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On the Generalized Poisson Transform on the Quaternionic Hyperbolic Space

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Abstract. Let $B(\mathbb{H}^n) = Sp(n,1)/Sp(n) \times Sp(1)$ be the quaternionic hyperbolic space. We consider a generalized Poisson transform $\mathcal{P}_{\lambda,l}$ associated with a character of a class of irreducible representations of $Sp(n) \times Sp(1)$. In this paper, we show that if f is a hyperfunction on the boundary of $B(\mathbb{H}^n)$, then f belongs to the space $L^p(\partial B(\mathbb{H}^n))$ if and only if either its generalized Poisson transform $\mathcal{P}_{\lambda,l} f$ satisfies a Hardy-type condition, or the modified admissible maximal function of $\mathcal{P}_{\lambda,l} f$ belongs to $L^p(\partial B(\mathbb{H}^n))$. In addition, we study the admissible convergence of the generalized Poisson transform $\mathcal{P}_{\lambda,l} f$ for $f \in L^1(\partial B(\mathbb{H}^n))$.

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1. Introduction and main results

Let G be a connected real semisimple noncompact Lie group with finite center and let K be a maximal compact subgroup of G. Let $G = KAN$ be an Iwasawa decomposition of G and let M be the centralizer of A in K. Then $X = G/K$ is a Riemannian symmetric space of noncompact type, with Furstenberg boundary K/M. A central result in harmonic analysis asserts that for λ generic in $\mathfrak{a}^*_{\mathbb{C}}$ (the complex dual of $\mathfrak{a} := Lie(A)$) the Poisson transform $\mathcal{P}_\lambda$ is an isomorphism from $\mathcal{A}'(K/M)$ the space of hyperfunctions on K/M onto $\mathcal{E}_\lambda(X)$ the space of joint-eigenfunctions on G/K (see [11]). After that, the authors characterized the image of L^p-functions on K/M under the Poisson transform (see e.g. [3, 19]). The generalization of these results to the homogenous vector bundle case is more recent. In [1], Ben Saïd characterized the range of the Poisson transform $\mathcal{P}_\lambda$ of L^p-functions ($1 < p \leq +\infty$) on the boundary K/M as a Hardy-type space on a homogeneous line bundle over a Hermitian symmetric space, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $\mathfrak{Re}(i\lambda)$ lies in the open positive Weyl chamber and the isotropy subgroup of λ and $\mathfrak{Im}(\lambda)$ in the Weyl group coincide. Later on, Boussejra et al. gave a characterization of the Poisson transform of L^2-sections for some $\lambda \in \mathfrak{a}^* \backslash \{0\}$, on a homogeneous line bundle over noncompact complex Grassmann manifolds (see [4]).

In [2], the authors characterized the L^p-range ($1 < p < +\infty$) of the Poisson transform for the vector bundle of p-forms. In this paper, we restrict ourselves to the case where X is the quaternionic hyperbolic space $B(\mathbb{H}^n) = G/K$, where G is

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the symplectic group $Sp(n,1)$ with maximal compact subgroup $K = \mathrm{Sp}(n)\times\mathrm{Sp}(1)$. Let $E_{\tau_l} = G\times_{\tau_l} K$ denote the homogeneous vector bundle over G/K associated to the unitary irreducible representation $\tau_l = 1\otimes\delta_l$ of $Sp(n)\times Sp(1)$, where δ_l is a unitary irreducible representation of $Sp(1)$ realized on a $(2l{+}1)$-dimensional Hilbert space V_l, where l is either an integer or a half-integer (i.e. $2l\in\mathbb{Z}^+$).

In [5], we developed some aspects of the L^2-harmonic analysis on the homogeneous vector bundles E_{τ_l}. As an application, we obtained a characterization of the L^2-range of the vector-valued Poisson transform for $\lambda\in\mathbb{R}\backslash\{0\}$. Additionally, in [15], we presented a characterization of the L^2-range of the vector-valued Poisson transform, along with an L^p-Hardy-type estimate for $\lambda\in\mathbb{C}$ such that $\Re(i\lambda)>0$.

All the above studies leave out the case $\lambda=0$. The aim of this paper is to study the L^p-range $(p\geq 2)$ of the generalized Poisson transform $\mathcal{P}_{\lambda,l}$ associated with the character χ_l of τ_l, for $\lambda=0$ or $\Re(i\lambda)>0$.

To introduce our main result, let us fix some notations (refer to section 2 for more details). We denote by $\mathcal{A}'(\partial B(\mathbb{H}^n))$ the space of hyperfunctions on the boundary $\partial B(\mathbb{H}^n)$ of $B(\mathbb{H}^n)$. For $f\in\mathcal{A}'(\partial B(\mathbb{H}^n))$ and $\lambda\in\mathbb{C}$, we define the generalized $\mathbb{C}$-valued Poisson transform $\mathcal{P}_{\lambda,l}f$ on $B(\mathbb{H}^n)$ by

$$\mathcal{P}_{\lambda,l}f(x) = \int_{\partial B(\mathbb{H}^n)} \mathcal{P}_{\lambda,l}(x,\omega)f(\omega)d\omega,$$

where $\mathcal{P}_{\lambda,l}(\cdot,\cdot)$ is the generalized $\mathbb{C}$-valued Poisson kernel defined on $B(\mathbb{H}^n)\times\partial B(\mathbb{H}^n)$ by

$$\mathcal{P}_{\lambda,l}(x,\omega) = \left(\frac{1-|x|^2}{|1-\langle x,\omega\rangle|^2}\right)^{\frac{i\lambda+2n+1}{2}} \chi_l\left(\frac{1-\langle x,\omega\rangle}{|1-\langle x,\omega\rangle|}\right).$$

When $\chi_l\equiv 1$, i.e. $l=0$, $\mathcal{P}_{\lambda,0}$ coincides with the classical Poisson transform over the unit ball $B(\mathbb{H}^n)$.

Recently, it is proved in [8] that for $\Re(i\lambda)>0$ and $f\in\mathcal{A}'(\partial B(\mathbb{H}^n))$, f belongs to the space $L^p(\partial B(\mathbb{H}^n))$ (for $p\geq 2$) if and only if its generalized Poisson transform $\mathcal{P}_{\lambda,l}f$ satisfies a Hardy-type growth condition.

In this paper, we extend the main result in [[8], Theorem 2.1] to the case $\lambda=0$ (see Theorem 4.1), and we investigate the behaviour of the generalized Poisson transform of L^p-functions on $\partial B(\mathbb{H}^n)$, in terms of the modified admissible maximal function.

