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Invariants in Varieties of Lie Algebras

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Abstract. For a positive integer n, with $n \geq 2$, let L_n be the free Lie algebra over a field K of characteristic 0 and let $P_n = L_n/V_1(L_n)$ and $Q_n = L_n/V_2(L_n)$ be relatively free Lie algebras, with $V_1(L_n) \subseteq V_2(L_n)$. For a non-trivial finite subgroup G of $\mathrm{GL}_n(K)$, let P_n^G and Q_n^G be the Lie subalgebras of invariants in P_n and Q_n, respectively. We give connections between P_n^G and Q_n^G. For $G = S_2$, we apply our methods to L_2/L_2'' and $R_2 = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))')$ (i.e., R_2 is a free (nilpotent of class 2)-by-abelian and abelian-by-(nilpotent of class 2) Lie algebra of rank 2). We give a basis and a minimal infinite generating set for $R_2^{S_2}$ and we find a presentation of $R_2^{S_2}$.

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1. Introduction

For a positive integer n, with $n \geq 2$, let L_n be the free Lie algebra of rank n over a field K of characteristic 0. Let G be a non-trivial finite subgroup of $\mathrm{Aut}(L_n)$. We say that an element $a \in L_n$ is a G-invariant if $a = \theta(a)$ for all $\theta \in G$. It is clear enough that the set L_n^G of all G-invariants in L_n is a Lie subalgebra of L_n. We call L_n^G the Lie subalgebra of G-invariants in L_n. It has been proved in [3] that L_n^G is infinitely generated and the generating series for the number of free generators of L_n^G is found in [14]. Let $\mathfrak{V}$ be a non-trivial variety of Lie algebras, that is, $\mathfrak{V}(L_n) \subseteq L_n'$, where $\mathfrak{V}(L_n)$ denotes the fully invariant ideal of L_n corresponding to $\mathfrak{V}$, and let $B_n = L_n/\mathfrak{V}(L_n)$. Thus, B_n is a relatively free Lie algebra of rank n. As in the case of the free Lie algebra, for a finite subgroup G of $\mathrm{Aut}(B_n)$, we say that an element $a \in B_n$ is a G-invariant in B_n if $a = \theta(a)$ for all $\theta \in G$ and we write B_n^G for the Lie subalgebra of G-invariants in B_n. It has been proved in [8] that if B_n is nilpotent, then B_n^G is finitely generated, whereas if B_n is free in a variety containing the metabelian variety, then B_n^G is infinitely generated. Analogous results hold when the field is of prime characteristic, see [3], [5], [8]. It is an interesting problem to find a minimal generating set and a presentation in terms of generators and defining relators of the Lie algebra of invariants B_n^G of a relatively free Lie algebra B_n.

Let $V_1(L_n)$ and $V_2(L_n)$ be fully invariant ideals of L_n such that $V_1(L_n) \subset V_2(L_n)$. Let $P_n = L_n/V_1(L_n)$ and $Q_n = L_n/V_2(L_n)$ be relatively free Lie algebras of rank n. Our main aim is to give a connection between P_n^G and Q_n^G (see Section 3 below).

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In the special case that $G = S_n$, we call an S_n-invariant element in B_n a symmetric polynomial in B_n. For $M_n = L_n/L_n''$ (that is, M_n is the free metabelian Lie algebra), a generating set for $M_n^{S_n}$ is given in [10] for $n = 2$ and in [9] for $n \geq 3$. In particular, in [10], a basis and a presentation for $M_2^{S_2}$ were given. In [11], a basis and a minimal generating set for the Lie algebra of symmetric polynomials in a free centre-by-metabelian Lie algebra of rank 2 were given. We apply our methods to extend the results of [10] in the case of a free (nilpotent of class 2)-by-abelian and abelian-by-(nilpotent of class 2) Lie algebra $R_2 = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))')$ of rank 2. In Section 6, we give a basis of $R_2^{S_2}$, we find an (infinite) minimal generating set of $R_2^{S_2}$ and we find a presentation of $R_2^{S_2}$ (see Theorem 2.1, Theorem 2.2 and Theorem 2.3).

2. Preliminaries

Let K be a field of characteristic zero. By a Lie algebra we mean a Lie algebra over K. We also write K-vector (sub)space instead of vector (sub)space over K. For any Lie algebra, we use the left-normed convention for Lie commutators. Let L be a Lie algebra. For $x, y \in L$ and for a non-negative integer κ, we use the notation $[x, {}_\kappa y]$ for the Lie commutator of x and κ copies of y (in the case $k = 0$, we write $[x, {}_0 y] = x$). For a positive integer c, we write $\gamma_c(L)$ for the c-th term of the lower central series of L. We write $L' = \gamma_2(L)$, that is, L' is the derived algebra of L, and $L'' = (L')'$. The automorphism group of L is denoted $\mathrm{Aut}(L)$. We write $\mathrm{IA}(L)$ for the normal subgroup of $\mathrm{Aut}(L)$ consisting of all automorphisms of L which induce the identity automorphism on L/L' (these are the so-called IA-automorphisms of L). For a non-empty subset X of L, we write $L(X)$ for the Lie algebra generated by the set X. We say that X is a minimal (or irredundant) generating set of $L(X)$ if for any $x \in X$, $x \notin L(X \setminus \{x\})$.

For a positive integer n, with $n \geq 2$, let L_n be the free Lie algebra of rank n freely generated by the set $\{\ell_1, \ldots, \ell_n\}$. Let $\mathfrak{V}$ be a non-trivial variety of Lie algebras and let $B_n = L_n/\mathfrak{V}(L_n)$ be a relatively free Lie algebra of rank n freely generated by the set $\{z_1, \ldots, z_n\}$, where $z_i = \ell_i + \mathfrak{V}(L_n)$ for $i = 1, \ldots, n$. In the case $\mathfrak{V}(L_n) = L_n''$, $M_n = L_n/L_n''$ is the free metabelian Lie algebra of rank n. For further information about varieties of Lie algebras, we refer to [1] or [2]. We point out that there is a well-known dichotomy for varieties of Lie algebras: a variety is either nilpotent or it contains the variety of all metabelian Lie algebras. (If $\mathfrak{V}$ is a non-trivial variety of Lie algebras which does not contain the variety of all metabelian Lie algebras and $B_n = L_n/\mathfrak{V}(L_n)$ is the free Lie algebra in the variety $\mathfrak{V}$, then $\mathfrak{V}(L_n) \not\subseteq L_n''$.

Hence, B_n satisfies an Engel identity (see, for example, [4, proof of Corollary 5.4]). Therefore, by [12], B_n is locally nilpotent (see also [13, Theorem 7.4]). Since B_n is finitely generated, it follows that B_n is nilpotent. (Since B_n is a Lie algebra over a field of characteristic 0 which satisfies an Engel identity, the required result follows also from [16].))

For a positive integer c, write $B_n^c = (L_n^c + \mathfrak{V}(L_n))/\mathfrak{V}(L_n)$. We point out that $B_n^c \cong L_n^c/(L_n^c \cap \mathfrak{V}(L_n))$ and B_n^c is the K-vector subspace of B_n spanned by all Lie commutators of total degree c in $z_1, \ldots, z_n$. Since $\mathfrak{V}(L_n) \subseteq L_n'$, the set $\{z_1, \ldots, z_n\}$ is a basis of the K-vector space B_n^1. Since K is infinite, we may see by a Vandermonde determinant argument that $B_n = \bigoplus_{c \geq 1} B_n^c$ (see, for example, [2, Section 4.2, Theorem 4.6]).

