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Harmonic Analysis on Inhomogeneous Nilpotent Lie Groups

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Abstract. Let G be a semi-direct product of a normal, vector subgroup by a connected, simply connected nilpotent Lie group. A detailed study of the coadjoint orbits of G in the dual space $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$ is motivated by classical harmonic analysis on solvable Lie groups, culminating in the work of Auslander and Kostant, and by more recent work on generalized continuous wavelets. We apply a procedure for matrix reduction to construct a stratification of the space of coadjoint orbits, where each layer of the stratification has an explicit fiber bundle structure, and provides a criterion for the property of regularity for a coadjoint orbit. Examination of the Zariski open layer Ω_0 then yields an algebraic characterization for regularity, and for both regularity and integrality, of every orbit in Ω_0. When the criterion for collective regularity holds, we construct a simple and explicit topological cross-section for the coadjoint orbits in Ω_0. When a criterion fails, then the corresponding property fails for a dense $\mathcal{G}_\delta$ set in Ω_0.

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1. Introduction

For connected, simply connected solvable Lie groups G, the *orbit method*, by which suitable coadjoint orbits of G are used to parametrize the dual of G, began with the thesis of A. A. Kirillov [11] (where G is nilpotent), and culminates in the paper of L. Auslander and B. Kostant [6]. An important generalization of the orbit method was shortly thereafter carried out by L. Pukanszky in [14, 15]. Applications of such groups have motivated further work where, for various subclasses of solvable Lie groups, explicit descriptions of the orbital parametrization of the dual are given and functorial properties are proved. Most notably for exponential groups, where the parametrization of the dual by coadjoint orbits is bijective, H. Leptin and J. Ludwig show in [12] that the orbital parametrization of the dual is a homeomorphism.

Relatively recent work on non-exponential solvable groups has focused on quite special subclasses of *inhomogeneous* groups: semi-direct products $G = TH$ where T is a normal vector group. The inhomogeneous abelian case – where H is abelian – was taken up in [3, 4, 5]. In subsequent work [7], it was shown that certain results for the inhomogeneous abelian case carry over to the inhomogeneous nilpotent case (meaning of course that H is now nilpotent), and further C^*-algebraic properties are proved for the inhomogeneous abelian case and subclasses therein. In the present work, we continue the study of inhomogeneous nilpotent groups.

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Let $\mathfrak{g}$ be the Lie algebra of G; the group G acts naturally on $\mathfrak{g}$ by the adjoint action, and on its linear dual $\mathfrak{g}^*$ by the coadjoint action. A point $\ell\in\mathfrak{g}^*$ is said to be

1. *regular* if the topology on its coadjoint orbit $\mathcal{O} = \mathrm{Ad}_G^*\ell$ relative to $\mathfrak{g}^*$ coincides with the quotient topology in the identification $\mathcal{O} \simeq G/G(\ell)$, and

2. *integral* if the unitary character χ_ℓ of the identity component $G(\ell)^0$ in the stability group $G(\ell)$ with differential $d\chi_\ell = \mathrm{i}\ell|_{\mathfrak{g}(\ell)}$ can be extended to a character χ of $G(\ell)$.

Both the regularity and integrality properties hold either for all points in a coadjoint orbit or none of them, and accordingly we may also say that a coadjoint orbit is regular or integral. To any integral point ℓ, we can associate a family $\{\pi_{\ell,\chi}\}$ of irreducible unitary representations indexed by the extensions χ of χ_ℓ to $G(\ell)$, and up to unitary equivalence, this family depends only upon the orbit $\mathcal{O}$ of ℓ. If ℓ is regular as well as integral, then the unitary equivalence classes $[\pi_{\ell,\chi}]$ are in fact of type 1. (This is a special case of a more general result in [14].) Thus a connected, simply connected solvable Lie group G is a type 1 group if and only if all the coadjoint orbits are both regular and integral, whence we have a parametrization of the unitary dual $\widehat{G}$ of G in terms of orbits and extensions of characters (see [6, 8, 1]).

Now suppose that G is an inhomogeneous nilpotent group; this means that $G = TH$ where T is a vector group and H is connected, simply connected nilpotent, acting on T by linear automorphisms. The space $\mathfrak{g}^*$ is identified with $V\times\mathfrak{h}^*$, with $V = T^*$, and the H-module V can be described as a real submodule of the complexification $V_{\mathbb{C}}$ having a basis for which the images of the elements of the Lie algebra $\mathfrak{h}$ in $\mathfrak{gl}(V)$ are simultaneously in Jordan form.

An even more special case, where H is a vector group, was studied in [5, 4, 3]. We find that

1. a coadjoint orbit $\mathcal{O}$ is regular if and only if its projected H-orbit is regular in V, and

2. each coadjoint orbit is integral.

Thanks to the explicit form of the H-action on V, a partition $\{\omega\}$ into invariant subsets of V is constructed in [4]. From [4, Proposition 3.9] and the discussion in [4] preceding this result, we see that when H is a vector group, for each ω, there is a closed subgroup Y of a torus acting freely on ω so that the quotient map $p : \omega \to \omega/Y$ is a continuous principal bundle, H acts naturally on ω/Y, the map p is H-equivariant, and each H-orbit in ω/Y is regular. The stability group $H(p(v))$ of $p(v)$ acts naturally on the fiber Y, and the question of regularity of the H-orbit of v is reduced to the regularity of the $H(p(v))$-orbit in Y. Moreover this last regularity is given by a concrete algebraic condition. Consequently, a non-empty H-invariant Zariski open subset ω of V is constructed such that either each orbit in ω is regular, or almost all orbits are non-regular. In [3] it is shown that when ω consists of regular orbits, the regular representation of G is type 1, and an explicit description of the Plancherel formula for G is proved.

For the somewhat more general inhomogeneous nilpotent case, the Jordan-type decomposition of V, the partition of V into the sets ω, and the determination of regularity of H-orbits in the Zariski open ω all continue to hold [7]. However,

1. a co-adjoint orbit $\mathcal{O}$ may be regular (resp. not regular) even as its projected H-orbit in V is not regular (resp. regular), and
2. coadjoint orbits are not necessarily integral.

Given a subset ω, denote its inverse projection in $\mathfrak{g}^*$ by Ω, so that $\mathfrak{g}^*$ is partitioned into finitely many such subsets. Instead of the bundle maps $\Omega \to \omega \to \omega/Y$, we now consider in Proposition 4.7 the topological fiber bundle structures for the natural maps $\Omega \to \Omega/T \rtimes Y \to \omega/Y$. These maps are G-equivariant, allowing a similar but more delicate criterion for regularity of orbits in each Ω, and motivating a further refinement of the partition. One of the refined stratum Ω_0 is Zariski open, and we show that either each orbit in Ω_0 is regular, or almost every orbit in Ω_0 are not regular (Corollary 4.8).

The algebraic construction also yields an explicit matrix criterion for regularity of every orbit in Ω_0 (Theorem 5.5) and in the regular case allows us to build an explicit topological cross-section for the G-action on Ω_0 (Proposition 5.9).

We then turn to the question of integrality for orbits in Ω_0, in the regular case. Integrality holds for each point ℓ in an open set U if and only if the Lie algebra of the stability group in H for the projection of ℓ in $\Omega_0/T \rtimes Y$ is subordinate to ℓ. This implies that either integrality holds for all points ℓ in Ω_0 or does not hold for any ℓ in a dense conull Borel subset of Ω_0 (Theorem 5.15).

In the case where all orbits in Ω_0 are both integral and regular, the nature of these results suggests the possibility of a concrete Plancherel formula for inhomogeneous nilpotent groups.

2. Coadjoint orbits of an inhomogeneous group

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We say that G is an inhomogeneous group if G admits a closed normal subgroup T and a closed subgroup H such that T is a nontrivial vector group, H is a connected, simply connected Lie group, and $T \cap H$ is the trivial subgroup. We realize T by identifying T with its Lie algebra $\mathfrak{t}$, so that the restriction to $\mathfrak{t}$ of the exponential mapping on $\mathfrak{g}$ is the identity. Thus for $x \in T$, $s \in G$, $sxs^{-1} = \mathrm{Ad}_s x$. The coadjoint representation of G is the map $\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*)$ defined by

$$\langle \mathrm{Ad}^*_s \ell, A\rangle = \langle \ell, \mathrm{Ad}_{s^{-1}} A\rangle, \quad s \in G, \ell \in \mathfrak{g}^*, A \in \mathfrak{g}.$$

Since $\mathfrak{t}$ is an ideal in $\mathfrak{g}$, there is a natural action of G on $\mathfrak{t}^*$ for which the restriction map $p : \mathfrak{g}^* \to \mathfrak{t}^*$ is G-equivariant. Moreover, if we consider the space $\mathfrak{g}^*/T$ of T-orbits $\mathrm{Ad}^*_T \ell$, then the map $G \times \mathfrak{g}^*/T \to \mathfrak{g}^*/T$ defined by $(s, \mathrm{Ad}^*_T \ell) \mapsto \mathrm{Ad}^*_T(\mathrm{Ad}^*_s \ell)$ is a smooth action of the group G on $\mathfrak{g}^*/T$ for which the quotient map $p_1 : \mathfrak{g}^* \to \mathfrak{g}^*/T$ is G-equivariant. Finally, since T is abelian, the restriction map $\mathfrak{g}^* \to \mathfrak{t}^*$ drops to a natural map $p_2 : \mathfrak{g}^*/T \to \mathfrak{t}^*$ that is also G-equivariant. Thus the projection p decomposes as the composition of the equivariant maps

$$\mathfrak{g}^* \xrightarrow{p_1} \mathfrak{g}^*/T \xrightarrow{p_2} \mathfrak{t}^* \tag{1}$$

In what follows, we slightly abuse notation by denoting the restriction of p to a subset of $\mathfrak{g}^*$ also by p; similarly for p_1, p_2, we also denote the various actions of G by a dot. For instance, $p(\mathrm{Ad}^*_s \ell)$ is simply $s \cdot p(\ell)$.

Fix $\ell \in \mathfrak{g}^*$, with $v = p(\ell)$, $\mathcal{O} = \mathrm{Ad}_G^*\ell$ and $o = G \cdot v = H \cdot v$. The G-orbit of the point $p_1(\ell) \in \mathfrak{g}^*/T$ is $\mathcal{O}/T$, and the stability group in G at $p_1(\ell)$ is $TG(\ell)$. When $\mathcal{O}$ and $\mathcal{O}/T$ are given the quotient topologies of $G/G(\ell)$ and $G/TG(\ell)$ respectively, it is shown in [16, Proposition 1] that $\mathcal{O}/T$ has the structure of a smooth fiber bundle over o with projection p_2 (called there a bundle of little-group orbits), and $\mathcal{O}$ has the structure of a smooth fiber bundle over $\mathcal{O}/T$ with projection p_1 :

$$\begin{array}{ccccc} \mathcal{O} & \xrightarrow{p_1} & \mathcal{O}/T & \xrightarrow{p_2} & o \\ \wr\wr & & \wr\wr & & \wr\wr \\ G/G(\ell) & & G/TG(\ell) & & G/G(v). \end{array} \tag{2}$$

Let us recall his construction in the present setting.

Let G be a locally compact group acting continuously on a locally compact Hausdorff space E, and fix $\ell \in E$. The G-orbit $G \cdot \ell$ is said to be regular if the quotient topology on $G \cdot \ell = G/G(\ell)$ coincides with the relative topology on $G \cdot \ell$ inherited from E. A fiber bundle $p : E \to B$ is said to be G-projectable if there are continuous G-actions on both E and B for which p is G-equivariant. Any G-projectable bundle denoted as above has the property that for each $b \in B$, the stability group $G(b)$ leaves the fiber $p^{-1}(b)$ invariant. If G is a Lie group, $p : E \to B$ is a smooth fiber bundle, and the actions of G are smooth, then we say that $p : E \to B$ is smoothly G-projectable.

The following fundamental result is proved in [5].

Proposition 2.1. *Let G be a Lie group and let $p : E \to B$ be smoothly G-projectable. Let $\ell \in E$, $\mathcal{O} = G \cdot \ell$, $o_B = p(\mathcal{O})$, $o_F = G(p(\ell)) \cdot \ell$, and suppose that o_B regular. Then $\mathcal{O}$ is regular if and only if o_F is regular.*

The preceding motivates interest in Ad_G^*-invariant subsets of $\mathfrak{g}^*$ for which the maps p_1 and p_2, or similar maps, are bundle maps.

For the moment we continue setting notation. Following [1, 4, 5, 16], we write $V = \mathfrak{t}^*$, and

$$\langle s \cdot v, x \rangle = \langle v, \mathrm{Ad}_{s^{-1}} x \rangle, \quad x \in \mathfrak{t}.$$

The stability subgroup of this action is $G(v)$, we put $H(v) = G(v) \cap H$. The action of the Lie algebras $\mathfrak{g}$ of G, $\mathfrak{h}$ of H on both $\mathfrak{t}$ and V are also written multiplicatively: for $A \in \mathfrak{g}$, $x \in \mathfrak{t}$, and $v \in V$,

$$\frac{d}{dt}\bigg|_{t=0} \exp tA \cdot v = A \cdot v, \quad \langle A \cdot v, x \rangle = -\langle v, [A, x] \rangle,$$

The Lie algebras of the stability groups $G(v)$ and $H(v)$ at v are

$$\mathfrak{g}(v) = \{A \in \mathfrak{g} : A \cdot v = 0\} \quad \text{and} \quad \mathfrak{h}(v) = \mathfrak{g}(v) \cap \mathfrak{h}.$$

We identify $\mathfrak{g}^*$ with $V \times \mathfrak{h}^*$ as follows: write elements ℓ of $\mathfrak{g}^*$ as $\ell = (v, \mathrm{f})$ with $v \in V$, $\mathrm{f} \in \mathfrak{h}^*$ and

$$\langle \ell, (x + A) \rangle = \langle v, x \rangle + \langle \mathrm{f}, A \rangle, \quad x \in \mathfrak{t}, A \in \mathfrak{h}.$$

Observe that $\mathfrak{h}^*$ is identified with the G-invariant subspace $\{\ell \in \mathfrak{g}^* : \ \ell|_{\mathfrak{t}} = 0\}$. With this identification, the coadjoint action of H on $\mathfrak{h}^*$ is the restriction to H of the coadjoint action of G, applied to elements of the invariant subspace $\mathfrak{h}^*$: for $h \in H$, $(\mathrm{Ad}_H)^*_h = \mathrm{Ad}^*_h|_{\mathfrak{h}^*}$.

Writing the coadjoint action of H on $\mathfrak{h}^*$ multiplicatively also, the coadjoint action on $\mathfrak{g}^*$ by elements $h \in H$ is a combination of the actions on V and $\mathfrak{h}^*$:

$$\mathrm{Ad}_h^*(v, \mathrm{f}) = (h \cdot v, h \cdot \mathrm{f}), \quad (v, \mathrm{f}) \in \mathfrak{g}^*.$$

To describe the coadjoint action of T on $\mathfrak{g}^*$, we use the notation of [16]: for $x \in \mathfrak{t}$ and $v \in V$, define $x \wedge v \in \mathfrak{h}^*$ by

$$\langle x \wedge v, A \rangle = -\langle A \cdot v, x \rangle, \quad A \in \mathfrak{h}.$$

It follows from the above observations and identifications that for $x \in \mathfrak{t}$ and $\ell = (v, \mathrm{f}) \in \mathfrak{g}^*$, $\mathrm{ad}_x^* \ell = x \wedge v$:

$$\langle \mathrm{ad}_x^* \ell, A \rangle = \langle \ell, [A, x] \rangle = \langle v, [A, x] \rangle = -\langle A \cdot v, x \rangle = \langle x \wedge v, A \rangle, \quad A \in \mathfrak{h}.$$

Hence
$$x \cdot \ell = \mathrm{Ad}_x^* \ell = (v, \mathrm{f} + x \wedge v), \quad \ell = (v, \mathrm{f}) \in \mathfrak{g}^*. \tag{3}$$

It is clear that for each $x \in T$, $x \wedge v$ belongs to $\mathfrak{h}(v)^\perp = \{\mathrm{f} \in \mathfrak{h}^* : \mathrm{f}|_{\mathfrak{h}(v)} = 0\}$. More precisely:

Lemma 2.2. [16, Lemma 1] *Let $\ell = (v, \mathrm{f}) \in \mathfrak{g}^*$. Then $\mathfrak{h}(v)^\perp = \{x \wedge v : x \in \mathfrak{t}\}$ and*

$$p_1(\ell) = T \cdot \ell = \{v\} \times \big(\mathrm{f} + \mathfrak{h}(v)^\perp\big).$$

Proof. By (3), $T \cdot \ell$ is a subset of $\{v\} \times \big(\mathrm{f} + \mathfrak{h}(v)^\perp\big)$, hence it remains to show the first part of the lemma. Now

$$(\mathfrak{h} \cdot v)^\perp = \{x \in \mathfrak{t} : \ \langle A \cdot v, x \rangle = 0, \ \forall A \in \mathfrak{h}\} = \{x \in \mathfrak{t} : \ x \wedge v = 0\}.$$

But since

$$\dim(\mathfrak{h}(v)^\perp) = \dim(\mathfrak{h}/\mathfrak{h}(v)) = \dim(\mathfrak{h} \cdot v) = \dim(\mathfrak{t}) - \dim(\mathfrak{h} \cdot v)^\perp = \dim\{x \wedge v : x \in \mathfrak{t}\},$$

then
$$\mathfrak{h}(v)^\perp = \{x \wedge v : \ x \in \mathfrak{t}\},$$

and the lemma follows. ■

Observe that for each $v \in V$, the map $\mathrm{f} + \mathfrak{h}(v)^\perp \mapsto \mathrm{f}|_{\mathfrak{h}(v)}$ defines a canonical isomorphism of $\mathfrak{h}^*/\mathfrak{h}(v)^\perp$ with $\mathfrak{h}(v)^*$. Accordingly, we make the identification

$$p_1(\ell) = \big(v, \mathrm{f}|_{\mathfrak{h}(v)}\big), \quad \ell = (v, \mathrm{f}) \in \mathfrak{g}^*,$$

which especially prove that the little-group orbits bundle in the proof of [16, Prop. 1] coincides with $\mathcal{O}/T$. With these notations, the coadjoint action is

$$xh \cdot (v, \mathrm{f}) = x \cdot (h \cdot v, h \cdot \mathrm{f}) = (h \cdot v, h \cdot \mathrm{f} + x \wedge (h \cdot v)), \tag{4}$$

while the action in $\mathfrak{g}^*/T$ is

$$xh \cdot p_1(v, \mathrm{f}) = h \cdot (v, \mathrm{f}|_{\mathfrak{h}(v)}) = (h \cdot v, (h \cdot \mathrm{f})|_{\mathfrak{h}(h \cdot v)}), \tag{5}$$

so the equivariance of p_1 can be exhibited as

$$p_1\big(xh \cdot (v, \mathrm{f})\big) = \big(h \cdot v, (h \cdot \mathrm{f} + x \wedge (h \cdot v))|_{\mathfrak{h}(h \cdot v)}\big) = (h \cdot v, (h \cdot \mathrm{f})|_{\mathfrak{h}(h \cdot v)}).$$

Lemma 2.3. *Given $\ell = (v, \mathrm{f}) \in \mathfrak{g}^*$, we have the following.*

(a) *The stability group $G(p_1(\ell))$ for the action of G on $\mathfrak{g}^*/T$ at the point $p_1(\ell)$ is $TH(p_1(\ell))$, where $H(p_1(\ell))$ is the stability group $H(v)(\mathrm{f}|_{\mathfrak{h}(v)})$ for the coadjoint action of $H(v)$ at the point $\mathrm{f}|_{\mathfrak{h}(v)}$.*

(b) *The stability group $G(\ell)$ for the coadjoint action at ℓ is the subgroup of $TH(p_1(\ell))$ defined by*

$$G(\ell) = \{xh \in G : h \in H(v),\ h \cdot \mathrm{f} = \mathrm{f} - x \wedge v\}.$$

Proof. The condition $xh{\cdot}p_1(v, \mathrm{f}) = p_1(v, \mathrm{f})$ means that $h \in H(v)$ and $h{\cdot}\mathrm{f}(A) = \mathrm{f}(A)$ for all $A \in \mathfrak{h}(v)$. This proves (a). The second statement follows from (4). ■

Consider the fibers for the bundle maps in the diagram (2). Let $\ell = (v, \mathrm{f}) \in \mathfrak{g}^*$ with $\mathcal{O} = G{\cdot}\ell$. Since p_1 is a quotient map, the fiber over $p_1(\ell)$ is simply $p_1^{-1}(p_1(\ell)) = T{\cdot}\ell$. The fiber $p_2^{-1}(v)$ inside $\mathcal{O}$ over v is $\mathcal{O} \cap (\{v\} \times \mathfrak{h}(v)^*) \simeq G(v)/G(p_1(\ell))$. By virtue of the first part of Lemma 2.3,

$$p_2^{-1}(v) \simeq G(v)/G(p_1(\ell)) \simeq H(v)/H(p_1(\ell)) \simeq H(v) \cdot \mathrm{f}|_{\mathfrak{h}(v)}.$$

In other words, the fiber $p_2^{-1}(v)$ is naturally identified with the $H(v)$-coadjoint orbit of the point $\mathrm{f}|_{\mathfrak{h}(v)}$ in $\mathfrak{h}(v)^*$. Let us call the orbit $H(v){\cdot}\mathrm{f}|_{\mathfrak{h}(v)}$ in $\mathfrak{h}(v)^*$ the 'little orbit' of ℓ. We can now exhibit an application of Proposition 2.1 to the case where H is a vector group.

