_3) = tAe_1 = tA_1^1e_1 + tA_1^2e_2, \quad \mu_s(Ae_1,e3) = A_1^1e_1 + sA_1^2e_2$$

and

$$\mu_s(e_1,Ae_3) = A_3^3e_1 + sA_3^4e_2.$$

Since $t \neq 1$, we have $A_1^1 = 0$, and so $A_3^3 = 0$ and also $(t-s)A_1^2 = 0$, $A_1^2 = A_3^4$ (recall $s \neq 0$). It follows immediately that $A_2^2 = 0$ and $A_4^4 = 0$ since

$$tA\mu_s(e_1, e_4) = tA(se_2) = tsA_2^2 e_2,$$

$\mu_s(Ae_1, e_4) = \mu_s(A_1^2 e_2, e_4) = 0$ and $\mu_s(e_1, Ae_4) = sA_4^4 e_2$ must be equal.

Therefore, if $t = s$, then A_1^2 is a free variable and $\mathcal{D}(t,1,0)(\mathfrak{A}_t)$ is a vector space generated by the nilpotent linear transformation D we just defined above, and if $t \neq s$, then $A = 0$ and so $\dim \mathcal{D}(t,1,0)(\mathfrak{A}_s) = 0$.

The case $s = 0$, which corresponds to the Lie algebra $\mathfrak{aff}(\mathbb{K}) \times \mathbb{K}^2$ (here $\mathfrak{aff}(\mathbb{K}^n)$ is the *affine Lie algebra* of $\mathbb{K}^n$), is straightforward or it can be solved by using the result of the Lemma 2.3. ∎

Notation. Let $\mathfrak{A} = (V, \mu)$ be an anti-commutative $\mathbb{K}$-algebra. Let us denote by $\Omega(\mathfrak{A})$ the set

$$\{T \in L(V;V) : \operatorname{Im}(T) \subseteq \mathcal{Z}(\mathfrak{A}) \text{ and } \mathfrak{A}^{(2)} \subseteq \operatorname{Ker}(T)\},$$

which is the intersection of the two vector spaces $\mathcal{D}(0,1,0)(\mathfrak{A})$ and $\mathcal{D}(1,0,0)(\mathfrak{A})$, and is isomorphic to the vector space $L(\mathfrak{A}/\mathfrak{A}^{(2)}; \mathcal{Z}(\mathfrak{A}))$. Moreover, note that $\Omega(\mathfrak{A})$ is contained in $\mathcal{D}(t,1,0)(\mathfrak{A})$, for any $t \in \mathbb{K}$.

The importance of the set $\Omega(\mathfrak{A})$ is in the elementary observation that any $(t,1,0)$-derivation is determined by its action on $\mathfrak{A}^{(2)}$ (modulo $\Omega(\mathfrak{A})$).

Proposition 2.2. *Let $\mathfrak{A} = (V, \mu)$ be an anti-commutative $\mathbb{K}$-algebra and let D_1 and D_2 be two $(t,1,0)$-derivations of $\mathfrak{A}$. Then D_1 and D_2 agree on $\mathfrak{A}^{(2)}$ if and only if $D_1 - D_2 \in \Omega(\mathfrak{A})$.*

Proof. Suppose that D_1 and D_2 are equal on $\mathfrak{A}^{(2)}$, we have $\mathfrak{A}^{(2)} \subseteq \operatorname{Ker}(D_1 - D_2)$. It remains only to show that $\operatorname{Im}(D_1 - D_2) \subseteq \mathcal{Z}(\mathfrak{A})$. Let $X, Y \in \mathfrak{A}$ be arbitrary. Then

$$\begin{aligned}
\mu((D_1 - D_2)X, Y) &= \mu(D_1X, Y) - \mu(D_2X, Y) && \text{(by linearity)}\\
&= tD_1\mu(X,Y) - tD_2\mu(X,Y) && \text{(since } D_1, D_2 \in \mathcal{D}(t,1,0)(\mathfrak{A}))\\
&= 0 && \text{(since } D_1, D_2 \text{ agree on } \mathfrak{A}^{(2)}). \quad \blacksquare
\end{aligned}$$

