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Left Invariant Semi Riemannian Metrics on Quadratic Lie Groups

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Abstract. To determine the Lie groups that admit a flat (eventually geodesically complete) left invariant semi-Riemannian metric is an open and difficult problem. The main aim of this paper is the study of the flatness of left invariant semi-Riemannian metrics on quadratic Lie groups i.e. Lie groups endowed with a bi-invariant semi-Riemannian metric. We give a useful necessary and sufficient condition that guarantees the flatness of a left invariant semi-Riemannian metric defined on a quadratic Lie group. All these semi-Riemannian metrics are complete. We show that there are no Riemannian flat left invariant metrics on non Abelian quadratic Lie groups. We study the Jacobi fields of any left invariant semi-Riemannian metric on a Lie group, using the notion of reflections. The case of Oscillator groups is addressed. This paper is a modification of a 2011 previous version due to the first two authors.

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1. Introduction

This article outlines some facts known by the authors about the semi Riemannian geometry of a Lie group provided with a semi Riemannian metric invariant under left translations.

Contents

When the relations between curvature or (geodesically) completeness of a semi Riemannian metric and other topological or geometrical properties are studied it is very useful to have many examples. This paper describes a rich collection of examples which are obtained by providing a Lie group G with a semi Riemannian metric invariant under left translations. It is well known that every left Riemannian

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metric is complete and in [16] Milnor described the Lie groups with flat left invariant Riemannian metrics. By contrast, the study of completeness and/or flatness of a non definite metric is in general very difficult. Even in the 3 dimensional non unimodular case, there is not in the literature a necessary and sufficient condition that guarantees the completeness of a left invariant Lorentzian metric. When the 3 dimensional Lie group is unimodular, the completeness of a left invariant Lorentzian metric is equivalent to the completeness of the geodesics of light type [6].

Our class of examples can be enlarged substantially, with no extra effort, as follows. If Γ is any discrete subgroup of G, then a left invariant semi Riemannian metric on G gives rise to a metric on the quotient space $\Gamma\backslash G$ with identical properties of curvature and (in)completeness. The case where $\Gamma\backslash G$ is compact is of particular interest.

The first section will survey general old and new results on left invariant semi Riemannian metrics on Lie groups. The principal and new result (Theorem 2.8) gives necessary and sufficient conditions for the flatness of a left invariant semi Riemannian metric on unimodular Lie groups. Under these conditions flatness implies completeness.

In Section 2, necessary and sufficient conditions that guarantees the existence of bi-invariant semi Riemannian metrics on Lie groups are given. These groups, called quadratic or orthogonal Lie groups (see [12]), are the central objects of our study.

Section 3 is devoted to the Jacobi vector fields corresponding to a left invariant semi Riemannian metric on Lie groups and on quadratic Lie groups in particular. The equation that defines the reflection on the Lie algebra of G of a such vector field is particularly simple when the metric is bi-invariant. The reflections of the Jacobi vector fields corresponding to the Lorentzian bi-invariant metrics on the oscillator Lie groups are determined.

Theorem 5.1, Theorem 5.12, and Theorem 5.15 are the main results of Section 4. All these results appeared in version [7]. The first one specializes Theorem 2.8 to the case of quadratic Lie groups. One of the consequences of Theorem 5.3 is that every left invariant Riemannian metric on a non Abelian quadratic Lie group is non flat. Theorem 5.12 shows the non existence of flat left invariant semi Riemannian metric on any indecomposable quadratic Lie group of dimension 4. The case of left invariant semi Riemannian metrics on quadratic nilpotent Lie groups is also treated. Theorem 5.15 shows that every 3 step nilpotent Lie group admits a flat left invariant connection given by an invertible f-derivation. This connection is the Levi Civita connection of a semi Riemannian metric if the group is quadratic (Theorem 5.16). A left semi Riemannian metric on a nilpotent quadratic Lie group (G, k) defined by a k symmetric linear isomorphism u is complete when u preserves the descending central sequence of the Lie algebra $\mathcal{G}$ (Proposition 9). If (G, k) is 2 step nilpotent and its corank is 0 then G admits many non isometric flat left invariant semi Riemannian metrics. If $\dim G > 8$ there are infinitely many non isometric such metrics (Theorem 6.1).

The following result is an important and final remark concerning the classical or generalized solutions of the Yang-Baxter equation on quadratic Lie groups and the relations with left invariant semi Riemannian metrics.

Theorem 1.1. [3] *Every solution of the classical Yang-Baxter equation on a quadratic Lie group induces a flat left invariant semi Riemannian metric on the dual Lie groups associated to the solution. Furthermore a solution of the generalized Yang-Baxter equation determines a left invariant semi Riemannian metric such that the covariant derivative of its curvature tensor vanishes.*

2. General results about left invariant semi Riemannian metrics on Lie groups

Let G be a Lie group, ε the unit element in G. A non degenerate symmetric bilinear form $\langle , \rangle$ on $\mathcal{G} := G_\varepsilon$ defines a left invariant semi Riemannian metric on G given by the formula

$$\langle v_\sigma, w_\sigma \rangle_\sigma := \langle (\mathrm{L}_{\sigma^{-1}})_{*,\sigma} v_\sigma, (\mathrm{L}_{\sigma^{-1}})_{*,\sigma} w_\sigma \rangle, \quad \sigma \in G, v_\sigma, w_\sigma \in G_\sigma$$

where $\mathrm{L}_\sigma : \tau \mapsto \sigma\tau$, and conversely.

The Levi-Civita connection ∇ of a semi Riemannian left invariant metric is left invariant, and defines a product on the Lie algebra given by the formula

$$xy^+ := \nabla_{x^+} y^+,$$

where x^+ stands for the left invariant vector field with infinitesimal generator $x \in \mathcal{G}$. This product, called the *Levi-Civita product*, verifies the Koszul formula

$$2\langle xy, z \rangle = \langle [x, y], z \rangle - \langle [y, z], x \rangle + \langle [z, x], y \rangle.$$

By means of
$$x(t) := \left(\mathrm{L}_{\sigma(t)^{-1}}\right)_{*,\sigma(t)} \sigma'(t),$$
the equation for the geodesics of the semi Riemannian metric becomes, in the Lie algebra,
$$\dot{x} = -xx. \tag{1}$$

Since the Levi-Civita connection is torsion free, the Levi-Civita product satisfies

$$xy - yx = [x, y].$$

Moreover the Koszul formula implies that the map $L_x : y \mapsto xy$ is $\langle \, , \, \rangle$ skew symmetric.

Many features of the geometry of left invariant semi Riemannian metrics on Lie groups can be studied in the Lie algebra.

A semi Riemannian metric is called *complete* when its geodesics are defined for all $t \in \mathbb{R}$. Notice that a left invariant semi Riemannian metric is complete if and only if the solutions of equation (1) are defined for all values of the parameter.

The exponential map associated to a semi Riemannian metric with base point $\sigma \in G$ is denoted by Exp_σ. This map is defined on all of G_σ for all σ whenever the semi Riemannian metric is complete. Notice that in general Exp_ε differs from the exponential map in Lie theory (see Remark 3.4 bellow).

Definition 2.1. A semi Riemannian metric is called *flat* if the curvature tensor vanishes. ∎

The Levi-Civita product for a flat left invariant semi Riemannian metric on a Lie Group G is a left symmetric product on $\mathcal{G}$, compatible with the Lie bracket, that is

$$(xy)z - x(yz) = (yx)z - y(xz),$$

and

$$xy - yx = [x, y].$$

As a partial converse we have that a left symmetric product compatible with the Lie bracket induces a flat left invariant connection on G.

The existence of a flat left invariant metric on a Lie group imposes serious restrictions on the group as the following result shows

Theorem 2.2. (Theorem 1.5 [16]) *A Lie group has a left invariant flat Riemannian metric if and only if its Lie algebra decomposes as a semidirect product of an Abelian Lie algebra with an Abelian ideal, the Abelian algebra acting on the Abelian ideal by infinitesimal isometries.*

The existence of a flat left invariant metric on a quadratic Lie group imposes even more restrictive conditions on the group. In fact, under this hypothesis, the group is Abelian (see Proposition 5.7). In the same line of thought Proposition 7 states that on non Abelian quadratic Lie groups there are no flat left invariant Lorentzian metrics.

Theorem 2.3. *A flat left invariant semi Riemannian metric is complete if and only if the Lie group is unimodular.*

For the proof see [1].