Corollary 2.4. *Let G be an inhomogeneous group for which H is a vector group. Suppose that $E \subset \mathfrak{g}^*$ is locally compact and G-invariant, and that $p : E \to B \subset V$ is a topological fiber bundle that is G-projectable. Then a coadjoint orbit $\mathcal{O}$ contained in E is regular if and only if $p(\mathcal{O}) \subset V$ is regular.*

Proof. Let $\mathcal{O} = G \cdot \ell$ with $\ell = (v, \mathrm{f}) \in E$, and suppose that $p(\mathcal{O})$ is regular. Since H is abelian, the little orbit of ℓ is trivial, hence regular. Hence by Proposition 2.1, the orbit $\mathcal{O}/T$ is regular. Now by part 1 of Lemma 2.3, $G(p_1(\ell)) = TH(p_1(\ell))$ and for $h \in H(p_1(\ell))$, $h \cdot \mathrm{f} - \mathrm{f} \in \mathfrak{h}(v)^\perp$. It follows that the $G(p_1(\ell))$-orbit of ℓ in the fiber $p_1^{-1}(p_1(\ell))$ is the fiber itself, $T \cdot \ell = p_1^{-1}(p_1(\ell))$, which is also regular. Thus Proposition 2.1 applies to p_1, and we get that $\mathcal{O}$ is regular.

For the converse, suppose that $\mathcal{O} = G \cdot \ell$ is regular. Then $\mathcal{O}$ is locally closed, so we have an open subset U of $\mathfrak{g}^*$ such that $\overline{\mathcal{O}} \cap U = \mathcal{O}$. Define the open subset U' of V by $U' \times \{\mathrm{f}\} = U \cap (V \times \{\mathrm{f}\})$. We show that $\overline{p(\mathcal{O})} \cap U' = p(\mathcal{O})$.

Given $v' \in \overline{p(\mathcal{O})} \cap U'$ we have a sequence $v_n = h_n{\cdot}v \in p(\mathcal{O}) \cap U'$ with $v_n \to v'$. Then $(v_n, \mathrm{f}) \in U' \times \{\mathrm{f}\}$ and $(v_n, \mathrm{f}) \to (v', \mathrm{f})$. Now since H is abelian, $(v_n, \mathrm{f}) = h_n \cdot \ell \in \mathcal{O}$ for all n. Hence (v', f) belongs to the set

$$\overline{\mathcal{O}} \cap (U' \times \{\mathrm{f}\}) = \overline{\mathcal{O}} \cap U \cap (V \times \{\mathrm{f}\}) \subset \overline{\mathcal{O}} \cap U = \mathcal{O},$$

Showing that $v' \in p(\mathcal{O})$. Thus $\overline{p(\mathcal{O})} \cap U' \subset p(\mathcal{O})$.

For the opposite inclusion, H abelian implies $p(\mathcal{O}) \times \{\mathrm{f}\} \subset \mathcal{O}$, and so

$$p(\mathcal{O}) \times \{\mathrm{f}\} \subset \mathcal{O} \cap (V \times \{\mathrm{f}\}) \subset \overline{\mathcal{O}} \cap U \cap (V \times \{\mathrm{f}\}) = \overline{\mathcal{O}} \cap (U' \times \{\mathrm{f}\}).$$

But it is easily seen that $\overline{\mathcal{O}} \cap (U' \times \{\mathrm{f}\}) = (\overline{p(\mathcal{O})} \cap U') \times \{\mathrm{f}\}$, so $p(\mathcal{O}) \subset \overline{p(\mathcal{O})} \cap U'$. This completes the proof. ■

In order to obtain regularity results for the case where H is nilpotent, we will apply Proposition 2.1 to a modified version of the sequence (1). (See the relation (9) below.)

For the question of integrality, the little orbit is of particular interest. Observe that, in the general case, $H(v)$ may not be connected, but we can speak of its coadjoint orbits, their regularity, and even integrality, using a simple extension of the definitions to non-connected Lie groups.

Proposition 2.5. ([16, Corollary of Prop. 2]) *For each* $\ell = (v, \mathrm{f}) \in \mathfrak{g}^*$, *the orbit* $\mathcal{O} = G\cdot\ell$ *is integral if and only if the corresponding little orbit* $H(v)\cdot\mathrm{f}|_{\mathfrak{h}(v)}$ *is integral.*

Proof. We first show that the map $e^v : (x, h) \mapsto e^{\mathrm{i}\langle v,x\rangle}$ is a character of $TH(v)$, since

$$e^{\mathrm{i}v}((x,h)(x',h')) = e^{\mathrm{i}v}(x + h\cdot x', hh') = e^{\mathrm{i}\langle v,x\rangle + \mathrm{i}\langle v, h\cdot x'\rangle}$$
$$= e^{\mathrm{i}\langle v,x\rangle + \mathrm{i}\langle h^{-1}\cdot v, x'\rangle} = e^{\mathrm{i}\langle v, x+x'\rangle} = e^{\mathrm{i}v}(x,h)e^{\mathrm{i}v}(x',h').$$

Hence the restriction of $e^{\mathrm{i}v}$ to $G(p_1(\ell)) \subset TH(v)$, still denoted $e^{\mathrm{i}v}$, is a character whose differential at 1 is $(x, A) \mapsto \mathrm{i}\langle v, x\rangle$.

Now suppose that $\ell = (v, \mathrm{f})$ integral and let χ be a character of $G(\ell)$ with differential $\mathrm{i}\ell$. Since $T \cap G(\ell)$ is a vector group, and $d\chi|_{\mathfrak{g}(\ell)\cap\mathfrak{t}} = \mathrm{i}v|_{\mathfrak{g}(\ell)\cap\mathfrak{t}}$, we have $\chi(x, 1) = e^{\mathrm{i}\langle v,x\rangle} = e^{\mathrm{i}v}(x, 1)$ for each $(x, 1) \in G(\ell) \cap T$, and therefore $\chi e^{-\mathrm{i}v}$ is a character of $G(\ell)$, trivial on $T \cap G(\ell)$. But by Lemma 2.3,

$$T \cap G(\ell) = \{x \in T :\ x \wedge v = 0\} = (\mathfrak{h}\cdot v)^{\perp},$$

and since $TG(\ell) = G(p_1(\ell)) = TH(p_1(\ell))$, we have

$$G(\ell)/(T \cap G(\ell)) = TG(\ell)/T = H(p_1(\ell)).$$

Let $\pi : G(\ell) \to H(p_1(\ell))$ be the quotient map, let η be the map

$$\eta : H(v)(\mathrm{f}|_{\mathfrak{h}(v)}) = H(p_1(\ell)) \to \mathbb{C} \quad \text{defined by} \quad \eta\circ\pi = \chi e^{-\mathrm{i}v}.$$

By construction, η is a character of $H(v)(\mathrm{f}|_{\mathfrak{h}(v)})$ with differential $\mathrm{i}\mathrm{f}|_{\mathfrak{h}(v)(\mathrm{f}|_{\mathfrak{h}(v)})}$. Therefore, the little orbit is integral.

Conversely, if the little orbit is integral, there is a character η of $H(p_1(\ell))$ with differential $\mathrm{i}\mathrm{f}|_{\mathfrak{h}(p_1(\ell))}$. In consequence $\chi = (\eta\circ\pi)e^{\mathrm{i}v}$ is a character of $G(\ell)$ such that $d\chi = \mathrm{i}v|_{\mathfrak{g}(\ell)} + \mathrm{i}\mathrm{f}|_{\mathfrak{g}(\ell)} = \ell|_{\mathfrak{g}(\ell)}$, and thus $\mathcal{O} = G\cdot\ell$ is integral. ∎

The following example first appears in [9]; see also [1, 8].

Example 2.6. Here we have $V = \mathbb{R}^4$, identified with $\mathbb{C}^2$ with basis $\{f_1, f_2\}$, and $\mathfrak{h} = (A_1, A_2, A_3)_{\mathbb{R}}$ is the Heisenberg Lie algebra, $[A_3, A_2] = A_1$, acting on V by $A_2\cdot f_1 = \mathrm{i}f_1$, $A_3\cdot f_2 = \mathrm{i}f_2$. We choose $v = z_1 f_1 + z_2 f_2$, with $z_1 z_2 \neq 0$ and f such that $\mathrm{f}(A_1) = \alpha \neq 0$. Then $H(v)$ is not connected, since

$$H(v) = \{\exp A :\ A = a_1A_1 + 2\pi n_2 A_2 + 2\pi n_3 A_3,\ a_1 \in \mathbb{R},\ n_2, n_3 \in \mathbb{Z}\}.$$

Since $\mathfrak{h}(v) = (A_1)_{\mathbb{R}}$ is central in $\mathfrak{h}$, $H(v)(\mathrm{f}|_{\mathfrak{h}(v)}) = H(v)$, but in the Heisenberg group, the relation $[\exp A, \exp A'] = \exp[A, A']$ holds, then;

$$[H(v)(\mathrm{f}|_{\mathfrak{h}(v)}), H(v)(\mathrm{f}|_{\mathfrak{h}(v)})] = \exp 4\pi^2\mathbb{Z}A_1.$$

Suppose that there is a character η of the stability group $H(v)(\mathrm{f}|_{\mathfrak{h}(v)})$ with differential if. Then $\eta(\exp tA_1) = e^{\mathrm{i}t}$ for all t, therefore the existence of η is equivalent to the relation

$$1 = \eta(\exp 4\pi^2 A_1) = e^{\mathrm{i}\alpha 4\pi^2}.$$

The orbit $H(v)\cdot \mathrm{f}|_{\mathfrak{h}(v)}$ is thus integral if and only if $\alpha \in \frac{1}{2\pi}\mathbb{Z}$. Hence by Proposition 2.5, the coadjoint orbit of (v, f) is integral if and only if $\alpha \in \frac{1}{2\pi}\mathbb{Z}$, the coadjoint orbits are generically not integral. ■

Note that there is no such result relating regularity of the coadjoint orbit and its little orbit. Indeed, in the case where H is a vector group as above, the little orbit is always regular since it is trivial, while coadjoint orbits may be regular or non-regular.

In the next section, we shall describe the group $H(p_1(\ell))$ more precisely, when H is nilpotent. We conclude this section with a final notation. If U is a module over a ring $\mathbb{K} = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and $u_1, \ldots, u_m$ are elements of U, we put:

$$(u_1, \ldots, u_m)_{\mathbb{K}} = \mathrm{span}_{\mathbb{K}}\{u_1, \ldots, u_m\} = \sum_{k=1}^{m} \mathbb{K}u_k.$$

3. Fiber bundles in the *H*-nilpotent case

From now on we suppose that H is a connected, simply connected nilpotent Lie group. Then H is exponential: the map $\exp : \mathfrak{h} \to H$ is a bijection.

Consider the dual action of $\mathfrak{h}$ on V, giving the representation $\mu : \mathfrak{h} \to \mathfrak{gl}(V)$. Using weight-spaces for the H-module V, V is linearly isomorphically realized as a product so that the matrices for the elements $\mu(A)$ have lower-triangular block form [1, 7]. We adopt the notation used in [4] (where H is assumed to be abelian): V is realized as $\prod_{k=1}^{m} V^{(k)}$, where each $V^{(k)}$ is an invariant subspace of $V_{\mathbb{C}}$, and corresponds to a weight $_k$ for the representation μ. If $_k$ is real-valued on $\mathfrak{h}$, then we have independent elements $f_1^{(k)}, \ldots, f_{n_k}^{(k)} \in V$ such that $V^{(k)} = (f_1^{(k)}, \ldots, f_{n_k}^{(k)})_{\mathbb{R}}$, and so that the matrix of $\mu(A)|_{V^{(k)}}$ with respect to this basis has the form $_k(A)I_{V^{(k)}} + n^{(k)}(A)$, with $n^{(k)}(A)$ a strictly lower triangular matrix with real entries. In this case put $\mathbb{K}_k = \mathbb{R}$. If $_k(A) \notin \mathbb{R}$ for some $A \in \mathfrak{h}$, then we have independent elements $f_1^{(k)}, \ldots, f_{n_k}^{(k)} \in V_{\mathbb{C}} \setminus V$ such that $V^{(k)} = (f_1^{(k)}, \ldots, f_{n_k}^{(k)})_{\mathbb{C}}$, and so that the matrix of $\mu(A)|_{V^{(k)}}$ with respect to this basis has the form $_k(A)I_{V^{(k)}} + n^{(k)}(A)$, with $n^{(k)}(A)$ a strictly lower triangular matrix with complex entries. In this case put $\mathbb{K}_k = \mathbb{C}$.

Write $v = (v^{(1)}, \ldots, v^{(m)})$ where $v^{(k)} \in V^{(k)}$. Thus for each $A \in \mathfrak{h}$, $\mu(A)$ is given by a matrix of the form $d(A) + n(A)$, where $d(A)$ is a diagonal matrix and $n(A)$ is strictly lower triangular.

For each k let α_k be the real part of the weight $_k$, and β_k its imaginary part. Each of the scalar-valued functions $A \mapsto {}_k(A)$, $A \mapsto \alpha_k(A)$ and $A \mapsto \beta_k(A)$ are Lie algebra homomorphisms (meaning that they vanish on $[\mathfrak{h}, \mathfrak{h}]$).

Remark 3.1. Observe that:

$$\begin{aligned} n^{(k)}([A, B]) &= \mu([A, B])|_{V^{(k)}} - {}_k([A, B])I_{V^{(k)}} = [\mu(A), \mu(B)]|_{V^{(k)}} \\ &= [{}_k(A)I_{V^{(k)}} + n^{(k)}(A), {}_k(B)I_{V^{(k)}} + n^{(k)}(B)] = [n^{(k)}(A), n^{(k)}(B)]. \end{aligned}$$

In other words, each map $A \mapsto n^{(k)}(A)$ from $\mathfrak{h}$ into $\mathfrak{gl}(n_k, \mathbb{K}_k)$ is a Lie algebra homomorphism.

Moreover, for each $A, B \in \mathfrak{h}$, with $h = \exp B$, we have

$$n(\mathrm{Ad}_h A) = e^{n(B)} n(A) e^{-n(B)}. \qquad \blacksquare$$

For each k, there is a natural decreasing flag of $\mathfrak{g}$-submodules in $V^{(k)}$, namely:

$$V_0^{(k)} = V^{(k)} \supset V_1^{(k)} = (f_2^{(k)}, \dots, f_{n_k}^{(k)})_{\mathbb{K}_k} \supset \cdots \supset V_{n_k}^{(k)} = \{0\}.$$

Associated to this flag, we define a partition $V^{(k)} = \sqcup_{a=0}^{n_k} \omega_a^{(k)}$ of $V^{(k)}$ by semi-algebraic, invariant subsets, putting $\omega_0^{(k)} = \{0\}$, and if $0 < a \le n_k$,

$$\omega_a^{(k)} = \{v^{(k)} \in V_{a-1}^{(k)} : \ v_a^{(k)} \neq 0\}.$$

Then for each integer sequence $\mathbf{a} = (a_1, \dots, a_m)$ with $0 \le a_k \le n_k$, we put $\omega_{\mathbf{a}} = \prod_k \omega_{a_k}^{(k)}$ and $\Omega_{\mathbf{a}} = \{(v, \mathrm{f}) \in \mathfrak{g}^* : \ v \in \omega_{\mathbf{a}}\}$. This gives a partition of V, and of $\mathfrak{g}^*$, by semi-algebraic, invariant subsets. Observe that the unique $\omega_{\mathbf{a}}$, $\Omega_{\mathbf{a}}$ which are Zariski open are $\omega_{\mathbf{1}}$, $\Omega_{\mathbf{1}}$, where $\mathbf{1} = (1, 1, \dots, 1)$.

Simultaneously, we realize $\mathfrak{t} = V^*$ as the product $\mathfrak{t} = \prod_k \mathfrak{t}^{(k)}$, where, for each $k = 1, \dots, m$, $\mathfrak{t}^{(k)} = V^{(k)*}$ is the $\mathbb{K}_k$ vector space with basis $\{e_1^{(k)}, \dots, e_{n_k}^{(k)}\}$ dual to the basis $\{f_1^{(k)}, \dots, f_{n_k}^{(k)}\}$. Then the adjoint action of $\mathfrak{h}$ on $\mathfrak{t}$ is given by $-{}^t\mu(A)$: for each $x^{(k)} \in \mathfrak{t}^{(k)}, 1 \le k \le m$,

$$[A, x^{(k)}] = -{}_k(A) x^{(k)} - \ {}^t n^{(k)}(A) x^{(k)}, \qquad A \in \mathfrak{h}.$$

Note that ${}^t n^{(k)}(A)$ is a strictly upper triangular matrix. Thus, for each k, there is a increasing natural flag of ideals:

$$\mathfrak{t}_0^{(k)} = \{0\} \subset \mathfrak{t}_1^{(k)} = (e_1^{(k)})_{\mathbb{K}_k} \subset \ldots \subset \mathfrak{t}_{n_k}^{(k)} = \mathfrak{t}^{(k)}.$$

Moreover, for each integer sequence $\mathbf{a} = (a_1, \dots, a_m)$ with $0 \le a_k \le n_k$, we put $\mathfrak{a}_{\mathbf{a}} = \prod_k \mathfrak{t}_{a_k}^{(k)}$. Clearly, $\mathfrak{a}_{\mathbf{a}}$ is an abelian ideal in $\mathfrak{g}$ such that, for any $\ell = (v, \mathrm{f}) \in \Omega_{\mathbf{a}}$, $\mathfrak{a}_{\mathbf{a}} \subset \ker \ell$. We also put $A_{\mathbf{a}} = \exp \mathfrak{a}_{\mathbf{a}}$, $G_{\mathbf{a}} = G/A_{\mathbf{a}}$, and $\mathfrak{g}_{\mathbf{a}} = \mathfrak{g}/\mathfrak{a}_{\mathbf{a}}$. The proof of the following is routine and left to the reader:

Lemma 3.2. *Let $\ell = (v, \mathrm{f})$ in $\Omega_{\mathbf{a}}$. Denote by $p : G \to G_{\mathbf{a}}$ the canonical projection and define $\ell_{\mathbf{a}}$ in $\mathfrak{g}_{\mathbf{a}}^*$ by $\ell = \ell_{\mathbf{a}} \circ p$. Then each of the following holds.*

(a) *$G_{\mathbf{a}}$ is the semi-direct product $T_{\mathbf{a}} H$ where $T_{\mathbf{a}}$ is the quotient $T/A_{\mathbf{a}}$, and $G_{\mathbf{a}}(\ell_{\mathbf{a}}) = G(\ell)/A_{\mathbf{a}}$.*

(b) *Denote by $V_{\mathbf{a}}$ the dual of $\mathfrak{t}_{\mathbf{a}}$; then $V_{\mathbf{a}}$ is the $\mathfrak{h}$-module:*

$$V_{\mathbf{a}} = \prod_{a_k > 0} V_{a_k - 1}^{(k)} = \prod_{a_k > 0} (f_{a_k}^{(k)}, \dots, f_{n_k}^{(k)})_{\mathbb{K}_k}.$$

(c) *Using the basis defined in* (b)*, $\ell_{\mathbf{a}}$ is in $\Omega_{\mathbf{1}} \subset \mathfrak{g}_{\mathbf{a}}^*$, the orbits $G \cdot \ell$ and $G_{\mathbf{a}} \cdot \ell_{\mathbf{a}}$ are isomorphic, $G \cdot \ell$ is regular if and only if $G_{\mathbf{a}} \cdot \ell_{\mathbf{a}}$ is regular and $G \cdot \ell$ is integral if and only if $G_{\mathbf{a}} \cdot \ell_{\mathbf{a}}$ is integral.*

Thanks to this lemma, we may confine our attention to the Zariski open sets $\omega = \omega_{\mathbf{1}}$ and $\Omega = \Omega_{\mathbf{1}}$. We write also:

$$\omega^{(k)} = \{v^{(k)} \in V^{(k)} : v_1^{(k)} \neq 0\}.$$

As described in [5, 4], each subset $\omega^{(k)}$ can be written as a product in such a way that the action of H splits. For each k, put $Z^{(k)} = (0, +\infty)$, $Y^{(k)} = \{y \in \mathbb{K}_k : |y| = 1\}$, and

$$W^{(k)} = \{w^{(k)} \in V^{(k)} : \ w_1^{(k)} = 1\}.$$

For each k, H acts on both $Z^{(k)}$ and $Y^{(k)}$: for $A \in \mathfrak{h}$, $\exp A \cdot z^{(k)} = e^{\alpha_k(A)} z^{(k)}$ and $\exp A \cdot y^{(k)} = e^{\mathrm{i}\beta_k(A)} y^{(k)}$. Note also that the Lie algebra homomorphism $A \mapsto n(A)$ gives an action of H on $W^{(k)}$: for $A \in \mathfrak{h}$,

$$\exp A \cdot w^{(k)} = e^{n^{(k)}(A)} w^{(k)}.$$

The map $\eta^{(k)} : \omega^{(k)} \to Z^{(k)} \times W^{(k)} \times Y^{(k)}$ defined by

$$\eta^{(k)}(v^{(k)}) = \big(|v_1^{(k)}|, v^{(k)}/v_1^{(k)}, \mathrm{sign}(v_1^{(k)})\big)$$

is a homeomorphism with inverse $(z^{(k)}, w^{(k)}, y^{(k)}) \mapsto z^{(k)} y^{(k)} w^{(k)}$. Note that $W^{(k)}$ is invariant under the unipotent maps $e^{n^{(k)}(A)}$, $A \in \mathfrak{h}$. If we now write $\eta^{(k)}\big(v^{(k)}\big) = (z^{(k)}, w^{(k)}, y^{(k)})$ we obtain:

$$\eta^{(k)}\big(\exp A \cdot v^{(k)}\big) = \big(e^{\alpha_k(A)} z^{(k)}, e^{n^{(k)}(A)} w^{(k)}, e^{\beta_k(A)} y^{(k)}\big).$$

Writing $Z = \prod_k Z^{(k)}$, $Y = \prod_k Y^{(k)}$, $W = \prod_k W^{(k)}$, we have the product homeomorphism $\eta : \omega \to Z \times W \times Y$ with inverse

$$\eta^{-1}(z, w, y) = zyw := (z^{(1)} y^{(1)} w^{(1)}, z^{(2)} y^{(2)} w^{(2)}, \ldots, z^{(m)} y^{(m)} w^{(m)}).$$

For brevity, we write here $v = (z, w, y)$, $\omega = Z \times W \times Y$, $e^{\alpha(A)} z = (e^{\alpha_k(A)} z_k)$, $e^{\beta(A)} y = (e^{\beta_k(A)} y_k)$, and $\exp A \cdot w = e^{n(A)} w = (e^{n^{(k)}(A)} w^{(k)})$. Thus we have actions of H on Z, Y, and W, respectively, and the action of H on ω splits: for $v \in \omega$,

$$\exp A \cdot v = (e^{\alpha(A)} z, e^{n(A)} w, e^{i\beta(A)} y), \quad A \in \mathfrak{h}. \tag{6}$$

Note that for all $v = (z, w, y) \in \omega$, the stability group for the action of H at z is the normal subgroup $H_0 = \exp \mathfrak{h}_0$, where

$$\mathfrak{h}_0 = \bigcap_k \ker \alpha_k.$$

In [4] where H is abelian, Proposition 2.1 is used to prove a criterion for regularity of an H-orbit in ω by considering the stability group at $(z, w) \in Z \times W$ and its action on the fiber Y. In the present work we are primarily interested in the coadjoint orbits in Ω, but these same stability groups will nevertheless play an essential role. The following is immediate.