Lemma 2.3. *Let $\mathfrak{A}$ be an anti-commutative $\mathbb{K}$-algebra and suppose $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}$ is the direct sum of the subalgebras $\mathfrak{B}$ and $\mathcal{Z}(\mathfrak{A})$. Then $\mathcal{D}(t,1,0)(\mathfrak{A})$ is a vector space isomorphic to $\mathcal{D}(t,1,0)(\mathfrak{B}) \times L(\mathfrak{A}/\mathfrak{A}^{(2)}; \mathcal{Z}(\mathfrak{A}))$ if $t \neq 0$.*

Proof. Define a linear transformation $\varphi : \mathcal{D}(t,1,0)(\mathfrak{A}) \to \mathcal{D}(t,1,0)(\mathfrak{B})$ given by $D \mapsto \pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}}$. To see that φ is well defined; i.e. $\pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}}$ is a $(t,1,0)$-derivation of $\mathfrak{B}$, let $X, Y \in \mathfrak{B}$ be arbitrary:

$$\begin{aligned}
\pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}}(t\mu(X,Y)) &= \pi_{\mathfrak{B}} \circ D(t\mu(X,Y)) && \text{(since } \mathfrak{B} \text{ subalgebra of } \mathfrak{A})\\
&= \pi_{\mathfrak{B}}(\mu(DX,Y)) && \text{(since } D \in \mathcal{D}(t,1,0)(\mathfrak{A}))\\
&= \mu(DX,Y) && \text{(since } \mathfrak{A}^{(2)} = \mathfrak{B}^{(2)})\\
&= \mu(D \circ \iota_{\mathfrak{B}}(X), Y) && \text{(since } X \in \mathfrak{B})
\end{aligned}$$

$$= \mu(\pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}}(X), Y) + \mu(\pi_{\mathcal{Z}(\mathfrak{A})} \circ D \circ \iota_{\mathfrak{B}}(X), Y) \quad (\text{since } \mathfrak{A} = \mathfrak{B} \oplus \mathcal{Z}(\mathfrak{A}))$$
$$= \mu(\pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}}(X), Y) \quad (\pi_{\mathcal{Z}(\mathfrak{A})} \circ D \circ \iota_{\mathfrak{B}}(X) \in \mathcal{Z}(\mathfrak{A}))$$

We note that φ passes to the quotient $\mathcal{D}(t,1,0)(\mathfrak{A})/\Omega(\mathfrak{A})$, yielding a linear transformation $\widetilde{\varphi} : \mathcal{D}(t,1,0)(\mathfrak{A})/\Omega(\mathfrak{A}) \to \mathcal{D}(t,1,0)(\mathfrak{B})$. To see this, let D_1 and D_2 in $\mathcal{D}(t,1,0)(\mathfrak{A})$ such that $D_1 - D_2 \in \Omega(\mathfrak{A})$, we need to show that $\pi_{\mathfrak{B}} \circ D_1 \circ \iota_{\mathfrak{B}}$ is equal to $\pi_{\mathfrak{B}} \circ D_2 \circ \iota_{\mathfrak{B}}$. For this purpose, let $X \in \mathfrak{B}$. We have that $\pi_{\mathfrak{B}} \circ (D_1 - D_2) \circ \iota_{\mathfrak{B}}(X)$ is equal to $\pi_{\mathfrak{B}} \circ (D_1 - D_2)(X)$ and since $\operatorname{Im}(D_1 - D_2) \subseteq \mathcal{Z}(\mathfrak{A})$, we get $\pi_{\mathfrak{B}} \circ (D_1 - D_2) \circ \iota_{\mathfrak{B}}(X) = 0$.

Next we show that $\widetilde{\varphi}$ is actually an isomorphism of vector spaces. To prove that it is injective, suppose $D \in \mathcal{D}(t,1,0)(\mathfrak{A})$ is such that $\pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}} = 0$ and let us prove $D \in \Omega(\mathfrak{A})$. We show first that $DX \in \mathcal{Z}(\mathfrak{A})$ for any $X \in \mathfrak{A}$. Let $Y \in \mathfrak{A}$ be arbitrary,

$$\begin{aligned} \mu(DX, Y) &= tD\mu(X, Y) \\ &= t\pi_{\mathfrak{B}} \circ D(\mu(X, Y)) && (\text{since } D \text{ preserves } \mathfrak{A}^{(2)}, \text{ and } \mathfrak{A}^{(2)} = \mathfrak{B}^{(2)}) \\ &= t\pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}}(\mu(X, Y)) && (\text{again, since } \mathfrak{A}^{(2)} = \mathfrak{B}^{(2)}) \\ &= 0. \end{aligned}$$

It remains to check $\mathfrak{A}^{(2)} \subseteq \operatorname{Ker}(D)$, which also follows from the preceding calculation, since $t \neq 0$.