In [9] the simply connected Lie groups with flat, complete left invariant Lorentz metrics are characterized. The nilpotent case was treated alternaternatively by means of the double extension in [1].

The Jacobi fields measure the variation of geodesics: If $t \mapsto \tau(t)$ is a geodesic, the vector field $t \mapsto Y(t)$ defined on the curve τ is a *Jacobi vector field* provided that it satisfies the second order differential equation

$$\frac{D^2Y}{dt^2} = \mathrm{R}_{Y\tau'}(\tau') \tag{2}$$

where DY/dt stands for the affine covariant derivative of (G, ∇) of the vector field Y on the curve τ and R is the curvature tensor. Hence if the semi Riemannian metric is flat, then a vector field Y on a geodesic is a Jacobi field if and only if the vector field DY/dt is parallel along the geodesic, that is if and only if $D^2Y/dt^2 = 0$.

Then a necessary condition for a left invariant semi Riemannian metric to be flat is that the second covariant derivative of any right invariant vector field along every geodesic vanishes.

Notice that every right invariant vector field is a Jacobi vector field along any geodesic, because it is a Killing vector field ([10]).

Definition 2.4. Let $\tau : [a, b] \to G$ a geodesic. The points $\tau(a)$, $\tau(b)$ are called *conjugate points* if there is a Jacobi field Y on τ such that $Y(a) = Y(b) = 0$.

Proposition 2.5. *Let $\tau : [a, b] \to G$ be a geodesic. Then $\tau(a)$ and $\tau(b)$ are conjugate if and only if the rank of* $\mathrm{Exp}_{\tau(a)}$ *at $(b - a)\tau'(a)$ is less than dim G.*

Lemma 2.6. *Let $(M, \langle\, , \rangle)$ a flat semi Riemannian manifold. Let U be a connected neighborhood of $0 \in M_\sigma$ where Exp_σ is defined. If the semi Riemannian metric is flat then Exp_σ is a local isometry.*

Proof. Let $x \in M_\sigma$, $v, w \in (M_\sigma)_x \approx M_\sigma$. Let J_v, J_w the unique Jacobi fields along a geodesic $\tau : [0,1] \to G$, $\tau(0) = \sigma, \tau(1) = \rho$ such that

$$J_v(0) = J_w(0) = \sigma, \ \frac{DJ_v}{dt}(0) = v, \ \frac{DJ_w}{dt}(0) = w.$$

The derivatives of the map $\varphi(t) := \langle J_v(t), J_w(t)\rangle$ are

$$\varphi'(t) = \langle \frac{DJ_v}{dt}(t), J_w(t)\rangle + \langle J_v(t), \frac{DJ_w}{dt}(t)\rangle,$$

$$\varphi''(t) = 2\langle \frac{DJ_v}{dt}, \frac{DJ_w}{dt}\rangle, \varphi'''(t) = 0$$

since J_v, J_w are parallel along τ. In consequence $\varphi''(t) = 2\langle v, w\rangle$, $\varphi'(t) = 2t\,\langle v, w\rangle$, $\varphi(t) = t^2\,\langle v, w\rangle$, and

$$\langle d\mathrm{Exp}_\sigma(0)v, d\mathrm{Exp}_\sigma(0)w\rangle = \varphi(1) = \langle v, w\rangle. \qquad \blacksquare$$

As a corollary we have that a complete semi Riemannian flat metric has no conjugate points. Furthermore

Lemma 2.7. *Let $(M, \langle\, , \rangle)$ be a flat semi Riemannian manifold. If Exp_p is defined for all $v \in M_p$, then $\mathrm{Exp}_p : (M_p, \langle\langle\, , \rangle\rangle) \to (M, \langle\, , \rangle)$ is a semi Riemannian covering, where $\langle\langle\, , \rangle\rangle$ is the affine metric induced by $\langle\, , \rangle_p$.*

Proof. We have to show that Exp_p has the lifting property for geodesics. Let $\tau : [0,1] \to M$ a geodesic and $x_0 \in M_p$ such that $\mathrm{Exp}_\sigma(x_0) = \tau(0)$. By lemma 2.6, there are neighborhoods U and V of x_0 in M_p and $\tau(0)$ in M such that Exp_p defined on U onto V is an isometry. If t satisfies $\tau([0,t)) \subset V$, then $s \mapsto \mathrm{Exp}_p^{-1}\tau(s)$ is a geodesic in M_p. By hypothesis this geodesic is defined in $[0,1]$ and it is a lifting of σ. The conclusion follows from Theorem 28.7 in [18]. ■

The following theorem puts together some scattered results.

Theorem 2.8. *Let G be a connected unimodular Lie group and $\langle\, , \rangle$ a left invariant semi Riemannian metric on G. Then the following assertions are equivalent:*

(i) *$\langle\, , \rangle$ is flat and complete.*

(ii) *Exp_ε is a local isometry (hence, for every σ in G, Exp_σ is a local isometry).*

(iii) *$\langle\, , \rangle$ is flat.*

In any case G is solvable, and $(G, \langle\, , \rangle)$ has no conjugate points.

Proof. By Theorem 2.3 a flat left invariant semi Riemannian metric defined on an unimodular Lie group is complete. The hypothesis of $\langle\, , \rangle$ being flat implies that G is locally symmetric and that Exp_q is a local isometry that has the lifting property for geodesics. Hence it is a semi Riemannian covering.

To prove that G is solvable notice that the hypothesis implies that the Levi-Civita connection defined by the semi Riemannian metric is a left invariant affine structure.

Then there is a representation θ of G by affine transformations of $\mathcal{G}$ with an open orbit and discrete isotropy (see [11]). The action of G on $\mathcal{G}$ induced by θ is transitive because the metric is complete. Hence the restriction of the representation to a Levi subalgebra is completely reducible. This contradiction implies the solvability. ■

Example 2.9. Let $G = \mathbb{R}^2 \rtimes \mathrm{SO}(\mathbb{R}^2)$ be the connected component of the unit element of the group of rigid motions of the plane. The product on G is given by

$$(x, y, \alpha)(x', y', \beta) = (x + x' \cos\alpha - y' \sin\alpha, y + x' \sin\alpha + y' \cos\alpha, \alpha + \beta).$$

Let $\mathcal{G} = \mathrm{Span}\{e_1, e_2\} \rtimes \mathbb{R}\, e_3$, where $\mathrm{e}_1, \mathrm{e}_2$ is an Abelian Lie ideal and $[e_3, e_1] = e_2$, $[e_3, e_2] = -e_1$. Define a left invariant semi Riemannian metric by the Lorentzian quadratic form on $\mathcal{G}$:

$$q(x_1e_1 + x_2e_2 + x_3e_3) = x_1^2 + x_2^2 - x_3^2.$$

Some straightforward calculations show that

$$L_{e_1} = L_{e_2} = 0 \qquad L_{e_3} = \mathrm{ad}_{e_3}.$$

Then $L_{[x,y]} = 0$ and $L_xL_y = L_yL_x$. Hence the Lorentzian metric is flat. Equation (1) is in this case

$$\dot{x_1} = x_2x_3, \quad \dot{x_2} = -x_1x_3, \quad \dot{x_3} = 0.$$

The solution to this equation with initial condition (x_1, x_2, x_3) is

$$x(t) = (x_1 \cos(x_3t) - x_2 \sin(x_3t), x_1 \sin(x_3t) + x_2 \cos(x_3t), x_3)\,.$$

The geodesic on G starting at ε with initial velocity (x_1, x_2, x_3) is for $x_3 = 0$

$$\gamma(t) = (tx_1, tx_2, 0),$$

and when $x_3 \neq 0$: $\gamma(t) =$

$$= \Big(-\frac{x_2}{2x_3} + \frac{x_2}{2x_3}\cos(2x_3t) + \frac{x_1}{2x_3}\sin(2x_3t), \frac{x_1}{2x_3} - \frac{x_1}{2x_3}\cos(2x_3t) + \frac{x_2}{2x_3}\sin(2x_3t), x_3t\Big).$$

Hence the exponential map based at ε is, for $x_3 = 0$,

$$\mathrm{Exp}_\varepsilon(x_1, x_2, x_3) = (x_1, x_2, 0)$$

and for $x_3 \neq 0$,

$$\Big(-\frac{x_2}{2x_3} + \frac{x_2}{2x_3}\cos(2x_3) + \frac{x_1}{2x_3}\sin(2x_3), \frac{x_1}{2x_3} - \frac{x_1}{2x_3}\cos(2x_3) + \frac{x_2}{2x_3}\sin(2x_3), x_3\Big)\,.$$

Hence Exp_ε is a global isometry.