We will deal with the following objects defined for a continuous function F on $B(\mathbb{H}^n)$:

(a) The modified admissible maximal function $M_\alpha^\lambda[F]$, defined on $\partial B(\mathbb{H}^n)$ by

$$M_\alpha^\lambda[F](\omega) = \begin{cases} \sup\limits_{z\in D_\alpha(\omega)} (1-|z|^2)^{\frac{\Re(i\lambda)-(2n+1)}{2}}|F(z)| & \text{for } \lambda\in\mathbb{C}\backslash\{0\}, \\ \sup\limits_{z\in D_\alpha(\omega)} \frac{(1-|z|^2)^{-\frac{(2n+1)}{2}}}{1+d(0,z)}|F(z)| & \text{for } \lambda=0, \end{cases}$$

where $D_\alpha(\omega)$ is the admissible region given for $\alpha>1$ by

$$D_\alpha(\omega) = \{z\in B(\mathbb{H}^n);\quad |1-\langle z,\omega\rangle| < \frac{\alpha}{2}(1-|z|^2)\}. \tag{1}$$

Note that as $\alpha\to+\infty$, the region $D_\alpha(\omega)$ fills $B(\mathbb{H}^n)$ for every $\omega\in\partial B(\mathbb{H}^n)$, and for $\alpha\leq 1$ the above region defines the empty set.

(b) The Hardy-type norm

$$\|F\|_{\lambda,p}=\begin{cases}\sup\limits_{t\geq 0}(1-\mathrm{th}^2t)^{\frac{Re(i\lambda)-(2n+1)}{2}}\Big(\int\limits_{\partial B(\mathbb{H}^n)}|F(\mathrm{th}t\,\theta)|^p d\theta\Big)^{\frac{1}{p}} & \text{for } \lambda\in\mathbb{C}\backslash\{0\},\\ \sup\limits_{t\geq 0}\frac{(1-\mathrm{th}^2t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th}t\,e)}\Big(\int\limits_{\partial B(\mathbb{H}^n)}|F(\mathrm{th}t\,\theta)|^p d\theta\Big)^{\frac{1}{p}} & \text{for } \lambda=0,\end{cases}$$

where $d(0,z)=\frac{1}{2}\log(\frac{1+|z|}{1-|z|})$ denotes the distance from the origin 0 to z in $B(\mathbb{H}^n)$, $e=(1,0,...,0)$ is the unit vector of $\mathbb{H}^n$, and "th" denotes the hyperbolic tangent function "tanh". Let π^+ be the set defined by

$$\pi^+=\{z\in\mathbb{C}:\quad \Re(iz)>0\}\cup\{0\}.$$

Our first main result can be stated as follows.

Theorem 1.1. *Let $\alpha>2$ and let $\lambda\in\pi^+$ be such that $i\lambda+2n\pm(2l+1)\notin 2\mathbb{Z}^-$. For $f\in\mathcal{A}'(B(\mathbb{H}^n))$, the following assertions are equivalent for $p\geq 2$:*

(i) $f\in L^p(\partial B(\mathbb{H}^n))$,

(ii) $M_\alpha^\lambda[\mathcal{P}_{\lambda,l}f]\in L^p(\partial B(\mathbb{H}^n))$,

(iii) $\|\mathcal{P}_{\lambda,l}f\|_{\lambda,p}<+\infty$.

Remark 1.2. The condition $i\lambda+2n\pm(2l+1)\notin 2\mathbb{Z}^-$ in Theorem 1.1 guarantees the injectiveness of the generalized Poisson transform $\mathcal{P}_{\lambda,l}$. ■

In [12], Korányi introduced the admissible region (1) to ensure the existence of the admissible convergence of Poisson-Szegö integrals. In [13], he extended the concept of admissible convergence to the Poisson transform on rank one Riemannian symmetric spaces of noncompact type. This has led to a wide range of studies exploring extensions and generalizations of his result [12] (see e.g. [14, 18, 20, 21]).

Our second main result addresses the admissible convergence of the generalized Poisson transform $\mathcal{P}_{\lambda,l}f$ for $f\in L^1(\partial B(\mathbb{H}^n))$.

For a $\mathbb{C}$-valued function F on $B(\mathbb{H}^n)$ and a function f on $\partial B(\mathbb{H}^n)$ we say that F converges to f admissibly almost everywhere if, for every $\alpha>1$,

$$\lim_{\substack{z\to\omega\\ z\in D_\alpha(\omega)}} F(z)=f(\omega)\quad \text{for almost all } \omega\in\partial B(\mathbb{H}^n).$$

The second main result we prove in this paper can be stated as follows.

Theorem 1.3. *Let $f\in L^1(\partial B(\mathbb{H}^n))$ and let $\lambda\in\mathbb{C}$ be such that $i\lambda+2n\pm(2l+1)\notin 2\mathbb{Z}^-$.*

(i) *If $Re(i\lambda)>0$, then $(1-|\cdot|^2)^{\frac{i\lambda-2n-1}{2}}\mathcal{P}_{\lambda,l}f(\cdot)$ converges admissibly almost everywhere to $\mathbf{c}_l(\lambda)f$.*

(ii) *If $\lambda=0$, then $\dfrac{(1-|\cdot|^2)^{\frac{i\lambda-2n-1}{2}}}{1+d(0,\cdot)}\mathcal{P}_{0,l}f(\cdot)$ converges admissibly almost everywhere to $\mathbf{c}_0 f$, where $\mathbf{c}_l(\lambda)$ is the analogue of the Harish-Chandra $\mathbf{c}$-function given by*

$$\mathbf{c}_l(\lambda)=\frac{\pi(2l+1)\Gamma(i\lambda)\Gamma(2n-2)}{4\Gamma(\frac{i\lambda+2n+1}{2}+l)\Gamma(\frac{i\lambda+2n+1}{2}-l-1)},\quad \textit{and}\quad \mathbf{c}_0=\frac{\pi(2l+1)\Gamma(2n-2)}{4\Gamma(\frac{2n+1}{2}+l)\Gamma(\frac{2n+1}{2}-l-1)}.$$

Now, we describe the plan of the paper. In section 2, we recall some basic known results of harmonic analysis on the quaternionc hyperbolic space and define the generalized Poisson transform $\mathcal{P}_{\lambda,l}$. In section 3, we give a uniform asymptotic behavior of the generalized spherical function $\Phi_{0,l}^{(p,q)}(|z|)$. In section 4, we show that a hyperfunction $f \in \mathcal{A}'(\partial B(\mathbb{H}^n))$ belongs to the space $L^p(\partial B(\mathbb{H}^n))$ for $p \geq 2$ if and only if its Poisson transform $\mathcal{P}_{0,l}f$ satisfies a Hardy-type growth condition. Section 5 will be devoted to prove our main results.