To show surjectivity, let $\widehat{D}$ be a $(t,1,0)$-derivation of $\mathfrak{B}$. Given the assumption that $\mathfrak{A} = \mathfrak{B} \oplus \mathcal{Z}(\mathfrak{A})$, we define $D = \iota_{\mathfrak{B}} \circ \widehat{D} \oplus \Theta_{\mathcal{Z}(\mathfrak{A});\mathfrak{A}}$. Recall that D is defined by $D \circ \iota_{\mathfrak{B}} = \iota_{\mathfrak{B}} \circ \widehat{D}$ and $D \circ \iota_{\mathcal{Z}(\mathfrak{A})} = \Theta_{\mathcal{Z}(\mathfrak{A});\mathfrak{A}}$. We want to check that D is a $(t,1,0)$-derivation of $\mathfrak{A}$. Let $X, Y \in \mathfrak{A}$ be arbitrary:

$$\begin{aligned} &\mu(DX, Y) = \mu(D \circ \iota_{\mathfrak{B}} \circ \pi_{\mathfrak{B}}(X), Y) + \mu(D \circ \iota_{\mathcal{Z}(\mathfrak{A})} \circ \pi_{\mathcal{Z}(\mathfrak{A})}(X), Y) \\ &= \mu(\iota_{\mathfrak{B}} \circ \widehat{D} \circ \pi_{\mathfrak{B}}(X), Y) + \mu(\Theta_{\mathcal{Z}(\mathfrak{A});\mathfrak{A}} \circ \pi_{\mathcal{Z}(\mathfrak{A})}(X), Y) && (\text{by definition of } D) \\ &= \mu(\iota_{\mathfrak{B}} \circ \widehat{D} \circ \pi_{\mathfrak{B}}(X), Y) \\ &= \mu(\iota_{\mathfrak{B}} \circ \widehat{D} \circ \pi_{\mathfrak{B}}(X), \pi_{\mathfrak{B}}(Y)) + \mu(\iota_{\mathfrak{B}} \circ \widehat{D} \circ \pi_{\mathfrak{B}}(X), \pi_{\mathcal{Z}(\mathfrak{A})}(Y)) && (\text{since } \mathfrak{A} = \mathfrak{B} \oplus \mathcal{Z}(\mathfrak{A})) \\ &= \mu(\widehat{D} \circ \pi_{\mathfrak{B}}(X), \pi_{\mathfrak{B}}(Y)) && (\text{since } \widehat{D} \circ \pi_{\mathfrak{B}}(X) \in \mathfrak{B}) \\ &= t\widehat{D}\mu(\pi_{\mathfrak{B}}(X), \pi_{\mathfrak{B}}(Y)) && (\text{since } \widehat{D} \in \mathcal{D}(t,1,0)(\mathfrak{B})) \\ &= t\iota_{\mathfrak{B}} \circ \widehat{D}(\mu(\pi_{\mathfrak{B}}(X), \pi_{\mathfrak{B}}(Y))) && (\text{since } \operatorname{Im} \widehat{D} \subseteq \mathfrak{B}) \\ &= tD \circ \iota_{\mathfrak{B}}(\mu(\pi_{\mathfrak{B}}(X), \pi_{\mathfrak{B}}(Y))) && (\text{by definition of } D) \\ &= tD\mu(\pi_{\mathfrak{B}}(X), \pi_{\mathfrak{B}}(Y)) && (\mathfrak{B} \text{ is subalgebra of } \mathfrak{A}) \\ &= tD\mu(\pi_{\mathfrak{B}}(X) + \pi_{\mathcal{Z}(\mathfrak{A})}(X), \pi_{\mathfrak{B}}(Y) + \pi_{\mathcal{Z}(\mathfrak{A})}(Y)) \\ &= tD\mu(X, Y). \end{aligned}$$

Since $\varphi(D) = \pi_{\mathfrak{B}} \circ D \circ \iota_{\mathfrak{B}} = \pi_{\mathfrak{B}} \circ \iota_{\mathfrak{B}} \circ \widehat{D} = \widehat{D}$, we have $\widetilde{\varphi}([D]) = \widehat{D}$; where $[D]$ stands for the equivalence class of D in $\mathcal{D}(t,1,0)(\mathfrak{A})/\Omega(\mathfrak{A})$. ∎

Although the Lemma 2.3 can be improved, it is quite sufficient to prove one of our main results.