Lemma 3.3. *Let $v = (z, w, y) \in \omega$.*

1. *The stability group for the action of H on $Z \times W$ at (z, w) is $H_0(w) = \exp \mathfrak{h}_0(w)$ where $\mathfrak{h}_0(w) = \{A \in \mathfrak{h}_0 : n(A) w = 0\}$.*
2. *The stability group $H(v)$ at v coincides with the stability group for the action of $H_0(w)$ on Y at y, and is given by*
$$H(v) = \{\exp A \in H_0(w) : \ \beta(A) \in (2\pi\mathbb{Z})^m\}.$$
3. *The Lie algebra of $H(v)$ is*
$$\mathfrak{h}(v) = \{A \in \mathfrak{h}_0(w) : \beta(A) = 0\}.$$

Put $\mathfrak{k} = \bigcap_{k=1}^{m} \ker_k$; observe that $\mathfrak{k}$ is an ideal in $\mathfrak{h}$ included in $\mathfrak{h}_0$, and

$$\mathfrak{h}(v) = \{A \in \mathfrak{k} : n(A)w = 0\} = \mathfrak{k}(w).$$

In what follows we write the more explicit notation $\mathfrak{k}(w)$ rather than $\mathfrak{h}(v)$. Though $\mathfrak{k}(w)$ is not necessarily an ideal in $\mathfrak{h}$, it is important to observe that $\mathfrak{k}(w)$ is an ideal in $\mathfrak{h}_0(w)$. Indeed, recall that we identify $\mathfrak{g}^*/T$ with the set of points $(v, \mathrm{f}|_{\mathfrak{k}(w)})$. In order to apply Proposition 2.1 to the regularity of coadjoint orbits in Ω, we will consider the action of the group $H_0(w)$ at the point $\mathrm{f}|_{\mathfrak{k}(w)} \in \mathfrak{k}(w)^*$, and only then, the action of the stability group $H_0(w)(\mathrm{f}|_{\mathfrak{k}(w)})$ on Y, at the point y. Accordingly, we write $\Omega = Z \times W \times \mathfrak{h}^* \times Y$ and

$$\ell = (v, \mathrm{f}) = (z, w, \mathrm{f}, y) \quad (\ell \in \Omega).$$

Similarly a point $(v, \mathrm{f}|_{\mathfrak{h}(v)}) \in \Omega/T$ is written as

$$(v, \mathrm{f}|_{\mathfrak{h}(v)}) = (z, w, \mathrm{f}|_{\mathfrak{k}(w)}, y).$$

The coadjoint action is given by

$$\begin{cases} \exp A \cdot \ell = (e^{\alpha(A)}z, e^{n(A)}w, \mathrm{Ad}^*_{\exp A}\mathrm{f}, e^{i\beta(A)}y), \\ \quad x \cdot \ell = (z, w, \mathrm{f} + x \wedge zyw, y). \end{cases} \tag{7}$$

Now denote the multiplicative group of non-zero elements of $\mathbb{K}_k$ by $\mathbb{K}_k^\times$, and observe that Y is a compact subgroup of $\prod_k \mathbb{K}_k^\times$. Then Y acts on ω by $y_0 \cdot (z, w, y) = (z, w, y_0 y)$, so that the projection $Z \times W \times Y \to Z \times W$ is an explicit form for the quotient map $\omega \to \omega/Y$. Similarly, Y acts on Ω/T and the projection $(z, w, \mathrm{f}|_{\mathfrak{k}(w)}, y) \mapsto (z, w, \mathrm{f}|_{\mathfrak{k}(w)})$ is the quotient map $\Omega/T \to (\Omega/T)/Y$. From now on we write $(\Omega/T)/Y = \Omega/(T \rtimes Y)$, as the combined action of T and Y is the action of a semi-direct product $T \rtimes Y$. Thus elements of $\Omega/(T \rtimes Y)$ may be written as (z, w, g), where $\mathrm{g} \in \mathfrak{k}(w)^*$. The actions of G on Ω/T (5) and on $\Omega/(T \rtimes Y)$ are written explicitly as

$$\begin{aligned} x \exp A \cdot (z, w, \mathrm{g}, y) &= (e^{\alpha(A)}z, e^{n(A)}w, \exp A \cdot \mathrm{g}, e^{\mathrm{i}\beta(A)}y), \\ x \exp A \cdot (z, w, \mathrm{g}) &= (e^{\alpha(A)}z, e^{n(A)}w, \exp A \cdot \mathrm{g}). \end{aligned} \tag{8}$$

Here it is understood that for $\mathrm{g} \in \mathfrak{k}(w)^*$, $\exp A \cdot \mathrm{g} = \mathrm{g} \circ \mathrm{Ad}_{\exp -A}$ belongs to $\mathfrak{k}(e^{n(A)}w)^*$. Recalling the relation (1) restricted to Ω:

$$\Omega \xrightarrow{p_1} \Omega/T \xrightarrow{p_2} \omega$$

we similarly have

$$\Omega \xrightarrow{q_1} \Omega/(T \rtimes Y) \xrightarrow{q_2} \omega/Y \tag{9}$$

given by

$$(z, w, \mathrm{f}, y) \longmapsto (z, w, \mathrm{f}|_{\mathfrak{k}(w)}) \longmapsto (z, w).$$

From (8) it is clear that q_1 and q_2 are G-equivariant. Here are further details concerning the sequence (9).

Proposition 3.4. (a) *Let $(z, w) \in Z \times W = \omega/Y$. The q_2-fiber over (z, w) is*

$$q_2^{-1}(z, w) = \{(z, w)\} \times \mathfrak{k}(w)^*.$$

The stability subgroup for the action of G on $Z \times W$ at (z, w) is $TH_0(w)$ and acts on $q_2^{-1}(z, w)$ by:

$$xh \cdot (z, w, \mathrm{g}) = (z, w, h \cdot \mathrm{g}) \quad (x \in T,\ h \in H_0(w)).$$

(b) *The stability group for the action of* G *on* $\Omega/(T \rtimes Y)$ *at the point* (z, w, g) *is* $TH_0(w)(\mathrm{g})$*, where* $H_0(w)(\mathrm{g})$ *is the stability group for the action of* $H_0(w)$ *on* $\mathfrak{k}(w)^*$ *at the point* g*. Moreover,* $H_0(w)(\mathrm{g}) = \exp \mathfrak{h}_0(w, \mathrm{g})$ *where*

$$\mathfrak{h}_0(w, \mathrm{g}) = \{A \in \mathfrak{h}_0(w) : \langle \mathrm{g}, [A, \mathfrak{k}(w)] \rangle = 0\}.$$

(c) *Let* $(z, w, \mathrm{g}) \in \Omega/(T \rtimes Y)$ *and* $\mathrm{f} \in \mathfrak{h}^*$ *such that* $\mathrm{f}|_{\mathfrak{k}(w)} = \mathrm{g}$*. The* q_1*-fiber over* (z, w, g) *is*

$$q_1^{-1}(z, w, \mathrm{g}) = \{(z, w)\} \times (\mathrm{f} + \mathfrak{k}(w)^\perp) \times Y.$$

(d) *Denote by* $\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$ *the set:*

$$\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)}) = \{A \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) : \beta(A) \in (2\pi\mathbb{Z})^m\}.$$

The stability group in H *for the quotient action on* Ω/T *at* $(v, \mathrm{f}|_{\mathfrak{k}(w)})$ *is*

$$H(v, \mathrm{f}|_{\mathfrak{k}(w)}) = \exp \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)}).$$

Its Lie algebra is:

$$\mathfrak{h}(v, \mathrm{f}|_{\mathfrak{k}(w)}) = \{A \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}),\ \beta(A) = 0\} = \mathfrak{k}(w, \mathrm{f}|_{\mathfrak{k}(w)}).$$

Proof. (a) The description of $q_2^{-1}(z, w)$ is immediate. Using Lemma 3.3, we see that $H(z, w)$ is connected and simply connected, and $H(z, w) = H_0(w)$, hence $G(z, w) = TH_0(w)$.

(b) For the action of $TH_0(w)$ on (z, w, g), recall that for any $v = (z, w, y)$, $x \wedge v \in \mathfrak{h}(v)^\perp = \mathfrak{k}(w)^\perp$, then $x \wedge v|_{\mathfrak{k}(w)} = 0$. This gives the value of $xh \cdot (z, w, \mathrm{g})$, and $G(z, w, \mathrm{g})$. Here too, since $\mathfrak{h}$ is nilpotent, $H_0(w)(\mathrm{g}) = \exp(\mathfrak{h}_0(w, \mathrm{g}))$.

(c) Observe that for each $\mathrm{f} \in \mathfrak{h}^*$, we have $\mathrm{f} + \mathfrak{k}(w)^\perp = \{\mathrm{f}' \in \mathfrak{h}^* : \mathrm{f}|_{\mathfrak{k}(w)} = \mathrm{f}'|_{\mathfrak{k}(w)}\}$, and this gives the form of $q_1^{-1}(z, w, \mathrm{g})$.

(d) The formula (7) gives the form of the fiber action of $TH_0(w)$. Now $\exp A$ stabilizes (v, f) if and only if A is in $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ and $\beta(A)$ in $(2\pi\mathbb{Z})^m$. This gives the form of $H(v, \mathrm{f}|_{\mathfrak{k}(w)})$ and its Lie algebra $\mathfrak{h}(v, \mathrm{f}|_{\mathfrak{k}(w)})$. ∎

The maps q_1 and q_2 are not necessarily fiber bundle maps, as the following example shows.

Example 3.5. Let $V = (f_1, f_2, f_3, f_4)_{\mathbb{R}}$ and $\mathfrak{h} = (A_1, A_2)_{\mathbb{R}}$ with $\mu(A_i)$ given by

$$\mu(A_1) = \begin{bmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \qquad \mu(A_2) = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{bmatrix}.$$

There is only one weight, $\lambda = 0$, so there is only one block $V = V^{(1)}$. For $v \in V$ write

$$v = v_1 f_1 + v_2 f_2 + v_3 f_3 + v_4 f_4 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Here
$$\omega = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} : v_1 \neq 0 \right\} = Z \times W \times Y$$

with
$$Z = (0, +\infty), \quad W = \left\{ \begin{bmatrix} 1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \right\}, \quad Y = \{\pm 1\}.$$

The coadjoint action of H on $\Omega = Z \times W \times \mathfrak{h}^* \times Y$ is given by

$$\exp(a_1A_1 + a_2A_2) \cdot (z, w, \mathrm{f}, y) = \big(z, \begin{bmatrix} 1 \\ w_2+a_1 \\ w_3 \\ w_4+a_2w_3 \end{bmatrix}, \mathrm{f}, y\big).$$

Suppose that $w_3 \neq 0$. Then $\mathfrak{k}(w) = \{0\}$, so $q_1(z, w, \mathrm{f}, y) = (z, w, 0)$ and the fiber $q_1^{-1}(z, w, 0) = \{(z, w)\} \times \mathfrak{h}^* \times Y$ is two-dimensional. But if $w_3 = 0$, then $\mathfrak{k}(w) = (A_2)_{\mathbb{R}}$, $q_1(z, w, \mathrm{f}, y) = (z, w, \alpha A_2^*)$ where $\alpha = \mathrm{f}(A_2)$. So in this case $q_1^{-1}(z, w, \alpha A_2^*)$ is one-dimensional:

$$q_1^{-1}(z, w, \alpha A_2^*) = \{(z, w)\} \times (\mathrm{f} + (A_1^*)_{\mathbb{R}}) \times Y.$$

Thus $q_1 : \Omega \to \Omega/(T \rtimes Y)$ is not a fiber bundle map. Similarly, if $w_3 \neq 0$, then $q_2^{-1}(z, w) = \{(z, w, 0)\}$ while if $w_3 = 0$ then $q_2^{-1}(z, w) = \{(z, w)\} \times (A_2^*)_{\mathbb{R}}$, showing that q_2 is also not a fiber bundle map. ∎

Note that if, in Example 3.5, we restrict the maps q_i to $\{(z, w, \mathrm{f}, y) : w_3 \neq 0\}$ then both q_1 and q_2 are fiber bundle maps. The same is true for their restrictions to $\{(z, w, \mathrm{f}, y) : w_3 = 0\}$. This illustrates a general result proved in the next section: there is a partition of Ω into finitely many invariant semi-algebraic subsets $\Omega_{\mathbf{i},\mathbf{j}}$, to each of which the restrictions of q_1 and q_2 are fiber bundle maps. A second application of Proposition 2.1 provides motivation for such a partition. See also [4] or [7].

Theorem 3.6. *Let $G = TH$ be an inhomogeneous group with H connected, simply connected nilpotent. Suppose that there is an Ad_G^*-invariant, locally closed subset E of Ω that is also Y-invariant, and such that the restrictions of q_1 to E, and q_2 to $E/(T \rtimes Y)$, are topological bundle maps. Let $\mathcal{O} = G \cdot \ell$ be a coadjoint orbit in E, and write $\ell = (z, w, \mathrm{f}, y)$. Then the following are equivalent.*

1. *$\mathcal{O}$ is regular.*
2. *The orbit $e^{\beta(\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)}))}y$ is regular in Y.*
3. *The rationality condition (here $1 \leq s \leq m$) holds:*

$$\dim_{\mathbb{R}}(\beta_s|_{\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})})_{\mathbb{R}} = \dim_{\mathbb{Q}}(\beta_s|_{\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})})_{\mathbb{Q}} \tag{10}$$

Proof. Since E is locally closed, both $q_1(E) = E/(T \rtimes Y)$ and $q_2(E/(T \rtimes Y))$ are locally closed also. But an orbit in a locally closed space X is regular if and only if it is regular in a locally closed subset of X. Hence we may confine our attention to E.

Consider first the orbit of $q_2(\ell) = (z, w)$. For $x \exp A \in G$, we have

$$x \exp A \cdot (z, w) = (e^{\alpha(A)}z, e^{n(A)} \cdot w),$$

so the action of G on $q_2(E/(T \rtimes Y))$ is of exponential type, and it is well-known that any orbit of an action of exponential type is regular. The stability group at $q_2(q_1(\ell)) = (z, w)$ is $TH_0(w)$, which acts on the fiber $\{(z, w)\} \times \mathfrak{k}(w)^*$ by a unipotent action, so the fiber orbit is regular also. Hence by Proposition 2.1, the orbit $G \cdot (z, w, \mathrm{f}|_{\mathfrak{k}(w)^*}) = q_1(\mathcal{O}) = \mathcal{O}/(T \rtimes Y)$ is regular.

Now apply Proposition 2.1 to the bundle $q_1 : E \to E/(T \rtimes Y)$. The stability group at $q_1(\ell) = (z, w, \mathrm{f}|_{\mathfrak{h}(v)})$ is $TH_0(w)(\mathrm{f}|_{\mathfrak{k}(w)})$ where $H_0(w)(\mathrm{f}|_{\mathfrak{k}(w)}) = \exp \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ is the stability group for the action of $H_0(w)$ on $\mathfrak{k}(w)^*$ at the point $\mathrm{f}|_{\mathfrak{k}(w)}$. The fiber over $(z, w, \mathrm{f}|_{\mathfrak{k}(w)})$ is $\{(z, w)\} \times (\mathrm{f} + \mathfrak{h}(v)^{\perp}) \times Y$.

Now for $A \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ and $x \in T$,

$$x \exp A \cdot (z, w, \mathrm{f}, y) = x \cdot (z, w, \mathrm{f}, e^{\beta(A)}y) \quad = (z, w, \mathrm{f} + x \wedge e^{\beta(A)} zyw, e^{\beta(A)}y).$$

Observe that for each $A \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$,

$$\{\mathrm{f} + x \wedge e^{\beta(A)} zyw : \ x \in T\} = \mathrm{f} + \mathfrak{k}(w)^{\perp}.$$

Hence the fiber orbit $TH_0(w)(\mathrm{f}|_{\mathfrak{k}(w)}) \cdot (z, w, \mathrm{f}, y)$ is the product

$$\{(z, w)\} \times \mathrm{f} + \mathfrak{k}(w)^{\perp} \times \{e^{\beta(A)}y : A \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})\} \tag{11}$$

and $\mathcal{O}$ is regular if and only if the product (11) is regular. Applying Proposition 2.1 to (11) we have $\mathcal{O}$ is regular if and only if the orbit $\{e^{\beta(A)}y : A \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})\}$ in the torus Y is regular. It is well known (see for instance [4]) that this regularity is equivalent to the compactness of the orbit, or to the rationality condition:

$$\dim_{\mathbb{R}}(\beta_s|_{\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})} : \ 1 \le s \le m)_{\mathbb{R}} = \dim_{\mathbb{Q}}(\beta_s|_{\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})} : \ 1 \le s \le m)_{\mathbb{Q}}. \qquad \blacksquare$$

4. Bundle structures

In this section we present an explicit construction of a partition of Ω whose elements $\Omega_{\mathbf{i},\mathbf{j}}$ have the desired invariance and projectable bundle properties. Then we refine this partition in smaller subsets on which the $H_0(w)$ orbits in $\mathfrak{k}(w)^*$ are isomorphic.

4.1. Real echelon form

We now describe the sequence of *real* column operations defined in [4] by which a complex matrix M is put into an echelon form. First some notation.

Let S be an $r \times s$ complex matrix, $S = [x_{i,j}]_{1 \le i \le r, 1 \le j \le s}$, let i_* be any row index, and j_* any column index. The row of S indexed by i_* is denoted by S_{i_*} and the column indexed by j_* is denoted by S^{j_*}. Put

$$S_{1,2,\ldots,i_*} = [x_{i,j}]_{1 \le i \le i_*, 1 \le j \le s}, \quad S^{1,2,\ldots,j_*} = [x_{i,j}]_{1 \le i \le r, 1 \le j \le j_*},$$

and

$$S^{1,2,\ldots,j_*}_{1,2,\ldots,i_*} = [x_{i,j}]_{1 \le i \le i_*, 1 \le j \le j_*}$$

The real span of the columns of S is denoted by

$$\mathrm{col}_{\mathbb{R}} S = (S^1, S^2, \ldots, S^s)_{\mathbb{R}}.$$

The *lexographic column basis* for S is the ordered basis $S^{j_1}, S^{j_2}, \ldots, S^{j_u}$ for $\mathrm{col}_{\mathbb{R}} S$ defined recursively by

$$j_1 = \min\{1 \le j \le s : S^j \ne 0\}, \quad j_2 = \min\{1 \le j \le s : S^j \notin \mathbb{R}S^{j_1}\},$$
$$j_3 = \min\{1 \le j \le s : S^j \notin (S^{j_1}, S^{j_2})_{\mathbb{R}}\}, \ \text{ and so on.}$$

The proof of the following is left to the reader.

Lemma 4.1. *Let S be an $r \times s$ complex matrix with lexographic column basis $S^{j_1}, S^{j_2}, \ldots, S^{j_u}$, let U be an $r \times r$ complex non-singular matrix, and V a real $s \times s$ unipotent upper triangular matrix. Then the lexographic column basis for USV is $(USV)^{j_1}, (USV)^{j_2}, \ldots, (USV)^{j_u}$. Hence for each $1 \le j \le s$ we have*

$$\dim \mathrm{col}_{\mathbb{R}}(USV)^{1,2,\ldots,j} = \dim \mathrm{col}_{\mathbb{R}} S^{1,2,\ldots,j}.$$

Let now M be an $n\times p$ complex matrix. Put $\rho_0=0$ and for each $1\le i\le n$ put

$$\rho_i=\dim\operatorname{col}_{\mathbb{R}}(M_{1,2,\dots,i}).$$

Observe that $\rho_i-\rho_{i-1}\le 2$. For $1\le j\le p$ put

$$\rho_i^j=\dim\operatorname{col}_{\mathbb{R}}\left(M_{1,2,\dots,i}^{1,2,\dots,j}\right)$$

and put $\rho_i^0=0$. Write $\mathbf{i}=\{i_1<i_2<\cdots<i_d\}=\{1\le i\le n:\rho_{i-1}<\rho_i\}$. For each $1\le r\le d$, define

$$\tilde{J}_r=\{1\le j\le p:\rho_{i_r}^{j-1}<\rho_{i_r}^j\}$$

and put $\tilde{J}_0=\emptyset$. Also, put $J_0=\emptyset$ and $J_r=\tilde{J}_r\setminus\tilde{J}_{r-1}$ $(r>0)$. Observe that for $1\le r<d$, if $i_r\le i<i_{r+1}$, then $\rho_i=\rho_{i_r}$ and for any j, $\rho_i^j=\rho_{i_r}^j$, hence $\tilde{J}_r$ is the index set for the lexographic column basis of $M_{1,2,\dots,i}$.