The general linear group $\mathrm{GL}_n(K)$ acts naturally on the K-vector subspace of L_n spanned by $\ell_1, \ldots, \ell_n$. We extend this action so that $\mathrm{GL}_n(K)$ becomes a group of algebra automorphisms of L_n. It is clear that the K-vector subspace $\mathfrak{V}(L_n)$ is $\mathrm{GL}_n(K)$-invariant. Therefore, $\mathrm{GL}_n(K)$ acts as a group of algebra automorphisms of B_n such that each B_n^k is a (left) $K\mathrm{GL}_n(K)$-module. Since $\mathrm{GL}_n(K)$ acts faithfully on B_n, we may regard $\mathrm{GL}_n(K)$ as a subgroup of $\mathrm{Aut}(B_n)$. The natural map from B_n onto B_n/B_n' induces a natural group homomorphism, say β_{B_n}, from $\mathrm{Aut}(B_n)$ into $\mathrm{Aut}(B_n/B_n')$. Since, by [6], $\mathrm{Aut}(B_n/B_n') \cong \mathrm{GL}_n(K)$ as groups, it is clear that β_{B_n} is onto. We point out that the kernel of β_{B_n} is $\mathrm{IA}(B_n)$. Since $\mathrm{GL}_n(K) \cap \mathrm{IA}(B_n) = \{1\}$, $\mathrm{Aut}(B_n)$ is the split extension of $\mathrm{IA}(B_n)$ by $\mathrm{GL}_n(K)$. If B is a residually nilpotent Lie algebra (that is, $\cap_{i\geq 1}\gamma_i(B) = \{0\}$) over a field K of arbitrary characteristic and G is a non-trivial finite subgroup of $\mathrm{IA}(B)$, then K has prime characteristic p and G is a p-group (see [5, Lemma 2.1], [8, Lemma 2.1]). Hence, in what follows, we study the Lie subalgebra of G-invariants in a relatively free Lie algebra for a non-trivial finite subgroup G of $\mathrm{GL}_n(K)$.

Let $R_2 = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))')$ be the free (nilpotent of class 2)-by-abelian and abelian-by-(nilpotent of class 2) Lie algebra of rank 2 over a field K of characteristic 0 freely generated by the set $\{x_1, x_2\}$ and let $R_2^{S_2}$ be the Lie subalgebra of symmetric polynomials in R_2. We point out that $R_2^{S_2}$ may be identified with the Lie algebra of invariants of the $\mathbb{Z}_2$-action permuting x_1 and x_2. For non-negative integers m and n, we use the following notation:

$$z_{m,n} = [x_2, x_1, {}_m x_1, {}_n x_2] - [x_2, x_1, {}_m x_2, {}_n x_1],$$
$$w_{m,n} = [x_2, x_1, {}_m x_1, {}_n x_2, [x_2, x_1]] + [x_2, x_1, {}_m x_2, {}_n x_1, [x_2, x_1]].$$

It is straightforward to verify that $z_{m,n}, w_{m,n} \in R_2^{S_2}$ for all $m, n \geq 0$.
By the above notation, we prove the following results.

Theorem 2.1. *The set*

$$\{x_1 + x_2, z_{m_1,n_1}, w_{m_2,n_2} \ : \ m_1 > n_1 \geq 0, \ m_2 \geq n_2 \geq 0, \ m_2 + n_2 \geq 1\}$$

is a basis of $R_2^{S_2}$.

Theorem 2.2. *The set* $\{x_1 + x_2, z_{2n-1,0}, w_{1,0}, w_{2n,0} \ : \ n \geq 1\}$ *is a minimal generating set of* $R_2^{S_2}$.

Theorem 2.3. $R_2^{S_2}$ *has a presentation* $\langle X | \mathcal{R} \rangle$, *where*

$$X = \{x_1 + x_2, z_{2n-1,0}, w_{1,0}, w_{2n,0} : n \geq 1\},$$
$$\mathcal{R} = \{[u, \ {}_r x_1 + x_2, v] = 0 : u, v \in \{[z_{2n-1,0}, w_{1,0}, w_{2n,0}] : n \geq 1\}, r \geq 0\}.$$

3. The subalgebra of invariants in relatively free Lie algebras

Let W be a K-vector space and G be a group acting (from the left) on W. We say that $w \in W$ is a G-invariant if $g \cdot w = w$ for all $g \in G$. We write W^G for the set of all G-invariants in W, that is, $W^G = \{w \in W : g \cdot w = w \text{ for all } g \in G\}$ and we point out that W^G is a K-vector subspace of W. Let W be a finite dimensional vector space and G be a finite group acting (from the left) on W.

Then, the Reynolds operator

$$w \to \frac{1}{|G|}\sum_{g\in G} g\cdot v$$

maps W on W^G and acts identically on W^G (see [15, Proposition 2.1.2]). If W_1 and W_2 are finite dimensional K-vector spaces, with $W_2 \subset W_1$, and G is a finite group acting (from the left) on W_1 and such that $G(W_2) = W_2$ (where $G(W_2) = \{g\cdot w : g\in G, w\in W_2\}$), then, by Maschke's Theorem, $W_1 \cong (W_1/W_2)\oplus W_2$ and $W_1^G \cong (W_1/W_2)^G \oplus W_2^G$ as (left) KG-modules.

For a positive integer n, with $n\geq 2$, let L_n be the free Lie algebra of rank n, let $B_n = L_n/\mathfrak{V}(L_n)$ be a relatively free Lie algebra freely generated by the set $\{z_1,\ldots,z_n\}$ and let G be a non-trivial finite subgroup of $\mathrm{GL}_n(K)$. $(B_n^c)^G = B_n^G \cap B_n^c$ is the K-vector subspace of B_n^G spanned by all G-invariants of total degree c in $z_1,\ldots,z_n$ and $B_n^G = \bigoplus_{c\geq 1}(B_n^c)^G$ is the Lie algebra of G-invariants in B_n. By the above arguments, for $W = B_n^c$, with $c\geq 1$, we may deduce that

$$B_n^G = \{\textstyle\sum_{\theta\in G}\theta(u) : u\in B_n\}.$$

Let $V_1(L_n)$ and $V_2(L_n)$ be fully invariant ideals of L_n such that $V_1(L_n)\subset V_2(L_n)$. Let $P_n = L_n/V_1(L_n)$ and $Q_n = L_n/V_2(L_n)$ be relatively free Lie algebras freely generated by the sets $\{a_1,\ldots a_n\}$ and $\{b_1,\ldots b_n\}$, respectively, and let P_n^G and Q_n^G be the Lie algebras of G-invariants in P_n and Q_n, respectively. We point out that $P_n^G = \bigoplus_{c\geq 1}(P_n^c)^G$, $Q_n^G = \bigoplus_{c\geq 1}(Q_n^c)^G$, $(V_2(L_n)/V_1(L_n))^c = P_n^c \cap (V_2(L_n)/V_1(L_n))$ for all $c\geq 1$ and $(V_2(L_n)/V_1(L_n))^G = \bigoplus_{c\geq 1}((V_2(L_n)/V_1(L_n))^c)^G$. Since we have $V_1(L_n)\subset V_2(L_n)$, there is a natural Lie algebra epimorphism π_{P_n,Q_n} from P_n onto Q_n, defined by $\pi(a_i) = b_i$, $i = 1,\ldots,n$. By the above arguments, for $W_1 = P_n^c$ and $W_2 = (V_2(L_n)/V_1(L_n))^c$, with $c\geq 1$, and for a non-trivial finite subgroup G of $\mathrm{GL}_n(K)$, we point out that $G((V_2(L_n)/V_1(L_n))^c) = (V_2(L_n)/V_1(L_n))^c$ and hence, it follows that $P_n^c \cong Q_n^c \oplus (V_2(L_n)/V_1(L_n))^c$ and $(P_n^c)^G \cong (Q_n^c)^G \oplus ((V_2(L_n)/V_1(L_n))^c)^G$ as (left) KG-modules. Therefore, we may deduce the following result.

Proposition 3.1. (1) *Let $\pi^*_{P_n,Q_n}$ be the restriction of π_{P_n,Q_n} on P_n^G. Then, $\pi^*_{P_n,Q_n}((P_n^c)^G) = (Q_n^c)^G$ for all $c\geq 1$ and $\pi^*_{P_n,Q_n}$ is a Lie algebra epimorphism from P_n^G onto Q_n^G with kernel* $\ker(\pi^*_{P_n,Q_n}) = (V_2(L_n)/V_1(L_n))^G$.

(2) *If the set $C = \{c_i : i = 1,2,\ldots\}$ is a basis of Q_n^G,*

$$D = \{d_i \in P_n^G :\ \pi^*_{P_n,Q_n}(d_i) = c_i,\ i = 1,2,\ldots\} \subseteq P_n^G,$$

and the set D' is a basis of $\ker(\pi^*_{P_n,Q_n})$*, then $D\cup D'$ is basis of P_n^G.*

In the following, we give conditions to extend a (minimal) generating set of Q_n^G to a (minimal) generating set of P_n^G.