3. Quadratic Lie groups

Definition 3.1. A Lie group G with a bi-invariant semi Riemannian metric k is called *orthogonal* or *quadratic* Lie group. The pair $(\mathcal{G}, k)$, where $\mathcal{G}$ is the corresponding Lie algebra and k is the restriction of k to $\mathcal{G}$, is called *orthogonal* or *quadratic* Lie algebra.

Let $(\mathcal{G}, k)$ be a quadratic Lie algebra. Then k is a non degenerate quadratic form and ad_x is k skew symmetric for all $x \in \mathcal{G}$:

$$k(\mathrm{ad}_x\, y, z) + k(y, \mathrm{ad}_x\, z) = 0.$$

For every left invariant semi Riemannian metric $\langle\, , \rangle$ on G there is a k symmetric isomorphism u of the vector space underlying $\mathcal{G}$ such that, for all $x, y \in \mathcal{G}$

$$\langle x, y\rangle = k(u(x), y).$$

Equation (1) becomes in this case $u(\dot{x}) = [u(x), x]$.

The following propositions characterize quadratic Lie groups.

Proposition 3.2. [15] *A Lie group is quadratic if and only if the adjoint and co-adjoint actions are isomorphic.*

Proposition 3.3. *The Lie group G is quadratic if and only if the linear Poisson structure on $\mathcal{G}^*$ given by the Lie bracket of $\mathcal{G}$ has a quadratic, non degenerate Casimir.*

Proof. Let $(\mathcal{G}, k)$ be an orthonormal Lie algebra. Denote by $\Phi : \mathcal{G} \to \mathcal{G}^*$ the symmetric isomorphism $\Phi(x) := k(x, \cdot)$. For $x \in \mathcal{G}$, define $\hat{x} \in (\mathcal{G}^*)^*$ by $\hat{x}(\xi) = \xi(x)$, $\xi \in \mathcal{G}^*$.

Let $f : \mathcal{G}^* \to \mathbb{R}$ be given by $f(\xi) = \xi(\Phi^{-1}(\xi))$. Clearly f is a non degenerate quadratic form, hence its differential is $(d\,f)_\xi(\xi') = 2\xi'\,(\Phi^{-1}(\xi))$ or, equivalently, $(d\,f)_\xi = 2\Phi^{-1}(\xi)^\wedge$. If $x_0 = \Phi^{-1}(\xi)$, we get, using the ad invariance of k, that

$$\{f, \hat{x}\}_\xi := \xi[2\Phi^{-1}(\xi), x] = \Phi(x_0)[2x_0, x] = k(x_0, [2x_0, x]) = 0,$$

for all $x \in \mathcal{G}$. Therefore f is a Casimir for the Lie-Poisson bracket $\{\cdot, \cdot\}$.

Conversely, let $f(\alpha) = \mathbf{b}(\alpha, \alpha)$ be a Casimir, $\mathbf{b}$ being a quadratic, symmetric, and non degenerate quadratic form. Define $k : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ by $k(x, y) := \Psi^{-1}(\hat{x})y$, where $\Psi(\alpha) := \mathbf{b}(\alpha, \cdot)$. Since Ψ is a symmetric isomorphism, so is k. Moreover, f being a Casimir, we get that for all $x \in \mathcal{G}$, $\{f, \hat{x}\} = 0$, that is,

$$\forall\, \alpha \in \mathcal{G}^*, \quad \forall\, x \in \mathcal{G},\ 0 = \{(df)_\alpha, \hat{x}\}_\alpha = \{\Psi(\alpha), \hat{x}\}_\alpha.$$

Hence for all $y, z \in \mathcal{G}$, $\qquad k(z, [z, y]) = 0$.

Replacing z by $a + b$, $a, b \in \mathcal{G}$, we get that

$$\forall\, a, b, y \in \mathcal{G},\ \ k(a, [b, y]) + k(b, [a, y]) = 0.$$

Hence k is a quadratic structure on $\mathcal{G}$. ∎

Remark 3.4. The Levi-Civita product and the curvature of a bi-invariant semi Riemannian metric on a Lie group are given (resp.) by:

$$xy = \frac{1}{2}[x, y], \qquad \mathrm{R}(x, y) = \frac{1}{4}\mathrm{ad}_{[x,y]}$$

Hence every semi Riemannian bi-invariant metric is geodesically complete, the geodesics through the unit element ε of G are the 1-parameter subgroups of G, and the bi-invariant metric is flat if and only if the group is 2-step nilpotent.

4. Jacobi fields on quadratic Lie groups, conjugate points

Every vector field X on a Lie group G defines a map $\tilde{X} : G \to \mathcal{G}$ given by $\sigma \mapsto (\mathrm{L}_{\sigma^{-1}})_{*,\sigma} X_\sigma$. Obviously, a vector field is left invariant if and only if the associated map is constant.

Given a curve $\sigma : [t_0, t_1] \to G$, every vector field Y on σ defines a curve in $\mathcal{G}$:

$$\tilde{Y}(t) = \left(\mathrm{L}_{\sigma(t)^{-1}}\right)_{*,\sigma(t)} Y(t)$$

and conversely, every curve in $\mathcal{G}$ defined on $[0,1]$ determines a vector field on σ. We say that one is the *reflection* of the other and we write either $y^\sim = Y$ or $Y^\sim = y$.

Notice that $y(t) = (Y(t))^\sim$ is equivalent to $y(t)^+_{\sigma(t)} = Y(t)$.

The following proposition describes the Jacobi fields for a left invariant semi Riemannian metric defined on a Lie group.

Proposition 4.1. *Let G be a Lie group, ∇ a left invariant torsion free connection on G and let $\sigma : [0,1] \to G$ be a geodesic such that $\sigma(0) = \varepsilon$. Then the vector field on σ, $t \mapsto Y(t)$ is a Jacobi vector field if and only if its reflection $y := \tilde{Y}$ satisfies the differential equation*

$$\ddot{y} + 2x\dot{y} = [y,x]x + x[y,x] + [xx,y] \tag{3}$$

where, as before, $x(t) = \left(\mathrm{L}_{\sigma^{-1}(t)}\right)_{*,\sigma(t)} \sigma'(t)$.

This proposition is a consequence of the following technical result.

Lemma 4.2. *With the notations introduced in the previous proposition, the first and second covariant derivatives of the vector field Y on σ are given by*

$$\frac{DY}{dt} = (y' + xy)^\sim, \qquad \frac{D^2Y}{dt^2} = (y'' + 2xy' + x'y + x(xy))^\sim$$

Proof. Let G be a n dimensional Lie with Lie algebra $\mathcal{G}$ and $\{e_i,\ 1 \le i \le n\}$ be a basis for the vector space underlying $\mathcal{G}$. Expressing Y by means of e_i^+ $(i = 1, \cdots, n)$, as in proposition 4.1, we get

$$\begin{aligned}\frac{DY}{dt} &= \frac{D}{dt}\sum_{i=1}^n y_i(t) e^+_{i,\sigma(t)} = \sum_{i=1}^n y_i'(t) e^+_{i,\sigma(t)} + \sum_{i=1}^n y_i(t) \nabla_{\sigma'(t)} e_i^+ \\ &= (y'(t))^\sim + \sum_{i=1}^n y_i(t) \nabla_{\sigma'(t)} e_i^+.\end{aligned}$$

Since $\nabla_{\sigma'(t)} e_i^+ = (x(t)e_i)^+_{\sigma(t)}$,

$$\sum_{i=1}^n y_i(t) \nabla_{\sigma'(t)} e_i^+ = \sum_{i=1}^n y_i(t)(x(t)e_i)^+_{\sigma(t)} = (x(t)y(t))^+_{\sigma(t)} = (x(t)y(t))^\sim,$$

that is

$$\frac{D}{dt}(\tilde{y}) = (y' + xy)^\sim.$$

Hence

$$\begin{aligned}\frac{D^2Y}{dt^2} &= \frac{D}{dt}(y' + xy)^\sim = ((y' + xy)' + x(y' + xy))^\sim \\ &= (y'' + x'y + xy' + xy' + x(xy))^\sim = (y'' + x'y + 2xy' + x(xy))^\sim . \quad \blacksquare\end{aligned}$$