Real column operations leading to the column echelon form MT set forth in [4] may be described as follows. Assume that $M\neq 0$. We define $T(1),T(2),\dots,T(d)$ with $M(r)=MT(1)T(2)\cdots T(r), 1\le r\le d$, inductively as follows.

Let $M(0)=M$ and $T(0)$ the $p\times p$ identity matrix. Given $1\le r\le d$, assume that $M(r-1)=MT(0)T(1)\cdots T(r-1)$ is defined, so that if $k\notin\tilde{J}_{r-1}$ and $i<i_r$, then the entry $M(r-1)_i^k=0$. Write $M(r-1)=[m_{i,j}]$, and denote the complex conjugate of $m_{i,j}$ by $\overline{m}_{i,j}$.

Case 1: $\rho_{i_r}-\rho_{i_{r-1}}=1$.

Here J_r has one element and we write $J_r=\{j_r\}$. Consider a real $p\times p$ unipotent upper triangular matrix $T(r)=[t_{j,k}]$ ($t_{j,k}=0$ if $k<j$, $t_{j,j}=1$) whose only non-zero entries $t_{j,k}$ to the right of the diagonal ($j<k$) lie in the row of $T(r)$ indexed by j_r ($j=j_r$). The matrix $M(r-1)T(r)$ is the result of performing column operations whereby for each $k>j_r$, the real multiple $t_{j_r,k}M(r-1)^{j_r}$ of the column $M(r-1)^{j_r}$ is added to the column $M(r-1)^k$. In this case we choose the unique such matrix $T(r)$ so that $t_{j_r,k}=0$ if $k\in\tilde{J}_r$ and $(M(r-1)T(r))_{i_r}^k=0$ if $k\notin\tilde{J}_r$. More explicitly, observe that by construction $m_{i_r,j_r}\neq 0$, and put for each $k>j_r$, $k\notin\tilde{J}_r$,

$$t_{j_r,k}=-\frac{m_{i_r,k}}{m_{i_r,j_r}}. \tag{12}$$

Thus we have
$$\big(M(r-1)T(r)\big)_{1,2,\dots,i_r}^j=0\quad\text{for } j\notin\tilde{J}_r, \tag{13}$$

and
$$\big(M(r-1)T(r)\big)_{1,2,\dots,i_r}^j=M(r-1)_{1,2,\dots,i_r}^j\quad\text{for } j\in\tilde{J}_r. \tag{14}$$

Case 2: $\rho_{i_r}-\rho_{i_{r-1}}=2$.

Here J_r has two elements and we write $J_r=\{j_r<j_r'\}$. Consider a real $p\times p$ unipotent upper triangular matrix $T(r)=[t_{j,k}]$ ($t_{j,k}=0$ if $k<j$, $t_{j,j}=1$) whose only non-zero entries $t_{j,k}$ to the right of the diagonal ($j<k$) lie in the rows of $T(r)$ indexed by j_r and j_r' ($j\in J_r$). Now the matrix $M(r-1)T(r)$ is the result of performing column operations whereby for each k, with $j_r<k<j_r'$, the real multiple $t_{j_r,k}$ of the column $M(r-1)^{j_r}$ is added to the column $M(r-1)^k$ and for each k, with $j_r'<k$, the columns $t_{j_r,k}M(r-1)^{j_r}$ and $t_{j_r',k}M(r-1)^{j_r'}$ are added to the column $M(r-1)^k$. We choose the unique such matrix $T(r)$ so that $t_{j_r,k}=t_{j_r',k}=0$ if $k\in\tilde{J}_r$, and if $k\notin\tilde{J}_r$, $(M(r-1)T(r))_{i_r}^k=0$.

Then for $j = j_r$, $k > j_r$, $k \notin \tilde{J}_r$: $t_{j_r,k} = -\dfrac{\Im(m_{i_r,k}\,\overline{m}_{i_r,j'_r})}{\Im(m_{i_r,j_r}\,\overline{m}_{i_r,j'_r})}$, (15)

and for $j = j'_r$, $k > j'_r$, $k \notin \tilde{J}_r$: $t_{j'_r,k} = -\dfrac{\Im(m_{i_r,k}\,\overline{m}_{i_r,j_r})}{\Im(m_{i_r,j'_r}\,\overline{m}_{i_r,j_r})}$. (16)

(Note that if $j_r < k < j'_r$, then the expression for $t_{j_r,k}$ simplifies to that of the first case.) As in Case 1 we have

$$\big(M(r-1)T(r)\big)^j_{1,2,\dots,i_r} = 0, \quad j \notin \tilde{J}_r,$$

and
$$\big(M(r-1)T(r)\big)^j_{1,2,\dots,i_r} = M(r-1)^j_{1,2,\dots,i_r}, \quad j \in \tilde{J}_r.$$

Put $T = T(1)T(2)\cdots T(d)$, $J = \tilde{J}_d$.

Lemma 4.2. *We have*

(a) $(MT)^j = 0$ *for* $j \notin J$, *so* $\{T^j : j \notin J\}$ *is a basis for the nullspace of* M,

(b) $\{M^j : j \in J\}$ *is the lexographic column basis for* M, *and*

(c) *for* $j \in J_r$, *the first non-zero entry in the column* $(MT)^j$ *is in the* i_r*-row.*

Remark 4.3. Suppose that the matrix M has real entries. Then each subset J_r is a singleton, $J_r = \{j_r\}$, and the ordered set $\mathbf{j} = \{\tilde{J}_1 \subset \tilde{J}_2 \subset \cdots \subset \tilde{J}_d\}$ is:

$$\mathbf{j} = \{\{j_1\} \subset \{j_1, j_2\} \subset \cdots \subset \{j_1, j_2, \dots, j_d\}\}.$$

Thus $\mathbf{j}$ is entirely characterized by the finite sequence $(j_1, j_2, \dots, j_d)$. In fact, it is easily seen that in the case where M is real, the procedure described above is equivalent with the procedure described in [1, Section 2.1].

4.2. Classification of orbits in ω

Recall the development of Section 3: for $v = (z, w, y) \in \omega$, $h = \exp A \in H$,

$$\exp A \cdot v = \left(e^{\alpha(A)} z, e^{n(A)} w, e^{\mathrm{i}\beta(A)} y\right).$$

The stability group for the action of H on Z at z is H_0, but the stability group $H_0(w) = \exp \mathfrak{h}_0(w)$ for the quotient action of H on $\omega/\mathbb{T}$ at (z, w) may vary considerably with $w \in W$ due to the unipotent action of H_0 on W. The reduction procedure for real echelon form is applied here to classify elements in ω accordingly. Choose a strong Malcev basis

$$(A_1, A_2, \dots A_k, \dots, A_p \dots, A_q)$$

for $\mathfrak{h}$, so that $(A_1, \dots, A_k)$ is a basis for $\mathfrak{k}$ and $(A_1, \dots, A_p)$ is a basis for $\mathfrak{h}_0$.

Recall the realization $V = V^{(1)} \times V^{(2)} \times \cdots \times V^{(m)}$, and for each $1 \le k \le m$, the ordered basis $f_1^{(k)}, \dots, f_{n_k}^{(k)}$ for $V^{(k)}$. Recall also that if ${}_k(A) \notin \mathbb{R}$ for some k, then $f_j^{(k)} \in V_{\mathbb{C}} \setminus V$, $1 \le j \le n_k$. We now collate this basis into a single ordered basis $f_1, f_2, \dots, f_n$ for V, while preserving the ordering of each $V^{(k)}$-basis. For $v = (z, w, y) \in \omega$, write $n(A_j)w$ as an $n \times 1$ column vector using the ordered basis $f_1, \dots, f_n$ for V, and define the $n \times p$ complex matrix $M(w)$ by

$$M(w) = [n(A_1)w, n(A_2)w, \dots, n(A_p)w].$$

Observe that nullspace$(M(w)) = \{A \in \mathfrak{h}_0 : n(A)w = 0\} = \mathfrak{h}_0(w)$.

Apply the preceding subsection to the matrix $M(w)$, with the notation $\rho_i^j(w), \mathbf{i}(w)$, and so on. Thus for each w we have

row indices	$\mathbf{i}(w) = \{i_1(w) < i_2(w) < \cdots < i_d(w)\}$,
column index sets	$\mathbf{j} = \{\tilde{J}_1(w) \subset \tilde{J}_2(w) \subset \cdots \subset \tilde{J}_d(w)\}$

(with $\tilde{J}_r(w) = J_1(w) \cup \ldots \cup J_r(w)$), and by induction, real $p \times p$ unipotent upper triangular matrices $T(1,w), T(2,w), \ldots, T(d,w)$ satisfying the conditions above. The formulae (12), (15), and (16) show that the entries of the matrices $T(1,w), T(2,w), \ldots, T(d,w)$, and therefore of

$$T(w) = T(1,w)T(2,w)\cdots T(d,w),$$

are rational in $w, \overline{w}$. Let $\mathbf{i} = \{i_1 < \cdots < i_d\}$ be an increasing sequence of row indices, and consider column index sets $\mathbf{j} = \{\tilde{J}_1 \subset \tilde{J}_2 \subset \cdots \subset \tilde{J}_d\}$. Define

$$W_{\mathbf{i},\mathbf{j}} = \{w \in W : \mathbf{i}(w) = \mathbf{i}, \mathbf{j}(w) = \mathbf{j}\} \quad \text{and} \quad \omega_{\mathbf{i},\mathbf{j}} = \{(z,w,y) \in \omega : \ w \in W_{\mathbf{i},\mathbf{j}}\}.$$

The collection of non-empty $\omega_{\mathbf{i},\mathbf{j}}$ is a partition of ω. Observe that for each (z,w,y) in $\omega_{\mathbf{i},\mathbf{j}}$, the dimension of the orbit $H_0 \cdot w$ is $\#\tilde{J}_d$. With

$$\Omega_{\mathbf{i},\mathbf{j}} = \{(z,w,\mathrm{f},y) \in \Omega : \ (z,w,y) \in \omega_{\mathbf{i},\mathbf{j}}\},$$

the non-empty $\Omega_{\mathbf{i},\mathbf{j}}$ constitute a partition of Ω. That these sets are invariant is a consequence of the following lemma.

Lemma 4.4. *Let $h = \exp B \in H_0$. Let S_h be the matrix for Ad_h with respect to the basis $A_1, \ldots, A_p$. Then for each $w \in W$,*

$$M(h \cdot w) = e^{n(B)} M(w) S_{h^{-1}}.$$

Proof. Let $w \in W$ and define $\varphi_w : \mathfrak{h}_0 \to V$ by $\varphi_w(A) = n(A)w$. Then $M(w)$ is the matrix for φ_w with respect to the bases $A_1, \ldots, A_p$ for $\mathfrak{h}_0$ and $f_1, \ldots, f_n$ for V. Now using Remark 3.1,

$$\big(\varphi_w \circ \mathrm{Ad}_{h^{-1}}\big)(A) = n(Ad_{h^{-1}}A)w = e^{-n(B)}n(A)e^{n(B)}w = e^{-n(B)}\varphi_{h\cdot w}(A)$$

Hence the matrix for $\varphi_{h\cdot w}$ coincides with the matrix for the map $e^{n(B)} \circ \varphi_w \circ \mathrm{Ad}_{h^{-1}}$, namely $e^{n(B)}M(w)S_{h^{-1}}$. ■

Proposition 4.5. *The sets $\omega_{\mathbf{i},\mathbf{j}}$, $\Omega_{\mathbf{i},\mathbf{j}}$ are semi-algebraic and H-invariant. Moreover, there is a pair $\mathbf{i}_0$ and $\mathbf{j}_0$ such that $\omega_{\mathbf{i}_0,\mathbf{j}_0}$ and $\Omega_{\mathbf{i}_0,\mathbf{j}_0}$ are Zariski open, respectively in V and $\mathfrak{g}^*$.*

Proof. By construction, $(z,w,y) \in \omega_{\mathbf{i},\mathbf{j}}$ is equivalent to a relation on the rank of submatrices in $M(w)$, hence $\omega_{\mathbf{i},\mathbf{j}}$ and $\Omega_{\mathbf{i},\mathbf{j}}$ are semi-algebraic sets. Now let $h = \exp B \in H$, $(z,w,y) \in \omega_{\mathbf{i},\mathbf{j}}$.

By Lemma 4.4, and using the fact that $e^{n(B)}$ is lower triangular,

$$M(h\cdot w)_{1,2,\ldots,i} = \big(e^{n(B)}\big)_{1,2,\ldots,i}^{1,2,\ldots,i} \ M(w)_{1,2,\ldots,i} \ S_{h^{-1}}$$

holds for each $1 \le i \le n$, so that by, Lemma 4.1, we have $\rho_i(e^{n(B)}w) = \rho_i(w)$.

Hence $\mathbf{i}(h\cdot w)=\mathbf{i}(w)$. Now $S_{h^{-1}}$ is upper triangular, so for each value i_r of $\mathbf{i}$, $1\le j\le p$,

$$M(h\cdot w)^{1,2,\dots,j}_{1,2,\dots,i_r}=\left(e^{n(B)}\right)^{1,2,\dots,i_r}_{1,2,\dots,i_r}\ M(w)^{1,2,\dots,j}_{1,2,\dots,i_r}\ (S_{h^{-1}})^{1,2,\dots,j}_{1,2,\dots,j}.$$

Again the hypothesis of Lemma 4.1 is fulfilled, so now we have $\rho^j_{i_r}(h\cdot w)=\rho^j_{i_r}(w)$ for all j, hence $\tilde{J}_r(h\cdot w)=\tilde{J}_r(w)$.

Finally we construct $\mathbf{i}_0$ and $\mathbf{j}_0$ so that $\omega_{\mathbf{i}_0,\mathbf{j}_0}$ is Zariski open. Let

$$i_1=\min\{i_1(w):w\in\omega\}\quad\text{and}\quad\omega_{i_1}=\{(z,w,y)\in\omega:i_1(w)=i_1\}.$$

Let $j_1=\min\{j_1(w):w\in\omega_{i_1}\}$ and put

$$\omega_{i_1,j_1}=\{(z,w,y)\in\omega:i_1(w)=i_1,j_1(w)=j_1\}.$$

By minimality of i_1, j_1, we have

$$\omega_{i_1,j_1}=\{(z,w,y)\in\omega:\langle f_{i_1},n(A_{j_1})w\rangle\neq 0\}.$$

If $\rho_{i_1}(w)=1$ for all $(z,w,y)\in\omega_{i_1}$ put $J_1=\{j_1\}$ and $\omega_{i_1,J_1}=\omega_{i_1,j_1}$.
If $\rho_{i_1}(w)=2$ for some $w\in\omega_{i_1}$ then define

$$j'_1=\min\{j'_1(w):(z,w,y)\in\omega_{i_1,j_1}\},\quad J_1=\{j_1,j'_1\},$$

and

$$\omega_{i_1,J_1}=\{(z,w,y)\in\omega:i_1(w)=i_1,\quad J_1(w)=J_1\}.$$

By minimality,

$$\omega_{i_1,J_1}=\{(z,w,y)\in\omega:\Im\left(\overline{\langle f_{i_1},n(A_{j_1})w\rangle}\langle f_{i_1},n(A_{j'_1})w\rangle\right)\neq 0\}.$$

In both cases, ω_{i_1,J_1} is Zariski open.

Suppose that for some r, we have row indices $i_1<i_2<\dots<i_r$, and for each $1\le s\le r$ a column index set J_s, so that

$$\omega_{i_1,J_1,\dots,i_r,J_r}=\{(z,w,y)\in\omega:i_s(w)=i_s,J_s(w)=J_s,1\le s\le r\}$$

is Zariski open. Now denote the entries of the matrix $M(w)T(1,w)\dots T(r,w)$ by $m(w)_{i,j}$. By construction they are rational functions in w and $\overline{w}$, regular on $\omega_{i_1,J_1,\dots,i_r,J_r}$. If $\rho_i(w)=\rho_{i_r}=\#(\tilde{J}_r)$ for all $(z,w,y)\in\omega_{i_1,J_1,\dots,i_r,J_r}$, $i\ge i_r$, then $\omega_{\mathbf{i}_0,\mathbf{j}_0}=\omega_{i_1,J_1,\dots,i_r,J_r}$ is a Zariski open set.

Otherwise, repeat the preceding constructions:

let

$$i_{r+1}=\min\{i_{r+1}(w):(z,w,y)\in\omega_{i_1,J_1,\dots,i_r,J_r}\},$$

$$\omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1}}=\{(z,w,y)\in\omega_{i_1,J_1,\dots,i_r,J_r}:i_{r+1}(w)=i_{r+1}\}.$$

Let $j_{r+1}=\min\{j_{r+1}(w):(z,w,y)\in\omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1}}\}$, and

$$\omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1},j_{r+1}}=\{(z,w,y)\in\omega_{i_1,J_1,\dots,i_r,J_r}:m_{i_{r+1},j_{r+1}}(w)\neq 0\}.$$

By minimality of i_{r+1}, j_{r+1} this set is Zariski open. Then consider the following two cases: $\rho_{i_{r+1}}(w) - \rho_{i_r} = 1$ for all $(z,w,y) \in \omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1}}$; then put

$$J_{r+1} = \{j_{r+1}\} \quad \text{and} \quad \omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1},J_{r+1}} = \omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1},j_{r+1}}.$$

If $\rho_{i_{r+1}}(w) - \rho_{i_r} = 2$ for some $(z,w,y) \in \omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1}}$, then define, as above,

$$j'_{r+1} = \min\{j'_{r+1}(w) : (z,w,y) \in \omega_{i_1,J_1,\dots,i_r,J_r,i_{r+1},j_{r+1}}\} \quad \text{and} \quad J_{r+1} = \{j_{r+1}, j'_{r+1}\},$$

$$\omega_{i_1,J_1,\dots,i_{r+1},J_{r+1}} = \{(z,w,y) \in \omega_{i_1,J_1,\dots,i_r,J_r} : i_{r+1}(w) = i_{r+1}, J_{r+1}(w) = J_{r+1}\}.$$

By minimality,

$$\omega_{i_1,J_1,\dots,i_{r+1},J_{r+1}} = \{(z,w,y) \in \omega_{i_1,J_1,\dots,i_r,J_r} : \Im\left(\overline{m(w)}_{i_{r+1},j_{r+1}} m(w)_{i_{r+1},j'_{r+1}}\right) \neq 0\}.$$

By induction this proves that $\omega_{\mathbf{i}_0,\mathbf{j}_0}$ is Zariski open. ∎

Since the set of points w in W such that the dimension of $H_0 \cdot w$ is maximal is Zariski open, then all the H_0-orbits in the Zariski open $\omega_{\mathbf{i}_0,\mathbf{j}_0}$ have maximal dimension.

Fix an index pair $(\mathbf{i},\mathbf{j})$ such that $\omega_{\mathbf{i},\mathbf{j}} \neq \emptyset$ and put $J = \tilde{J}_d$. For $w \in W_{\mathbf{i},\mathbf{j}}$, put $T(w) = T(1,w) \cdots T(d,w)$; the entries $t_{i,j}(w)$ of $T(w)$ are well-defined rational functions on $W_{\mathbf{i},\mathbf{j}}$ in w and $\overline{w}$. By part a. of Lemma 4.2,

$$\mathfrak{h}_0(w) = \text{nullspace}(M(w)) = (T(w)A_h : h \notin J, 1 \leq h \leq p)_{\mathbb{R}},$$

and
$$\mathfrak{k}(w) = (T(w)A_e : e \notin J, 1 \leq e \leq k)_{\mathbb{R}}.$$

Let $\mathbf{k} = \{1 \leq e \leq k : e \notin J\}$, and consider the basis $B_1(w), \dots, B_q(w)$ of $\mathfrak{h}$ defined by

$$B_j(w) = \begin{cases} T(w)A_j, & \text{if } j \in \mathbf{k}, \\ A_j, & \text{otherwise.} \end{cases}$$

Recall that the ordered basis $\{A_1, \dots, A_q\}$ is a strong Malcev basis for $\mathfrak{h}$, and denote its dual basis by $\{A_1^*, A_2^*, \dots, A_q^*\}$. For each $w \in W_{\mathbf{i},\mathbf{j}}$, since $T(w)$ is upper triangular, $\{B_1(w), B_2,(w), \dots, B_q(w)\}$ is also a strong Malcev basis for $\mathfrak{h}$. Denote by $\tau(w)$ the transformation of $\mathfrak{h}$ defined by $\tau(w)A_j = B_j(w)$; just as for $T(w)$, the entries of the matrix for $\tau(w)$ with respect to the basis $\{A_1, \dots, A_q\}$ are well-defined rational functions on $W_{\mathbf{i},\mathbf{j}}$ in w and $\overline{w}$. The matrix with respect to the dual basis $\{B_1(w)^*, \dots, B_q(w)^*\}$ for the transformation ${}^t\tau(w) : \mathfrak{h}^* \to \mathfrak{h}^*$ such that ${}^t\tau(w)B_j(w)^* = A_j^*, 1 \leq j \leq q$ is its transpose.

Continue to fix $(\mathbf{i},\mathbf{j})$ as above, and obtain $\mathfrak{h} = \mathfrak{f} \oplus \mathfrak{k}(w)$ by setting

$$\mathfrak{f} = (A_j : \ k < j \leq q, \text{ or } j \in J)_{\mathbb{R}}. \tag{17}$$

Lemma 4.6. *Recall* $\mathbf{k} = \{1 \leq e \leq k : e \in J^c\}$, *put* $\mathbb{R}^{\mathbf{k}} = \{(a_e)_{e \in \mathbf{k}}\}$.