Proposition 3.2. *If C is a generating set of Q_n^G, D is a subset of P_n^G such that $\pi^*_{P_n,Q_n}(D) = C$ and D' is a subset of $(V_2(L_n)/V_1(L_n))^G$, then $D\cup D'$ is a generating set of P_n^G if and only if $u\in L(D\cup D')$ for all $u\in (V_2(L_n)/V_1(L_n))^G$.*

Proof. For the non-trivial part of the proof, assume that $u\in L(D\cup D')$ for all $u\in (V_2(L_n)/V_1(L_n))^G$. We need to show that $P_n^G = L(D\cup D')$. Clearly, it suffices to show that $P_n^G \subseteq L(D\cup D')$. Let $v\in P_n^G$. Then, by Proposition 3.1(1), $\pi^*_{P_n,Q_n}(v)\in Q_n^G$.

Since C is a generating set of Q_n^G, $Q_n^G = L(C)$ and therefore, $\pi^*_{P_n,Q_n}(v) \in L(C)$. Since $\pi^*_{P_n,Q_n}(D) = C$, it follows that $\pi^*_{P_n,Q_n}(v) \in L(\pi^*_{P_n,Q_n}(D))$. Since, by Proposition 3.1(1), $\pi^*_{P_n,Q_n}$ is a homomorphism of Lie algebras, it follows that $\pi^*_{P_n,Q_n}(v) = \pi^*_{P_n,Q_n}(z)$ for some $z \in L(D)$. Since $D \subseteq D \cup D'$, we point out that $z \in L(D \cup D')$. Furthermore, $v = z + w$ for some $w \in \mathrm{Ker}(\pi^*_{P_n,Q_n})$. Hence, by Proposition 3.1(1), $w \in (V_2(L_n)/V_1(L_n))^G$ and therefore, $w \in L(D \cup D')$. Since $z, w \in L(D \cup D')$ and $v = z + w$, we get $v \in L(D \cup D')$. Therefore, $P_n^G \subseteq L(D \cup D')$ and hence, we obtain the required result. ■

Proposition 3.3. *If $C = \{c_i : i = 1, 2, \ldots\}$ is a minimal generating set of Q_n^G, $D = \{d_i \in P_n^G : \pi^*_{P_n,Q_n}(d_i) = c_i,\ i = 1, 2, \ldots\} \subseteq P_n^G$ and D' is a subset of $(V_2(L_n)/V_1(L_n))^G$, then $D \cup D'$ is a minimal generating set of P_n^G if and only if the following conditions hold:*

(1) *$u \in L(D \cup D')$ for all $u \in (V_2(L_n)/V_1(L_n))^G$.*

(2) *For any $u \in D'$, $u \notin L(D \cup D' \setminus \{u\})$.*

Proof. Assume at first that $D \cup D'$ is a minimal generating set of P_n^G. Then, Condition (1) is satisfied by Proposition 3.2 and Condition (2) is trivially satisfied. Conversely, assume that Conditions (1) and (2) are satisfied. Then, by Condition (1) and Proposition 3.2, $D \cup D'$ is a generating set of P_n^G. Suppose on the contrary that $D \cup D'$ is not a minimal generating set of P_n^G. Therefore, there is some $v \in D \cup D'$ such that $v \in L(D \cup D' \setminus \{v\})$. It follows from Condition (2) that $v \notin D'$. Hence, we have $v \in D$, say $v = d_j$ for some $j \geq 1$, such that $d_j \in L(D \cup D' \setminus \{d_j\})$. Then, $c_j = \pi^*_{P_n,Q_n}(d_j) \in \pi^*_{P_n,Q_n}(L(D \cup D' \setminus \{d_j\}))$. Since $\pi^*_{P_n,Q_n}(d_i) = c_i$ for all $i \geq 1$, $D' \subseteq (V_2(L_n)/V_1(L_n))^G$ and, by Proposition 3.1(1), $\pi^*_{P_n,Q_n}$ is a Lie algebra homomorphism with $\ker(\pi^*_{P_n,Q_n}) = (V_2(L_n)/V_1(L_n))^G$, it may be easily verified that $c_j \in L(C \setminus \{c_j\})$. Since C is a minimal generating set of Q_n^G, we get the required contradiction. ■

For positive integers n and c, with $n \geq 2$, let $B_{n,c} = B_n/\gamma_{c+1}(B_n)$. Since we have $B_n = L_n/\mathfrak{V}(L_n)$, we obtain $B_{n,c} \cong L_n/(\mathfrak{V}(L_n) + \gamma_{c+1}(L_n))$ and $(B_n)^d \cong (B_{n,c})^d$ for all $d \geq 1$. Therefore, we may easily verify the following result.

Proposition 3.4. (1) *If the set D is a basis of B_n^G, then $\pi^*_{B_n,B_{n,c}}(D)$ is basis of $B_{n,c}^G$.*

(2) *If D is a generating set of B_n^G, then $\pi^*_{B_n,B_{n,c}}(D)$ is a generating set of $B_{n,c}^G$. If, in addition, D is minimal, then $\pi^*_{B_n,B_{n,c}}(D)$ is minimal.*

(3) *If B_n^G has a presentation $B_n^G = \langle D | \mathcal{R} \rangle$, then $B_{n,c}^G$ has a presentation $B_{n,c}^G = \langle \pi^*_{B_n,B_{n,c}}(D) \mid \pi^*_{B_n,B_{n,c}}(\mathcal{R}) \rangle$.*

4. The Lie algebra $R_2 = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))')$

At first, we show some identities in a Lie algebra and in a (nilpotent of class 2)-by-abelian and abelian-by-(nilpotent of class 2) Lie algebra.

Lemma 4.1. *Let L be a Lie algebra. Then, for any $a, b, c \in L$ and $s \geq 0$,*

$$[a, [b, {}_s c]] = \sum_{j=0}^{s} (-1)^j \binom{s}{j} [a, {}_j c, b, {}_{s-j} c].$$

Proof. We use induction on s. We point out that the result is trivially true for $s=0$ and we assume that the result is true for some $s \geq 0$, that is, we assume that

$$[a,[b,\ {}_s c]] = \sum_{j=0}^{s}(-1)^j \binom{s}{j} [a,\ {}_j c, b,\ {}_{s-j} c] \text{ for all } a,b,c \in L.$$

Let $a,b,c \in L$. By using the Jacobi identity, we have

$$[a,[b,\ {}_{s+1}c]] = [a,[b,\ {}_s c],c] - [a,c,[b,\ {}_s c]].$$

Therefore, by our inductive argument,

$$\begin{aligned}
[a,[b,\ {}_{s+1}c]] &= \sum_{j=0}^{s}(-1)^j\binom{s}{j}[a,\ {}_j c, b,\ {}_{s+1-j}c] - \sum_{j=0}^{s}(-1)^j\binom{s}{j}[a,\ {}_{j+1}c, b,\ {}_{s-j}c] \\
&= \sum_{j=0}^{s}(-1)^j\binom{s}{j}[a,\ {}_j c, b,\ {}_{s+1-j}c] + \sum_{j=0}^{s}(-1)^{j+1}\binom{s}{j}[a,\ {}_{j+1}c, b,\ {}_{s-j}c] \\
&= \sum_{j=0}^{s}(-1)^j\binom{s}{j}[a,\ {}_j c, b,\ {}_{s+1-j}c] + \sum_{j=1}^{s+1}(-1)^j\binom{s}{j-1}[a,\ {}_j c, b,\ {}_{s+1-j}c] \\
&= [a,b,\ {}_{s+1}c] + \sum_{j=1}^{s}(-1)^j\{\binom{s}{j}+\binom{s}{j-1}\}[a,\ {}_j c, b,\ {}_{s+1-j}c] + (-1)^{s+1}[a,\ {}_{s+1}c, b] \\
&= [a,b,\ {}_{s+1}c] + \sum_{j=1}^{s}(-1)^j\binom{s+1}{j}[a,\ {}_j c, b,\ {}_{s+1-j}c] + (-1)^{s+1}[a,\ {}_{s+1}c, b] \\
&= \sum_{j=0}^{s+1}(-1)^j\binom{s+1}{j}[a,\ {}_j c, b,\ {}_{s+1-j}c]
\end{aligned}$$

and hence, the result follows. ■

Lemma 4.2. *Let G be a (nilpotent of class 2)-by-abelian and abelian-by-(nilpotent of class 2) Lie algebra, that is, $\gamma_3(G') + (\gamma_3(G))' = \{0\}$. Then:*