Theorem 2.4. *For all $n \in \mathbb{N}$ with $n \geq 4$, the familiy $\{\phi_{n,t}\}_{t \in \mathbb{K}}$ is infinite.*

Proof. If $n = 4$, the result follows from Proposition 2.1. When $n = 4 + m$, with $m \in \mathbb{N}$, we consider the one-parameter family of algebras $\mathfrak{A}_s \times \mathbb{K}^m$, with $s \in \mathbb{K} \setminus \{0\}$, where $\mathfrak{A}_s$ is the four-dimensional $\mathbb{K}$-algebra defined in Propoposition 2.1. Let $t \in \mathbb{K}$, $t \neq 0, 1$. By the preceding lemma, $\mathcal{D}(t,1,0)(\mathfrak{A}_s \times \mathbb{K}^m)$ is isomorphic to $\mathcal{D}(t,1,0)(\mathfrak{A}_s) \times L(\mathrm{span}\{e_3, e_4\} \oplus \mathbb{K}^m; \mathbb{K}^m)$ (since $s \neq 0$). And so, $\phi_{n,t}(\mathfrak{A}_t \times \mathbb{K}^m) = 1 + (m+2)m$ and if $s \neq t$, $\phi_{n,t}(\mathfrak{A}_s \times \mathbb{K}^m) = 0 + (m+2)m$, which implies $\phi_{n,t} \neq \phi_{n,u}$ if $t \neq u$ (with $t, u \in \mathbb{K} \setminus \{0, 1\}$). ■

Remark 2.5. It is a simple matter to see that the preceding theorem does not hold with $n = 3$. Let $\mathfrak{A}$ be an anti-commutative algebra of dimension 3 over $\mathbb{K}$ and let $t \in \mathbb{K} \setminus \{0, 1\}$. If $\mathfrak{A}$ is centerless, then one can see that $\mathcal{D}(t,1,0)(\mathfrak{A}) = \{0\}$. If $\mathcal{Z}(\mathfrak{A}) \neq \{0\}$, then $\mathfrak{A}$ is a Lie algebra which is isomorphic to either $\mathbb{K}^3$, $\mathfrak{aff}(\mathbb{K}) \times \mathbb{K}$ or the Heisenberg Lie algebra $\mathfrak{h}_3(\mathbb{K})$, and an easy computation shows that $\phi_{3,t}(\mathfrak{h}_3(\mathbb{K})) = 3$ and $\phi_{3,t}(\mathfrak{aff}(\mathbb{K}) \times \mathbb{K}) = 2$.

3. The family $\{\widehat{\phi}_{n,t}\}_{t \in \mathbb{K}}$ is finite and has three elements

Notation. We denote by $\mathcal{L}(V)$ to the *algebraic subset* of $C^2(V; V)$ defined by

$$\mathcal{L}(V) = \{\lambda \in C^2(V; V) : \mathfrak{A} = (V, \lambda) \text{ is a Lie algebra}\},$$

which is usually called the *variety of Lie algebras on* V.

We start by proving an interesting algebraic identity which is satisfied by $(t,1,0)$-derivations of Lie algebras, with $t \neq 0, 1$.

Lemma 3.1. *Let $t \in \mathbb{K} \setminus \{0, 1\}$ and let $\mathfrak{g} = (V, \mu)$ be a Lie algebra. If D is a $(t,1,0)$-derivation of $\mathfrak{g}$ then for all $X, Y, Z \in \mathfrak{g}$*

$$D\mu(X, \mu(Y, Z)) = 0.$$

Proof. Let $X, Y, Z \in \mathfrak{g}$ be arbitrary

$$\begin{aligned}
D\mu(Z, \mu(X,Y)) &= \frac{1}{t}\mu(DZ, \mu(X,Y)) && (\text{since } D \in \mathcal{D}(t,1,0)(\mathfrak{g}))\\
&= \frac{1}{t}\mu(Z, D\mu(X,Y)) && (\text{since } D \in \mathcal{D}(0,1,-1)(\mathfrak{g}))\\
&= \frac{1}{t}\mu(Z, \frac{1}{t}\mu(DX,Y)) && (\text{since } D \in \mathcal{D}(t,1,0)(\mathfrak{g}))\\
&= -\frac{1}{t^2}\mu(DX, \mu(Y,Z)) - \frac{1}{t^2}\mu(Y, \mu(Z, DX)) && (\text{by Jacobi Identity})\\
&= -\frac{t}{t^2}D\mu(X, \mu(Y,Z)) - \frac{t^2}{t^2}D\mu(Y, \mu(Z,X))\\
& \qquad (\text{since } D \in \mathcal{D}(t,1,0)(\mathfrak{g}) \subseteq \mathcal{D}(0,1,-1)(\mathfrak{g})).
\end{aligned}$$