2. Notation and Preliminaries

2.1. Generalities

We use the classical notation ${}_2F_1(a,b;c;z)$ to denote the Gaussian hypergeometric function

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, $$

with $c \neq$ 0,-1,-2,..., where

$$ (a)_0 = 1, \qquad (a)_k = (a)(a+1)...(a+k-1) \quad \text{for} \quad k \geq 1. $$

We refer to [[7], Chapter II] for the properties of these functions. Here, we list those that will be needed in the sequel. For $\Re(c-a-b) > 0$, we have

$$ \lim_{z\to 1} {}_2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2} $$

Moreover,
$$ \lim_{z\to 1^-} \frac{{}_2F_1(a,b;a+b;z)}{\log(1-z)} = \frac{-\Gamma(a+b)}{\Gamma(a)\Gamma(b)}, \tag{3} $$

and
$$ a\,{}_2F_1(a+1,b;c;z) - b\,{}_2F_1(a,b+1;c;z) = (b-a)\,{}_2F_1(a,b;c;z). \tag{4} $$

We denote $f(x) \underset{a}{\sim} g$ if $f(x) - g(x) \underset{x\to a}{=} o(g(x))$. Additionally, we use the notation $X \lesssim Y$ if there exists some constant $C > 0$ such that $X \leq CY$.

2.2. Bounded realization of $Sp(n,1)/Sp(n) \times Sp(1)$.

Let $n \geq 2$ be an integer. On $\mathbb{H}^{n+1}$ considered as a right vector space over $\mathbb{H}$, we consider the following quadratic form:

$$ J(x_1,...,x_{n+1}) = \sum_{j=1}^{n} |x_j|^2 - |x_{n+1}|^2, $$

with $|x|^2 = x\overline{x}$, where $x \to \overline{x}$ is the standard involution of $\mathbb{H}$.

Let $G = Sp(n,1)$ be the group of all $\mathbb{H}$-linear transformations g on $\mathbb{H}^{n+1}$ keeping the quadratic form J invariant, then the group G acts on the unit ball $B(\mathbb{H}^n)$ by fractional transformations

$$ g.z = (Az+B)(Cz+d)^{-1}, $$

for $g = \begin{pmatrix} A & B \\ C & d \end{pmatrix} \in Sp(n,1)$, $A \in \mathbb{H}^{n\times n}$, $B \in \mathbb{H}^{n\times 1}$, $C \in \mathbb{H}^{1\times n}$ and $d \in \mathbb{H}$.

This action of G on $B(\mathbb{H}^n)$ is transitive and as a homogeneous space we have the identification $B(\mathbb{H}^n) = G/K$, where K is the stabilizer of $0 \in B(\mathbb{H}^n)$. More precisely,

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in Sp(n,1) \right\} \simeq Sp(n) \times Sp(1),$$

where $Sp(n)$ denotes the group of all $n \times n$ matrices over $\mathbb{H}$ keeping the inner product $\langle u, v \rangle = \sum_{j=1}^{n} \overline{u}_j\, v_j$ on $\mathbb{H}^n$ invariant, and $Sp(1)$ is identified with the group of quaternions of norm equal to one.

The action of G mentioned above extends naturally to $\overline{B(\mathbb{H}^n)}$ and under this action, K is transitive on the boundary $\partial B(\mathbb{H}^n) = \{\omega \in \mathbb{H}^n; \quad |\omega| = 1\}$ of $B(\mathbb{H}^n)$. Let e denote the unit vector of $\mathbb{H}^n$ given by $e = (1, 0, ..., 0)$, and M be the stabilizer in K of e. Then, the group M can be identified to $Sp(n-1) \times Sp(1)$ and as a homogenous space we have $\partial B(\mathbb{H}^n) = K/M$.

2.3. The generalized Poisson transform $\mathcal{P}_{\lambda,l}$

Let $E_{\tau_l} = G \times_{\tau_l} K$ be the homogeneous vector bundle associated to the unitary irreducible representation (τ_l, V_l) of K. As usual we shall identify the space of smooth sections of E_τ with

$$C^\infty Ind_K^G(\tau) = \{F \in C^\infty(G, V_l); \quad F(gk) = \tau_l(k)^{-1}F(g), \quad \text{for all } g \in G,\ k \in K\}.$$

Let $C^\infty Ind_P^G(\sigma)$ be the space of smooth sections of the homogeneous vector bundle corresponding to the representation $\sigma = \tau_{l|M} \otimes a^{i\lambda-\rho} \otimes 1$ of the minimal parabolic subgroup $P = MAN$. The Poisson transform is the map $\mathcal{P}_\lambda^{\tau_l}$ defined from $C^\infty Ind_P^G(\sigma)$ to $C^\infty Ind_K^G(\tau_l)$, by

$$\mathcal{P}_\lambda^{\tau_l} f(g) = \int_K \tau_l(k) f(gk) dk,$$

where $\lambda \in \mathfrak{a}_{\mathbb{C}}^* \simeq \mathbb{C}$ and $\rho = 2n+1$.

Let $C^\infty Ind_M^K(\tau_{l|M})$ be the space of V_l-valued C^∞-functions on K such that

$$f(km) = \tau_l(m)^{-1} f(k), \quad \text{for all } k \in K,\ m \in M.$$

Then the restriction map from G to K gives an isomorphism from $C^\infty Ind_P^G(\sigma)$ onto $C^\infty Ind_M^K(\tau_{l|M})$. Lifting elements in $C^\infty Ind_M^K(\tau_{l|M})$ to elements in $C^\infty Ind_P^G(\sigma)$, we easily show that the Poisson transform of $f \in C^\infty Ind_M^K(\tau_{l|M})$ is given by

$$\mathcal{P}_\lambda^{\tau_l} f(g) = \int_K e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_l(\kappa(g^{-1}k)) f(k) dk,$$

where $H : G \to \mathfrak{a} = Lie(A)$ and $\kappa : G \to K$ are the projections with respect to the Iwasawa decomposition of $G = KAN$.

As $C^\infty Ind_M^K(\tau_{|M}) \simeq C^\infty(K/M, V_l)$ via the trivialisation map $\Lambda f(kM) = \tau_l(k) f(k)$, it follows that the Poisson transform $\mathcal{P}_\lambda^{\tau_l} f$ of $f \in C^\infty(K/M, V_l)$, is given by

$$\mathcal{P}_\lambda^{\tau_l} f(g) = \int_K e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_l(\kappa(g^{-1}k)k^{-1}) f(k) dk,$$

Next, we shall write the Poisson transform $\mathcal{P}_{\lambda,\tau_l} f$ on G/K. To this end, we use the Cartan decomposition $G = (\exp \mathfrak{p})K$ to define the map B_{τ_l} from $C^\infty Ind_K^G(\tau_l)$ into

$C^\infty(G/K, V_l)$ by $B_{\tau_l}\phi(gK) = \tau(U(g))\phi(g)$, where $U(g)$ is the K-component of g with respect to the Cartan decomposition $G = (\exp \mathfrak{p})K$.