(1) *For all $r \geq 2$, $z_1,\ldots,z_r \in G$, $\sigma \in S_r$ and $v_1,v_2 \in G'$,*

$$[v_1,z_1,\ldots,z_r,v_2] = [v_1,z_{\sigma(1)},\ldots,z_{\sigma(r)},v_2].$$

(2) *For all $z \in G$, $v_1 \in \gamma_3(G)$ and $v_2 \in G'$, $[v_1,v_2,z] = [v_1,z,v_2]$.*

(3) *For all $u,v \in G'$, $x,y \in G$ and $k \geq 1$,*

$$[u,\ {}_k x+y, v] = \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{k-i}x,\ {}_i y, v].$$

(4) *For all $u,v \in G'$, $x,y \in G$, $k \geq 1$ and $m,n \geq 0$, with $m+n \geq 1$,*

$$[u,\ {}_m x,\ {}_n y, v,\ {}_k x+y] = \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{m+k-i}x,\ {}_{n+i}y, v].$$

Proof. (1) By the Jacobi identity in the form $[a,b,c] = [a,c,b] + [a,[b,c]]$ and since $\gamma_3(G') = \{0\}$, the result may be easily obtained.

(2) By the Jacobi identity in the form $[a,b,c] = [a,c,b] + [a,[b,c]]$ and since $(\gamma_3(G))' = \{0\}$, we get the result.

(3) Let $u,v \in G'$ and $x,y \in G$. We use induction on k. For $k=1$, the result is trivially true. Assume that the result is true for some $k \geq 1$, that is, assume that $[u,\ {}_k x+y,v] = \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{k-i}x,\ {}_i y,v]$. By Lemma 4.2(2), $[u,\ {}_{k+1}x+y,v] = [u,\ {}_k x+y,v,x+y]$ and hence, by our inductive argument, $[u,\ {}_{k+1}x+y,v] = \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{k-i}x,\ {}_i y,v,x+y]$. Therefore, by Lemma 4.2(1)-(2),

$$[u,\ {}_{k+1}x+y,v] = \sum_{i=0}^{k}\binom{k}{i}([u,\ {}_{k-i+1}x,\ {}_i y,v] + [u,\ {}_{k-i}x,\ {}_{i+1}y,v]).$$

Thus,

$$\begin{aligned}
[u,\ {}_{k+1}x+y,v] &= \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{k-i+1}x,\ {}_i y,v] + \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{k-i}x,\ {}_{i+1}y,v] \\
&= \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{k-i+1}x,\ {}_i y,v] + \sum_{i=1}^{k+1}\binom{k}{i-1}[u,\ {}_{k-i+1}x,\ {}_i y,v] \\
&= [u,\ {}_{k+1}x,v] + \sum_{i=1}^{k}(\binom{k}{i} + \binom{k}{i-1})[u,\ {}_{k-i+1}x,\ {}_i y,v] + [u,\ {}_{k+1}y,v] \\
&= [u,\ {}_{k+1}x,v] + \sum_{i=1}^{k}\binom{k+1}{i}[u,\ {}_{k-i+1}x,\ {}_i y,v] + [u,\ {}_{k+1}y,v] \\
&= \sum_{i=0}^{k+1}\binom{k+1}{i}[u,\ {}_{k+1-i}x,\ {}_i y,v]
\end{aligned}$$

and the result follows.

(4) Let $u,v \in G'$, $x,y \in G$ and let $m,n \geq 0$, with $m+n \geq 1$, and $k \geq 1$. By Lemma 4.2(2),

$$[u,\ {}_m x,\ {}_n y,v,\ {}_k x+y] = [u,\ {}_m x,\ {}_n y,\ {}_k x+y,v].$$

Therefore, by Lemma 4.2(3) and Lemma 4.2(1),

$$\begin{aligned}
[u,\ {}_m x,\ {}_n y,v,\ {}_k x+y] &= \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_m x,\ {}_n y,\ {}_{k-i}x,\ {}_i y,v] \\
&= \sum_{i=0}^{k}\binom{k}{i}[u,\ {}_{m+k-i}x,\ {}_{n+i}y,v]
\end{aligned}$$

and hence, we get the result. ∎

Recall that $R_2 = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))')$ is freely generated by the set $\{x_1,x_2\}$. Since $L_2/L_2' \cong R_2/R_2'$ in a natural way, the set $\{x_1+R_2', x_2+R_2'\}$ is a basis of the quotient R_2/R_2'. Since $L_2'/L_2'' \cong R_2'/R_2''$ in a natural way, it follows from [2, Section 2.4.2, p. 64] that the set $\{[x_2,x_1,\ {}_{m_1}x_1,\ {}_{m_2}x_2] + R_2'' : m_1,m_2 \geq 0\}$ is a basis of the quotient R_2'/R_2''. For a positive integer c, with $c \geq 4$, we write $(R_2'')^c$ for the K-vector subspace of R_2'' spanned by all Lie commutators of R_2'' of homogeneous degree c.

By [7, Lemma 2.1], $(R_2'')^c$ has a basis

$$\{[x_2, x_1, {}_{m_1}x_1, {}_{m_2}x_2, [x_2, x_1]] : m_1, m_2 \geq 0, m_1 + m_2 = c - 4\}.$$

In the following result, we summarize the aforementioned results.

Proposition 4.3. (1) *The set* $\{x_1, x_2\}$ *is a basis of the* K*-vector subspace* R_2^1.

(2) *For* $2 \leq d \leq 4$, *the set* $\{[x_2, x_1, {}_m x_1, {}_n x_2] : m, n \geq 0, m + n = d - 2\}$ *is a basis of the* K*-vector subspace* R_2^d.

(3) *For* $d \geq 5$, *the set* $\{[x_2, x_1, {}_m x_1, {}_n x_2, [x_2, x_1]] : m, n \geq 0, m + n = d - 4\}$ *is a basis of the* K*-vector subspace* $(R_2'')^d$ *and the set*

$$\left\{\begin{matrix}[x_2, x_1, {}_{m_1}x_1, {}_{n_1}x_2], [x_2, x_1, {}_{m_2}x_1, {}_{n_2}x_2, [x_2, x_1]] : \\ m_1, n_1, m_2, n_2 \geq 0, m_1 + n_1 = d - 2, m_2 + n_2 = d - 4\end{matrix}\right\}$$

is a basis of the K*-vector subspace* R_2^d.

(4) *The set*

$$\{x_1, x_2, [x_2, x_1, {}_{m_1}x_1, {}_{n_1}x_2], [x_2, x_1, {}_{m_2}x_1, {}_{n_2}x_2, [x_2, x_1]] : m_1, n_1, m_2, n_2 \geq 0\}$$

is a basis of R_2. *In particular, the set* $\{x_1 + R_2', x_2 + R_2'\}$ *is a basis of the quotient* R_2/R_2', *the set* $\{[x_2, x_1, {}_m x_1, {}_n x_2] + R_2'' : m, n \geq 0\}$ *is a basis of the quotient* R_2'/R_2'' *and the set* $\{[x_2, x_1, {}_{m_1}x_1, {}_{m_2}x_2, [x_2, x_1]] : m_1, m_2 \geq 0\}$ *is a basis of* R_2''.

By Proposition 4.3, we may give a proof of the following useful result.

Lemma 4.4. *If* $[w, x_1 + x_2] = 0$ *for some* $w \in R_2''$, *then* $w = 0$.