The last equality implies $-\frac{1}{t}D\mu(X, \mu(Y,Z))$ is equal to

$$D\mu(Z, \mu(X,Y)) + D\mu(Y, \mu(Z,X)) = -D\mu(X, \mu(Y,Z))$$

(by linearity and the Jacobi Identity), and since $t \neq 1$, it follows that $D\mu(X, \mu(Y,Z))$ must be 0. ■

The next theorem follows immediately from the preceding lemma.

Theorem 3.2. *Let* $t \in \mathbb{K} \setminus \{0,1\}$ *and let* $\mathfrak{g}$ *be a perfect Lie algebra over* $\mathbb{K}$*; i.e.* $\mathfrak{g}^{(2)} = \mathfrak{g}$*. Then* $\mathcal{D}(t,1,0)(\mathfrak{g}) = \{0\}$*.*

Remark 3.3. Note that we can obtain other interesting observations from Lemma 3.1, for instance, any $(t,1,0)$-derivation of a Lie algebra $\mathfrak{g}$ sends $\mathfrak{g}^{(2)}$ to $\mathfrak{g}^{(2)} \cap \mathcal{Z}(\mathfrak{g})$ (if $t \neq 0,1$). ■

Now, we focus our attention on non-perfect Lie algebras.

Theorem 3.4. *Let* $t \in \mathbb{K} \setminus \{0,1\}$ *and let* $s \in \mathbb{K} \setminus \{0\}$*. Let* $\mathfrak{g}$ *be a non-perfect Lie algebra over* $\mathbb{K}$*. Then* $\mathcal{D}(t,1,0)(\mathfrak{g})$ *can be embedded in* $\mathcal{D}(s,1,0)(\mathfrak{g})$*.*

Proof. Since $\mathfrak{g}$ is non-perfect, let $\mathfrak{a}$ be a complementary subspace of $\mathfrak{g}^{(2)}$ in $\mathfrak{g}$; i.e. $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{g}^{(2)}$. Given $D \in \mathcal{D}(t,1,0)(\mathfrak{g})$, consider the linear transformations $\frac{s}{t} D \circ \iota_{\mathfrak{g}^{(2)}}$ and $D \circ \iota_{\mathfrak{a}}$, and let $\widehat{D} = \frac{s}{t} D \circ \iota_{\mathfrak{a}} \oplus D \circ \iota_{\mathfrak{g}^{(2)}}$ be the unique linear transformation in $L(\mathfrak{g},\mathfrak{g})$ such that $\widehat{D} \circ \iota_{\mathfrak{a}} = \frac{s}{t} D \circ \iota_{\mathfrak{a}}$ and $\widehat{D} \circ \iota_{\mathfrak{g}^{(2)}} = D \circ \iota_{\mathfrak{g}^{(2)}}$. We will show that $\widehat{D}$ is a $(s,1,0)$-derivation of $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$ and suppose first that $X \in \mathfrak{a}$:

$$\begin{aligned}
\mu(\widehat{D}X, Y) &= \mu(\widehat{D} \circ \iota_{\mathfrak{a}} X, Y) \\
&= \mu\left(\frac{s}{t} D \circ \iota_{\mathfrak{a}} X, Y\right) && \text{(by definition of } \widehat{D}) \\
&= \frac{s}{t} t D \mu(X,Y) && \text{(since } X \in \mathfrak{a} \text{ and } D \in \mathcal{D}(t,1,0)(\mathfrak{g}) \text{)} \\
&= sD \circ \iota_{\mathfrak{g}^{(2)}} \mu(X,Y) \\
&= s\widehat{D} \circ \iota_{\mathfrak{g}^{(2)}} \mu(X,Y) && \text{(by definition of } \widehat{D}) \\
&= s\widehat{D}\mu(X,Y).
\end{aligned}$$