Using the isomorphism B_{τ_l}, the Poisson transform may be rewritten as a map (denoted also by $\mathcal{P}^{\tau_l}_\lambda$) from $C^\infty(K/M, V_l)$ to the space $C^\infty(G/K, V_l)$. Namely, we have

$$\mathcal{P}^{\tau_l}_\lambda f(gK) = \int_{K/M} \mathcal{K}_{\lambda,l}(gK, kM) f(kM) dkM,$$

where $\mathcal{K}_{\lambda,l}(gK, kM)$ is the vector-valued Poisson kernel defined on $G/K \times K/M$ by

$$\mathcal{K}_{\lambda,l}(gK, kM) = e^{-(i\lambda+\rho)H(g^{-1}k)} \tau_l(U(g))\tau_l(\kappa(g^{-1}k)k^{-1}).$$

Using the identities $U(gk) = U(g)k$ and $\kappa(kg) = k\kappa(g)$, $\forall g \in G \quad and \quad \forall k \in K$, we can easily see that the Poisson kernel is well defined on $G/K \times K/M$.

We define the generalized Poisson kernel $\mathcal{P}_{\lambda,l}(gK, kM)$ on $G/K \times K/M$ by taking the traces, namely

$$\mathcal{P}_{\lambda,l}(gK, kM) = Tr\big(\mathcal{K}_{\lambda,l}(gK, kM)\big).$$

A straightforward computation shows that the generalized Poisson kernel may be realized on $B(\mathbb{H}^n) \times \partial B(\mathbb{H}^n)$ as follows

$$\mathcal{P}_{\lambda,l}(x, \omega) = \left(\frac{1 - |x|^2}{|1 - \langle x, \omega\rangle|^2}\right)^{\frac{i\lambda+2n+1}{2}} \chi_l\left(\frac{1 - \langle x, \omega\rangle}{|1 - \langle x, \omega\rangle|}\right),$$

where $\chi_l = Tr[\tau_l]$.

For $f \in \mathcal{A}'(\partial B(\mathbb{H}^n))$, we define the generalized Poisson transform of f by

$$\mathcal{P}_{\lambda,l} f(x) = \int_{\partial B(\mathbb{H}^n)} \mathcal{P}_{\lambda,l}(x, \omega) f(\omega) d\omega.$$

Now, we describe in brief the Peter-Weyl decomposition of $L^2(\partial B(\mathbb{H}^n))$, the space of all square integrable $\mathbb{C}$-valued functions on $\partial B(\mathbb{H}^n)$ with respect to the normalized superficial measure σ of $\partial B(\mathbb{H}^n)$. For simplification, $d\sigma(\omega)$ will be noted by $d\omega$. The group K acts on $L^2(\partial B(\mathbb{H}^n))$ by composition $f \to f \circ k, k \in K$. The Peter-Weyl decomposition of $L^2(\partial B(\mathbb{H}^n))$ is given by

$$L^2(\partial B(\mathbb{H}^n)) = \sum_{(p,q)\in\Omega} V_{p,q},$$

where $V_{p,q}$ is the finite linear span of $\phi_{p,q} \circ k, k \in K$, the $\phi_{p,q}$ are the zonal spherical functions, and the parameter set Ω is given by $\Omega = \{(p, q) \in (\mathbb{Z}^+)^2 : \ q - p \in 2\mathbb{Z}^+\}$ (for instance see [9, 10]).

We finish this section by recalling the action of the Poisson transform $\mathcal{P}_{\lambda,l}$ on $L^2(\partial B(\mathbb{H}^n))$.

Proposition 2.1. [8] *If $f \in V_{p,q}$, then for $(t,\theta) \in \mathbb{R}^+ \times \partial B(\mathbb{H}^n)$ we have*

$$\mathcal{P}_{\lambda,l} f(\operatorname{th}t\,\theta) = \Phi_{\lambda,l}^{(p,q)}(\operatorname{th}t) f(\theta), \tag{5}$$

where the generalized spherical function $\Phi_{\lambda,l}^{(p,q)}$ is given by

$$\Phi_{\lambda,l}^{(p,q)}(\operatorname{th}t) = \frac{\pi}{4(p+1)} \frac{\Gamma(2n-2)}{\Gamma(q+2n)} (1-\operatorname{th}^2 t)^s \operatorname{th}^p(t) \times$$

$$\Big\{ (s+l)_{\frac{p+q}{2}+1} (s-l-1)_{\frac{q-p}{2}} \, {}_2F_1(s+1+l+\frac{p+q}{2}, s-l-1+\frac{q-p}{2}; q+2n; \operatorname{th}^2 t)$$

$$-(s-l-1)_{\frac{p+q}{2}+1} (s+l)_{\frac{q-p}{2}} \, {}_2F_1(s+l+\frac{q-p}{2}, s-l+\frac{q+p}{2}; q+2n; \operatorname{th}^2 t) \Big\},$$

with $s = \frac{i\lambda+2n+1}{2}$.

3. Asymptotic behavior of the generalized spherical function $\Phi_{0,l}^{(p,q)}(|z|)$

In this section, we will establish an asymptotic behavior of some linear combination of two Gaussian hypergeometric functions. As a consequence, we will get a uniform asymptotic behavior of the generalized spherical function $\Phi_{0,l}^{(p,q)}(|z|)$ on p,q, which is the key of the proof of Theorem 4.4.

Theorem 3.1. *Let $\alpha, \beta \in \mathbb{N}$ and $a,b \in \mathbb{C}\backslash\mathbb{Z}^-$ with*

$$\begin{cases} (a-b) \in \mathbb{N}. \\ a+\alpha\ ,\ a+\beta \notin \mathbb{Z}^- \text{ and } b+\alpha\ ,\ b+\beta \notin \mathbb{Z}^-. \\ a+b+\alpha+\beta-1 \notin \mathbb{Z}^-. \end{cases}$$

We have

$$(a)_\alpha (b)_\beta {}_2F_1(a+\alpha, b+\beta; a+b+\alpha+\beta-1; z) - (a)_\beta (b)_\alpha {}_2F_1(a+\beta, b+\alpha; a+b+\alpha+\beta-1; z)$$

$$\underset{z\to 1^-}{\sim} \frac{-\Gamma(a+b+\alpha+\beta-1)}{\Gamma(a)\Gamma(b)} (a-b)(\alpha-\beta)\log(1-z).$$

Proof. Denote by $F_{\alpha,\beta,\gamma}(z)$ the hypergeometric functions ${}_2F_1(a+\alpha, b+\beta; c+\gamma; z)$, then we have the following identities (see [16], page 407):

$$F_{\alpha,\beta,\gamma}(z) = F_{\alpha-1,\beta,\gamma}(z) + \frac{b+\beta}{c+\gamma} z F_{\alpha,\beta+1,\gamma+1}(z). \tag{6}$$

Put $c = a+b+\alpha+\beta-1$. Then, by (6),

$$(a)_\alpha (b)_\beta F_{\alpha,\beta,0}(z) = (a)_\alpha (b)_\beta F_{\alpha-1,\beta,0}(z) + \frac{(a)_\alpha (b)_{\beta+1}}{c} z F_{\alpha,\beta+1,1}(z).$$