Proof of Proposition 4.1. Recall that

$$\mathrm{R}_{x^+y^+}z^+ = ([x,y]z - x(yz) + y(xz))^+ .$$

A straightforward calculation shows that,

$$\mathrm{R}_{Y\sigma'(t)}\sigma'(t) = ([y,x]z - y(xz) + x(yz))^\sim.$$

Thus $Y = y^\sim$ is a Jacobi vector field if and only if

$$\ddot{y} + \dot{x}y + 2x\dot{y} + x(xy) = [y,x]z - y(xz) + x(yz). \tag{4}$$

Since σ is a geodesic we have $\dot{x} = -xx$, and since the connection is torsion free, $xy - yx = [x,y]$, then equation (4) becomes

$$\begin{aligned}\ddot{y} + 2x\dot{y} &= [y,x]x + (xx)y - y(xx) + x[y,x]\\ &= [y,x]x + x[y,x] + [(xx),y]\end{aligned}$$

■

Remark 4.3. If $x_0 \neq 0$ is a solution of $xx = 0$, then the geodesic σ through ε with velocity x_0 is the one-parameter subgroup of G with infinitesimal generator x_0 and the reflections of Jacobi fields on σ are the solutions of the equation

$$\ddot{y} + 2x_0\dot{y} = [y,x_0]x_0 + x_0[y,x_0].$$

Corollary 4.4. *If ∇ is the Levi-Civita connection defined by a bi-invariant semi Riemannian metric, then a vector field Y on a geodesic $\sigma : [0,1] \to G$ with $\sigma(0) = \varepsilon$, is a Jacobi vector field is and only if its reflection curve $y = Y^\sim$ is a solution of the differential equation*

$$\ddot{y} = [\dot{y}, x_0],$$

where x_0 is the initial velocity of the geodesic.

The proof follows immediately from the fact that the Levi-Civita product of a bi-invariant metric is given by $xy = (1/2)[x,y]$.

Corollary 4.5. *Let ∇ be the Levi-Civita connection defined by a bi-invariant semi Riemannian metric on a nilpotent Lie group. Then the reflection of Jacobi fields along a geodesic $\sigma : [0,1] \to G$ are polynomial.*

Proof. Let Y be a Jacobi field along a geodesic σ, and y its reflection. Then by the previous corollary $y^{(2)}(t) = [y'(t), x_0]$. Hence $y^{(k+1)}(t) = [y^{(k)}(t), x_0]$, and $y^{(m+1)} \equiv 0$, where m is such that $\mathcal{G}^{(m)} = 0$. ■

4.1. Jacobi fields on the oscillator groups

Consider the quadratic Lie algebra $(\mathbb{R}^{2n}, k_0)$, where k_0 is an Euclidean inner product. Let $\lambda := (\lambda_1, \dots, \lambda_n)$ where each λ_i is a positive real number and θ the antisymmetric isomorphism of $(\mathbb{R}^{2n}, k_0)$ given by the matrix (with respect to an orthonormal basis) $\mathcal{B} = \{e_1, \dots, e_n, \check{e}_1, \dots, \check{e}_n\}$

$$M_{\mathcal{B}}\theta = \begin{pmatrix} 0 & -\mathrm{diag}(\lambda_1,\dots,\lambda_n) \\ \mathrm{diag}(\lambda_1,\dots,\lambda_n) & 0 \end{pmatrix}$$

where $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ stands for the diagonal matrix with $\lambda_1, \dots, \lambda_n$ on the main diagonal.

Obviously θ defines a representation of the Lie algebra $\mathbb{R}$ by endomorphisms of $\mathcal{G}$, noted also by θ.

Let $\mathcal{G}(\lambda)$ be Lie algebra obtained by a process of double extension (see for example [13]) of $(\mathbb{R}^{2n}, k_0)$ by $\mathbb{R}$ by means of θ. This means that the algebra is obtained by a central extension $\mathbb{R}\, e_0 \times_\omega \mathbb{R}^{2n}$ of $\mathbb{R}^{2n}$ by $\mathbb{R}$ by means of the scalar 2-cocycle $\omega(x,y) := k_0(\theta(x), y)$, then by semi-direct product of $\mathbb{R}\, e_{-1}$ by $\mathbb{R}\, e_0 \times_\omega \mathbb{R}^{2n}$ where the action is given by

$$[e_{-1}, e_0] = 0, \;\; [e_{-1}, x] = \theta x, \text{ for } x \in \mathbb{R}^{2n}.$$

This algebra has a quadratic structure k that extends k_0 and is given in the Minkowski plane, $V = \mathrm{Span}\{e_{-1}, e_0\}$, by

$$k(e_0, e_0) = k(e_{-1}, e_{-1}) = 0, \quad k(e_{-1}, e_0) = 1$$

and is orthogonal to $\mathbb{R}^{2n}$.

Since the algebra $\mathcal{G}(\lambda)$ is solvable, the connected and simply-connected Lie group with Lie algebra $\mathcal{G}(\lambda_1, \ldots, \lambda_n)$ can be identified as a manifold to $\mathbb{R}^{2n+2} \equiv \mathbb{R} \times \mathbb{C}^n \times \mathbb{R}$ with product

$$(s, z_1, \ldots, z_n, t) \cdot (s', z'_1, \ldots, z'_n, t')$$
$$= (s + s' + \frac{1}{2} \sum_{j=1}^{n} \mathrm{Im} \bar{z}_j \exp(i\, t\lambda_j) z'_j, z_1 + \exp(i\, t\lambda_1) z'_1, \ldots, z_n + \exp(i\, t\lambda_n) z'_n, t + t')$$

Definition 4.6. The groups $\mathrm{G}(\lambda)$ are called *Oscillator groups* and the corresponding Lie algebras $\mathcal{G}(\lambda)$ are called *Oscillator algebras.*

The equation that defines the reflection of a Jacobi field in the oscillator algebra is given by

$$\begin{array}{rcl} y''_{-1} &=& 0 \\ y''_0 &=& \check{x}_1 y'_1 + \cdots + \check{x}_n y'_n - x_1 \check{y}'_1 - \cdots - x_n \check{y}'_n \\ y''_1 &=& -\lambda_1 \check{x}_1 y'_{-1} + x_{-1} \lambda_1 \check{y}'_1 \\ & \cdots & \\ y''_n &=& -\lambda_n \check{x}_n y'_{-1} + x_{-1} \lambda_n \check{y}'_n \\ \check{y}''_1 &=& \lambda_1 x_1 y'_{-1} - x_{-1} \lambda_1 y'_1 \\ & \cdots & \\ \check{y}''_n &=& \lambda_n x_n y'_{-1} - x_{-1} \lambda_n y'_n, \end{array}$$

where $x(0) = \sum_{i=-1}^{n} x_i\, e_i + \sum_{i=1}^{n} \check{x}_i\, \check{e}_i$. In order to find the conjugate points to ε, it is also necessary that $y(0) = 0$ and $y(t_1) = 0$ for some $t_1 \neq 0$. This implies that $y_{-1} \equiv 0$ and the system is equivalent to the system

$$\begin{array}{rcl} y''_0 &=& \check{x}_1 y'_1 + \cdots + \check{x}_n y'_n - x_1 \check{y}'_1 - \cdots - x_n \check{y}'_n \\ y''_j &=& x_{-1} \lambda_j \check{y}'_j \\ \check{y}''_j &=& -x_{-1} \lambda_j y'_j \end{array} ,$$

$1 \leq j \leq n$. When $x_{-1} \neq 0$, the solutions are, since $y(0) = 0$,

$$y_j(t) = \frac{r_j}{x_{-1}\lambda_j} \sin(x_{-1} \lambda_j\, t)$$
$$\check{y}_j(t) = \frac{r_j}{x_{-1}\lambda_j} \left(1 - \cos(x_{-1} \lambda_j\, t)\right).$$

In order to have $y(t_1)=0$, it is necessary that

$$x_{-1}\lambda_j\, t_1 = 2\pi k, \quad k\in\mathbb{Z}, \quad \text{or} \quad r_j = 0$$

for all $1\le j\le n$. Hence, letting $r_j=0$ for $j\neq i$, y_0 must be a solution to the equation

$$y_0'' = \check{x}_i y_i' - x_i \check{y}_i' = r_i\left(\check{x}_i\cos(x_{-1}\lambda_i\, t) - x_i\sin(x_{-1}\lambda_i\, t)\right)$$

which implies that

$$y_0'(t) = c + \frac{r_i}{x_{-1}\lambda_i}\left(x_i\cos(x_{-1}\lambda_i\, t) + \check{x}_i\sin(x_{-1}\lambda_i\, t)\right)$$

and thus $\quad y_0(t) = d + ct + \dfrac{r_i}{(x_{-1}\lambda_i)^2}\left(x_i\sin(x_{-1}\lambda_i\, t) - \check{x}_i\cos(x_{-1}\lambda_i\, t)\right).$