And if $X \in \mathfrak{g}^{(2)}$, we have

$$\begin{aligned}
\mu(\widehat{D}X, Y) &= \mu(\widehat{D} \circ \iota_{\mathfrak{g}^{(2)}} X, Y) \\
&= \mu(D \circ \iota_{\mathfrak{g}^{(2)}} X, Y) && \text{(by definition of } \widehat{D}) \\
&= \mu(DX, Y) \\
&= tD\mu(X,Y) \\
&= 0 && \text{(by Lemma 3.1, since } X \in \mathfrak{g}^{(2)})
\end{aligned}$$

and

$$\begin{aligned}
s\widehat{D}\mu(X,Y) &= s\widehat{D} \circ \iota_{\mathfrak{g}^{(2)}} \mu(X,Y) \\
&= sD \circ \iota_{\mathfrak{g}^{(2)}} \mu(X,Y) && \text{(by definition of } \widehat{D}) \\
&= sD\mu(X,Y) \\
&= 0 && \text{(by Lemma 3.1, since } X \in \mathfrak{g}^{(2)}).
\end{aligned}$$

Thus, $s\widehat{D}\mu(X,Y) = \mu(\widehat{D}X, Y)$ when $X \in \mathfrak{g}^{(2)}$. It follows by linearity that $\widehat{D}$ is a $(s,1,0)$-derivation of $\mathfrak{g}$.

And so, the function $\varphi : \mathcal{D}(t,1,0)(\mathfrak{g}) \to \mathcal{D}(s,1,0)(\mathfrak{g})$ that sends a D to $\widehat{D}$ is well defined and is an injective linear map. ■

The preceding results yield our second objective.

Theorem 3.5. *Let $t, s \in \mathbb{K} \setminus \{0,1\}$ be arbitrary. Then the functions $\widehat{\phi}_{n,t}$ and $\widehat{\phi}_{n,s}$ are equal.*

Remark 3.6. Because of this theorem, the family $\{\widehat{\phi}_{n,t}\}_{t\in\mathbb{K}}$ “collapses” to the set $\{\widehat{\phi}_{n,-1}, \widehat{\phi}_{n,0}, \widehat{\phi}_{n,1}\}$, and it is easy to check that these three functions are different from each other by evaluating the functions at a Lie algebra law. Consider the Lie algebra $\mathfrak{g} := \mathfrak{aff}(\mathbb{K}) \times \mathbb{K}^n$ (with $n \in \mathbb{N}$)). It is clear that $\mathcal{D}(0,1,0)(\mathfrak{g})$ is isomorphic to $L(\mathbb{K}^2 \times \mathbb{K}^n, \mathbb{K}^n)$, and by Lemma 2.3 we have $\mathcal{D}(1,1,0)(\mathfrak{g})$ is isomorphic to $\mathcal{D}(1,1,0)(\mathfrak{aff}(\mathbb{K})) \oplus L(\mathbb{K} \times \mathbb{K}^n, \mathbb{K}^n)$ and $\mathcal{D}(-1,1,0)(\mathfrak{g})$ is isomorphic to $\mathcal{D}(-1,1,0)(\mathfrak{aff}(\mathbb{K})) \oplus L(\mathbb{K} \times \mathbb{K}^n, \mathbb{K}^n)$. Since $\mathcal{D}(1,1,0)(\mathfrak{aff}(\mathbb{K})) = \operatorname{span}\{\mathrm{Id}_{\mathfrak{aff}(\mathbb{K})}\}$ and $\mathcal{D}(-1,1,0)(\mathfrak{aff}(\mathbb{K})) = \{0\}$, where $\mathrm{Id}_{\mathfrak{aff}(\mathbb{K})}$ denotes the *identity map* of $\mathfrak{aff}(\mathbb{K})$, then the statement follows. ■

The Theorem 3.4 gives us a interesting result about the *Centroid* of a Lie algebra, usually denoted by $\mathcal{C}(\mathfrak{g})$, and which is the space $\mathcal{D}(1,1,0)(\mathfrak{g})$.