Using again (6), we get

$$(a)_\alpha (b)_\beta F_{\alpha,\beta,0}(z) = (a)_\alpha (b)_\beta F_{\alpha-1,\beta,0}(z) + \frac{(a)_\alpha (b)_{\beta+1}}{c} z F_{\alpha-1,\beta+1,1}(z)$$
$$+ \frac{z^2}{c(c+1)} (a)_\alpha (b)_{\beta+2} F_{\alpha,\beta+2,2}(z).$$

By repeating this process $(a-b)$ times, we obtain

$$(a)_\alpha(b)_\beta F_{\alpha,\beta,0}(z)=\sum_{i=0}^{a-b-1}\frac{z^i}{(c)_i}(a)_\alpha(b)_{\beta+i}F_{\alpha-1,\beta+i,i}(z)+\frac{z^{a-b}}{(c)_{a-b}}(a)_\alpha(b)_{\beta+a-b}F_{\alpha,\beta+a-b,a-b}(z).$$

Similarly, we find that

$$(a)_\beta(b)_\alpha F_{\beta,\alpha,0}(z)=\sum_{i=0}^{a-b-1}\frac{z^i}{(c)_i}(a)_\beta(b)_{\alpha+i}F_{\beta-1,\alpha+i,i}(z)+\frac{z^{a-b}}{(c)_{a-b}}(a)_\beta(b)_{\alpha+a-b}F_{\beta,\alpha+a-b,a-b}(z).$$

The last terms on the right-hand side of the two expressions above are equal, leading us to

$$(a)_\alpha(b)_\beta\,{}_2F_1(a+\alpha,b+\beta,a+b+\alpha+\beta-1;z)-(a)_\beta(b)_\alpha\,{}_2F_1(a+\beta,b+\alpha;a+b+\alpha+\beta-1;z)$$

$$=\sum_{i=0}^{a-b-1}\left(\frac{z^i}{(c)_i}(a)_\alpha(b)_{\beta+i}F_{\alpha-1,\beta+i,i}(z)-\frac{z^i}{(c)_i}(a)_\beta(b)_{\alpha+i}F_{\beta-1,\alpha+i,i}(z)\right).$$

By (3), for each $i\in[0,a-b-1]$, we find

$$\frac{z^i}{(c)_i}(a)_\alpha(b)_{\beta+i}F_{\alpha-1,\beta+i,i}(z)\underset{1^-}{\sim}-\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(a+\alpha-1)\log(1-z),\tag{7}$$

and similarly

$$\frac{z^i}{(c)_i}(a)_\beta(b)_{\alpha+i}F_{\beta-1,\alpha+i,i}(z)\underset{1^-}{\sim}-\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}(a+\beta-1)\log(1-z).\tag{8}$$

By combining (7) and (8), we get the desired asymptotics. ■

As an immediate consequence of the above theorem, we get a uniform asymptotic behavior of the generalized spherical function $\Phi_{0,l}^{(p,q)}(|z|)$.

Lemma 3.2. *Let $(p,q)\in\mathbb{N}^2$ such that $q-p\in2\mathbb{N}$. Then*

$$\Phi_{0,l}^{(p,q)}(|z|)\underset{z\to1^-}{\sim}\frac{\pi}{4}\frac{\Gamma(2n-2)(2l+1)}{\Gamma(\frac{2n+1}{2}+l)\Gamma(\frac{2n-1}{2}-l)}(1-|z|^2)^{\frac{2n+1}{2}}(1+d(0,z)),\tag{9}$$

uniformly in p,q.

Proof. We recall the expression of the generalized spherical function

$$\Phi_{0,l}^{(p,q)}(\mathrm{th}t)=\frac{\pi}{4(p+1)}\frac{\Gamma(2n-2)}{\Gamma(q+2n)}(1-\mathrm{th}^2t)^s\mathrm{th}^pt\times$$

$$\Big\{(s+l)_{\frac{p+q}{2}+1}(s-l-1)_{\frac{q-p}{2}}\,{}_2F_1(s+1+l+\frac{p+q}{2},s-l-1+\frac{q-p}{2};q+2n;\mathrm{th}^2t)$$

$$-(s-l-1)_{\frac{p+q}{2}+1}(s+l)_{\frac{q-p}{2}}\,{}_2F_1(s+l+\frac{q-p}{2},s-l+\frac{q+p}{2};q+2n;\mathrm{th}^2t)\Big\},$$

where $s=\frac{2n+1}{2}$.

According to Theorem 3.1 with $a=\frac{2n+1}{2}+l$, $b=\frac{2n-1}{2}-l$, $\alpha=\frac{p+q}{2}+1$, and $\beta=\frac{q-p}{2}$, we find that

$$\begin{aligned}\Phi^{(p,q)}_{0,l}(|z|) &\underset{1^-}{\sim} \frac{\pi}{2}\frac{\Gamma(2n-2)(2l+1)}{\Gamma(\frac{2n+1}{2}+l)\Gamma(\frac{2n-1}{2}-l)}(1-|z|^2)^{\frac{2n+1}{2}}\log(1-|z|^2)\\ &\underset{1^-}{\sim} \frac{\pi}{2}\frac{\Gamma(2n-2)(2l+1)}{\Gamma(\frac{2n+1}{2}+l)\Gamma(\frac{2n-1}{2}-l)}(1-|z|^2)^{\frac{2n+1}{2}}(1+d(0,z)),\end{aligned}$$

uniformly in p,q. ∎

Let $\mathbb{B}(0,t)$ be the geodesic ball of radius t centred at 0 and let $d\mu$ be the $Sp(n,1)$-invariant metric on $B(\mathbb{H}^n)$ (explicitly, $d\mu(z)=(1-|z|^2)^{-2n-2}dm(z)$, where $dm(z)$ is the Lebesgue measure). From (9), we easily get a uniform asymptotic behavior of the integral of the generalized spherical function $\Phi^{(p,q)}_{0,l}$.

Corollary 3.3. *Let $(p,q)\in\mathbb{N}^2$ such that $q-p\in 2\mathbb{N}$. Then*

$$\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}\Phi^{(p,q)}_{0,l}(|z|)^2 d\mu(z)=\frac{\pi^2}{48}\left(\frac{\Gamma(2n-2)(2l+1)}{\Gamma(\frac{2n+1}{2}+l)\Gamma(\frac{2n-1}{2}-l)}\right)^2, \tag{10}$$

uniformly in p,q.

4. Characterisation of the L^p-range by the generalized Poisson transform $\mathcal{P}_{0,l}$

In this section, we will establish the equivalence between assertions (i) and (iii) in the case $\lambda=0$ of the main result (Theorem 1.1). The central result established in this section is as follows.