Since $y_0(0)=y_0(t_1)=0$,

$$y_0(t) = \frac{r_i}{(x_{-1}\lambda_i)^2}\left(x_i\sin(x_{-1}\lambda_i\, t) + \check{x}_i(1-\cos(x_{-1}\lambda_i\, t))\right)$$

and the vector field $Y=y^{\sim}$, where

$$y(t) = y_0(t)e_0 + \frac{r_i}{x_{-1}\lambda_i}\sin(x_{-1}\lambda_i t)e_i + \frac{r_i}{x_{-1}\lambda_i}\left(1-\cos(x_{-1}\lambda_i\, t)\right)\check{e}_i,$$

is a Jacobi field along the geodesic $t\mapsto \exp(tx(0))$. The points

$$\exp\left(\frac{\pi k}{x_{-1}\lambda_i}x(0)\right), \quad k\in\mathbb{Z},\ 1\le i\le n,$$

are conjugate to ε. Notice that when $x_{-1}=0$ then a Jacobi field along the geodesic $t\mapsto\exp(tx(0))$, with $y(0)=0$ vanishes everywhere.

In what follows, G is a connected Lie group.

5. On flat left invariant semi Riemannian metrics on quadratic Lie groups

Let G be a quadratic Lie group and $\langle,\rangle$ a flat left invariant semi Riemannian metric. Notice that a quadratic Lie group is unimodular because ad_x is k skew-symmetric, for all $x\in\mathcal{G}$, hence, by Theorem 2.3, every flat left invariant semi Riemannian metric on a quadratic Lie group is complete, and we have a companion theorem of Theorem 2.8,

Theorem 5.1. *Let G be a connected quadratic Lie group, $\langle\,,\,\rangle$ a left invariant semi Riemannian metric on G. Then the following assertions are equivalent*

(i) *$\langle\,,\,\rangle$ is flat.*

(ii) *the exponential map relative to $\langle\,,\,\rangle$ is a local isometry.*

(iii) *$\langle\,,\,\rangle$ is flat and complete.*

In any case, G is solvable, and $(G,\langle\,,\rangle)$ has no conjugate points.
Moreover G viewed as a group of transformations of $\mathcal{G}$ has non trivial central 1-parameter subgroups of translations.

Proof. A quadratic Lie group is unimodular, hence the first part of the theorem follows from Theorem 2.8.

As for the existence of non-trivial 1-parameter subgroups of translations, since the metric is complete, the Levi-Civita product has no non trivial idempotents, hence there is an element $x_0 \in \mathcal{G}$, $x_0 \neq 0$ such that $x_0 x_0 = 0$. The 1-parameter subgroup with infinitesimal generator x_0 is a geodesic for the metric. ■

Definition 5.2. The *index* of a non degenerate quadratic form q defined on a real vector space V is the maximal dimension of a q totally isotropic subspace of V.

We have the following result

Theorem 5.3. *If a connected non Abelian quadratic Lie group (G, k) admits a flat left invariant semi Riemannian metric $\langle , \rangle$, then $\langle , \rangle$ is geodesically complete, G is solvable, the index of $\langle , \rangle$ is ≥ 1 and the universal covering of G viewed as a group of affine transformations contains central 1-parameter groups of translations.*

The two first assertions of the theorem follow from Theorem 2.8. The other assertions will follow from a series of lemmas and propositions.

Lemma 5.4. *The center of a quadratic Lie group with a flat left invariant semi Riemannian metric is non trivial.*

Proof. By Theorem 2.8 the Lie group is solvable. The result follows from the observation that $\mathcal{Z}(\mathcal{G})^{\perp_k} = [\mathcal{G}, \mathcal{G}]$. ■

Proposition 5.5. *Let (G, k) be a quadratic Lie group with a flat left invariant semi Riemannian metric. Then then for all $e \in \mathcal{Z}(\mathcal{G})$, $\nabla_e^2 = 0$. If the metric is either Riemannian or Lorentzian, then for all $e \in \mathcal{Z}(\mathcal{G})$, $\nabla_e = 0$. In this case, if $u \in \mathrm{Gl}(\mathcal{G})$ is the k symmetric isomorphism induced by the semi Riemannian metric, then $\mathcal{Z}(\mathcal{G})$ is invariant by u.*

Proof. Consider the Levi-Civita product induced by the semi Riemannian metric. It is immediate from the Koszul formula that $ee' = L_e e' = 0$ for $e, e' \in \mathcal{Z}(\mathcal{G})$. The following string of equalities

$$\begin{aligned} \langle L_e L_e x, y\rangle = -\langle L_e x, L_e y\rangle &= -\langle L_e x, L_y e\rangle = \langle L_y L_e x, e\rangle \\ &= \langle L_e L_y x, e\rangle = -\langle L_y x, L_e e\rangle = 0, \end{aligned}$$

implies the first assertion and it also implies that the subspace $\mathrm{Im}\,(L_e)$ is totally isotropic. If the metric is either Riemannian or Lorentzian $\dim \mathrm{Im}\,(L_e) \leq 1$. Suppose that there exists $e \in \mathcal{Z}(\mathcal{G})$ such that $L_e \neq 0$, and let x such that $L_e x \neq 0$. Then for every $y \in \mathcal{G}$, there is a $\lambda \in \mathbb{R}$ such that $L_e y = \lambda L_e x$. Notice that $\langle x, L_e x\rangle = 0$ because L_e is $\langle\,,\rangle$ skew symmetric. Then

$$0 = \langle L_e y, x\rangle + \langle y, L_e x\rangle = \lambda\langle L_e x, x\rangle + \langle y, L_e x\rangle = \langle y, L_e x\rangle,$$

thus, for all $y \in \mathcal{G}$, we have $\qquad \langle y, L_e x\rangle = 0.$

This equality implies that $L_e x = 0$, because the semi Riemannian metric is non degenerate, contrary to the assumption.

For the second part of the assertion, the Koszul formula

$$\langle xy, z\rangle = \frac{1}{2}\left(\langle [x,y],z\rangle - \langle [y,z],x\rangle + \langle [z,x],y\rangle\right)$$

for $x = e$, $e \in \mathcal{Z}(\mathcal{G})$, reduces to

$$\langle L_e y, z\rangle = -\frac{1}{2}\langle [y,z],e\rangle$$

for all $y, z \in \mathcal{G}$. Let u be the k symmetric isomorphism of the underlying vector space of $\mathcal{G}$ induced by the semi Riemannian metric $\langle\, ,\rangle$. Then, using that $L_e = 0$,

$$0 = \langle L_e(y), z\rangle = -\frac{1}{2}\langle [y,z],e\rangle = -\frac{1}{2}k([y,z],u(e)) = -\frac{1}{2}k([u(e),y],z).$$

The former equality implies that

$$0 = [u(e), y],$$

because k is non degenerate. Thus, for a flat left invariant semi Riemannian metric of index < 2, $u(e) \in \mathcal{Z}(\mathcal{G})$. ■

Corollary 5.6. *Under the hypothesis of proposition* 5.5 *if G is viewed as a group of affine transformations of $\mathcal{G}$, it has one-parameter subgroups of translations.*

As announced in Section 2, the existence of flat left invariant Riemannian metrics on quadratic Lie groups imposes severe restrictions on the group:

Proposition 5.7. *A quadratic Lie group with a flat left invariant Riemannian metric is Abelian.*

Proof. By Lemma 5.4, $\mathcal{Z}(\mathcal{G}) \neq 0$. Consider the map $L : \mathcal{G} \to gl(\mathcal{G})$ defined by $x \mapsto L_x$. The fact that $[x,y] = L_x y - L_y x$ implies that $\ker(L)$ is an Abelian ideal of $\mathcal{G}$. By Proposition 5.5, $\mathcal{Z}(\mathcal{G}) \subset \ker(L)$. Since the metric is flat, by Theorem 1.5 in [16]

$$\mathcal{G} = \ker(L) \oplus \mathcal{H}$$

where $\mathcal{H} := \ker(L)^{\perp}$ and $\mathcal{H}$ acts on $\mathrm{Ker}(L)$ by adjoints. Hence $[\mathcal{G},\mathcal{G}] \subset \ker(L)$. Since $[\mathcal{G},\mathcal{G}]$ is orthogonal to $\mathcal{Z}(\mathcal{G})$ relative to $\langle\, ,\,\rangle$ (if $e \in \mathcal{Z}(\mathcal{G})$, and $x, y \in \mathcal{G}$ then $\langle [x,y],e\rangle = k([x,y],u(e)) = k(x,[y,u(e)]) = 0$, because $\mathcal{Z}(\mathcal{G})$ is invariant by u. Since $\mathcal{G}$ is quadratic,

$$\dim\mathcal{Z}(\mathcal{G}) + \dim[\mathcal{G},\mathcal{G}] = \dim\mathcal{G}.$$

Hence $\mathcal{H} = (0)$. ■

Corollary 5.8. *A flat left invariant semi Riemannian metric on a non Abelian quadratic Lie group has index* ≥ 1.