Corollary 3.7. *Let $t \in \mathbb{K} \setminus \{0,1\}$ and let $\mathfrak{g}$ be a non-perfect Lie algebra over $\mathbb{K}$. Then $\mathcal{D}(t,1,0)(\mathfrak{g})$ can be embedded into the centroid of $\mathfrak{g}$.*

4. Upper and lower bounds of $\widehat{\phi}_{n,t}$, $t \neq 0, 1$

Theorem 4.1. *Let $\mathfrak{g} = (V, \mu)$ be a non-perfect Lie algebra and let $t \in \mathbb{K}$ with $t \neq 0, 1$. Then*

$$\dim(\Omega(\mathfrak{g})) \leq \widehat{\phi}_{n,t}(\mathfrak{g}) \leq \dim(\Omega(\mathfrak{g})) + \dim(L(\mathfrak{g}^{(2)}; \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)})).$$

Proof. The idea of the proof is similar to that of the Lemma 2.3. Let D be a $(t,1,0)$-derivation of $\mathfrak{g}$ and consider the linear transformation $\widehat{D} : \mathfrak{g}^{(2)} \to \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}$ defined by $\widehat{D}X = DX$ (formally and rigorously, $\widehat{D}$ is not $D \circ \iota_{\mathfrak{g}^{(2)}}$). The function $\widehat{D}$ is well defined because D preserves $\mathfrak{g}^{(2)}$, and by Lemma 3.1, D sends $\mathfrak{g}^{(2)}$ to $\mathcal{Z}(\mathfrak{g})$. Now, consider the function $\varphi : \mathcal{D}(t,1,0)(\mathfrak{g}) \to L(\mathfrak{g}^{(2)}; \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)})$ that sends D to $\widehat{D}$. The function φ is a linear transformation that passes to the quotient $\mathcal{D}(t,1,0)(\mathfrak{g})/\Omega(\mathfrak{g})$ and so, it defines a linear transformation

$$\widetilde{\varphi} : \mathcal{D}(t,1,0)(\mathfrak{g})/\Omega(\mathfrak{g}) \to L(\mathfrak{g}^{(2)}; \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}).$$

The only thing that remains to be proved is that $\widetilde{\varphi}$ is a injective function. Suppose $D \in \mathcal{D}(t,1,0)(\mathfrak{g})$ such that $\widehat{D}$ is the zero map from $\mathfrak{g}^{(2)}$ to $\mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}$. We will show that $D \in \Omega(\mathfrak{g})$. Clearly D satisfies the condition $\mathfrak{g}^{(2)} \subseteq \operatorname{Ker}(D)$. Now we show that $\operatorname{Im}(D) \subseteq \mathcal{Z}(\mathfrak{g})$. Let X, Y be arbitrary. Since D is $(t,1,0)$-derivation of $\mathfrak{g}$, we have $\mu(DX, Y) = tD\mu(X,Y)$, and since $\mu(X,Y) \in \mathfrak{g}^{(2)}$, it follows that $\mu(DX, Y) = t\widehat{D}\mu(X,Y) = 0$; and thus $DX \in \mathcal{Z}(\mathfrak{g})$. ■

Remark 4.2. The preceding theorem is sharp in all dimensions. For instance, take $\mathfrak{g}$ to be $\mathfrak{h}_3(\mathbb{K}) \times \mathbb{K}^n$, with $n \in \mathbb{N} \cup \{0\}$. If $\{e_1, e_2, e_3\}$ is a basis of $\mathfrak{h}_3(\mathbb{K})$ and $\mu(e_1, e_2) = e_3$, we consider the linear transformation $D \in L(\mathfrak{g};\mathfrak{g})$ defined by $e_1 \mapsto te_1$, $e_2 \mapsto te_2$, $e_3 \mapsto e_3$ and $\mathbb{K}^n \mapsto 0$ (when $n \neq 0$). It is clear that D is a $(t,1,0)$-derivation of $\mathfrak{g}$ which is not in $\Omega(\mathfrak{g})$. Thus,

$$1 + \dim(\Omega(\mathfrak{g})) \leq \widehat{\phi}_{3+n,t}(\mathfrak{g}),$$

and since $\mathfrak{g}^{(2)} = \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}$ is a 1-dimensional vector space, it follows from Theorem 4.1 that the inequality becomes equality; i.e. $\widehat{\phi}_{3+n,t}(\mathfrak{g})$ is equal to

$$\dim(\Omega(\mathfrak{g})) + \dim(L(\mathfrak{g}^{(2)}; \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)})).$$

On the other hand, as we mentioned above, if $\mathfrak{h} = \mathfrak{aff}(\mathbb{K}) \times \mathbb{K}^m$, with $m \in \mathbb{N}$, then $\widehat{\phi}_{2+m,t}(\mathfrak{h}) = \dim(\Omega(\mathfrak{h}))$. ■