Theorem 4.1. *Let $p\geq 2$ and $2n\pm(2l+1)\notin 2\mathbb{Z}^-$. Then, for $f\in\mathcal{A}'(\partial B(\mathbb{H}^n))$ we have*

$$\begin{aligned}&f\in L^p(\partial B(\mathbb{H}^n))\\ &\Leftrightarrow \|\mathcal{P}_{0,l}f\|_{0,p}=\sup_{t\geq 0}\frac{(1-\mathrm{th}^2 t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th}t\, e)}\Big(\int\limits_{\partial B(\mathbb{H}^n)}|\mathcal{P}_{0,l}f(\mathrm{th}t\,\theta)|^p d\theta\Big)^{\frac{1}{p}}<+\infty.\end{aligned} \tag{11}$$

Moreover, for every $f\in L^p(\partial B(\mathbb{H}^n))$ the following estimates hold.

$$\|f\|_{L^p(\partial B(\mathbb{H}^n))}\lesssim\|\mathcal{P}_{0,l}f\|_{0,p}\lesssim\|f\|_{L^p(\partial B(\mathbb{H}^n))}.$$

This extends the result of Ghanimi [[8], Theorem 2.1] to the case $\lambda=0$.

The proof of the necessary condition follows from the following result.

Proposition 4.2. *Let $f\in L^p(\partial B(\mathbb{H}^n))$ with $p>1$. Then*

$$\sup_{t\geq 0}\frac{(1-\mathrm{th}^2 t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th}t e)}\Big(\int\limits_{\partial B(\mathbb{H}^n)}|\mathcal{P}_{0,l}f(\mathrm{th}t\,\theta)|^p d\theta\Big)^{\frac{1}{p}}\lesssim\|f\|_{L^p(\partial B(\mathbb{H}^n))}. \tag{12}$$

Proof. Let $f\in L^p(\partial B(\mathbb{H}^n))$. Since K acts transitively on $\partial B(\mathbb{H}^n)$, for any $\theta\in\partial B(\mathbb{H}^n)$, we may write $\theta=h\cdot e$ with $h\in K$ and $e=(1,0,...,0)\in\partial B(\mathbb{H}^n)$.

We take a Haar measure dk on K so that

$$\int_{\partial B(\mathbb{H}^n)} g(\theta)d\theta = \int_K g(k\cdot e)dk, \quad \text{for all } g\in L^1(\partial B(\mathbb{H}^n)).$$

Next, we define the function $P^t_{0,l}$ on K by $\mathcal{P}^t_{0,l}(h)=\mathcal{P}_{0,l}(h\cdot(\mathrm{th}t\ e),e)$.
Then, the Poisson transform $\mathcal{P}_{0,l}f$ can be written as a convolution over the compact group K, namely

$$\mathcal{P}_{0,l}f(\mathrm{th}t\theta)=\mathcal{P}_{0,l}f(\mathrm{th}(t)\ (h\cdot e))=\int_K \mathcal{P}_{0,l}(h\cdot(\mathrm{th}t\ \ e),k\cdot e)f(k\cdot e)dk = f*\mathcal{P}^t_{0,l}(h).$$

By applying the Hausdorff-Young inequality, we obtain $\|f*\mathcal{P}^t_{0,l}\|_p\le \|f\|_p\|\mathcal{P}^t_{0,l}\|_1$.
Next, we estimate

$$\left\|\mathcal{P}^t_{0,l}\right\|_{L^1(K)}=\int_K |\mathcal{P}_{0,l}(h\cdot(\mathrm{th}\, te),e)|\,dh=\int_{\partial B(\mathbb{H}^n)}|\mathcal{P}_{0,l}(\mathrm{th}\, t\theta,e)|\,d\theta.$$

Using the Poisson kernel estimate

$$|\mathcal{P}_{0,l}(\mathrm{th}\, t\theta,e)|\lesssim\left(\frac{1-\mathrm{th}^2\, t}{|1-\langle \mathrm{th}\, t\theta,e\rangle|^2}\right)^{\frac{2n+1}{2}},$$

we obtain

$$\int_{\partial B(\mathbb{H}^n)}\left(\frac{1-\mathrm{th}^2\, t}{|1-\langle \mathrm{th}\, t\theta,e\rangle|^2}\right)^{\frac{2n+1}{2}}d\theta\lesssim\left(1-\mathrm{th}^2\, t\right)^{\frac{2n+1}{2}}{}_2F_1\left(\frac{2n+1}{2},\frac{2n+1}{2}-1;2n;\mathrm{th}^2\, t\right).$$

Now, by the known property of the hypergeometric function (3), we find that

$${}_2F_1(\frac{2n+1}{2},\frac{2n+1}{2}-1;2n;\mathrm{th}^2 t)\lesssim(1+d(0,\mathrm{th}t\, e)),\qquad \text{for all}\quad t>0.$$

Thus, for all $t\ge 0$, it follows that

$$\frac{(1-\mathrm{th}^2 t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th}t\, e)}\left(\int_{\partial B(\mathbb{H}^n)}|\mathcal{P}_{0,l}f(\mathrm{th}t\,\theta)|^p d\theta\right)^{\frac{1}{p}}\lesssim\|f\|_p.$$

This completes the proof. ■

By combining Proposition 4.2 and (5), we deduce a uniform pointwise estimate for the generalized spherical function $\Phi^{(p,q)}_{0,l}(|z|)$.

Corollary 4.3. *There exists a positive constant C, depending only on n, such that*

$$\sup_{(p,q)\in\Omega}|\Phi^{(p,q)}_{0,l}(|z|)|\le C(1-|z|^2)^{\frac{(2n+1)}{2}}(1+d(0,z)). \tag{13}$$

The main difficulty of Theorem 4.1 lies in establishing the sufficient condition. In order to prove it, we first address the case $p=2$, and we establish an L^2-type inversion formula for the generalized Poisson transform $\mathcal{P}_{0,l}$.

Theorem 4.4. *Let* $2n \pm (2l+1) \notin 2\mathbb{Z}^-$.

(1) *Let* $f \in \mathcal{A}'(\partial B(\mathbb{H}^n))$. *The following equivalence holds:*

$$f \in L^2(\partial B(\mathbb{H}^n))$$
$$\Leftrightarrow \|\mathcal{P}_{0,l} f\|_{0,2} = \sup_{t\geq 0} \frac{(1-\mathrm{th}^2 t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th} t\, e)} \Big(\int\limits_{\partial B(\mathbb{H}^n)} |\mathcal{P}_{0,l} f(\mathrm{th} t\,\theta)|^2 d\theta \Big)^{\frac{1}{2}} < +\infty. \tag{14}$$

Moreover, the L^2*-boundary value is given by the following formula:*

$$f(\omega) = \lim_{t\to+\infty} \gamma(n,l)^{-1} \frac{1}{t^3} \int\limits_{\mathbb{B}(0,t)} \mathcal{P}_{0,l}(z,\omega) \mathcal{P}_{0,l} f(z) d\mu(z) \;\; in \; L^2(\partial B(\mathbb{H}^n)), \tag{15}$$

where $\gamma(n,l) = \dfrac{\pi^2}{48} \Big(\dfrac{\Gamma(2n-2)(2l+1)}{\Gamma(\frac{2n+1}{2}+l)\Gamma(\frac{2n-1}{2}-l)} \Big)^2.$