(a) *The map* $\varphi : \Omega_{\mathbf{i},\mathbf{j}}/T \to \omega_{\mathbf{i},\mathbf{j}} \times \mathbb{R}^{\mathbf{k}}$ *defined by* $\varphi(v, \mathrm{g}) = \left(v, (\langle \mathrm{g}, B_e(w) \rangle)_{e \in \mathbf{k}}\right)$ *is a homeomorphism.*

(b) *The map* $\psi : \Omega_{\mathbf{i},\mathbf{j}} \to \Omega_{\mathbf{i},\mathbf{j}}/T \times \mathfrak{f}^*$ *defined by* $\psi(v, \mathrm{f}) = \left(p_1(v, \mathrm{f}), \mathrm{f}|_{\mathfrak{f}}\right)$ *is a homeomorphism.*

Proof. Suppose that $(v_n, \mathrm{g}_n) \to (v_0, \mathrm{g}_0) \in \Omega_{\mathbf{i},\mathbf{j}}/T$; then $v_n \to v_0$ and there is some $\mathrm{f}_n \in \mathfrak{h}^*$, $\mathrm{f}_n|_{\mathfrak{k}(w_n)} = \mathrm{g}_n$ and $\mathrm{f}_n \to \mathrm{f}_0$, $\mathrm{f}_0|_{\mathfrak{k}(w_0)} = \mathrm{g}_0$. Since $T(w_n)A_e \to T(w_0)A_e$ then $(\langle \mathrm{f}_n, B_e(w_n)\rangle)_{e\in\mathbf{k}} \to (\langle \mathrm{f}_0, B_e(w_0)\rangle)_{e\in\mathbf{k}}$. Thus φ is continuous.

Let $b_n = (b_{n,e})_{e\in\mathbf{k}}$ and $v_n \in \omega_{\mathbf{i},\mathbf{j}}$ such that $b_n \to b_0$ and $v_n \to v_0$. Define $\mathrm{f}_n \in \mathfrak{f}^\perp$ by $\langle \mathrm{f}_n, B_e(w_n)\rangle = b_{n,e}$. Thus with respect to the basis $\{B_1(w)^*, \ldots, B_q(w)^*\}$ for $\mathfrak{h}^*$, f_n is given by the column vector $[\mathrm{f}_{n,j}(w)]_{1\le j\le q}$ where

$$\mathrm{f}_{n,j}(w) = \begin{cases} b_{n,j}, & \text{if } j = e \in \mathbf{k}, \\ 0, & \text{otherwise.} \end{cases}$$

Hence
$$\mathrm{f}_n(A_i) = \begin{cases} \sum_{j=1}^{q} t_{i,j}(w_n) b_{n,j}, & \text{if } i = e \in \mathbf{k}, \\ 0, & \text{otherwise.} \end{cases}.$$

Now $w_n \to w_0$ implies $t_{i,j}(w_n) \to t_{i,j}(w_0)$ so $\mathrm{f}_n(A_i) \to \mathrm{f}_0(A_i)$, showing that $(v_n, \mathrm{f}_n) \to (v_0, \mathrm{f}_0)$ in $\Omega_{\mathbf{i},\mathbf{j}}$. Define $\mathrm{g}_n = \mathrm{f}_n|_{\mathfrak{k}(w)}$, and we have $(v_n, \mathrm{g}_n) \to (v_0, \mathrm{g}_0)$ in $\Omega_{\mathbf{i},\mathbf{j}}/T$. This completes the proof of (a).

For (b), suppose $(v_n, \mathrm{f}_n) \to (v_0, \mathrm{f}_0)$ in $\Omega_{\mathbf{i},\mathbf{j}}$. Then $p_1(v_n, \mathrm{f}_n) \to p_1(v_0, \mathrm{f}_0)$ and $\mathrm{f}_n|_{\mathfrak{f}} \to \mathrm{f}_0|_{\mathfrak{f}}$ so ψ is continuous.

Let $(v_n, \mathrm{g}_n) \in \Omega_{\mathbf{i},\mathbf{j}}/T$ and $\mathrm{h}_n \in \mathfrak{f}^*$ such that $(v_n, \mathrm{g}_n) \to (v_0, \mathrm{g}_0)$ and $\mathrm{h}_n \to \mathrm{h}_0$. Define $\mathrm{f}_n \in \mathfrak{h}^*$ by

$$\langle \mathrm{f}_n, B_j(w_n)\rangle = \begin{cases} \langle \mathrm{g}_n, B_j(w_n)\rangle, & \text{if } j = e \in \mathbf{k}, \\ \langle \mathrm{h}_n, A_j\rangle, & \text{otherwise.} \end{cases}.$$

As above we see that $\mathrm{f}_n \to \mathrm{f}_0$, $\psi(v_n, \mathrm{f}_n) = ((v_n, \mathrm{g}_n), \mathrm{h}_n)$, and $(v_n, \mathrm{f}_n) \to (v_0, \mathrm{f}_0)$. ∎

The following is almost immediate.

Proposition 4.7. *Each of the following maps has a natural structure as a G-projectable fiber bundle.*

(a) $p_2 : \Omega_{\mathbf{i},\mathbf{j}}/T \to \omega_{\mathbf{i},\mathbf{j}}$ *with fiber* $\mathbb{R}^{\mathbf{k}}$.

(b) $p_1 : \Omega_{\mathbf{i},\mathbf{j}} \to \Omega_{\mathbf{i},\mathbf{j}}/T$, *with fiber* $\mathfrak{f}^*$.

(c) $q_2 : \Omega_{\mathbf{i},\mathbf{j}}/(T \rtimes Y) \to \omega_{\mathbf{i},\mathbf{j}}/Y$ *with fiber* $\mathbb{R}^{\mathbf{k}}$.

(d) $q_1 : \Omega_{\mathbf{i},\mathbf{j}}/T \to \Omega_{\mathbf{i},\mathbf{j}}/(T \rtimes Y)$ *with fiber* $\mathfrak{f}^* \times Y$.

Proof. We have seen that the maps p_1, p_2, q_1 and q_2 are G-equivariant. We need only show that each of (a) through (d) has the structure of a topological fiber bundle. For the bundles (a) and (b), this follows immediately from Lemma 4.6.

The trivialization maps for (c) and (d) are the same except with v replaced by (z, w) and p_1 replaced by q_1. For (c), $\varphi : \Omega_{\mathbf{i},\mathbf{j}}/(T \rtimes Y) \to \omega_{\mathbf{i},\mathbf{j}}/Y \times \mathbb{R}^{\mathbf{k}}$ is defined by

$$\varphi(z, w, \mathrm{g}) = \big(z, w, (\langle \mathrm{g}, B_e(w)\rangle)_{e\in\mathbf{k}}\big).$$

For (d), $\psi : \Omega_{\mathbf{i},\mathbf{j}} \to \Omega_{\mathbf{i},\mathbf{j}}/(T \rtimes Y) \times \mathfrak{f}^*$ is defined by

$$\psi(z, w, \mathrm{f}, y) = (z, w, \mathrm{f}|_{\mathfrak{k}(w)}, \mathrm{f}|_{\mathfrak{f}}) = \big(q_1(z, w, \mathrm{f}, y), \mathrm{f}|_{\mathfrak{f}}\big).$$ ∎

Combining Theorem 3.6 with parts (c) and (d) of Proposition 4.7 we have:

Corollary 4.8. *Let $\mathcal{O}$ be a coadjoint orbit of a nilpotent inhomogeneous nilpotent Lie group. Then the conditions of Theorem* 3.6 *are equivalent for $\mathcal{O}$.*

Remark 4.9. Referring to Lemma 4.6(b), each continuous map $\sigma:\Omega_0/T \to \Omega_0/T$ can be lifted to a continuous map $\sigma_0 : \Omega_0 \to \Omega_0$, by putting:

$$\sigma_0(v, \mathrm{f}) = \psi^{-1}(\sigma(v, \mathrm{f}|_{\mathfrak{k}(w)}), 0).$$

Observe that $\sigma_0(v, \mathrm{f}) = (v', \mathrm{f}')$, with $\mathrm{f}' \in \mathfrak{f}^{\perp}$. ∎

Example 4.10. Let us come back to Example 3.5. Suppose that $v \in \omega$, so that $v = (z, w, y)$ with $w = \begin{bmatrix} 1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$. Then

$$M(w) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & w_3 \end{bmatrix}.$$

Note that since the action of H is unipotent in this example, then $\mathfrak{k} = \mathfrak{h}_0 = \mathfrak{h}$ and $k = 2$. Observe that $M(w)$ is real so Remark 4.3 applies: If $w_3 \neq 0$, then $\mathbf{i} = \{2, 4\}$, and $\tilde{J_1} = \{1\}$, $\tilde{J_2} = \{1, 2\}$, $\mathbf{j}$ is characterized by the sequence $(j_1, j_2) = (1, 2)$, $J = \{1, 2\}$, $\mathbf{k} = \emptyset$, $\mathbb{R}^{\mathbf{k}} = \{0\}$, $\mathfrak{k}(w) = \{0\}$, and $\mathfrak{f} = \mathfrak{h}$. If $w_3 = 0$, then $\mathbf{i} = \{2\}$, $(j_1) = (1)$, $\mathbf{k} = \{2\}$, $\mathbb{R}^{\mathbf{k}} = \mathbb{R}$, $\mathfrak{k}(w) = (A_2)_{\mathbb{R}}$, and $\mathfrak{f} = (A_1^*)_{\mathbb{R}}$. Thus ω is the disjoint union of

$$\omega_{\{2,4\},(1,2)} = \{(z, w, y) :\ w_3 \neq 0\}, \quad \omega_{\{2\},(1)} = \{(z, w, y) :\ w_3 = 0\}.$$

In both cases, $T(w)$ is the identity matrix. In Example 3.5, we see directly the result of Proposition 4.7: if $w_3 \neq 0$, then $q_1^{-1}(z, w, 0) = \{(z, w)\} \times \mathfrak{f}^* \times Y$, while $q_2^{-1}(z, w) = \{(z, w, 0)\}$. If $w_3 = 0$, then $q_1^{-1}(z, w, \alpha A_2^*) = \{(z, w)\} \times (\mathrm{f} + (A_1^*)_{\mathbb{R}}) \times Y$, while $q_2^{-1}(z, w) = \{(z, w)\} \times (A_2^*)_{\mathbb{R}}$. ∎

4.3. Classification of orbits in each $\Omega_{\mathbf{i},\mathbf{j}}$

For each $v = (z, w, y) \in \omega_{\mathbf{i},\mathbf{j}}$, the classification method described in Subsections 4.1 and 4.2 is now applied to the action of $H_0(w) = \exp \mathfrak{h}_0(w)$ on $\mathfrak{k}(w)^*$. Now fix $v = (z, w, y) \in \omega_{\mathbf{i},\mathbf{j}}$, put $A_j(w) = T(w)A_j, 1 \leq j \leq p$, and put $J^c = \{1 \leq j \leq p : j \notin J\}$. Since $T(w)$ is upper triangular, the set $\{A_h(w) :\ h \in J^c\}$, when written in order of increasing indices, is a strong Malcev basis for $\mathfrak{h}_0(w)$. Similarly the ordered set $\{A_e(w) :\ e \in J^c,\ 1 \leq e \leq k\}$ is a strong Malcev basis for $\mathfrak{k}(w)$.

For each $\mathrm{g} \in \mathfrak{k}(w)^*$ and $h \in J^c$, write $A_h(w)\mathrm{g}$ as a column vector with respect to the ordered dual basis $\{A_e(w)^* : e \in J^c, 1 \leq e \leq k\}$; thus, with $\langle A_h(w)\mathrm{g}, A_e(w)\rangle = \langle g, [A_e(w), A_h(w)]\rangle$, we identify $A_h(w)\mathrm{g}$ with the column vector

$$A_h(w)\mathrm{g} = \Big[\langle g, [A_e(w), A_h(w)]\rangle\Big]_{e \in J^c, e \leq k}. \tag{18}$$

Write $J^c = \{h_1 < h_2 < \cdots < h_t\}$ and define the matrix $M'(w, \mathrm{g})$ by

$$M'(w, \mathrm{g}) = [A_{h_1}(w)\mathrm{g},\ A_{h_2}(w)\mathrm{g},\ \ldots,\ A_{h_t}(w)\mathrm{g}].$$

Thus $M'(w, \mathrm{g})$ is the matrix associated with the derived action of $\mathfrak{h}_0(w)$ on $\mathfrak{k}(w)^*$ with respect to the given bases, in exactly the same way that $M(w)$ is associated to the derived action of $n(\mathfrak{h}_0)$ on W. Especially, we have

$$\mathfrak{h}_0(w, \mathrm{g}) = \{A \in \mathfrak{h}_0(w) : \mathrm{g}[A, \mathfrak{k}(w)] = 0\} = \mathrm{nullspace}(M'(w, \mathrm{g})).$$

Continuing to fix $v = (z, w, y) \in \omega_{\mathbf{i},\mathbf{j}}$ and $\mathrm{g} \in \mathfrak{k}(w)^*$, apply the sequence of column operations described in Section 4.1 and the classification method described in Section 4.2, to the matrix $M'(w, \mathrm{g})$, replacing the indices as follows: the ordered set $\{1 \leq j \leq p\}$ is replaced by $J^c = \{h_1 < \cdots < h_t\}$, and the ordered set $\{1 \leq i \leq n\}$ by $J^c \cap \{1, \ldots, k\}$, also written in increasing order.

Note that since the matrix $M'(w, \mathrm{g})$ has real entries, Remark 4.3 applies. Thus we obtain row indices

$$\mathbf{e}(w, \mathrm{g}) = \big\{e_1(w, \mathrm{g}) < e_2(w, \mathrm{g}) < \cdots < e_u(w, \mathrm{g})\big\} \subset J^c \cap \{1, \ldots, k\},$$

and a corresponding sequence of column indices

$$\mathbf{h}(w, \mathrm{g}) = \big(h_1(w, \mathrm{g}), h_2(w, \mathrm{g}), \ldots, h_u(w, \mathrm{g})\big)$$

with values in J^c. The column operations are given by unipotent, upper triangular real matrices $T(w, \mathrm{g}, 1), T(w, \mathrm{g}, 2), \ldots, T(w, \mathrm{g}, u)$ and the resulting echelon form is $M'(w, \mathrm{g})T(w, \mathrm{g})$ where $T(w, \mathrm{g}) = T(w, \mathrm{g}, 1) \cdots T(w, \mathrm{g}, u)$.

The rows and columns of $T(w, \mathrm{g}, r)$ being indexed by J^c, the only non-vanishing entries of $T(w, \mathrm{g}, r)$ belong to the row indexed by $h_r(w, \mathrm{g})$, $1 \leq r \leq u$. Note that if $\mathbf{e}(w, \mathrm{g}) = \emptyset$, then $\mathbf{h}(w, \mathrm{g})$ is undefined, whence we write $\mathbf{h}(w, \mathrm{g}) = (\ )$.

For each $(z, w, y) \in \omega_{\mathbf{i},\mathbf{j}}$, and each such subset $\mathbf{e}$ of $J^c \cap \{1, \ldots, k\}$, and corresponding sequence $\mathbf{h}$, put

$$\Omega(w)_{\mathbf{e},\mathbf{h}} = \{\mathrm{g} \in \mathfrak{k}(w)^* : \ \mathbf{e}(w, \mathrm{g}) = \mathbf{e}, \ \mathbf{h}(w, \mathrm{g}) = \mathbf{h}\}.$$

Then put
$$\Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}} = \{(z, w, \mathrm{f}, y) \in \Omega_{\mathbf{i},\mathbf{j}} : \ \mathrm{f}|_{\mathfrak{k}(w)} \in \Omega(w)_{\mathbf{e},\mathbf{h}}\}.$$

It is clear that the non-empty sets $\Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ constitute a partition of $\mathfrak{g}^*$, and we may call these sets *layers*.

Define $\omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ as the set of all $v = (z, w, y) \in \omega_{\mathbf{i},\mathbf{j}}$ such that for some $\mathrm{f} \in \mathfrak{h}^*$ we have $(z, w, \mathrm{f}, y) \in \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$. By the Tarski-Seidenberg theorem [17, 18], $\omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ is a semi-algebraic subset of $\omega_{\mathbf{i},\mathbf{j}}$.

Remark 4.11. Observe that by definition, the entries of $M'(w, \mathrm{g})$ are linear functions in g with coefficients rational in w. Therefore, for any non zero real number c, $M'(w, c\mathrm{g}) = cM'(w, \mathrm{g})$, and by construction, $\mathbf{e}(w, \mathrm{g}) = \mathbf{e}(w, c\mathrm{g})$, $\mathbf{h}(w, \mathrm{g}) = \mathbf{h}(w, c\mathrm{g})$, and $T(w, \mathrm{g}) = T(w, c\mathrm{g})$. Hence $(z, w, \mathrm{f}, y) \in \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ implies $(z, w, c\mathrm{f}, y) \in \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ for $c \neq 0$.
Moreover, as with $T(w)$, the formulae (12), (15), and (16) show that $T(w, \mathrm{g})$ is rational in $w, \overline{w}$ and in particular, for each $j \in J^c$, $v \mapsto T(w, \mathrm{g})A_j(w)$ is continuous on $\omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$. ∎

Fix a layer $\Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$, and denote by $\mathbf{h}^c$ the set of j which are not values of $\mathbf{h}$. For each $\ell = (z, w, \mathrm{f}, y) \in \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ and $j \in J^c \cap \mathbf{h}^c$, put

$$A_j(\ell) = T(w, \mathrm{f}|_{\mathfrak{k}(w)})A_j(w).$$

Then for $\ell = (z, w, \mathrm{f}, y) \in \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$,

$$\mathrm{nullspace}(M'(w, \mathrm{g})) = \mathfrak{h}_0(w, \mathrm{g}) = \big(A_j(w, \mathrm{g}) : \ j \in J^c \cap \mathbf{h}(w, \mathrm{g})^c\big)_{\mathbb{R}}.$$

In fact, just as the ordered basis $\{A_h(w) : h \in J^c\}$ is a strong Malcev basis of $\mathfrak{h}_0(w)$, so also $\{A_j(w, \mathrm{g}) : j \in J^c \cap \mathbf{h}(w, \mathrm{g})^c\}$, ordered according to increasing order of indices in $J^c \cap \mathbf{h}(w, \mathrm{g})^c$, is a strong Malcev basis of $\mathfrak{h}_0(w, \mathrm{g})$.

Similarly, $\{A_j(w,\mathrm{g}) : j \in J^c \cap \mathbf{h}(w,\mathrm{g})^c \cap \{1,2,\ldots,k\}\}$ is a strong Malcev basis for $\mathfrak{k} \cap \mathfrak{h}_0(w,\mathrm{g})$, and the image of $\{A_j(w,\mathrm{g}) : j \in J^c \cap \mathbf{h}(w,\mathrm{g})^c \cap \{k+1,\ldots,p\}\}$ in $\mathfrak{h}_0(w,\mathrm{g})/(\mathfrak{h}_0(w,\mathrm{g}) \cap \mathfrak{k})$ under the quotient map is a strong Malcev basis for $\mathfrak{h}_0(w,\mathrm{g})/(\mathfrak{h}_0(w,\mathrm{g}) \cap \mathfrak{k})$.

We next show that each $\Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ is Ad^*_G-invariant. Put

$$\mathfrak{h}_0(w)_j = \mathfrak{h}_0(w) \cap (A_1, A_2, \ldots, A_j)_{\mathbb{R}},\ 1 \le j \le p\,;$$

note that since $\mathfrak{k} = (A_1, A_2, \ldots, A_k)_{\mathbb{R}}$, then we may write $\mathfrak{k}(w)_j = \mathfrak{h}_0(w)_j, 1 \le j \le k$. Since $T(w)$ is upper triangular, then $\{A_r(w) : r \in J^c, r \le j\}$ is a strong Malcev basis of $\mathfrak{h}_0(w)_j, 1 \le j \le p$.