5. Conclusions

Several of the results presented in this paper can be further refined or improved under additional hypotheses. For example, theorem 4.1 can be improved under additional assumptions about $\mathfrak{g}^{(2)}$. If $\mathfrak{g}_2 := \mu(\mathfrak{g}^{(2)}, \mathfrak{g})$ is a nontrivial proper subset of $\mathfrak{g}^{(2)}$ (for instance, if $\mathfrak{g}$ is a k-step nilpotent Lie algebra with $k \geq 3$), then we can modify $\dim(L(\mathfrak{g}^{(2)}; \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}))$ to $\dim(L(\mathfrak{g}^{(2)}/\mathfrak{g}_2; \mathcal{Z}(\mathfrak{g}) \cap \mathfrak{g}^{(2)}))$ in the conclusion of the theorem, due to Lemma 3.1.

Under the assumption that $\mathfrak{g}_2 := \mu(\mathfrak{g}^{(2)}, \mathfrak{g})$ is a nontrivial proper subset of $\mathfrak{g}^{(2)}$, since the identity map of $\mathfrak{g}$ is not in the image of the embedding of $\mathcal{D}(t,1,0)(\mathfrak{g})$ into $\mathcal{D}(1,1,0)(\mathfrak{g}) = \mathcal{C}(\mathfrak{g})$ given in the proof of Theorem 3.4, we also have the upper bound $\widehat{\phi}_{n,t}(\mathfrak{g}) + 1 \leq \widehat{\phi}_{n,1}(\mathfrak{g})$, with $t \neq 0, 1$. This bound is sharp in all dimensions: take $\mathfrak{g}$ to be the *standard filiform Lie algebra* of dimension n $(n \geq 4)$, which is the semidirect product $\mathbb{K} \ltimes_{\mathrm{T}} \mathbb{K}^n$ where $\mathrm{T} \in L(\mathbb{K}^n; \mathbb{K}^n)$ is any nilpotent linear transformation with *nilpotency index* n. An straightforward computation shows that $\mathcal{D}(1,1,0)(\mathfrak{g}) = \operatorname{span}\{\mathrm{Id}_{\mathfrak{g}}\} \oplus \Omega(\mathfrak{g})$ and $\mathcal{D}(1,t,0)(\mathfrak{g}) = \Omega(\mathfrak{g})$.

It is important to point out that there is a connection between $(t,1,0)$-derivations and *deformations* of Lie algebras. Let D be a $(t,1,0)$-derivation of a Lie algebra $\mathfrak{g} = (V, \mu)$ and let $\lambda \in C^2(V;V)$ be the skew-symmetric bilinear map defined by $\lambda(X,Y) = \mu(DX, Y)$. It is a simple matter to see that λ is a *trivial solution* to the *Maurer-Cartan (deformation) equation* of $\mathfrak{g}$

$$\delta_\mu \lambda + \frac{1}{2}[\lambda, \lambda] = 0;$$

which is to say that λ is a 2-cocycle for the adjoint representation of the Lie algebra $\mathfrak{g}$ and (V, λ) is a Lie algebra; and so $\mu + s\lambda \in \mathcal{L}(V)$ for all $s \in \mathbb{K}$. Thus, the results obtained provides information about *linear deformations* obtained by $(t,1,0)$-derivations.

An interesting infinite family of invariant functions on Lie algebras is given by $\{\widehat{\psi}_{n,s}\}_{s \in \mathbb{K}}$, where $\widehat{\psi}_{n,s} : \mathcal{L}(V) \to \{0, 1, \ldots, n^2\}$ is defined by

$$\widehat{\psi}_{n,s}(\mu) = \operatorname{Dim} \mathcal{D}(s,1,1)(\mathfrak{g}), \quad \text{with } \mathfrak{g} = (V, \mu).$$

It is straightforward to see that $\mathcal{D}(t,1,0)(\mathfrak{g}) \subseteq \mathcal{D}(2t,1,1)(\mathfrak{g})$. It would be worthwhile to explore the possibility of providing a practical description of $\mathcal{D}(2t,1,1)(\mathfrak{g})$ in terms of $\mathcal{D}(t,1,0)(\mathfrak{g})$ and other spaces of generalized derivations. We consider that the results obtained in this paper could help pave the way for answers in this direction.

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