Proof of Theorem 5.3. Let G be a quadratic non Abelian Lie group. By Theorem 5.1 every flat left invariant semi Riemannian metric on G is geodesically complete, and by Theorem 2.8, G is solvable. By Proposition 5.5, the geodesic through the unit of G with velocity e_0 is the 1-parameter subgroup of G with infinitesimal generator e_0, because $e_0 e_0 = 0$. Finally, Corollary 5.8 states that the index of the metric is ≥ 1. ■

The following corollaries are also consequences of Theorem 5.3:

Corollary 5.9. *Every left invariant semi Riemannian metric on a quadratic Lie group with non trivial Levi component (in particular when the group is reductive) is non flat.*

Definition 5.10. A quadratic Lie group (G, k) is called *undecomposable* if it has no non trivial normal Lie subgroups N such that the restriction of the bi-invariant metric k to N is non degenerate. At the algebra level, this means that every ideal of $\mathcal{G}$ is k degenerate.

Corollary 5.11. *Let (G, k) be an undecomposable quadratic Lie group that admits a flat left invariant semi Riemannian metric. Then the index of k is ≥ 1.*

Theorem 5.12. *Every undecomposable quadratic Lie group of dimension 4 has (flat) affine left invariant structures and no flat left invariant semi Riemannian metrics.*

Proof. There are two undecomposable quadratic connected Lie groups. The first one is the oscillator group of dimension 4 that was treated in [5]. The Lie algebra of the second one is obtained as follows.

Let $\mathcal{G}$ be the Abelian Lie algebra obtained by quadratic double extension from the Minkowski plane Span$\{e_1, e_2\}$ (viewed as an Abelian Lie algebra) by a central line $\mathbb{R}\, e_0$. This is a four-dimensional quadratic Lie algebra with an ad antisymmetric scalar product of index 2. It has a basis e_{-1}, e_0, e_1, e_2 with bracket

$$[e_1, e_2] = -e_0, \quad [e_{-1}, e_1] = e_2, \quad [e_{-1}, e_2] = e_1$$

and quadratic structure

$$k(e_{-1}, e_0) = k(e_1, e_1) = -k(e_2, e_2) = 1,$$

the non stated products are either given by antisymmetry/symmetry or are 0. Note that $\mathcal{Z}(\mathcal{G}) = \mathbb{R}e_0$.

We claim that $\mathcal{G}$ has no left invariant flat semi Riemannian metric. Suppose on the contrary that there exists a k symmetric isomorphism u of the vector space underlying $\mathcal{G}$ such that the metric

$$\langle x, y\rangle := k(x, u(y))$$

is flat. By Proposition 5.5 , we have that

$$\operatorname{im} L_{e_0} \subset \ker L_{e_0}.$$

Recall that, for a left invariant semi Riemannian metric, the Levi-Civita product is given by:

$$2xy = [x, y] + u^{-1}([x, u(y)] + [y, u(x)]).$$

Then
$$2L_{e_0} = -u^{-1} \circ \operatorname{ad}_{u(e_0)}$$

hence
$$\operatorname{im} \operatorname{ad}_{u(e_0)} = u(\operatorname{im} L_{e_0}) \subset u(\ker L_{e_0}) = u(\ker \operatorname{ad}_{u(e_0)}).$$

This equation implies that $\operatorname{rank}\mathrm{ad}_{u(e_0)} \leq 2$.
Let $u(e_0) = x_{-1}e_{-1} + x_0e_0 + x_1e_1 + x_2e_2$.
If $x_{-1} \neq 0$, then e_0 and $u(e_0)$ are linearly independent and

$$\operatorname{im}\mathrm{ad}_{u(e_0)} = u(\ker \mathrm{ad}_{u(e_0)}) = \mathrm{Span}\{u(e_0), u^2(e_0)\}. \tag{5}$$

It is easy to show that

$$\operatorname{im}\mathrm{ad}_{u(e_0)} = \mathrm{Span}\{u(e_1) = x_2e_0 + x_{-1}e_2, u(e_2) = -x_1e_0 + x_{-1}e_1\}.$$

By Equation (5), there exist A_1, B_1, A_2, B_2 such that

$$\begin{aligned} x_2e_0 + x_{-1}e_2 &= A_1u(e_0) + B_1u^2(e_0) \\ -x_1e_0 + x_{-1}e_1 &= A_2u(e_0) + B_2u^2(e_0) \end{aligned}$$

Then

$$\begin{aligned} 0 &= k(x_2e_0 + x_{-1}e_2, e_0) &&= k(A_1u(e_0) + B_1u^2(e_0), e_0) \\ 0 &= k(-x_1e_0 + x_{-1}e_1, e_0) &&= k(A_2u(e_0) + B_2u^2(e_0), e_0). \end{aligned}$$

and

$$\begin{aligned} 0 &= A_1k(u(e_0), e_0) + B_1k(u^2(e_0), e_0) \\ 0 &= A_2k(u(e_0), e_0) + B_2k(u^2(e_0), e_0). \end{aligned}$$

As a consequence, $A_1 = \lambda A_2$, $B_1 = \lambda B_2$, and

$$x_2e_0 + x_{-1}e_2 = A_1u(e_0) + B_1u^2(e_0) = \lambda(A_2u(e_0) + B_2u^2(e_0)) = \lambda(-x_1e_0 + x_{-1}e_1).$$

This equality contradicts the fact that e_0, e_1, e_2 are linearly independent.
If $x_{-1}=0$, and $x_1^2+x_2^2 \neq 0$, then $u(e_0) = x_0e_0+x_1e_1+x_2e_2$, and $\dim(\ker \mathrm{ad}_{u(e_0)}) = 2$.
Hence $\dim(\operatorname{im}\mathrm{ad}_{u(e_0)}) = 2$, and

$$\operatorname{im}(\mathrm{ad}_{u(e_0)}) = u(\ker \mathrm{ad}_{u(e_0)}).$$

Moreover $\ker(\mathrm{ad}_{u(e_0)}) = \mathrm{Span}\{e_0, u(e_0)\}$ and $e_0 \in \operatorname{im}\mathrm{ad}_{u(e_0)}$.
By (5), $e_0 = \lambda u(e_0)+\mu u^2(e_0)$ with $\mu \neq 0$, because we are supposing that $u(e_0) \notin \mathbb{R}e_0$.
Then

$$0 = k(e_0, e_0) = k(\lambda u(e_0) + \mu u^2(e_0), e_0) = \lambda k(u(e_0), e_0) + \mu k(u^2(e_0), e_0)$$

and the hypothesis implies that

$$0 = k(u^2(e_0), e_0) = k(u(e_0), u(e_0)) = x_1^2 - x_2^2.$$

Then $x_1 = \pm x_2$. Suppose first that $x_1 = x_2 = a$ ($a \neq 0$, because $u(e_0) \notin \mathbb{R}e_0$).
Then $u(e_0) = \alpha e_0 + a(e_1 + e_2)$. Consider a new basis of $\mathcal{G}$ consisting of

$$e_{-1},\ e_0,\ v_1 = e_1 + e_2,\ v_2 = e_1 - e_2.$$

Then

$$V := \ker \mathrm{ad}_{u(e_0)} = \mathrm{Span}\{e_0, v_1\}.$$

We have that $u(e_{-1}) = -x_2e_1 - x_1e_2 = -av_1$. Hence

$$\operatorname{im}\mathrm{ad}_{u(e_0)} = \mathrm{Span}\{e_0, u(e_1)\} = \mathrm{Span}\{e_0, v_1\}.$$

This implies that u leaves V invariant. In particular, $u(v_1) \in V$. Hence we have $u(v_1) = Ae_0 + Bv_1$, and

$$A = k(u(v_1), e_{-1}) = k(v_1, u(e_{-1})) = -ak(v_1, v_1) = 0.$$

Then $u(v_1)$ and $u(e_{-1})$ are linearly dependent, which is not possible.