Let $\pi_i : \mathfrak{k}(w)^* \to \mathfrak{k}(w)_i^* = \mathfrak{k}(w)^*/\mathfrak{k}(w)_i^{\perp}$ be the restriction map. Since each $\mathfrak{k}(w)_i$ is an ideal in $\mathfrak{h}_0(w)$, then $H_0(w)$ acts naturally on $\mathfrak{k}(w)_i^*$; in fact it is the quotient action on $\mathfrak{k}(w)^*/\mathfrak{k}(w)_i^{\perp}$. Now for each $\mathrm{g} \in \mathfrak{k}(w)^*$ put

$$\mathfrak{h}_0(w,\pi_i(\mathrm{g})) = \{A \in \mathfrak{h}_0(w) : A \cdot \pi_i(\mathrm{g}) = 0\} = \{A \in \mathfrak{h}_0(w) : \langle \mathrm{g}, [A, \mathfrak{k}(w)_i] \rangle = \{0\}\}.$$

For $1 \le i \le k$, put $\quad \rho_i(w,\mathrm{g}) = \dim \mathrm{col}_{\mathbb{R}}\left(M'(w,\mathrm{g})_{e \in J^c, e \le i}\right),$

and for $1 \le j \le p$, $\quad \rho_i^j(w,\mathrm{g}) = \dim \mathrm{col}_{\mathbb{R}}\left(M'(w,\mathrm{g})_{e \in J^c, e \le i}^{h \in J^c, h \le j}\right).$

The expression (18) shows that

$$\rho_i(w,\mathrm{g}) = \dim \mathfrak{h}_0(w,\pi_i(\mathrm{g})) \quad \text{and} \quad \rho_i^j(w,\mathrm{g}) = \dim\left(\mathfrak{h}_0(w,\pi_i(\mathrm{g})) \cap \mathfrak{h}(w)_j\right).$$

With these observations, we immediately get ([1, Lemma 2.2.2]):

Lemma 4.12. *The set* $\mathbf{e}(w,\mathrm{g})$ *is*

$$\mathbf{e}(w,\mathrm{g}) = \{1 \le i \le k : \ \mathfrak{h}_0(w,\pi_{i-1}(\mathrm{g})) \ne \mathfrak{h}_0(w,\pi_i(\mathrm{g}))\},$$

and for each $e \in \mathbf{e}(w,\mathrm{g})$, *the value* $h(w,\mathrm{g})_e$ *of the sequence* $\mathbf{h}(w,\mathrm{g})$ *is*

$$h(w,\mathrm{g})_e = \min\{1 \le j \le p : \ \mathfrak{h}_0(w)_j \cap \mathfrak{h}_0(w,\pi_{e-1}(\mathrm{g})) \not\subset \mathfrak{h}_0(w,\pi_e(\mathrm{g}))\}.$$

Let $h \in H$ and recall the definitions of $h \cdot w$ and $h \cdot \mathrm{g}$: the element $h \in H$ maps $\mathfrak{k}(w)$ onto $\mathfrak{k}(h \cdot w)$ and hence maps $\mathfrak{k}(w)^*$ onto $\mathfrak{k}(h \cdot w)^*$, so that for $\mathrm{f} \in \mathfrak{h}^*$, $(h \cdot \mathrm{f})|_{\mathfrak{k}(h \cdot w)} = h \cdot \left(\mathrm{f}|_{\mathfrak{k}(w)}\right)$. Using these definitions and the fact that $\mathrm{Ad}_h \mathfrak{h}_j = \mathfrak{h}_j$, is easy to check that

$$\mathfrak{k}(h \cdot w)_j = \mathrm{Ad}_h\big(\mathfrak{k}(w)_j\big), \quad 1 \le j \le k,$$
$$\mathfrak{h}_0(h \cdot w)_j = \mathrm{Ad}_h\big(\mathfrak{h}_0(w)_j\big), \quad 1 \le j \le p,$$

and
$$\mathfrak{h}_0(h \cdot w, \pi_j(h \cdot \mathrm{g})) = \mathrm{Ad}_h\big(\mathfrak{h}_0(w,\pi_j(\mathrm{g}))\big), \quad 1 \le j \le k.$$

Now for each set $\mathbf{e}$ and sequence $\mathbf{h}$, recall the following subset of $\mathfrak{k}(w)^*$:

$$\Omega(w)_{\mathbf{e},\mathbf{h}} = \{\mathrm{g} \in \mathfrak{k}(w)^* : \ \mathbf{e}(w,\mathrm{g}) = \mathbf{e},\ \mathbf{h}(w,\mathrm{g}) = \mathbf{h}\}.$$

Lemma 4.13. *One has* $h \cdot \big(\Omega(w)_{\mathbf{e},\mathbf{h}}\big) = \Omega(h \cdot w)_{\mathbf{e},\mathbf{h}}$, *more precisely,*

$$\mathrm{g} \in \Omega(w)_{\mathbf{e},\mathbf{h}} \ \textit{ if and only if } \ h \cdot \mathrm{g} \in \Omega(h \cdot w)_{\mathbf{e},\mathbf{h}}.$$

Proof. By Lemma 4.12:

$$\begin{aligned}\mathbf{e}(h\cdot w, h\cdot \mathrm{g}) &= \{1\le j\le k : \mathfrak{h}_0(h\cdot w, \pi_{j-1}(h\cdot \mathrm{g})) \neq \mathfrak{h}_0(h\cdot w, \pi_j(h\cdot \mathrm{g}))\}\\ &= \{1\le j\le k : \mathrm{Ad}_h(\mathfrak{h}_0(w,\pi_{j-1}(\mathrm{g}))) \neq \mathrm{Ad}_h(\mathfrak{h}_0(w,\pi_j(\mathrm{g})))\}\\ &= \{1\le j\le k : \mathfrak{h}_0(w,\pi_{j-1}(\mathrm{g})) \neq \mathfrak{h}_0(w,\pi_j(\mathrm{g}))\}\\ &= \mathbf{e}(w,\mathrm{g}).\end{aligned}$$

Similarly, for each $e\in \mathbf{e}(w,\mathrm{g})$,

$$\begin{aligned}&\mathbf{h}(h\cdot w, h\cdot \mathrm{g})_e =\\ &= \min\{1\le j\le p :\ \mathfrak{h}_0(h\cdot w)_j \cap \mathfrak{h}_0(h\cdot w, \pi_{e-1}(h\cdot \mathrm{g})) \not\subset \mathfrak{h}_0(h\cdot w, \pi_e(h\cdot \mathrm{g}))\}\\ &= \min\{1\le j\le p : \mathrm{Ad}_h\big(\mathfrak{h}_0(w)_j \cap \mathfrak{h}_0(w,\pi_{e-1}(\mathrm{g})) \not\subset \mathrm{Ad}_h\big(\mathfrak{h}_0(w,\pi_e(\mathrm{g}))\big)\big)\}\\ &= \min\{1\le j\le p : \mathfrak{h}_0(w)_j \cap \mathfrak{h}_0(w,\pi_{e-1}(\mathrm{g})) \not\subset \mathfrak{h}_0(w,\pi_e(\mathrm{g}))\}\\ &= \mathbf{h}(w,\mathrm{g})_e.\end{aligned}$$

■

The following summarizes much of this subsection.

Proposition 4.14. *Let $G = TH$ be an inhomogeneous nilpotent Lie group with Lie algebra $\mathfrak{g}$, and fix bases $f_1,\dots,f_n$ of $V=\mathfrak{t}^*$ and $A_1,\dots,A_q$ of $\mathfrak{h}$ with the properties described in Section* 3.

(a) *The column operation scheme and classification procedure described above defines a partition $\{\Omega_{\mathbf{i,j,e,h}}\}$ of $\mathfrak{g}^*$ into semi-algebraic, Ad^*_G-invariant sets.*

(b) *For each $\ell=(z,w,\mathrm{f},y)$ in $\Omega_{\mathbf{i,j,e,h}}$, put*

$$A_u(\ell) = A_u(w, \mathrm{f}|_{\mathfrak{k}(w)}) = T(w,\mathrm{f}|_{\mathfrak{k}(w)})A_u(w), \quad u\in J^c\cap \mathbf{h}^c.$$

The ordered set $\{A_u(\ell) :\ u\in J^c\cap\mathbf{h}^c\}$ is a strong Malcev basis for $\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$. Moreover, the coordinate functions of each map $\ell\mapsto A_u(\ell)$ are well-defined rational functions on $\Omega_{\mathbf{i,j,e,h}}$.

(c) *For each $\ell\in\Omega_{\mathbf{i,j,e,h}}$, define the real matrix*

$$N(\ell) = [\beta_i(A_u(\ell))]\,, \tag{19}$$

with rows indexed by $1\le i\le m$ and columns indexed by $J^c\cap\mathbf{h}^c$. Then ℓ is regular if and only if the real rank of $N(\ell)$ coincides with the rational rank of $N(\ell)$, that is,

$$\dim_{\mathbb{R}}(N(\ell)_i : 1\le i\le m)_{\mathbb{R}} = \dim_{\mathbb{Q}}(N(\ell)_i : 1\le i\le m)_{\mathbb{Q}}.$$

(d) *Among the sets $\Omega_{\mathbf{i,j,e,h}}$, there is one, denoted by Ω_0, that is Zariski open.*

Proof. Part (a) has already been verified. For part (b) it is enough to observe that $\{A_j : 1\le j\le q\}$ is a strong Malcev basis of $\mathfrak{h}$, and the transformation $T(w,\mathrm{f}|_{\mathfrak{k}(w)})T(w)$ maps $\mathfrak{h}$ onto $\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$ with full rank

$$d = \#(J^c\cap\mathbf{h}^c) = \dim \mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$$

and is upper triangular.

For part (c) observe that for each $\ell\in\Omega_{\mathbf{i,j,e,h}}$ we have the $\mathbb{R}$-linear isomorphism from $(\beta_i|_{\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})} : 1\le i\le m)_{\mathbb{R}}$ to the real row space of $N(\ell)$ such that $\beta_i|_{\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})}$ maps to $N(\ell)_i$.

Finally, following the definition [1, p. 78] and [1, Proposition 2.2.4], put

$$\begin{aligned} P_{\mathbf{e},\mathbf{h}}(w,\mathrm{f}) &= P_{\mathbf{e},\mathbf{h}}\big(M'(w,\mathrm{f}|_{\mathfrak{k}(w)})\big) \\ &= P_{e_1}^{h_1}\big(M'(w,\mathrm{f}|_{\mathfrak{k}(w)})\big)P_{e_1,e_2}^{h_1,h_2}\big(M'(w,\mathrm{f}|_{\mathfrak{k}(w)})\big)\cdots P_{e_1,\dots,e_r}^{h_1,\dots,h_r}\big(M'(w,\mathrm{f}|_{\mathfrak{k}(w)})\big) \end{aligned}$$

The function $(w,\mathrm{f}) \mapsto P_{\mathbf{e},\mathbf{h}}(w,\mathrm{f})$ is rational, non-singular in w and polynomial in f, and

$$\Omega_0 = \{(z,w,\mathrm{f},y) \in \mathfrak{g}^* : \ w \in \omega_{\mathbf{i},\mathbf{j}},\ P_{\mathbf{e},\mathbf{h}}(w,\mathrm{f}) \neq 0\}. \qquad \blacksquare \quad (20)$$

The matrix $N(\ell)$ of the preceding proposition will be called the regularity matrix. The following illustrates the results of Proposition 4.14.

Example 4.15. Let $\mathfrak{h} = (A_1,A_2,A_3,A_4,A_5)_{\mathbb{R}}$ with non-vanishing Lie brackets $[A_4,A_3] = A_1$, $[A_5,A_3] = A_2$. Let $V = \mathbb{C}^2$, with basis $\{f_1,f_2\}$ and $\mathfrak{h}$-action on V determined by $A_4\cdot f_1 = \mathrm{i}f_1$, $A_5\cdot f_2 = \mathrm{i}f_2$, while $\mu(A_1)=\mu(A_2)=\mu(A_3)=0$. Thus there are two weights and two 'blocks' $V^{(1)} = V^{(2)} = \mathbb{C}$. Following Lemma 3.2, we confine our attention to the points $(v,\mathrm{f}) \in \Omega$, where

$$\Omega = \{(v,\mathrm{f}) \in \mathfrak{g}^* : \ v_1v_2 \neq 0\}.$$

Accordingly, $W = \{(1,1)\}$, the H-action on $Z\times W$ is trivial, $M(w)$ is the 2×2 zero matrix, $T(w)$ is the identity matrix, $\mathbf{i} = \emptyset$, $\mathbf{j} = \emptyset$, and we have finally $\mathfrak{h}(v) = \mathfrak{k}(w) = \mathfrak{k} = (A_1,A_2,A_3)_{\mathbb{R}}$. However, $H(v)$ is not connected:

$$H(v) = \exp(2\pi\mathbb{Z}A_4 + 2\pi\mathbb{Z}A_5)\exp(A_1,A_2,A_3)_{\mathbb{R}}.$$

Put $\langle \mathrm{f}, A_j\rangle = \mathrm{f}_j$, $1\le j\le 5$ and similarly for $\mathrm{g}\in\mathfrak{k}(w)^* = \mathfrak{k}^*$. The matrix $M'(w,\mathrm{g})$ is

$$M'(w,\mathrm{g}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{g}_1 & -\mathrm{g}_2 \end{bmatrix}.$$

Then $\mathrm{g}_1 \neq 0$ if and only if $\mathbf{e}(w,\mathrm{g}) = \{3\}$ and $\mathbf{h}(w,\mathrm{g}) = (4)$; in this case we have $J^c\cap\mathbf{h}^c = \{1,2,3,5\}$, $\Omega(w)_{\{3\},(4)} = \{\mathrm{g}\in\mathfrak{k}^* : \mathrm{g}_1 \neq 0\}$, and

$$\Omega_{\emptyset,\emptyset,\{3\},(4)} = \{(v,\mathrm{f}) \in \Omega : \mathrm{f}_1 \neq 0\}.$$

For $(v,\mathrm{f}) \in \Omega_{\emptyset,\emptyset,\{3\},(4)}$ we have

$$\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)}) = (A_1,A_2,A_3,A_5(\ell))_{\mathbb{R}},$$

where
$$A_5(\ell) = T(w,\mathrm{f}|_{\mathfrak{k}(w)})A_5 = A_5 - (\mathrm{f}_2/\mathrm{f}_1)A_4.$$

Next, $\mathrm{g}_1 = 0$ and $\mathrm{g}_2 \neq 0$ if and only if $\mathbf{e}(w,\mathrm{g}) = \{3\}$, $\mathbf{h}(w,\mathrm{g}) = (5)$; so

$$\Omega(w)_{\{3\},(5)} = \{\mathrm{g}\in\mathfrak{k}^* : \mathrm{g}_1 = 0, \mathrm{g}_2 \neq 0\} \quad\text{and}\quad \Omega_{\emptyset,\emptyset,\{3\},(5)} = \{(v,\mathrm{f})\in\Omega : \mathrm{f}_1 = 0, \mathrm{f}_2 \neq 0\}.$$

In this case $J^c\cap\mathbf{h}^c = \{1,2,3,4\}$ and

$$\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)}) = (A_1,A_2,A_3,A_4)_{\mathbb{R}}.$$

Finally $\mathrm{g}_1 = \mathrm{g}_2 = 0$ if and only if $\mathbf{e}(w,\mathrm{g}) = \emptyset$, $\mathbf{h} = ()$, whence $J^c\cap\mathbf{h}^c = \{1,2,3,4,5\}$ and $\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)}) = \mathfrak{h}$.

We now further restrict our attention to the Zariski open layer $\Omega_0 = \Omega_{\emptyset,\emptyset,\{3\},(4)}$. Here the column operation matrix $T(w, \mathrm{g})$ is

$$T(w,\mathrm{g}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\mathrm{g}_2/\mathrm{g}_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the regularity matrix is

$$N(\ell) = \begin{bmatrix} 0 & 0 & 0 & \beta_1(A_5(\ell)) \\ 0 & 0 & 0 & \beta_2(A_5(\ell)) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{f}_2/\mathrm{f}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus we see that $\ell \in \Omega_0$ is regular if and only if f_2 is a rational multiple of f_1. ■

5. Regularity and integrality criteria

Let $G = TH$ be an inhomogeneous nilpotent Lie group with Lie algebra $\mathfrak{g}$, with $\{A_1, \ldots, A_q\}$ a strong Malcev basis for the Lie algebra $\mathfrak{h}$ of H such that $\{A_1, \ldots, A_p\}$ is a basis for $\mathfrak{h}_0$ and $\{A_1, \ldots, A_k\}$ a basis for $\mathfrak{k}$. We retain the notations and constructions of all of the above, so we have the Zariski open subset $\omega = \omega_{\mathbf{i},\mathbf{j}}$ of $\omega_{\mathbf{1}}$ identified with $Z \times W \times Y$, and the Zariski open layer $\Omega_0 = \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$ included in $\Omega = \omega \times \mathfrak{h}^*$. If every point $\ell \in \Omega_0$ is regular, then we say that Ω_0 is regular, or that we are in the regular case. In this section we show that if Ω_0 is not regular, then there is a dense $\mathcal{G}_\delta$ subset of Ω_0 consisting of non-regular points. We then give an algebraic characterization of the regular case. For integrality the situation is similar: we say that Ω_0 is integral, or that we are in the integral case, if every point in Ω_0 is integral, and show that if Ω_0 is not integral, then there is a dense $\mathcal{G}_\delta$ subset of Ω_0 consisting of non-integral points. Also similarly, we give an algebraic characterization of the integral case.

5.1. Criteria for the regular case

Recall the constructions of Section 4, as applied to the Zariski open set $\Omega_0 = \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}}$, and put $d = \#(J^c \cap \mathbf{h}^c)$. Specifically, for each $\ell \in \Omega_0$, we have the $m \times d$ regularity matrix

$$N(\ell) = [\beta_i(A_u(\ell))], \tag{21}$$

with rows indexed by $1 \leq i \leq m$ and columns indexed by $J^c \cap \mathbf{h}^c$. Put

$$\mathbf{r} = J^c \cap \mathbf{h}^c \cap \{k+1, \ldots, p\} \quad \text{and} \quad r = \#\mathbf{r}.$$

Writing now $\ell = (z, w, \mathrm{f}, y)$, observe that a strong Malcev basis for the quotient $(\mathfrak{k} + \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}))/\mathfrak{k}$ is $\{A_u(\ell) + \mathfrak{k} : u \in \mathbf{r}\}$. It follows that for all $\ell \in \Omega_0$, $r = \operatorname{rank} N(\ell)$. In particular, $N(\ell)$ is the zero matrix if and only if $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) = \mathfrak{k} \subset \ker \beta_i$, whence every orbit in Ω_0 is regular.

Let $\mathbf{s}(\ell) \subset \{1, 2, \ldots, m\}$ be the index sequence for the lexographic row basis of the real row space $(N(\ell)_i : 1 \leq i \leq m)_{\mathbb{R}}$ of $N(\ell)$, and let $\mathbf{s}$ be the minimum such sequence for the lexographic ordering:

$$\mathbf{s} = \min\{\mathbf{s}(\ell) : \ell \in \Omega_0\}. \tag{22}$$

Then $\Omega' = \{\ell \in \Omega_0 : \mathbf{s}(\ell) = \mathbf{s}\}$ is Zariski open. For each $\ell \in \Omega'$ and $1 \le i \le m$, the i-th row $N(\ell)_i$ of $N(\ell)$ is a linear combination of the rows indexed by $s \in \mathbf{s}$:

$$N(\ell)_i = \sum_{s \in \mathbf{s}} f_{i,s}(\ell) N(\ell)_s.$$

It is clear that each coefficient function $f_{i,s}$ is a well-defined rational function of $\ell \in \Omega'$. By Proposition 4.14, for each $\ell \in \Omega'$, $\mathcal{O} = \mathrm{Ad}_G^*\ell$ is regular if and only if $f_{i,s}(\ell) \in \mathbb{Q}$, $1 \le i \le m$. Put

$$\mathcal{R} = \{\ell \in \Omega_0 : \ \ell \text{ is regular}\}.$$

Lemma 5.1. *Assume that $r > 0$ and that $\mathcal{R} \cap \Omega' \neq \emptyset$. Either*
(a) *each coefficient function $f_{i,s}$ is constant on Ω' with its value in $\mathbb{Q}$, and $\Omega' \subset \mathcal{R}$, or*
(b) *$\mathcal{R} \cap \Omega'$ is a countable union of closed nowhere dense sets in Ω'.*

Proof. The preceding discussion shows that

$$\mathcal{R} \cap \Omega' = \bigcap_{\substack{1 \le i \le m \\ s \in \mathbf{s}}} \bigcup_{q \in \mathbb{Q}} \mathcal{F}(i,s,q)$$

where $\mathcal{F}(i,s,q)$ is the closed subset of Ω':

$$\mathcal{F}(i,s,q) = \{\ell \in \Omega' : \ f_{i,s}(w,\mathrm{f}) = q\}.$$

For each $1 \le i \le m$, $s \in \mathbf{s}$, $q \in \mathbb{Q}$, $\mathcal{F}(i,s,q) = \Omega'$ means that $f_{i,s}$ is constant on Ω' with value $q \in \mathbb{Q}$. If $\mathcal{F}(i,s,q) \neq \Omega'$, then our assumption that $\mathcal{R} \cap \Omega' \neq \emptyset$ implies that $f_{i,s}$ is not constant, and hence $\mathcal{F}(i,s,q)$ is nowhere dense and measure zero in Ω'. Hence $\Omega' \subset \mathcal{R}$ if and only if every $f_{i,s}$ is constant and rational-valued on Ω'.

If condition (a) of this lemma does not hold, then at least one coefficient function $f_{i,s}$ is not constant, and for every $q \in \mathbb{Q}$, $\mathcal{F}(i,s,q)$ is closed and nowhere dense in Ω'. Thus $\mathcal{R} \cap \Omega'$ is a union of a countable number of closed nowhere dense sets and $\Omega' \setminus \mathcal{R}$ is a dense $\mathcal{G}_\delta$ set. ■

Note that if (a) holds in Lemma 5.1, then by density of Ω' in Ω_0, $\Omega_0 = \mathcal{R}$.

Corollary 5.2. *Either each point in Ω_0 is regular, or there is a G-invariant, dense, conull $\mathcal{G}_\delta$ subset $\mathcal{U}$ in $\mathfrak{g}^*$ such that every point ℓ in $\mathcal{U}$ is not regular.*

Recall that Example 4.15 illustrates the 'non-regular' case, even though each G-orbit in V is regular.

Example 5.3. In Example 4.15, we have

$$\Omega_0 = \{(v,\mathrm{f}) : v_1 v_2 \neq 0, \langle \mathrm{f}, A_1 \rangle \neq 0\}, \quad \text{and} \quad N(\ell) = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{f}_2/\mathrm{f}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The minimal sequence $\mathbf{s}$ is $\mathbf{s} = (1)$, $\Omega' = \{\ell \in \Omega_0 : \mathrm{f}_2 \neq 0\}$ and for $\ell \in \Omega'$, $N(\ell)_2 = f_{2,1}(\ell) N_1(\ell)$ where $f_{2,1}(\ell) = -\mathrm{f}_1/\mathrm{f}_2$. Thus

$$\mathcal{U} = \Omega' \setminus \mathcal{R} = \{\ell \in \Omega' : \mathrm{f}_1/\mathrm{f}_2 \notin \mathbb{Q}\}.$$

Observe however that, for all v in V, $G\cdot v$ is closed and that in particular for $v\in\omega=\{v\in V: v_1v_2\neq 0\}$, $G\cdot v$ is a 2-torus. Thus $G\cdot v$ is always regular.

Next is an example of the regular case even while all G-orbits in ω are not regular.