Proof. Since $R_2'' = \bigoplus_{d \geq 5} (R_2'')^d$, it suffices to show the result for $w \in (R_2'')^d$, with $d \geq 5$. Throughout the proof, we write $u = [x_2, x_1]$. By Proposition 4.3(3), we may write

$$w = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[u, {}_m x_1, {}_n x_2, u] = \sum_{m=0}^{d-4} c_{m,d-4-m}[u, {}_m x_1, {}_{d-4-m}x_2, u].$$

Hence, by Lemma 4.2(2),

$$[w, x_1 + x_2] = \sum_{m=0}^{d-4} c_{m,d-4-m}([u, {}_{m+1}x_1, {}_{d-4-m}x_2, u] + [u, {}_m x_1, {}_{d-3-m}x_2, u])$$

$$= \sum_{m=1}^{d-3} c_{m-1,d-3-m}[u, {}_m x_1, {}_{d-3-m}x_2, u] + \sum_{m=0}^{d-4} c_{m,d-4-m}[u, {}_m x_1, {}_{d-3-m}x_2, u].$$

Since $[w, x_1 + x_2] = 0$, it follows from Proposition 4.3(3) that

$$c_{0,d-4} = 0, \; c_{m-1,d-3-m} + c_{m,d-4-m} = 0 \text{ for all } m = 1, ..., d-4, \text{ and } c_{d-4,0} = 0.$$

Hence, it follows that $c_{m,d-4-m} = 0$ for all $m = 0, \ldots, d-4$ and therefore, $w = 0$ as required. ■

5. The Lie algebra of symmetric polynomials in $R_2 = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))')$.

5.1. The Lie algebra $M_2^{S_2}$

Let $M_2 = L_2/L_2''$ be a free metabelian Lie algebra of rank 2 freely generated by the set $\{y_1, y_2\}$. We write π for the natural Lie algebra epimorphism from R_2 onto M_2, defined by $\pi(x_i) = y_i$, $i = 1, 2$. By Proposition 3.1(1), the restriction π^* of π on $R_2^{S_2}$ is a Lie algebra epimorphism from $R_2^{S_2}$ onto $M_2^{S_2}$. Since

$$L_2''/(\gamma_3(L_2') + (\gamma_3(L_2))') = (L_2/(\gamma_3(L_2') + (\gamma_3(L_2))'))'' = R_2'',$$

it follows from Proposition 3.1(1) that $\ker(\pi^*) = (R_2'')^{S_2}$.

For non-negative integers m and n, write $t_{m,n}$ for the element of $M_2^{S_2}$ defined by

$$t_{m,n} = [y_2, y_1, {}_m y_1, {}_n y_2] - [y_2, y_1, {}_m y_2, {}_n y_1].$$

Since M_2 is metabelian, it is easily verified that

$$t_{m,n} = [y_2, y_1, {}_m y_1, {}_n y_2] - [y_2, y_1, {}_n y_1, {}_m y_2].$$

The Lie algebra $M_2^{S_2}$ has been studied in [10]. It has been proved in [10, Corollary 3.2] that the set $\{y_1 + y_2, t_{m,n} : m > n \geq 0\}$ is a basis of $M_2^{S_2}$ and in [10, Theorem 3.4] that the set $\{y_1 + y_2, t_{2n-1,0} : n \geq 1\}$ is a generating set of $M_2^{S_2}$.

Remark 5.1. Since M_2 is metabelian, it is readily verified that

$$[t_{2m-1,0}, {}_r y_1 + y_2, [t_{2n-1,0}\ {}_s y_1 + y_2]] = 0 \text{ for all } m, n \geq 1 \text{ and } r, s \geq 0.$$

Since, in a Lie algebra L, $[a, [b, {}_s c]] = -[b, {}_s c, a]$ for all $a, b, c \in L$ and $s \geq 0$, and by Lemma 4.1, for $a, b, c \in L$ and $r, s > 0$,

$$[a, {}_r c, [b, {}_s c]] = \sum_{j=0}^{s} (-1)^j \binom{s}{j} [a, {}_{r+j} c, b, {}_{s-j} c],$$

we point out that the relations of the form $[t_{2m-1,0}, {}_r y_1 + y_2, [t_{2n-1,0}, {}_s y_1 + y_2]] = 0$ in $M_2^{S_2}$ follow from the relations of the form $[t_{2m-1,0}, {}_r y_1 + y_2, t_{2n-1,0}] = 0$. By [10, proof of Theorem 3.5], there are no other non-trivial relations in $M_2^{S_2}$.

We summarize the aforementioned results.

Proposition 5.2. (1) *The set* $\{y_1 + y_2, t_{m,n} : m > n \geq 0\}$ *is a basis of* $M_2^{S_2}$. *In particular, for a positive integer* d, *with* $d \geq 3$, *the set*

$$\{t_{m,n} : \ m > n \geq 0,\ m + n = d - 2\}$$

is a basis of the K*-vector subspace* $(M_2^d)^{S_2}$.

(2) $M_2^{S_2}$ *has a presentation*

$$\langle \{y_1 + y_2, t_{2n-1,0} : n \geq 1\} \mid \ [t_{2i-1}, {}_r x_1 + x_2, t_{2j-1}] = 0,\ i, j \geq 1,\ r \geq 0 \rangle.$$

By Proposition 5.2(2), we get directly the next result.

Corollary 5.3. *The set* $\{y_1 + y_2, t_{2n-1,0} : n \geq 1\}$ *is a minimal generating set of* $M_2^{S_2}$.

5.2. Relations in $R_2^{S_2}$

Since $[z_{m,n}\ _r x_1 + x_2], w_{m,n} \in \gamma_3(R_2)$ for all $m, n \geq 0$, with $m + n \geq 1$, and $r \geq 0$, and $(\gamma_3(R_2))' = \{0\}$, we get directly the following relations in $R_2^{S_2}$.

Lemma 5.4. *For all $u, v \in \{[z_{m,n}, w_{m,n} :\ m, n \geq 0,\ m + n \geq 1\}$ and $r, s \geq 0$,*

$$[u,\ _r x_1 + x_2, [v\ _s x_1 + x_2]] = 0.$$

Remark 5.5. By similar arguments as in Remark 5.1, we point out that the relations of the form $[u,\ _r x_1 + x_2, [v,\ _s x_1 + x_2]] = 0$ in Lemma 5.4 follow from the relations of the form $[u,\ _r x_1 + x_2, v] = 0$.

Lemma 5.6. (1) *For non-negative integers m and n, with $m+n \geq 1$, $w_{m,n} = w_{n,m}$.*

(2) *For non-negative integers m and n, with $m \geq n$ and $m+n \geq 1$, the elements $w_{m,n}$ are linearly independent over K.*

(3) *For non-negative integers r, m and n, with $m + n \geq 1$,*

$$[w_{m,n},\ _r x_1 + x_2] = \sum_{i=0}^{r} \binom{r}{i} w_{m+r-i,n+i}.$$

(4) *For $m \geq 1$ and $n \geq 0$, write $W_{m,n} = w_{m,n} + w_{m-1,n+1}$. Then:*

(a) $W_{m,n} = W_{n+1,m-1}$.

(b) *For positive integers r and m, $[w_{m,0},\ _r x_1 + x_2]$ belongs to the K-vector space spanned by the set $\{W_{m+r-j,j} : j = 0, \ldots, r-1\}$.*

Proof. (1) By Lemma 4.2(1),

$$\begin{aligned} w_{m,n} &= [x_2, x_1,\ _m x_1,\ _n x_2, [x_2, x_1]] + [x_2, x_1,\ _m x_2,\ _n x_1, [x_2, x_1]] \\ &= [x_2, x_1,\ _n x_2,\ _m x_1, [x_2, x_1]] + [x_2, x_1,\ _n x_1,\ _m x_2, [x_2, x_1]] = w_{n,m}. \end{aligned}$$

(2) Since, by Lemma 4.2(2), for $m \geq n \geq 0$, with $m + n \geq 1$,

$$\begin{aligned} w_{m,n} &= [u,\ _m x_1,\ _n x_2, u] + [u,\ _m x_2,\ _n x_1, u] \\ &= [u,\ _m x_1,\ _n x_2, u] + [u,\ _n x_1,\ _m x_2, u] \end{aligned}$$

and by Proposition 4.3(4), the elements $[u,\ _m x_1,\ _n x_2, u]$, with $m \geq n \geq 0$ and $m + n \geq 1$, are linearly independent over K, the result follows.