If $x_1 = -x_2 = a$, then the same proof applies, with v_2 playing the role of v_1.

Finally, suppose that $\operatorname{rank} \operatorname{ad}_{u(e_0)} = 0$, that is $u(e_0) \in \mathbb{R}e_0$. Without loss of generality, suppose that $u(e_0) = e_0$.

Some calculations show that

$$k\left(\mathrm{R}_{e_{-1},e_1} e_{-1}, e_2\right) = \frac{b(-1+a+d)}{b^2+ad}$$

where $b = k(u(e_2), e_1)$, $a = k(u(e_1), e_1)$, and $d = -k(u(e_2), e_2)$. The semi Riemannian metric being flat, either $b = 0$ or $a + d = 1$. If $a + d = 1$ then

$$k\left(\mathrm{R}_{e_{-1},e_1} e_{-1}, e_1\right) = 1.$$

Since the former equality contradicts the hypothesis, $b = 0$. Using this, we get

$$0 = k\left(\mathrm{R}_{e_{-1},e_1} e_{-1}, e_1\right) = \frac{1}{4ad}(a^2 + 2a(-1+d) - (-1+d)(1+3d)),$$

and $$0 = k\left(\mathrm{R}_{e_{-1},e_2} e_{-1}, e_2\right) = \frac{1}{4ad}(3a^2 - 2a(1+d) - (-1+d)^2).$$

Adding the two equations, we get

$$0 = a^2 - a - d(-1+d) = (a-d)(a+d-1).$$

It is easy to check that neither $a = d$ nor $a + d = 1$ satisfy

$$3a^2 - 2a(1+d) - (-1+d)^2 = 0.$$

In order to conclude the proof of Theorem 5.12, it is easy to check that the linear map $\mathcal{G} \to \mathrm{gl}(\mathcal{G})$ given by

$$x = x_{-1}e_{-1} + x_0e_0 + x_1e_1 + x_2e_2 \mapsto L_x = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & -x_1 \\ 0 & 0 & 0 & 2x_{-1} \\ 0 & 0 & 2x_{-1} & 0 \end{pmatrix}$$

is a left symmetric product on $\mathcal{G}$ compatible with the Lie bracket. ∎

Remark 5.13. In fact, for this latter quadratic group there is a left invariant affine structure which is holomorphic ([8]).

As a consequence, we have that for undecomposable quadratic Lie groups of dimension 4, no exponential map of a left invariant semi Riemannian metric is a local isometry.

The following results give flat left invariant semi Riemannian metrics on nilpotent Lie groups. Let f be an endomorphism of the underlying vector space of a Lie algebra $\mathcal{G}$, such that

$$f([x,y]) - [f(x), f(y)] \in \mathcal{Z}(\mathcal{G})$$

for all $x, y \in \mathcal{G}$. Such an endomorphism is called a *q-homomorphism* of Lie algebras. An endomorphism d of the linear space $\mathcal{G}$ is called an f derivation if

$$d[x,y] = [dx, fy] + [fx, dy],$$

for all $x, y \in \mathcal{G}$. In particular a derivation is a Id derivation.

Proposition 5.14. *Let $(\mathcal{G}, k)$ a quadratic Lie algebra with an invertible f derivation d. Then the semi Riemannian metric defined by*

$$\langle x, y\rangle = k(dx, dy)$$

is flat and the Levi-Civita product is given by

$$xy = d^{-1}[fx, dy].$$

Proof. The Levi-Civita product is given by the Koszul formula:

$$2\langle xy, z\rangle = \langle [x,y], z\rangle - \langle [y,z], x\rangle + \langle [z,x], y\rangle$$

By the definition of the metric,

$$2k(d(xy), dz) = k(d[x,y], dz) - k(d[y,z], dx) + k(d[z,x], dy).$$

Using the fact that d is a f derivation,

$$\begin{aligned} 2k(d(xy), dz) &= k([dx, fy], dz) - k([dy, fz], dx) + k([dz, fx], dy) \\ &\quad + k([fx, dy], dz) - k([fy, dz], dx) + k([fz, dx], dy) \\ &= 2k([fx, dy], dz). \end{aligned}$$

Hence
$$d(xy) = [fx, dy]$$

because k is non degenerate. A simple calculation shows that

$$(xy)z = d^{-1}[fx, [fy, dz]].$$

Therefore,

$$\begin{array}{rcl} (xy)z - (yx)z &=& d^{-1}([fx, [fy, dz]] - [fy, [fx, dz]]) \\ &=& d^{-1}[[fx, fy], dz] \end{array}.$$

Finally,
$$[x,y]z = d^{-1}([f[x,y], dz]) = d^{-1}([[fx, fy], dz])$$

because d is a f derivation and f is a q-homomorphism of Lie algebras. ■

Theorem 5.15. *Every 3 step nilpotent Lie group has an invertible f derivation d that induces a flat left invariant connection on G.*

Proof. Every 3 step nilpotent can be decomposed as

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where $\mathcal{G}_0 = [\mathcal{G}, [\mathcal{G}, \mathcal{G}]] \subset \mathcal{Z}(\mathcal{G})$, $\mathcal{G}_1$ is a supplement of $\mathcal{G}_0$ in $[\mathcal{G}, \mathcal{G}]$ and $\mathcal{G}_2$ is a supplement of $[\mathcal{G}, \mathcal{G}]$ in $\mathcal{G}$. Define

$$f(x) = a_i x \quad \text{for} \quad x \in \mathcal{G}_i,$$

where $a_1 = 4/9$, $a_2 = 2/3$, and

$$d(x) = \alpha_i x \quad \text{for} \quad x \in \mathcal{G}_i,$$

with $\alpha_0 = \alpha_1$. The conditions on the parameters in order that d is an f derivation are:

$$\begin{aligned} \alpha_0\alpha_2 &\neq 0 \\ \alpha_0 &= (4/9)\alpha_2 + \alpha_0 a_0 \\ \alpha_2 &= 1/3. \end{aligned}$$

The f derivation d is invertible and the product

$$xy := d^{-1}([fx, dy])$$

is left symmetric hence defines a flat left invariant connection on $\mathcal{G}$. ■

Theorem 5.16. *Every quadratic 3 step nilpotent Lie group admits a flat left invariant semi Riemannian metric induced by an invertible f derivation on its Lie algebra.*

Another general situation with flat left invariant complete semi Riemannian metric is for quadratic Lie groups with a left invariant symplectic form ([14]).

Proposition 5.17. *Let $(\mathcal{G}, k)$ be a nilpotent, quadratic Lie algebra and u a k symmetric isomorphism of the vector space underlying $\mathcal{G}$. If u preserves the descending central sequence $\mathcal{G}$ (and hence the ascending central sequence of $\mathcal{G}$) then the metric k_u is complete. Moreover, the solutions of the Euler equation (1) are polynomial.*

The proposition follows from the following lemma.

Lemma 5.18. *If $(\mathcal{G}, k)$ is a nilpotent, quadratic Lie algebra of degree m and $v \in \mathrm{End}(\mathcal{G})$ preserves the descending central sequence of $\mathcal{G}$, then the mth derivative of the vector field given by $\dot{x} = [x, v(x)]$ is zero.*

Proof. Let $t \mapsto \alpha(t)$ be a curve in $\mathcal{G}$. Define $\beta(t) := [\alpha(t), v(\alpha(t))]$. Then

$$\forall\, i \in \mathbb{N} \setminus \{0\}, \quad \beta^{(i)}(t) = \sum_{j=0}^{i} C_j^i \, [\alpha^{(j)}(t), v(\alpha^{(i-j)}(t))]$$

and $\beta^{(i)} \in \mathrm{C}^{(i+1)}(\mathcal{G})$. Hence, if $x : t \mapsto x(t)$ is a solution of (1) and if we have $\beta(t) := [x(t), v(x(t))]$, then $\beta^{(i)}(t) = x^{(i+1)}(t)$, $\forall i \in \mathbb{N}$. It follows that $x^{(m)} \in \mathrm{C}^m(\mathcal{G}) = \{0\}$. ■

Proof of Proposition 5.17. Simply consider in Lemma 5.18 the vector field $\dot{x} = [x, u^{-1}(x)]$. ■

Remark 5.19. There are incomplete left invariant semi Riemannian metrics on nilpotent quadratic Lie groups.