Example 5.4. Here again $V=\mathbb{C}^2$ with basis $\{f_1,f_2\}$, but now $\mathfrak{h}=(A_1,A_2,A_3)_{\mathbb{R}}$ is the Heisenberg Lie algebra, $[A_3,A_2]=A_1$ acting by $A_3f_1=\mathrm{i}f_1$, $A_3f_2=\mathrm{i}\sqrt{2}f_2$. Write $\ell=(v,\mathrm{f})$ with $v=v_1f_1+v_2f_2$. Then $\Omega=\{(v,\mathrm{f})\in\mathfrak{g}^*: v_1v_2\neq 0\}$. For each $(v,\mathrm{f})\in\Omega$, the orbit $G{\cdot}v$ is the 'rope' on the 2-torus: $G{\cdot}v=\{(v_1e^{\mathrm{i}t},v_2e^{\mathrm{i}\sqrt{2}t}):\ t\in\mathbb{R}\}$, which is not regular.

Here, $W=\{(1,1)\}$, and for $v=(z,w,y)\in\omega$, $M(w)$ is the 2×3 zero matrix, $T(w)$ is the identity matrix, $\mathbf{i}=\mathbf{j}=\emptyset$, $\mathfrak{h}_0(w)=\mathfrak{h}$, and $\mathfrak{k}(w)=\mathfrak{k}=(A_1,A_2)_{\mathbb{R}}$. Writing $\mathrm{g}_i=\langle \mathrm{g},A_i\rangle$, $i=1,2$, the matrix $M'(w,\mathrm{g})$ is now:

$$M'(w,\mathrm{g})=\begin{bmatrix}0&0&0\\0&0&-\mathrm{g}_1\end{bmatrix}.$$

We have $\mathbf{e}(w,\mathrm{g})=\{2\}$ and $\mathbf{h}(w,\mathrm{g})=(3)$ if and only if $\mathrm{g}_1\neq 0$, so

$$\Omega(w)=\{\mathrm{g}\in\mathfrak{k}^*:\mathrm{g}_1\neq 0\}\quad\text{and}\quad \Omega_0=\Omega_{\emptyset,\emptyset,\{2\},(3)}=\{(v,\mathrm{f})\in\Omega:\langle \mathrm{f},A_1\rangle\neq 0\}.$$

For $(z,w,\mathrm{f},y)\in\Omega_0$, $\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})=\mathfrak{k}$, so $N(\ell)$ is the zero matrix for all $\ell\in\Omega_0$, and $\Omega_0=\mathcal{R}$.

The following shows that the regular case is characterized by a normal form for $N(\ell)$.

Theorem 5.5. *Let Ω_0 be the Zariski open set defined in* (20), *and suppose that $N(\ell)$ is not the zero matrix for some (hence all) $\ell\in\Omega_0$. Then the following are equivalent.*

(a) *Ω_0 is regular.*

(b) *$\ell\mapsto N(\ell)$ is constant on Ω_0, and there are matrices $P=[p_{s,i}]\in PSL(m,\mathbb{Z})$ and $Q=[q_{u,t}]\in GL(d,\mathbb{R})$ such that*

$$PNQ=\begin{bmatrix}0&I_r\\0&0\end{bmatrix}. \tag{23}$$

Proof. Suppose that Ω_0 is regular and consider the matrix $N(\ell)$. By Lemma 5.1, we have a Zariski open set $\Omega'\subset\Omega_0$ and an index set $\mathbf{s}\subset\{1,2,\dots m\}$, and constants $f_{i,s}\in\mathbb{Q}, 1\le i\le m, s\in\mathbf{s}$, such that for $\ell\in\Omega'$, the rows $N(\ell)_s, s\in\mathbf{s}$ are independent, and for each $1\le i\le m$, $\ell\in\Omega'$,

$$N(\ell)_i=\sum_{s\in\mathbf{s}} f_{i,s}N(\ell)_s.$$

This implies that the set $\{\beta_s: s\in\mathbf{s}\}$ is independent. Since $\dim\mathfrak{h}=q$, then

$$\dim(\beta_i:\ 1\le i\le m)_{\mathbb{R}}=q-k,$$

and $\quad\dim(\beta_t-\sum_{s\in\mathbf{s}}f_{t,s}\beta_s: t\notin\mathbf{s})_{\mathbb{R}}=\dim(\beta_i:\ 1\le i\le m)_{\mathbb{R}}-r=q-k-r.$

Hence $$\dim\big((\beta_t - \sum_{s\in\mathbf{s}} f_{t,s}\beta_s :\ t\notin \mathbf{s})_{\mathbb{R}}\big)^{\perp} = k+r\,.$$

Now we also have

$$\dim\big(\mathfrak{k}+\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})\big) = \dim\big(\mathfrak{k}\oplus (A_u(\ell) : u\in \mathbf{r})_{\mathbb{R}}\big) = k+r$$

and by definition of $N(\ell)$, $\mathfrak{k}+\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)}) \subset \big((\beta_t - \sum_{s\in\mathbf{s}} f_{t,s}\beta_s :\ t\notin \mathbf{s})_{\mathbb{R}}\big)^{\perp}$. Hence

$$\mathfrak{k}+\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)}) = \big((\beta_t - \sum_{s\in\mathbf{s}} f_{t,s}\beta_s :\ t\notin \mathbf{s})_{\mathbb{R}}\big)^{\perp},$$

showing that $\mathfrak{k}+\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$ is independent of $\ell\in\Omega'$. Since $\ell\mapsto A_u(\ell)$ is a well-defined rational function on all of Ω_0, then the preceding equality holds for all $\ell\in\Omega_0$. Thus $\mathfrak{k}+\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$ is independent of all $\ell\in\Omega_0$.

Put $\mathfrak{h}_1 = \mathfrak{k}+\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$, and let $\ell,\ell'\in\Omega_0$. For each $u\in\mathbf{r}$, both $A_u(\ell)$ and $A_u(\ell')$ belong to $\mathfrak{h}_1$. By construction, both $A_u(\ell)$ and $A_u(\ell')$ also belong to $A_u + (A_j :\ j\in J\cup\mathbf{h},\ j<u)_{\mathbb{R}}$, so

$$A_u(\ell)-A_u(\ell')\in\mathfrak{h}_1\cap (A_j :\ j\in J\cup\mathbf{h},\ j<u)_{\mathbb{R}}.$$

We claim that $$\mathfrak{h}_1\cap (A_j :\ j\in J\cup\mathbf{h})_{\mathbb{R}}\subset\mathfrak{k}.$$

Indeed, let $A\in\mathfrak{h}_1\cap (A_j :\ j\in J\cup\mathbf{h})_{\mathbb{R}}$, $A = B+\sum_{v\in\mathbf{r}} c_vA_v(\ell)$, with $B\in\mathfrak{k}$. Then

$$A = B+\sum_{v\in\mathbf{r}} c_vA_v(\ell) = B+\sum_{v\in\mathbf{r}} c_vA_v + A_0$$

where $A_0\in (A_j :\ j\in J\cup\mathbf{h})_{\mathbb{R}}$. Since the elements A_j are independent, and since $A\in\mathfrak{h}_1\cap (A_j :\ j\in J\cup\mathbf{h})_{\mathbb{R}}$, we get $c_v=0$ for all $v\in\mathbf{r}$ and $A=B\in\mathfrak{k}$, verifying the claim. Therefore $A_u(\ell)-A_u(\ell')\in\mathfrak{k}$ and $\ell\mapsto\beta_i(A_u(\ell))$ is constant.

Denote by $N_{\mathbf{s}}^{\mathbf{r}}$ the submatrix of N with rows indexed by the ordered set $\mathbf{s}$ and columns by the ordered set $\mathbf{r}$. For each $\ell\in\Omega_0$ pick $C_s(\ell)\in (A_u(\ell) : u\in\mathbf{r})_{\mathbb{R}}$ such that $\beta_s(C_t(\ell)) = \delta_{s,t}, s,t\in\mathbf{s}$. Write

$$C_s(\ell) = \sum_{u\in\mathbf{r}} q^{(0)}_{u,s}(\ell)A_u(\ell),\quad s\in\mathbf{s}$$

and put $Q^{(0)}(\ell) = [q^{(0)}_{u,s}(\ell)]$ with rows indexed by $\mathbf{r}$ and columns indexed by $\mathbf{s}$. Then

$$I_r = N_{\mathbf{s}}^{\mathbf{r}}\ Q^{(0)}(\ell)$$

holds for all $\ell\in\Omega_0$. Since $N_{\mathbf{s}}^{\mathbf{r}}$ is invertible, we have $Q^{(0)} = Q^{(0)}(\ell)$ is independent of $\ell\in\Omega_0$. Put

$$C_v(\ell) = \begin{cases} A_v(\ell), & v\in J^c\cap\mathbf{h}^c\cap\{1,2,\ldots,k\},\\ C_s(\ell), & v=s\in\mathbf{s}.\end{cases}$$

Then $\{C_v(\ell) : v\in (J^c\cap\mathbf{h}^c\cap\{1,2,\ldots,k\})\cup\mathbf{s}\}$ is a basis for $\mathfrak{h}_0(w,\mathrm{f}|_{\mathfrak{k}(w)})$. Recalling $d=\#(J^c\cap\mathbf{h}^c)$, let $Q_0\in GL(d,\mathbb{R})$ be defined by

$$Q_0 = \begin{bmatrix} I_{d-r} & 0\\ 0 & Q^{(0)}\end{bmatrix}.$$

Note that the columns of Q_0 are now indexed by $(J^c \cap \mathbf{h}^c \cap \{1, 2, \ldots, k\}) \cup \mathbf{s}$. Let $P_0 \in PSL(m, \mathbb{Z})$ be a permutation matrix putting the ordered set $\mathbf{s}$ in $\{1, \ldots, r\}$. Then $P_0 N Q_0$ has the form

$$P_0 N Q_0 = \begin{bmatrix} 0 & I_r \\ 0 & E \end{bmatrix}.$$

By the condition (10) of Theorem 3.6, the entries of E are rational numbers.

Now, using the methods of [3, Section 5], we have matrices $P_1 \in PSL(m, \mathbb{Z})$ and $Q_1 \in GL(d, \mathbb{Q})$ such that

$$P_1 P_0 N Q_0 Q_1 = PNQ = \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}.$$

Thus (b) is proved.

For the converse, assume that we have $P \in PSL(m, \mathbb{Z})$ and $Q \in GL(d, \mathbb{R})$ such that (23) holds for all ℓ in Ω_0. Put

$$R = P^{-1} \begin{bmatrix} 0 & I_r \\ 0 & 0 \end{bmatrix}.$$

Since the entries of R belong to $\mathbb{Z} \subset \mathbb{Q}$, then

$$\dim_{\mathbb{R}} (R_i : 1 \leq i \leq m)_{\mathbb{R}} = \dim_{\mathbb{Q}} (R_i : 1 \leq i \leq m)_{\mathbb{Q}}.$$

Now $\ell \mapsto N(\ell)$ is constant and

$$N_i = R_i Q^{-1}, \quad 1 \leq i \leq m,$$

hence $\dim_{\mathbb{R}} (N_i : 1 \leq i \leq m)_{\mathbb{R}} = \dim_{\mathbb{Q}} (N_i : 1 \leq i \leq m)_{\mathbb{Q}}$. Hence by Proposition 4.14, every $\ell \in \Omega_0$ is regular. ■

The following is a simple example of the regular case, just to illustrate the preceding objects.

Example 5.6. Suppose $V = \mathbb{C}^2$ and $\mathfrak{h} = (A)_{\mathbb{R}}$ with the action given by

$$\mu(A) = \begin{bmatrix} 3ib & 0 \\ 0 & 2ib \end{bmatrix},$$

where b is a fixed non-zero real number. Here $V^{(1)} = V^{(2)} = \mathbb{C}$ and $W = \{(1, 1)\}$. Then for all $\ell = (z, w, \mathrm{f}, y) \in \mathfrak{g}^*$ we have $M(w) = 0$ and $M'(w, \mathrm{f}) = 0$, so that $\Omega_0 = \Omega = \{(v, \mathrm{f}) : v^{(1)} v^{(2)} \neq 0\}$. For $\ell \in \Omega_0$, we have $\mathfrak{k} = \{0\}$ and $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) = \mathfrak{h}$ so

$$N = \begin{bmatrix} 3b \\ 2b \end{bmatrix}.$$

Now $\mathbf{r} = \mathbf{s} = \{1\}$, and $\beta_2 = (2/3)\beta_1$ so $f_{2,1} = 2/3$. Turning to the proof of Theorem 5.5, $C_1 = A_1(\ell) = (1/3b)A$, $P_0 = I_2$, $Q_0 = [1/3b]$ and

$$P_0 N Q_0 = \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}.$$

Then $Q_1 = [3]$ and $Q = Q_0Q_1 = [b^{-1}]$, while

$$P = P_1 = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

so that $$PNQ = P_1P_0NQ_0Q_1 = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} [3] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ ■

5.2. Cross-sections in the regular case

Assume in this subsection that we are in the regular case. We say that a Borel subset $\Sigma_0 \subset \Omega_0$ is a Borel cross-section in Ω_0 if the restriction of the quotient map $\theta : \Omega_0 \to \Omega_0/\mathrm{Ad}_G^*$ to Σ_0 is a bijection, and Σ_0 is a topological cross-section if $\theta|_{\Sigma_0}$ is a homeomorphism. It is well-known that since Ω_0 is regular, it admits a Borel cross-section [10, Theorem 2.9]. (In [10] such a set Σ_0 is called a transversal.)
Recall the vector subspace of $\mathfrak{h}$:

$$\mathfrak{f} = (A_j : \ k < j \le q, \ \text{or } j \in J)_{\mathbb{R}}.$$

Lemma 5.7. *Let Σ be a topological cross-section for the H-orbits in Ω_0/T. Then*

$$\Sigma_0 = \{(v, \mathrm{f}) \in \Omega_0 : \ (v, \mathrm{f}|_{\mathfrak{k}(w)}) \in \Sigma \ \textit{and}\ \mathrm{f} \in \mathfrak{f}^{\perp}\}$$

is a topological cross-section for the Ad_G^*-orbits in Ω_0.

Proof. As usual write $v \in \omega_{\mathbf{1}}$ as $v = (z, w, y)$. Let $(v, \mathrm{f}) \in \Omega_0$, so that $\mathrm{f}|_{\mathfrak{k}(w)} \in \Omega(w)_{\mathbf{e},\mathbf{h}}$, and we have $h_0 \in H$ such that $h_0 \cdot (v, \mathrm{f}|_{\mathfrak{k}(w)}) \in \Sigma$. Recall $\mathfrak{h} = \mathfrak{k}(w) \oplus \mathfrak{f}$, so $\mathfrak{h}^* = \mathfrak{k}(w)^{\perp} \oplus \mathfrak{f}^{\perp}$. Let $p_w(\mathrm{f})$ be the projection of f in $\mathfrak{k}(w)^{\perp}$ parallel to $\mathfrak{f}^{\perp}$, so $\mathrm{f} - p_w(\mathrm{f}) \in \mathfrak{f}^{\perp}$. We have $x_w \in T$ such that $x_w \cdot (v, \mathrm{f}) = \big(v, \mathrm{f} - p_w(\mathrm{f})\big)$ belongs to $\mathrm{Ad}_G^*(v, \mathrm{f}) \cap \Sigma_0$. If (v_0, f_0) and (v_1, f_1) both belong to the same coadjoint orbit as well as to Σ_0, then $v_0 = v_1$, $\mathrm{f}_0|_{\mathfrak{k}(w)} = \mathrm{f}_1|_{\mathfrak{k}(w)}$, and both f_0 and f_1 belong to $\mathfrak{f}^{\perp}$, so

$$\mathrm{f}_1 - \mathrm{f}_0 \in \mathfrak{k}(w)^{\perp} \cap \mathfrak{f}^{\perp} = \{0\}.$$

Thus $(v_0, \mathrm{f}_0) = (v_1, \mathrm{f}_1)$, which proves that Σ_0 is a cross-section for the Ad_G^*-orbits in Ω_0.

To see that Σ_0 is topological, observe that a cross-section S for a group action on a space X is topological if and only if the cross-section map $s : X \to X$ is continuous. Define $\sigma_0 : \Omega_0 \to \Omega_0$ by $\{\sigma_0(\ell)\} = \mathrm{Ad}_G^*(\ell) \cap \Sigma_0$. Similarly let $\sigma : \Omega_0/T \to \Omega_0/T$ be the cross-section map for $(\Omega_0/T)/H$. By hypothesis, σ is continuous, and we must show that σ_0 is continuous. But by construction, the cross-section map σ_0 is the continuous lift of σ, as discussed in Remark 4.9. ■

We now construct a topological cross-section for the H-orbits in Ω/T: start by choosing a cross-section for H-orbits in Z. Since H acts on Z by real exponentials, a simple topological cross-section Σ_Z is obtained immediately: choose a basis α_{k_t} for the space of real forms $(\alpha_1, \ \alpha_2, \dots, \ \alpha_m)_{\mathbb{R}}$, and then put

$$\Sigma_Z = \{z \in Z : \ z_{k_t} = 1\}.$$

Recall that $\omega_{\mathbf{i},\mathbf{j}}$ is defined by condition on w only, and thus we have $\omega_{\mathbf{i},\mathbf{j}} = Z \times W_{\mathbf{i},\mathbf{j}} \times Y$. Turning to $W_{\mathbf{i},\mathbf{j}}$, we look for a cross-section for the H_0-orbits

$$\{e^{n(B)} w : B \in \mathfrak{h}_0,\ w \in W_{\mathbf{i},\mathbf{j}}\}.$$

There is a standard construction due to Pukanszky [13] of a topological cross-section for a unipotent action on a real vector space using a real basis, and we modify this construction in order to allow for complex (non-real) coordinates. Specifically, for some $i_r \in \mathbf{i}$, we may have $J_r = \{j_r\}$ while the coordinate w_{i_r} is nevertheless non-real, and so the condition that $w_{i_r} = 0$ for the Pukanszky cross-section is replaced by the following. Put

$$\mathbf{i}' = \{i \in \mathbf{i} : f_i \in V_{\mathbb{C}} \setminus V \text{ and } \#J_r = 1\}.$$

For $i_r \in \mathbf{i}'$ put $\nu_{i_r}(w) = \Re\big(\langle f_{i_r},\ n(A_{j_r}(w))w\rangle\ \overline{w}_{i_r}\big)$ and define

$$\Sigma_W = \{w \in W_{\mathbf{i},\mathbf{j}} :\ \text{if } i \in \mathbf{i}',\ \nu_i(w) = 0,\ \text{and if } i \in \mathbf{i} \setminus \mathbf{i}',\ w_i = 0\}.$$

It is shown in [4, Proposition 6.2] (see also [2]) that Σ_W is a topological cross-section for the H_0-orbits in $W_{\mathbf{i},\mathbf{j}}$.

For each $v = (z, w, y) \in \omega_{\mathbf{i},\mathbf{j}}$, recall the action of $H_0(w)$ on $\mathfrak{k}(w)^*$ and the partition of $\mathfrak{k}(w)^*$ into $H_0(w)$-invariant subsets $\Omega(w)_{\mathbf{e},\mathbf{h}} \subset \mathfrak{k}(w)^*$:

$$\Omega(w)_{\mathbf{e},\mathbf{h}} = \{\mathrm{g} \in \mathfrak{k}(w)^* :\ \mathbf{e}(w, \mathrm{g}) = \mathbf{e},\ \mathbf{h}(w, \mathrm{g}) = \mathbf{h}\}.$$

Again employing the construction of [13], the set

$$\Sigma(w) = \Sigma(w)_{\mathbf{e},\mathbf{h}} = \{\mathrm{g} \in \Omega(w)_{\mathbf{e},\mathbf{h}} :\ \langle \mathrm{g}, A_e(w)\rangle = 0,\ e \in \mathbf{e}\}$$

is a topological cross-section for the $H_0(w)$-orbits in $\mathfrak{k}(w)^*$. Recall (20) that there is a single index sequence $(\mathbf{i}, \mathbf{j}, \mathbf{e}, \mathbf{h})$, $\mathbf{e} = \{e_1 < e_2 < \cdots < e_r\}$, $\mathbf{h} = (h_1, h_2, \ldots, h_r)$, such that

$$\Omega_0 = \Omega_{\mathbf{i},\mathbf{j},\mathbf{e},\mathbf{h}} = \{(z, w, \mathrm{f}, y) :\ (z, w, y) \in \omega_{\mathbf{i},\mathbf{j}},\ \mathrm{f}|_{\mathfrak{k}(w)} \in \Omega(w)_{\mathbf{e},\mathbf{h}}\}.$$

To achieve the description of the H-orbit space in Ω_0/T, it remains to consider, for each (z, w, f, y), the action of $H_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ on Y: for $A \in H_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$,

$$\exp A \cdot y = e^{\mathrm{i}\beta(A)} \cdot y = \left(e^{\mathrm{i}\beta_1(A)} y_1, e^{\mathrm{i}\beta_2(A)} y_2, \ldots, e^{\mathrm{i}\beta_m(A)} y_m\right).$$

Recall relation (23) and write $P = [p_{s,i}], 1 \le s, i \le m$ and $Q = [q_{u,t}], u \in \mathbf{r}, t \in \mathbf{s}$.

Since $P \in PSL(m, \mathbb{Z})$, the isomorphism $x \mapsto x' = Px$ of $\mathbb{R}^m$ preserves $\mathbb{Z}^m$ and defines a bijection c of $\mathbb{R}^m/\mathbb{Z}^m$, whose coordinate functions c_i, $1 \le i \le m$, are rational: with the usual identification of $\mathbb{R}^m/\mathbb{Z}^m$ with

$$\mathbb{T}^m = \{y \in \mathbb{C}^m : |y_i| = 1, 1 \le i \le m\}, \quad c_i(y_1, y_2, \ldots, y_m) = y_1^{p_{i,1}} y_2^{p_{i,2}} \cdots y_m^{p_{i,m}}.$$

We show that $\qquad \Sigma_Y = \{y \in Y : c_s(y) = 1,\ 1 \le s \le r\}$

is a cross-section for the orbits $H_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) \cdot y$ in Y.