(3) Throughout the proof, we write $u = [x_2, x_1]$. Since, by Lemma 4.2(2),

$$w_{m,n} = [u,\ _m x_1,\ _n x_2, u] + [u,\ _n x_1,\ _m x_2, u],$$

it follows from Lemma 4.2(4) that

$$\begin{aligned} &[w_{m,n},\ _r x_1 + x_2] = \\ &= \sum_{i=0}^{r} \binom{r}{i} [u,\ _{m+r-i} x_1,\ _{n+i} x_2, u] + \sum_{i=0}^{r} \binom{r}{i} [u,\ _{n+r-i} x_1,\ _{m+i} x_2, u] \end{aligned}$$

$$= \sum_{i=0}^{r} \binom{r}{i} [u,\ {}_{m+r-i}x_1,\ {}_{n+i}x_2, u] + \sum_{i=0}^{r} \binom{r}{i} [u,\ {}_{n+i}x_1,\ {}_{m+r-i}x_2, u]$$
$$= \sum_{i=0}^{r} \binom{r}{i} ([u,\ {}_{m+r-i}x_1,\ {}_{n+i}x_2, u] + [u,\ {}_{n+i}x_1,\ {}_{m+r-i}x_2, u])$$
$$= \sum_{i=0}^{r} \binom{r}{i} w_{m+r-i,n+i}.$$

(4a) By Lemma 5.6(1) we have

$$W_{m,n} = w_{m,n} + w_{m-1,n+1} = w_{n,m} + w_{n+1,m-1} = W_{n+1,m-1}.$$

(4b) For $i \in \{0, \ldots, r\}$, write $\lambda_i = \binom{r}{i} - \binom{r}{i-1} + \ldots + (-1)^i \binom{r}{0}$ and observe that $\lambda_r = 0$ and $\lambda_i + \lambda_{i-1} = \binom{r}{i}$, $i = 1, \ldots, r$. Since, by Lemma 5.6(3),

$$[w_{m,0},\ {}_r x_1 + x_2] = \sum_{i=0}^{r} \binom{r}{i} w_{m+r-i,i},$$
$$[w_{m,0},\ {}_r x_1 + x_2] = \lambda_0 w_{m+r,0} + \sum_{i=1}^{r-1} (\lambda_i + \lambda_{i-1}) w_{m+r-i,i} + \lambda_{r-1} w_{m,r}$$
$$= \sum_{i=0}^{r-1} \lambda_i (w_{m+r-i,i} + w_{m+r-i-1,i+1}) = \sum_{i=0}^{r-1} \lambda_i W_{m+r-i,i}$$

and hence, we get the result. ■

Lemma 5.7. *Let B be the Lie subalgebra of $R_2^{S_2}$ generated by the set*

$$\{x_1 + x_2, z_{2m-1,0} : m \geq 1\}.$$

Then, $B^d \cap R_2'' = \{0\}$ for all $d \geq 1$.

Proof. The result is trivially true for $1 \leq d \leq 4$ and hence, we may assume that $d \geq 5$. Furthermore, we assume that d is odd and we point out that similar arguments may be applied in the case that d is even. Let $d = 2s+1$ for some $s \geq 2$ and suppose on the contrary that there exists some $u \in B^d \cap R_2''$ such that $u \neq 0$. It follows from Lemma 5.4 that u has the form

$$u = \lambda_{2s-1} z_{2s-1,0} + \lambda_{2s-3} [z_{2s-3,0},\ {}_2 x_1 + x_2] + \ldots + \lambda_1 [z_{1,0},\ {}_{2s-2} x_1 + x_2].$$

Since $u \in (R_2'')^{S_2}$, $\pi^*(u) = 0$. Hence, by applying π^* on the above equation and by the notation in Section 5.1, we get

$$0 = \lambda_{2s-1} t_{2s-1,0} + \lambda_{2s-3} [t_{2s-3,0},\ {}_2 y_1 + y_2] + \ldots + \lambda_1 [t_{1,0},\ {}_{2s-2} y_1 + y_2].$$

But, by Proposition 5.2(2), there are no non-trivial relations among the elements of the set $\{y_1 + y_2, t_{2m-1,0} : m \geq 1\}$. Hence, it follows that

$$\lambda_{2s-1} = \lambda_{2s-3} = \ldots = \lambda_1 = 0.$$

Therefore, $u = 0$, which is the required contradiction and hence, we obtain the required result. ■

6. The main results

6.1. A basis of the Lie algebra $R_2^{S_2}$

Proposition 6.1. *For a positive integer d, with $d \geq 5$, the set*

$$\{w_{m,n} : \ m \geq n \geq 0, \ m+n = d-4\}$$

is a basis of the K-vector space $((R_2'')^d)^{S_2}$.

Proof. Throughout the proof, we write $[x_2, x_1] = u$. Let $f(x_1, x_2)$ be a symmetric polynomial in $((R_2'')^d)^{S_2}$, with $d \geq 5$. Since $((R_2'')^d)^{S_2} \subseteq (R_2'')^d$, by Proposition 4.3(3) we may write

$$f(x_1, x_2) = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[u \ {}_m x_1, \ {}_n x_2, u],$$

where $c_{m,n} \in K$. Hence,

$$f(x_2, x_1) = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[x_1, x_2, \ {}_m x_2, \ {}_n x_1, [x_1, x_2]] = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_2, \ {}_n x_1, u]$$

and hence, by Lemma 4.2(1),

$$f(x_2, x_1) = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_n x_1, \ {}_m x_2, u] = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{n,m}[u, \ {}_m x_1, \ {}_n x_2, u].$$

Since $f(x_1, x_2) = f(x_2, x_1)$, we get

$$\sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_1, \ {}_n x_2, u] = \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{n,m}[u, \ {}_m x_1, \ {}_n x_2, u].$$

Since, by Proposition 4.3(3), the set $\{[u, \ {}_m x_1, \ {}_n x_2, u] : \ m, n \geq 0, \ m+n = d-4\}$ is a basis of $(R_2'')^d$, it follows from the above equation that $c_{m,n} = c_{n,m}$ for all $m, n \geq 0$, with $m+n = d-4$. Assume at first that d is even, with $d \geq 6$. Hence,

$$\begin{aligned}
f(x_1, x_2) &= \sum_{\substack{m,n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_1, \ {}_n x_2, u] \\
&= \sum_{\substack{m \geq n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_1, \ {}_n x_2, u] + \sum_{\substack{0 \leq m < n \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_1, \ {}_n x_2, u] \\
&= \sum_{\substack{m \geq n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_1, \ {}_n x_2, u] + \sum_{\substack{m > n \geq 0 \\ m+n=d-4}} c_{n,m}[u, \ {}_n x_1, \ {}_m x_2, u] \\
&= \sum_{\substack{m \geq n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_m x_1, \ {}_n x_2, u] + \sum_{\substack{m > n \geq 0 \\ m+n=d-4}} c_{m,n}[u, \ {}_n x_1, \ {}_m x_2, u] \\
&= c_{m,m}[u, \ {}_m x_1, \ {}_m x_2, u] + \sum_{\substack{m > n \geq 0 \\ m+n=d-4}} c_{m,n}([u, \ {}_m x_1, \ {}_n x_2, u] + [u, \ {}_n x_1, \ {}_m x_2, u]) \\
&= \frac{1}{2} c_{m,m} w_{m,m} + \sum_{\substack{m > n \\ m+n=d-4}} c_{m,n} w_{m,n}.
\end{aligned}$$

Assume now that d is odd. By similar arguments as above, we get

$$f(x_1, x_2) = \sum_{\substack{m>n \\ m+n=d-4}} c_{m,n} w_{m,n}.$$

Hence, by the above, the set $\{w_{m,n} : \ m \geq n \geq 0, \ m+n = d-4\}$ spans the K-vector space $((R_2'')^d)^{S_2}$. Furthermore, by Lemma 5.6(2), the elements $w_{m,n}$, with $m \geq n \geq 0$ and $m+n = d-4$, are linearly independent over K. Therefore, we get the required result. ∎

Since $(R_2'')^{S_2} = \bigoplus_{d\geq 1}((R_2'')^d)^{S_2}$, the next result follows directly from Proposition 6.1.

Corollary 6.2. *The set $\{w_{m,n} : \ m \geq n \geq 0, \ m+n \geq 1\}$ is a basis of the K-vector space $(R_2'')^{S_2}$.*

Proof of Theorem 2.1. Throughout the proof, we write

$$D = \{x_1 + x_2, z_{m,n} : \ m > n \geq 0\} \quad \text{and} \quad D' = \{w_{m,n} : \ m \geq n \geq 0, \ m+n \geq 1\}.$$

Since, by the notation in Section 5.1, $\pi^*(x_1 + x_2) = y_1 + y_2$ and $\pi^*(z_{m,n}) = t_{m,n}$, it follows from Proposition 5.2(1) that $\pi^*(D)$ is a basis of $M_2^{S_2}$. Furthermore, by Corollary 6.2, D' is a basis of the K-vector space $(R_2'')^{S_2}$. Therefore, by Proposition 3.1 (2), the set $D \cup D'$ is a basis of $R_2^{S_2}$ and so, we obtain the required result. ∎

Corollary 6.3. (1) *The set $\{x_1 + x_2\}$ is a basis of the K-vector subspace $(R_2^1)^{S_2}$.*

(2) $(R_2^2)^{S_2} = \{0\}$.