(2) *Let* $f \in L^2(\partial B(\mathbb{H}^n))$. *Then we have the asymptotic formula*

$$\lim_{t\to+\infty} \frac{1}{t^3} \int\limits_{\mathbb{B}(0,t)} |\mathcal{P}_{0,l} f(z)|^2 d\mu(z) = \gamma(n,l) \|f\|^2_{L^2(\partial B(\mathbb{H}^n))}.$$

Proof. (1) The necessary condition follows from Proposition 4.2.
Let $F = \mathcal{P}_{0,l} f$ with $f \in \mathcal{A}'(\partial(\mathbb{H}^n))$, such that

$$\|F\|_{0,2} = \sup_{t\geq 0} \frac{(1-\mathrm{th}^2 t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th} t\, e)} \Big(\int\limits_{\partial B(\mathbb{H}^n)} |\mathcal{P}_{0,l} f(\mathrm{th} t\,\theta)|^2 d\theta \Big)^{\frac{1}{2}} < +\infty.$$

Expanding f into K-type series $f(\theta) = \sum_{(p,q)\in\Omega} f_{p,q}(\theta)$ and using (5), we see that F may be written as

$$F(\mathrm{th} t\theta) = \sum_{(p,q)\in\Omega} \Phi^{(p,q)}_{0,l}(\mathrm{th} t) f_{p,q}(\theta). \tag{16}$$

Furthermore $\displaystyle\int\limits_{\partial B(\mathbb{H}^n)} |F(\mathrm{th} t\theta)|^2 d\theta = \sum_{(p,q)\in\Omega} \Phi^{(p,q)}_{0,l}(\mathrm{th} t)^2 \|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))}.$

Given that $\|F\|_{0,2} < +\infty$, for each $t > 0$ we obtain

$$\frac{(1-\mathrm{th}^2 t)^{-(2n+1)}}{(1+d(0,\mathrm{th} t\, e))^2} \sum_{(p,q)\in\Omega} \Phi^{(p,q)}_{0,l}(\mathrm{th} t)^2 \|f_{p,q}\|^2_{L^2(\partial(B\mathbb{H}^n))} \leq \|F\|^2_{0,2} < +\infty.$$

Let Λ be a finite subset in Ω. By using the asymptotic behavior of the generalized spherical function (9), we deduce that

$$3\gamma(n,l) \sum_{(p,q)\in\Lambda} \|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))} \leq \|F\|^2_{0,2} < +\infty.$$

Since Λ is arbitrary, we get

$$\|f\|^2_{L^2(\partial B(\mathbb{H}^n))} = \sum_{(p,q)\in\Omega} \|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))} \leq \frac{1}{3\gamma(n,l)} \|F\|^2_{0,2} < +\infty.$$

Thus $f \in L^2(\partial B(\mathbb{H}^n))$.

Next for $t>0$, we define the following $\mathbb{C}$-valued function on $\partial B(\mathbb{H}^n)$

$$g_t(\omega)=\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)} F(z)P_{0,l}(z,\omega)d\mu(z).$$

By expanding f into K-type series $f=\sum_{(p,q)\in\Omega} f_{p,q}$ and replacing F by its series expansion from (16), also using the property $\mathcal{P}_{0,l}(\mathrm{th}r\theta,\omega)=\mathcal{P}_{0,l}(\mathrm{th}r\omega,\theta)$, the function g_t can be written as

$$g_t(\omega)=\sum_{(p,q)\in\Omega}\Big(\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}\Phi_{0,l}^{(p,q)}(|z|)^2d\mu(z)\Big)f_{p,q}(\omega).$$

The $L^2(\partial B(\mathbb{H}^n))$-norm of g_t is given by

$$\|g_t\|^2_{L^2(\partial B(\mathbb{H}^n))}=\sum_{(p,q)\in\Omega}\left|\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}\Phi_{0,l}^{(p,q)}(|z|)^2d\mu(z)\right|^2\|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))}.$$

Now, the uniform pointwise estimate of $\Phi_{0,l}^{(p,q)}(|z|)$ given in (13) shows that the family $(g_t)_{t>0}$ becomes a bounded subset of $L^2(\partial B(\mathbb{H}^n))$. Furthermore, we have

$$\|\gamma(n,l)^{-1}g_t-f\|^2_{L^2(\partial B(\mathbb{H}^n))}=\sum_{(p,q)\in\Omega}\left|\frac{\gamma(n,l)^{-1}}{t^3}\int\limits_{\mathbb{B}(0,t)}\Phi_{0,l}^{(p,q)}(|z|)^2d\mu(z)-1\right|^2\|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))}.$$

Using the uniform asymptotic behavior (10), we obtain

$$\lim_{t\to+\infty}\|\gamma(n,l)^{-1}g_t-f\|_{L^2(\partial B(\mathbb{H}^n))}=0.$$

(2) Let $f\in L^2(\partial B(\mathbb{H}^n))$. By (5), we have

$$\mathcal{P}_{0,l}f(\mathrm{th}t\theta)=\sum_{(p,q)\in\Omega}\Phi_{0,l}^{(p,q)}(\mathrm{th}t)f_{p,q}(\theta).$$

It follows that

$$\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}|\mathcal{P}_{0,l}f(z)|^2d\mu(z)=\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}\sum_{(p,q)\in\Omega}\left(\Phi_{0,l}^{(p,q)}(|z|)^2\|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))}\right)d\mu(z).$$

By applying Beppo-Levi's theorem, we obtain

$$\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}|\mathcal{P}_{0,l}f(z)|^2d\mu(z)=\lim_{t\to+\infty}\sum_{(p,q)\in\Omega}\left(\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}\Phi_{0,l}^{(p,q)}(|z|)^2\|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))}\right)d\mu(z).$$

By the uniform pointwise boundedness of $\Phi_{0,l}^{(p,q)}(|z|)$ given in (13) we can interchange the limit and the summation

$$\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}|\mathcal{P}_{0,l}f(z)|^2d\mu(z)=\sum_{(p,q)\in\Omega}\left(\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}\Phi_{0,l}^{(p,q)}(|z|)^2d\mu(z)\right)\|f_{p,q}\|^2_{L^2(\partial B(\mathbb{H}^n))}.$$

By using (10), we ultimately obtain

$$\lim_{t\to+\infty}\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}|\mathcal{P}_{0,l}f(z)|^2d\mu(z)=\gamma(n,l)\|f\|^2_{L^2(\partial B(\mathbb{H}^n))}.\qquad\blacksquare$$

Now, we are able to prove Theorem 4.1.