Write $Q = [q_{s,t}]$ and for each $t \in J^c \cap \mathbf{h}^c$ put

$$A'_t(\ell) = \sum_{u \in J^c \cap \mathbf{h}^c} q_{u,t} A_u(\ell).$$

For $a = [a_t]_{t \in J^c \cap \mathbf{h}^c} \in \mathbb{R}^{J^c \cap \mathbf{h}^c}$, put $A'(a,\ell) = \sum_t a_t A'_t(\ell)$. Then

$$A'(a,\ell) = \sum_{u \in J^c \cap \mathbf{h}^c} (Qa)_u A_u(\ell).$$

Since we are in the regular case, for each $1 \le i \le m$, $\beta_i(A'(a,\ell))$ does not depend on ℓ and we can write

$$\beta_i\left(A'(a,\ell)\right) = (NQa)_i.$$

Hence for $y \in Y$,

$$\exp A'(a,\ell) \cdot y = e^{\mathrm{i}\beta(A'(a))} \cdot y = \left(e^{\mathrm{i}(NQa)_1} y_1, e^{\mathrm{i}(NQa)_2} y_2, \ldots, e^{\mathrm{i}(NQa)_m} y_m\right).$$

So for $1 \le i \le m$

$$\begin{aligned} c_i(\exp A'(a,\ell) \cdot y) &= \left(e^{\mathrm{i}(NQa)_1} y_1\right)^{p_{i,1}} \left(e^{\mathrm{i}(NQa)_2} y_2\right)^{p_{i,2}} \cdots \left(e^{\mathrm{i}(NQa)_m} y_m\right)^{p_{i,m}} \\ &= e^{\mathrm{i}(p_{i,1}(NQa)_1 + p_{i,2}(NQa)_2 + \cdots + p_{i,m}(NQa)_m} \; c_i(y) \\ &= e^{\mathrm{i}(PNQa)_i} \; c_i(y), \end{aligned}$$

and the relation (23) gives

$$c(\exp A'(a,\ell) \cdot y) = (e^{\mathrm{i}a_{s_1}} c_1(y), \ldots, e^{\mathrm{i}a_{s_r}} c_r(y), c_{r+1}(y) \ldots, c_m(y)), \tag{24}$$

where $s_j \in \mathbf{s}$ corresponds to $j, 1 \le j \le r$, by the permutation given by P_0. Thus for each $y \in Y$ we can choose $a \in \mathbb{R}^{J^c \cap \mathbf{h}^c}$ so that $\exp A'(a,\ell) \cdot y \in \Sigma_Y$, showing that the orbit $H_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) \cdot y$ meets Σ_Y. Now for $y \in \Sigma_Y$,

$$H_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) \cdot y = \left\{e^{\mathrm{i}NQa} \cdot y : \; a \in \mathbb{R}^{J^c \cap \mathbf{h}^c}\right\},$$

so if $y' \in \Sigma_Y \cap H_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) \cdot y$ then $y' = \exp A'(a,\ell) \cdot y$ with $a_s \in 2\pi\mathbb{Z}$ for all $s \in \mathbf{s}$, so (24) shows that $y' = y$. This proves that Σ_Y is a cross-section for the orbits $H_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) \cdot y$ in Y.

Example 5.8. Recall Example 5.6, where

$$P = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}.$$

So here $A'_1(\ell) = b^{-1}A$, $y'_1 = y_1 y_2^{-1}$, $y'_2 = y_1^{-2} y_2^3$, and $\Sigma_Y = \{(y_1, y_2) : \; y_1 = y_2\}$. ■

Now define

$$\Sigma = \{(v, \mathrm{g}) \in \Omega_0/T : \; v = (z, w, y) \in \Sigma_Z \times \Sigma_W \times \Sigma_Y \text{ and } \mathrm{g} \in \Sigma(w)\}.$$

Proposition 5.9. *Σ is a topological cross-section for the H-orbits in Ω_0/T.*

Proof. Let $(v_0, g_0) \in \Omega_0/T$. Choose $h_1 \in H$ so that $(v_1, g_1) := h_1 \cdot (v_0, g_0)$ satisfies $v_1 = (z_1, w_1, y_1) \in \Sigma_Z \times \Sigma_W \times Y$. Choose $h_2 \in H_0(w_1)$ so that $g_2 = h_2 \cdot g_1 \in \Sigma(w_1)$.

Then $(v_2, g_2) := h_2 \cdot (v_1, g_1)$ satisfies $v_2 = v_1 \in \Sigma_Z \times \Sigma_W \times Y$ and $g_2 \in \Sigma(w_1)$.

Choose $h_3 \in \exp(\mathfrak{h}_0(w_2, g_2))$ so that $h_3 \cdot v_2 = (z_3, w_3, y_3)$ satisfies $y_3 \in \Sigma_Y$. Then $z_3 = z_2 = z_1 \in \Sigma_Z$, $w_3 = w_2 = w_1 \in \Sigma_W$ and $g_3 = h_3 \cdot g_2 = g_2$ still belongs to $\Sigma(w_1) = \Sigma(w_3)$. Thus $\Sigma \cap H \cdot (v_0, g_0) \neq \emptyset$.

Now suppose that $(v_0, g_0) \in \Sigma$, let $h \in H$, and put $(v_1, g_1) = h \cdot (v_0, g_0)$. If $(v_1, g_1) \in \Sigma$, then $h \cdot v_0 \in \Sigma_Z \times \Sigma_W \times Y$ so $h \in H_0(w_0)$. Now $g_1 = h \cdot g_0$; but g_0 is the only element of $H_0(w_0) \cdot g_0$ that belongs to $\Sigma(w_0)$ so $g_1 = g_0$ and $h \in H_0(w_0)(g_0)$. Finally, y_0 is the only element of Σ_Y that belongs to $H_0(w_0)(g_0) \cdot y_0$ so $y_1 = h \cdot y_0 = y_0$ and thus, Σ meets each H-orbit in Ω_0/T exactly once.

Since each cross-section map in Z, W, Y and $\Omega(w)$ are continuous, the cross-section map for Σ is also continuous, and this achieves the proof. ■

Example 5.10. Let us come back to Example 5.4. Following prior notation, $v = (v_1, v_2) \in V = \mathbb{C}^2$, $\mathfrak{h} = (A_1, A_2, A_3)_{\mathbb{R}}$ is the Heisenberg Lie algebra, $\omega = \omega_{\mathbf{i},\mathbf{j}} = \{v \in V : v_1 v_2 \neq 0\}$, and ω is identified with $Z \times W \times Y$ where $W = \{(1,1)\}$. Since the H-action on $Z \times W$ is trivial, $\mathfrak{h}_0(w) = \mathfrak{h}$ and $\Sigma_Z \times \Sigma_W = Z \times W$. Recall that $\mathfrak{k} = (A_1, A_2)_{\mathbb{R}}$, so $k = 2$, $\mathfrak{k}(w) = \mathfrak{k}$, $\mathfrak{f} = (A_3)_{\mathbb{R}}$. Writing $g \in \mathfrak{k}^*$ as $g = (g_1, g_2)$, where $g_i = \langle g, A_i \rangle$, then for $v = (z, w, y) \in \omega$, $\Omega(w) = \{g \in \mathfrak{k}^* : g_1 \neq 0\}$. We have $\Sigma(w) = \{(g_1, 0) : g_1 \neq 0\}$. Now

$$\Omega_0 = \{(v, f) : \ v_1 v_2 \neq 0, \langle f, A_1 \rangle \neq 0\}.$$

For any $(v, f) \in \Omega_0$, one verifies that $\mathfrak{h}_0(w, f|_{\mathfrak{k}(w)}) = \mathfrak{k}$, so $\mathbf{r} = \emptyset$ and $\Sigma_Y = Y$. Hence $\Sigma = \{(v, g) : \ v_1 v_2 \neq 0, g_2 = 0\}$. For the cross-section Σ_0 of Ω_0 (Lemma 5.7), since $\mathfrak{f} = (A_3)_{\mathbb{R}}$, and the condition $(v, f|_{\mathfrak{k}}) \in \Sigma$ requires $\langle f, A_2 \rangle = 0$, we get

$$\Sigma_0 = \{(v, f) \in \Omega_0 : \ \langle f, A_2 \rangle = \langle f, A_3 \rangle = 0\}.$$

5.3. Integrality

In this subsection, we study the property of integrality for orbits in Ω_0. Recall that a point $(v, f) \in \mathfrak{g}^*$ is said to be integral if there is a character χ of the stability group $G(v, f)$ whose differential at the identity is $(v, f)|_{\mathfrak{g}(v,f)}$, and that integrality is an Ad^*_G-invariant property. Recall Proposition 2.5: (v, f) is integral if and only if the little orbit $H(v) \cdot f|_{\mathfrak{k}(w)}$ is integral. When H is abelian, $H(v)$ is abelian and the little orbit is integral; thus in the case where H is abelian, all points are integral. However, we saw in Example 2.6 that integrality does not necessarily hold if H is nilpotent.

Denote by $I(\Omega_0)$ the set of all integral points in Ω_0. Let $\ell = (v, f) = (z, w, f, y) \in \Omega_0$. Recall that the group $H(v)(f|_{\mathfrak{k}(w)})$ is generally not connected: it is the set

$$\exp \tilde{\mathfrak{h}}(v, f|_{\mathfrak{k}(w)}), \quad \text{where} \quad \tilde{\mathfrak{h}}(v, f|_{\mathfrak{k}(w)}) = \{A \in \mathfrak{h}_0(w, f|_{\mathfrak{k}(w)}) : \ \beta(A) \in (2\pi\mathbb{Z})^m\}.$$

The Lie algebra of $H(v)(f|_{\mathfrak{k}(w)})$ is $\mathfrak{k}(w, f|_{\mathfrak{k}(w)}) = \mathfrak{h}_0(w, f|_{\mathfrak{k}(w)}) \cap \mathfrak{k}$.

Denote by $\mathfrak{l}(v, f)$ the vector space spanned by $\tilde{\mathfrak{h}}(v, f|_{\mathfrak{k}(w)})$:

$$\mathfrak{l}(v, f) = \big(\tilde{\mathfrak{h}}(v, f|_{\mathfrak{k}(w)})\big)_{\mathbb{R}} = (A \in \mathfrak{h}_0(w, f|_{\mathfrak{k}(w)}) : \ \beta(A) \in (2\pi\mathbb{Z})^m)_{\mathbb{R}}.$$

Since $$[\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}), \mathfrak{l}(v, \mathrm{f})] \subset \mathfrak{k}(w, \mathrm{f}|_{\mathfrak{k}(w)}) \subset \mathfrak{l}(v, \mathrm{f}),$$

$\mathfrak{l}(v, \mathrm{f})$ is an ideal in $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$. Put $L(v, \mathrm{f}) = \exp \mathfrak{l}(v, \mathrm{f})$.

Proposition 5.11. *Let* $\ell = (v, \mathrm{f}) \in \Omega_0$.

(a) *If* ℓ *is regular, then* $\mathfrak{l}(v, \mathrm{f}) = \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$.

(b) *If* $\langle \mathrm{f}, \big[\mathfrak{l}(v, \mathrm{f}), \mathfrak{l}(v, \mathrm{f})\big]\rangle = 0$*, then* ℓ *is integral.*

Proof. To prove (a), it is enough to show that if ℓ is regular, then there is a basis of $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ whose elements belong to $\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$. Recall the index set $\mathbf{r} = J^c \cap \mathbf{h}^c \cap \{k+1, \ldots, p\}$ and $r = \#\mathbf{r}$; if $r = 0$, then $\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)}) = \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$. Assume that $r > 0$.

Recall the vectors $A'_u(\ell) = \sum_{t \in \mathbf{r}} q_{t,u} A_t(\ell)$ used in the discussion before Example 5.8. By construction, they form a basis of $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ modulo $\mathfrak{k}(w, \mathrm{f}|_{\mathfrak{k}(w)})$, with the property that for each $u \in \mathbf{r}$, $2\pi A'_u(\ell)$ belongs to $\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$. Then

$$\{2\pi A'_u(\ell) : \ 1 \leq u \leq r\} \cup \{A_j(\ell) : j \in J^c \cap \mathbf{h}^c \cap \{1, \ldots, k\}\}$$

is the desired basis.

Now suppose that $\langle \mathrm{f}, [\mathfrak{l}(v, \mathrm{f}), \mathfrak{l}(v, \mathrm{f})]\rangle = 0$, then the map $\exp A \mapsto e^{\mathrm{i}\mathrm{f}(A)}$ (where $\exp A \in L(v, \mathrm{f})$) is a character of $L(v, \mathrm{f})$. The restriction of this map to $H(v)(\mathrm{f}[_{\mathfrak{k}(w)})$ is a character of $H(v)(\mathrm{f}[_{\mathfrak{k}(w)})$, with differential $\mathrm{i}\mathrm{f}$. Hence by Proposition 2.5, (v, f) is integral. ■

Denote by $S(\Omega_0)$ the set of points (v, f) in Ω_0 such that $\langle \mathrm{f}, \big[\mathfrak{l}(v, \mathrm{f}), \mathfrak{l}(v, \mathrm{f})\big]\rangle = 0$, so that by Proposition 5.11, $S(\Omega_0) \subset I(\Omega_0)$.

For A and A' in $\mathfrak{h}$, define $C(A, A')$ in $\mathfrak{h}$ by:

$$[\exp A, \exp A'] = \exp A \exp A' \exp -A \exp -A' = \exp C(A, A'),$$

Since H is nilpotent, the map $(A, A') \mapsto C(A, A')$ is polynomial, and:

$$C(A, A') = [A, A'] + \sum (\text{ terms of degree} > 2).$$

Lemma 5.12. *Let* $(v, \mathrm{f}) \in \Omega_0$ *and suppose that* $\langle \mathrm{f}, C(A, A')\rangle = 0$ *holds for all* $A, A' \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$*. Then* (v, f) *is in* $S(\Omega_0)$.

Proof. Let $A, A' \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$. For any natural number n, $nA \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$, so $\langle \mathrm{f}, C(nA, A')\rangle = 0$ for any natural number n. Thus the polynomial function $P(s) = \langle \mathrm{f}, C(sA, A')\rangle$ satisfies $P(n) = 0$ for each n, hence $P = 0$. The same argument proves that $\langle \mathrm{f}, C(sA, s'A')\rangle = 0$ for all $s, s' \in \mathbb{R}$, and therefore

$$\langle \mathrm{f}, [A, A']\rangle = \frac{\partial^2}{\partial s \partial s'}|_{s=s'=0} \langle \mathrm{f}, C(sA, s'A')\rangle = 0$$

holds for all $A, A' \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$, and $\langle \mathrm{f}, [\mathfrak{l}(v, \mathrm{f}), \mathfrak{l}(v, \mathrm{f})]\rangle = 0$. ■

Let us now denote by $I^0(\Omega_0)$ the set of interior points in $I(\Omega_0)$.

Lemma 5.13. *Suppose that* (v, f) *is integral. Then*

(a) $\langle \mathrm{f}, C(A, A')\rangle \in 2\pi\mathbb{Z}$ *holds for all* A, A' *in* $\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$.

(b) *If* $(v, \mathrm{f}) \in I^0(\Omega_0)$, *then* $\langle \mathrm{f}, C(A, A')\rangle = 0$ *holds for all* $A, A' \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$.

Proof. To prove (a), let A, $A' \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$, so that $\exp A$, $\exp A' \in H(v)(\mathrm{f}|_{\mathfrak{k}(w)})$. Then $C(A, A') \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)}) \cap \mathfrak{k} = \mathfrak{k}(w, \mathrm{f}|_{\mathfrak{k}(w)})$. Since H is nilpotent, connected and simply connected, $\exp C(A, A')$ belongs to the identity component in $H(v)(\mathrm{f}|_{\mathfrak{k}(w)})$, and for each character η of $H(v)(\mathrm{f}|_{\mathfrak{k}(w)})$ with differential $\mathrm{if}|_{\mathfrak{k}(w, \mathrm{f}|_{\mathfrak{k}(w)})}$,

$$\eta\big([\exp A, \exp A']\big) = \eta\big(\exp C(A, A')\big) = e^{\mathrm{i}\langle \mathrm{f}, C(A, A')\rangle}.$$

But $\eta([\exp A, \exp A']) = 1$, so $\langle \mathrm{f}, C(A, A')\rangle$ belongs to $2\pi\mathbb{Z}$.

For (b), suppose that there is $\mathcal{U}$ open in $\mathfrak{g}^*$ such that $(v, \mathrm{f}) \in \mathcal{U} \subset I(\Omega_0)$, and let A, $A' \in \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$. For c near 1, $(v, c\mathrm{f})$ is in $\mathcal{U}$, hence in $I(\Omega_0)$. Moreover, by definition, $\mathfrak{h}_0(w, c\mathrm{f}|_{\mathfrak{k}(w)}) = \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$, and similarly, $\tilde{\mathfrak{h}}(v, c\mathrm{f}|_{\mathfrak{k}(w)}) = \tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)})$. Now by (a), $\langle c\mathrm{f}, C(A, A')\rangle \in 2\pi\mathbb{Z}$ for all c in a neighborhood of 1. Thus $\langle \mathrm{f}, C(A, A')\rangle = 0$. ■

Combining Lemmas 5.13 and 5.12, the following is immediate.

Proposition 5.14. *Any interior point in* $I(\Omega_0)$ *is in* $S(\Omega_0)$ *or:*

$$I^0(\Omega_0) \subset S(\Omega_0) \subset I(\Omega_0).$$

We are now ready to prove the main result for integrality.

Theorem 5.15. *Let* $G = TH$ *be an inhomogeneous nilpotent Lie group with Lie algebra* $\mathfrak{g}$. *Fix bases for* $\mathfrak{g}$ *and* $V = \mathfrak{t}^*$ *as in Section 3, and let* Ω_0 *be the Zariski open layer of Proposition 4.14 (see* (20) *in the proof of loc.cit.). Suppose that* Ω_0 *is regular. Then either*

(a) *there is an invariant dense, conull* $\mathcal{G}_\delta$*-subset* $\mathcal{Z}$ *of* Ω_0 *consisting of non-integral points, or*

(b) *every point* ℓ *in* Ω_0 *is integral.*

Moreover, the second case occurs if and only if for each $\ell = (v, \mathrm{f})$ *in* Ω_0, *and each* $t, u \in J^c \cap \mathbf{h}^c$,

$$\langle \mathrm{f}, [A_t(\ell), A_u(\ell)]\rangle = 0.$$

Proof. Put $\mathfrak{m} = (A_j : j \in J^c \cap \mathbf{h}^c)_{\mathbb{R}}$, and for $A = \sum_j a_j A_j \in \mathfrak{m}$ define $A(\ell) \in \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ by

$$A(\ell) = \sum_j a_j A_j(\ell).$$

For each A, $A' \in \mathfrak{m}$, consider the well-defined rational function $\varphi_{A,A'} : \Omega_0 \to \mathbb{R}$ defined by

$$\varphi_{A,A'}(\ell) = \langle \mathrm{f}, C(A(\ell), A'(\ell))\rangle.$$

Suppose that one of these functions $\varphi_{A,A'}$ is not constant on Ω_0. Then the set

$$\mathcal{Z} = \{\ell \in \Omega_0 :\ \varphi_{A,A'}(\ell) \notin 2\pi\mathbb{Z}\}$$

is a dense, conull, $\mathfrak{G}_\delta$ set. By part (a) of Lemma 5.13, every ℓ in $\mathcal{Z}$ is not integral. Otherwise each of the functions $\varphi_{A,A'}$ is constant. In consequence of Remark 4.11, $A(v, c\mathrm{f}) = A(v, \mathrm{f})$ for each A in $\mathfrak{m}$, $c \neq 0$, we have $\varphi_{A,A'}(v, c\mathrm{f}) = c\varphi_{A,A'}(v, \mathrm{f})$ for $c \neq 0$, and it follows that for every A, A' in $\mathfrak{m}$,

$$\langle \mathrm{f}, C(A(\ell), A'(\ell))\rangle = 0.$$

Since $\tilde{\mathfrak{h}}(v, \mathrm{f}|_{\mathfrak{k}(w)}) \subset \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) = \{A(\ell) :\ A \in \mathfrak{m}\}$, by Lemma 5.12, $\ell \in S(\Omega_0)$. Therefore $\Omega_0 = S(\Omega_0) = I(\Omega_0)$. In particular,

$$\langle \mathrm{f}, [A_t(\ell), A_u(\ell)]\rangle = 0$$

for each $t,\ u \in J^c \cap \mathbf{h}^c$.

Conversely, suppose that $\langle \mathrm{f}, [A_t(\ell), A_u(\ell)]\rangle = 0$ for each $t,\ u \in J^c \cap \mathbf{h}^c$, any $\ell \in \Omega_0$. Since $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})$ is spanned by the elements $A_u(\ell)$,

$$\langle \mathrm{f}, [\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}), \mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)})]\rangle = 0$$

holds for all $\ell = (v, \mathrm{f}) = (z, w, \mathrm{f}, y) \in \Omega_0$. Since every $\ell \in \Omega_0$ is regular, part (a) of Proposition 5.11 gives $\mathfrak{h}_0(w, \mathrm{f}|_{\mathfrak{k}(w)}) = \mathfrak{l}(v, \mathrm{f})$, and hence by part (b) of Proposition 5.11, ℓ is integral. ∎

Define the $d \times d$ matrix

$$R(\ell) = \Big[\langle \mathrm{f}, [A_t(\ell), A_u(\ell)]\rangle\Big],$$

with rows and columns indexed by the ordered set $J^c \cap \mathbf{h}^c$. Results of Section 5 are summarized briefly as follows.

Corollary 5.16. *Every point in Ω_0 is both regular and integral if and only if both of the following hold.*

(a) The regularity matrix $N = N(\ell)$ is independent of $\ell \in \Omega_0$ and if $N \neq 0$, there are $P \in PSL(m, \mathbb{Z})$ and $Q \in GL(d, \mathbb{R})$ satisfying the relation (23).

(b) For all $\ell \in \Omega_0$, $R(\ell)$ is the zero matrix.

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