(3) *For $3 \leq d \leq 4$, the set $\{z_{m,n} : \ m > n \geq 0, \ m+n = d-2\}$ is a basis of the K-vector subspace $(R_2^d)^{S_2}$.*

(4) *For $d \geq 5$, the set*

$$\{z_{m_1,n_1}, w_{m_2,n_2} : \ m_1 > n_1 \geq 0, \ m_1 + n_1 = d-2, \ m_2 > n_2 \geq 0, \ m_2 + n_2 = d-4\}$$

is a basis of the K-vector subspace $(R_2^d)^{S_2}$.

6.2. A minimal generating set of the Lie algebra $R_2^{S_2}$

Lemma 6.4. *Let $X_1 = \{x_1 + x_2, z_{2m-1,0}, w_{m,0} : m \geq 1\}$ and let $L(X_1)$ be the Lie subalgebra of $R_2^{S_2}$ generated by the set X_1. Then, $R_2^{S_2} = L(X_1)$.*

Proof. Throughout the proof, we write $X_1 = D_1 \cup D_2$, where

$$D_1 = \{x_1 + x_2, z_{2m-1,0} : m \geq 1\} \quad \text{and} \quad D_2 = \{w_{m,0} : m \geq 1\}.$$

We point out that $D_2 \subseteq (R_2'')^{S_2}$ and, by Proposition 5.2(2), $\pi^*(D_1)$ is a generating set of $M_2^{S_2}$. Hence, by Proposition 3.2, to complete the proof, it suffices to show that $u \in L(D_1 \cup D_2) = L(X_1)$ for all $u \in (R_2'')^{S_2}$. Since, by Corollary 6.2, the set $\{w_{m,n} : \ m \geq n \geq 0, \ m+n \geq 1\}$ is a basis of the K-vector space $(R_2'')^{S_2}$, it suffices to show that $w_{m,n} \in L(X_1)$ for all m, n, with $m \geq n \geq 0$, $m+n \geq 1$.

By Lemma 5.6(3) we have $[w_{m,0}, x_1 + x_2] = w_{m+1,0} + w_{m,1}$. Since, by definition, $[w_{m,0}, x_1 + x_2]$, $w_{m+1,0} \in L(X_1)$, it follows that $w_{m,1} \in L(X_1)$ for all $m \geq 1$. Similarly, by Lemma 5.6(3), $[w_{m,1}, x_1 + x_2] = w_{m+1,1} + w_{m,2}$. Since, by the above, $[w_{m,1}, x_1 + x_2], w_{m+1,1} \in L(X_1)$, it follows that $w_{m,2} \in L(X_1)$ for all $m \geq 1$. By continuing the same process, $w_{m,n} \in L(X_1)$ for all m, n, with $m \geq n \geq 0$, $m+n \geq 1$ and hence, we get the result. ∎

Proposition 6.5. *Let $X_2 = \{x_1 + x_2, z_{2m-1,0}, w_{1,0}, w_{2m,0} : n \geq 1\}$ and let $L(X_2)$ be the Lie subalgebra of $R_2^{S_2}$ generated by the set X_2. Then, $R_2^{S_2} = L(X_2)$.*

Proof. Since, by Lemma 6.4, the set $X_1 = \{x_1 + x_2, z_{2m-1,0}, w_{m,0} : m \geq 1\}$ is a generating set of $R_2^{S_2}$, it suffices to show that $w_{2m+1,0} \in L(X_2)$ for all $m \geq 1$. We use induction on m. Since $w_{1,0}, w_{2,0}, x_1 + x_2 \in L(X_2)$, by Lemma 5.6(3)

$$[w_{2,0}, x_1 + x_2] = w_{3,0} + w_{2,1}, \qquad [w_{1,0}, {}_2x_1 + x_2] = w_{3,0} + 2w_{2,1} + w_{1,2}$$

and by Lemma 5.6(2), $w_{2,1} = w_{1,2}$, it follows that

$$w_{3,0} + w_{2,1} \equiv 0 \mod (L(X_2)), \qquad w_{3,0} + 3w_{2,1} \equiv 0 \mod (L(X_2)).$$

Since $\mathrm{char}(K) = 0$, it follows that $w_{3,0} \in L(X_2)$ and thus, the result is true for $m = 1$. Let $m > 1$ and assume that $w_{2k+1,0} \in L(X_2)$ for all $1 \leq k < m$. We show that $w_{2m+1,0} \in L(X_2)$.

We claim that $w_{2m-t,t+1} \equiv (-1)^{t+1} w_{2m+1,0} \mod (L(X_2))$ for all $t = 0, \ldots, m$. We prove this claim by using the above inductive argument and by induction on t. Since, by Lemma 5.6(3), $[w_{2m,0}, x_1 + x_2] = w_{2m+1,0} + w_{2m,1}$ and $w_{2m,0}, x_1 + x_2 \in L(X_2)$, we get $w_{2m,1} \equiv -w_{2m+1,0} \mod (L(X_2))$, that is, our claim is true for $t = 0$. Assume that our claim is true for all $0 \leq t < m$, that is, assume that

$$w_{2m-t,t+1} \equiv (-1)^{t+1} w_{2m+1,0} \mod (L(X_2)) \quad \text{for all} \quad 0 \leq t < m. \tag{1}$$

By Lemma 5.6(3),

$$[w_{2m-t-1,0}, {}_{t+2}x_1 + x_2] = \sum_{i=0}^{t+2} \binom{t+2}{i} w_{2m+1-i,i}.$$

Since $2m - t - 1 < 2m$, by our inductive argument we get $w_{2m-t-1,0} \in L(X_2)$ and therefore, $[w_{2m-t-1,0}, {}_{t+2}x_1 + x_2] \in L(X_2)$. Thus, by working modulo $L(X_2)$,

$$\sum_{i=0}^{t+2} \binom{t+2}{i} w_{2m+1-i,i} \equiv 0 \mod (L(X_2)),$$

that is,

$$w_{2m+1,0} + \sum_{i=0}^{t} \binom{t+2}{i+1} w_{2m-i,i+1} + w_{2m-t-1,t+2} \equiv 0 \mod (L(X_2)),$$

Hence, by Equation (1),

$$w_{2m+1,0} + \sum_{i=0}^{t} \binom{t+2}{i+1} (-1)^{i+1} w_{2m+1,0} + w_{2m-t-1,t+2} \equiv 0 \mod (L(X_2)).$$

But, $\sum_{i=0}^{t}\binom{t+2}{i+1}(-1)^{i+1}=-1-(-1)^{t+2}$. Therefore, by the above equation,

$$w_{2m-t-1,t+2}\equiv(-1)^{t+2}w_{2m+1,0}\mod(L(X_2)).$$

Thus, our claim is also true for $t+1$ and therefore, our claim is true for all $t=0,\dots,m$, that is,

$$w_{2m-t,t+1}\equiv(-1)^{t+1}w_{2m+1,0}\text{ for all }t=0,\dots,m.$$

Hence, for $t=m-1$,

$$w_{m+1,m}\equiv(-1)^{m}w_{2m+1,0}\mod(L(X_2))$$

and for $t=m$,

$$w_{m,m+1}\equiv(-1)^{m+1}w_{2m+1,0}\mod(L(X_2)).$$

Since, by Lemma 5.6(2), $w_{m+1,m}=w_{m,m+1}$, it follows from the above equations that $2w_{2m+1,0}\equiv0\mod(L(X_2))$. Hence, $w_{2m+1,0}\equiv0\mod(L(X_2))$ and thus, we get the result. ■

Proof of Theorem 2.2. Throughout the proof, we write

$$D_1=\{x_1+x_2,z_{2n-1,0}:n\geq1\}\quad\text{and}\quad D_2=\{w_{1,0},w_{2n,0}:n\geq1\}.$$