The following is an example of this situation.

Example 5.20. Let $\mathcal{G} = \mathrm{Span}\{e_0, e_1, e_2, e_3, e_4\}$ with Lie bracket

$$[e_4, e_1] = e_2; \quad [e_4, e_2] = e_3; \quad [e_1, e_2] = e_0,$$

the non stated products are obtained either by antisymmetry or are zero. For $x \in \mathcal{G}$, let $x = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$. The Lie algebra $\mathcal{G}$ is 3 step nilpotent and for k given by

$$k(x, x) := 2(x_0\, x_4 - x_1\, x_3) + x_2^2,$$

$(\mathcal{G}, k)$ is quadratic. Let $u \in \mathrm{GL}(\mathcal{G})$ with the following matrix given in the basis $\mathcal{B} = \{e_0, e_1, e_2, e_3, e_4\}$,

$$\mathrm{M}_{\mathcal{B}}(u) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

It is easy to check that u is k symmetric. Equation (1) is (in the same coordinate system)

$$\begin{aligned} \dot{x}_0 &= -x_2 x_0 + x_1 x_2 \\ \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_4 x_0 + x_1 x_3 \\ \dot{x}_3 &= x_2 x_4 + x_2 x_3 \\ \dot{x}_4 &= 0. \end{aligned}$$

The curve

$$x_0(t) = \frac{-2}{(1+t)^2}, \quad x_1(t) = 0, \quad x_2(t) = \frac{2}{1+t}, \quad x_3(t) = c(1+t)^2 - 1, \quad x_4(t) = 1$$

is a non complete solution of Equation (1), see [4]. Hence the semi Riemannian metric defined by u is not flat.

Notice that the quadratic Lie algebra $(\mathcal{G}, k)$ given in the example above is a 3 step nilpotent quadratic algebra. The flat metric given by Theorem 5.16 is of signature $(2, 3)$. This algebra is undecomposable.

Proof of the undecomposability. We remark that $\mathcal{Z}(\mathcal{G})$ is totally isotropic and that any ideal of dimension 1 is central. Let $\mathcal{I}$ be an ideal of dimension 2. Since $\mathcal{I} \cap \mathcal{Z}(\mathcal{G}) \neq (0)$ we have $\mathcal{I} = \mathrm{Span}\{x, y\}$, where $x = x_0 e_0 + x_3 e_3$ and $y = y_0 e_0 + y_1 e_1 + y_2 e_2 + y_3 e_3 + y_4 e_4$. The following vectors are in $\mathcal{I}$:

$$\begin{aligned} & x_0 e_0 + x_3 e_3 \\ y_2 e_0 - y_4 e_2 &= [e_1, y] \\ -y_1 e_0 - y_4 e_3 &= [e_2, y] \\ y_1 e_2 + y_2 e_3 &= [e_4, y]. \end{aligned}$$

Since we are assuming that the ideal is of dimension 2, the matrix

$$\begin{pmatrix} x_0 & 0 & x_3 \\ y_2 & -y_4 & 0 \\ -y_1 & 0 & -y_4 \\ 0 & y_1 & y_2 \end{pmatrix}$$

has rank at most 2. This implies that

$$\begin{aligned} y_4(x_3y_1 - x_0y_4) &= 0 \\ y_2(x_3y_1 - x_0y_4) &= 0 \\ y_1(x_3y_1 - x_0y_4) &= 0 \end{aligned}$$

If $x_3y_1 - x_0y_4 \neq 0$, $y_1 = y_2 = y_4 = 0$ and $y \in \mathcal{Z}(\mathcal{G})$. If $x_3y_1 - x_0y_4 = 0$, then

$$\begin{aligned} k(x,x) &= 0 \\ k(x,y) &= k(x_0e_0 + x_3e_3, y_0e_0 + y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4) \\ &= x_0y_4 - x_3y_1 = 0. \end{aligned}$$

and the ideal $\mathcal{I}$ is degenerate. ■

6. Quadratic 2-step nilpotent Lie groups

Let $(\mathcal{G}, k)$ be a quadratic 2-step nilpotent Lie algebra with 0 corank, that is such that $[\mathcal{G},\mathcal{G}] = \mathcal{Z}(\mathcal{G})$. Under this hypothesis the Lie algebra $(\mathcal{G}, k)$ is isomorphic to a quadratic Lie algebra $(V \oplus V^*, \theta, k)$ where $V^* = [\mathcal{G},\mathcal{G}]$, $\theta \in \Lambda^3(V)$, $\operatorname{rank}\theta = \dim V$, the bracket is given by

$$[(x,\alpha),(y,\beta)] = (0, \theta(x,y,\cdot)),$$

and
$$k((x,\alpha),(y,\beta)) = \alpha(y) + \beta(x).$$

Let $\phi \in \mathrm{Gl}(V)$ and define $u : V \oplus V^* \to V \oplus V^*$ by

$$u(x,\alpha) := (\phi(x), {}^t\phi(\alpha)) = (\phi(x), \alpha \circ \phi).$$

It is easily verified that $u \in \mathrm{GL}(V \oplus V^*)$. Denote by $\langle\,,\rangle_\phi$ the bilinear form induced on $V \oplus V^*$ via k by u (hence by δ). Then $\langle\,,\rangle_\phi$ is non degenerate and u is $\langle\,,\rangle_\phi$ symmetric. We have

Theorem 6.1. *Let (G,k) be a quadratic Lie group with Lie algebra $\mathcal{G} := (V \oplus V^*, \theta, k)$, as above. Then for every $\phi \in \mathrm{Gl}(V)$ the metric $\langle\,,\rangle_\phi$ defines a flat and geodesically complete semi Riemannian metric on G, and $(G, \langle\,,\rangle_\phi)$, $G, \langle\,,\rangle_{\phi'})$ are isometric if and only if $\phi' = \psi^{-1}\phi\psi$ for some $\psi \in \mathrm{Gl}(V)$. Moreover, if $\dim V \geq 9$ there are infinitely many non isometric such metrics.*

Proof. The Levi-Civita product associated to $\langle\,,\rangle_\phi$ is given by

$$2ab := 2L_ab = [a,b] + u^{-1}([a,u(b)] + [b,u(a)]).$$

In order to prove that $L_{[a,b]} = [L_a, L_b]$ notice that, since $V^* = [\mathcal{G},\mathcal{G}] = \mathcal{Z}(\mathcal{G})$ and u^{-1} invariant, then

$$a(bc) = b(ac) = [a,b]c = 0, \quad \text{for all } a,b,c \in \mathcal{G}.$$

Hence $\langle\, ,\rangle_\phi$ is flat, and geodesically complete because $\mathcal{G}$ is unimodular.

A straightforward calculation shows that $(\mathcal{G}, \theta, k, \phi)$ and $(\mathcal{G}, \theta, k, \phi')$ are isometric if and only if there exists $\psi \in \mathrm{Gl}(V)$ such that $\phi' = \psi^{-1}\phi\psi$.

Finally the Vinberg-Elashvili Classification Theorem (see [20]) implies that there are infinitely many non degenerate and non conjugate 3-linear forms on V when $\dim V \geq 9$. Consequently, there are infinitely many non isometric flat left invariant semi Riemannian metrics on G. ∎

Theorem 6.1 can be used to construct flat compact semi Riemannian nilmanifolds as is shown by the following example.

Example 6.2. Consider the Lie algebra A_d with basis $\{e_1, e_2, e_3, f_1, f_2\}$ and Lie bracket

$$[e_1, e_2] = f_2, \quad [e_3, e_4] = f_2, \quad [e_1, e_3] = f_1, \quad [e_2, e_4] = df_2$$

where d is a square free integer. It is clear that A_d is a 2-step nilpotent Lie algebra of 0 corank. Moreover if $d \neq d'$ the $\mathbb{Q}$ algebras A_d and $A_{d'}$ are not isomorphic (see [19]). Hence the simply connected Lie group G of Lie algebra ${}^*tA_d := A_d^* \rtimes_{coadj} A_d$ has lattices. Consequently the manifold $M = \Gamma\backslash G$, where Γ is a lattice, has many flat semi Riemannian metrics.

For more details on 2-step nilpotent quadratic Lie algebras, see [17].

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