We point out that by Proposition 6.5 the set $D_1\cup D_2$ is a generating set of $R_2^{S_2}$, by Corollary 5.3 $\pi^*(D_1)$ is a minimal generating set of $M_2^{S_2}$ and by Proposition 6.5, $D_1\cup D_2$ is a generating set of $R_2^{S_2}$. Hence, to show that $D_1\cup D_2$ is minimal, by Proposition 3.3 it suffices to show that for any $u\in D_2$, $u\notin L(D_1\cup D_2\setminus\{u\})$. Since $(R_2'')^{S_2}=\bigoplus_{d\geq1}((R_2'')^d)^{S_2}$, it suffices to show that

$$w_{1,0}\notin L(D_1\cup D_2\setminus\{w_{1,0}\})\cap((R_2'')^5)^{S_2}\text{ and }w_{2n,0}\notin L(D_1\cup D_2\setminus\{w_{2n,0}\})\cap((R_2'')^{2n+4})^{S_2}$$

for all $n\geq1$. By Lemma 5.7, it is straightforward to verify that

$$w_{1,0}\notin L(D_1\cup D_2\setminus\{w_{1,0}\})\cap((R_2'')^5)^{S_2}.$$

Assume that $n\geq1$ and suppose on the contrary that

$$w_{2n,0}\in L(D_1\cup D_2\setminus\{w_{2n,0}\})\cap((R_2'')^{2n+4})^{S_2}.$$

In consequence of Lemma 5.7 and Lemma 5.4, the only nontrivial elements in $L(D_1\cup D_2\setminus\{w_{2n,0}\})\cap((R_2'')^{2n+4})^{S_2}$ are the elements of the form $[w_{1,0},\ {}_{2n-1}x_1+x_2]$ and $[w_{2n-2k,0},\ {}_{2k}x_1+x_2]$, with $1\leq k\leq n-1$ (with $n\geq2$). Therefore, $w_{2n,0}$ may be written as a K-linear combination of such elements. Hence, by Lemma 5.6(4b), $w_{2n,0}$ belongs to the K-vector subspace spanned by the set

$$\{W_{2n-k,k}=w_{2n-k,k}+w_{2n-k-1,k+1}:k=0,\dots,2n-2\}.$$

Hence, by Lemma 5.6(4a), $w_{2n,0}$ belongs to the K-vector subspace spanned by the set $\{W_{2n-k,k}:k=0,\dots,n-1\}$. Thus, we may write

$$w_{2n,0}=\sum_{k=0}^{n-1}\lambda_kW_{2n-k,k}=\sum_{k=0}^{n-1}\lambda_k(w_{2n-k,k}+w_{2n-k-1,k+1}).$$

Therefore, by Proposition 6.1, we get

$$\lambda_0=1,\lambda_k+\lambda_{k+1}=0,k=0,\dots,n-2,\lambda_{n-1}=0$$

and so, we get the required contradiction. ■

6.3. A presentation of the Lie algebra $R_2^{S_2}$

Proof of Theorem 2.3. By Theorem 2.2, X is a generating set of $R_2^{S_2}$ and, by Lemma 5.4, $\mathcal{R}$ is a set of relations in $R_2^{S_2}$. We show that there is no other non-trivial relation among the elements of X. Write $D_1 = \{x_1 + x_2, z_{2n-1,0} : n \geq 1\}$ and $D_2 = \{x_1 + x_2, w_{2n-1,0} : n \geq 1\}$. If there is another non-trivial relation in $L(D_1)$, then, (by the notation in Section 5.1) by applying the mapping π^*, we get a non-trivial relation among the elements $y_1 + y_2$, $t_{1,0}, \ldots, t_{2n-1,0}$ in $M_2^{S_2}$, which, by Proposition 5.2(2) is the required contradiction. By Lemma 5.7, by Remark 5.5 and by the set of relations $\mathcal{R}$, it is clear enough that it suffices to show that there is no other non-trivial relation in $L(D_2)$. Since $R_2^{S_2}$ is a graded Lie algebra, it suffices to show that there is no non-trivial relation among the elements of $(L(D_2))^d$ for all $d \geq 5$. Since this is trivially true for $d = 5$, we may assume that $d \geq 6$.

At first, assume that d is even, $d = 2k$, with $k \geq 3$. We show the result by induction on k. For $k = 3$, we may consider a relation $c_1 w_{2,0} + c_2[w_{1,0}, x_1 + x_2] = 0$, where $c_1, c_2 \in K$. By Lemma 5.6(3), we get $(c_1 + c_2)w_{2,0} + c_2 w_{1,1} = 0$. Hence, by Proposition 6.1, $c_1 = c_2 = 0$, as required. Assume that the result is true for some $k \geq 3$, that is, assume that there is no other non-trivial relation among the elements of $(L(D_2))^{2k}$. Suppose that there is a non-trivial relation in $(L(D_2))^{2k+2}$. Hence, by the set of relations $\mathcal{R}$ and by Remark 5.5, such a relation has the form

$$\lambda_{2k-1}[w_{1,0},\ {}_{2k-3}x_1 + x_2] + \sum_{j=1}^{k-1} \lambda_{2j-2}[w_{2k-2j,0},\ {}_{2j-2}x_1 + x_2] = 0.$$

It follows from Theorem 2.2 that $\lambda_0 = 0$ and hence, we have

$$\lambda_{2k-1}[w_{1,0},\ {}_{2k-3}x_1 + x_2] + \sum_{j=2}^{k-1} \lambda_{2j-2}[w_{2k-2j,0},\ {}_{2j-2}x_1 + x_2] = 0,$$

that is,

$$[\lambda_{2k-1}[w_{1,0},\ {}_{2k-5}x_1 + x_2] + \sum_{j=2}^{k-1} \lambda_{2j-2}[w_{2k-2j,0},\ {}_{2j-4}x_1 + x_2],\ {}_{2}x_1 + x_2] = 0.$$

Hence, by Lemma 4.4, we get

$$\lambda_{2k-1}[w_{1,0},\ {}_{2k-5}x_1 + x_2] + \sum_{j=2}^{k-1} \lambda_{2j-2}[w_{2k-2j,0},\ {}_{2j-4}x_1 + x_2] = 0 \text{ in } (L(D_2))^{2k}.$$

Thus, by our inductive argument,

$$\lambda_{2k-1} = \lambda_2 = \ldots = \lambda_{2k-4} = 0$$

and therefore, we get the required contradiction. Hence, it follows that the result is true for all d, with d even.

Assume now that d is odd, $d = 2k + 1$, with $k \geq 3$ and suppose on the contrary that there is a non-trivial relation in $(L(D_2))^{2k+1}$. By the set of relations $\mathcal{R}$ and by Remark 5.5, such a relation has the form

$$\mu_{2k-4}[w_{1,0},\ {}_{2k-4}x_1 + x_2] + \sum_{j=2}^{k-1} \mu_{2j-3}[w_{2k-2j,0},\ {}_{2j-3}x_1 + x_2] = 0.$$

Thus,

$$[\mu_{2k-4}[w_{1,0},\ _{2k-5}x_1 + x_2] + \sum_{j=2}^{k-1} \mu_{2j-3}[w_{2k-2j,0},\ _{2j-4}x_1 + x_2], x_1 + x_2] = 0.$$

Therefore, by Lemma 4.4, we get

$$\mu_{2k-4}[w_{1,0},\ _{2k-5}x_1 + x_2] + \sum_{j=2}^{k-1} \mu_{2j-3}[w_{2k-2j,0},\ _{2j-4}x_1 + x_2] = 0 \text{ in } (L(D_2))^{2k}.$$

Hence, as shown above,

$$\mu_{2k-4} = \mu_1 = \ldots = \mu_{2k-5} = 0$$

and thus, we obtain the required result. ∎

Remark 6.6. Let $R_{2,c} = L_2/(\gamma_3(L_2') + (\gamma_3(L_2))' + \gamma_{c+1}(L_2))$ be the free (nilpotent of class 2)-by-abelian and abelian-by-(nilpotent of class 2) and nilpotent Lie algebra of rank 2 and class c, with $c \geq 1$. Then, by Proposition 3.4 and Theorem 2.1, Theorem 2.2 and Theorem 2.3, we get a basis, a minimal generating set and a presentation of $R_{2,c}^{S_2}$.

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