Proof. The necessary condition follows from Proposition 4.2.
For the sufficient condition, let $F=\mathcal{P}_{0,l}f$ with $f\in\mathcal{A}'(\partial B(\mathbb{H}^n))$, such that

$$\|F\|_{0,p}=\sup_{t\geq 0}\frac{(1-\mathrm{th}^2t)^{-\frac{(2n+1)}{2}}}{1+d(0,\mathrm{th}t\,e)}\Bigg(\int\limits_{\partial B(\mathbb{H}^n)}|\mathcal{P}_{0,l}f(\mathrm{th}t\,\theta)|^pd\theta\Bigg)^{\frac{1}{p}}<+\infty. \tag{17}$$

It is clear that $\|\mathcal{P}_{0,l}f\|_{0,2}\lesssim\|\mathcal{P}_{0,l}f\|_{0,p}<+\infty$.
Thus, by (14) we have $f\in L^2(\partial B(\mathbb{H}^n))$. Moreover, by the L^2-inversion formula (15), f is expressed as

$$f(\omega)=\lim_{t\to+\infty}\gamma(n,l)^{-1}g_t(\omega)\ \text{ in } L^2(\partial B(\mathbb{H}^n)),$$

where $g_t(\omega)=\dfrac{1}{t^3}\displaystyle\int_{\mathbb{B}(0,t)}F(z)\mathcal{P}_{0,l}(z,\omega)d\mu(z)$.
Next we claim that $f\in L^p(\partial B(\mathbb{H}^n))$. Indeed, for $\phi\in L^q(\partial B(\mathbb{H}^n))$ and with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\int\limits_{\partial B(\mathbb{H}^n)}f(\omega)\phi(\omega)d\omega=\lim_{t\to+\infty}\gamma(n,l)^{-1}\int\limits_{\partial B(\mathbb{H}^n)}g_t(\omega)\phi(\omega)d\omega.$$

Furthermore

$$\int\limits_{\partial B(\mathbb{H}^n)}g_t(\omega)\phi(\omega)d\omega=\int\limits_{\partial B(\mathbb{H}^n)}\Bigg(\frac{1}{t^3}\int\limits_{\mathbb{B}(0,t)}F(z)\mathcal{P}_{0,l}(z,\omega)d\mu(z)\Bigg)\phi(\omega)d\omega$$

$$=\int\limits_{\partial B(\mathbb{H}^n)}\frac{1}{t^3}\int_0^t\int\limits_{\partial B(\mathbb{H}^n)}F(\mathrm{th}r\theta)\mathcal{P}_{0,l}(\mathrm{th}r\theta,\omega)d\theta(1-\mathrm{th}^2r)^{-(2n+1)}\mathrm{th}^{4n-1}rdr\phi(\omega)d\omega.$$

By applying Fubini's theorem, we obtain

$$\int\limits_{\partial B(\mathbb{H}^n)}g_t(\omega)\phi(\omega)d\omega$$

$$=\frac{1}{t^3}\int_0^t\left(\int_{\partial B(\mathbb{H}^n)}\mathcal{P}_{0,l}\phi(\mathrm{th}r\theta)F(\mathrm{th}r\theta)d\theta\right)(1-\mathrm{th}^2r)^{-(2n+1)}(\mathrm{th}r)^{4n-1}dr. \tag{18}$$

Using Hölder's inequality, we find

$$\Bigg|\int\limits_{\partial B(\mathbb{H}^n)}\mathcal{P}_{0,l}\phi(\mathrm{th}r\theta)F(\mathrm{th}r\theta)d\theta\Bigg|\leq\Bigg(\int\limits_{\partial B(\mathbb{H}^n)}|\mathcal{P}_{0,l}\phi(\mathrm{th}r\theta)|^qd\theta\Bigg)^{1/q}\Bigg(\int\limits_{\partial B(\mathbb{H}^n)}|F(\mathrm{th}r\theta)|^pd\theta\Bigg)^{1/p}.$$

On the one hand, by (12) we have for every $\phi \in L^q(\partial B(\mathbb{H}^n))$

$$\left(\int\limits_{\partial B(\mathbb{H}^n)} |\mathcal{P}_{0,l}\phi(\mathrm{th}r\theta)|^q d\theta \right)^{1/q} \lesssim (1+d(0,\mathrm{th}t\, e))(1-\mathrm{th}^2 t)^{\frac{2n+1}{2}} \|\phi\|_{L^q(\partial B(\mathbb{H}^n))}. \tag{19}$$

On the other hand, by (17) we have

$$\left(\int_{\partial B(\mathbb{H}^n)} |F(\mathrm{th}r\theta)|^p d\theta \right)^{1/p} \lesssim (1+d(0,\mathrm{th}t\, e))(1-\mathrm{th}^2 t)^{\frac{2n+1}{2}} \|F\|_{0,p}. \tag{20}$$

From the estimates in (19), (20) and (18), we deduce that for every $\phi \in L^q(\partial B(\mathbb{H}^n))$

$$\left| \int\limits_{\partial B(\mathbb{H}^n)} f(\omega)\phi(\omega) d\omega \right| \lesssim \|F\|_{0,p} \|\phi\|_{L^q(\partial B(\mathbb{H}^n))}.$$

Next, taking the supremum over all continuous ϕ with $\|\phi\|_{L^q(\partial B(\mathbb{H}^n))} = 1$ in the above inequality, we get

$$\|f\|_{L^p(\partial B(\mathbb{H}^n))} \lesssim \|F\|_{0,p} < +\infty.$$

Thus, the proof is finished. ■

5. Proof of the main result

First, we briefly study the L^p-boundedness of the modified admissible maximal function of the generalized Poisson transform. Specifically, we establish the following estimate.

Proposition 5.1. *Let $\lambda \in \pi^+$ and $p > 1$. If $f \in L^p(\partial B(\mathbb{H}^n))$, then the following estimate holds*

$$\|M_\alpha^\lambda[\mathcal{P}_{\lambda,l} f]\|_{L^p(\partial B(\mathbb{H}^n)} \lesssim \|f\|_{L^p(\partial B(\mathbb{H}^n)}. \tag{21}$$

To proceed, we introduce the non-isotropic metric ρ on $\partial B\left(\mathbb{H}^n\right)$. For any two points $z_1, z_2 \in \overline{B(\mathbb{H}^n)}$, we define ρ as:

$$\rho(z_1, z_2) = |1 - \langle z_1, z_2 \rangle|^{1/2}.$$

For $\delta > 0$ and $\omega \in \partial B(\mathbb{H}^n)$, let $B(\omega, \delta)$ denotes the non-isotropic metric ball at ω with radius δ, i.e.;

$$B(\omega, \delta) = \{\theta \in \partial B(\mathbb{H}^n) : \quad \rho(\theta, \omega) < \delta\}.$$

It is clear that $B(\omega, \delta) = \partial B(\mathbb{H}^n)$ when $\delta \geq \sqrt{2}$.

The properties of the non-isotropic metric ρ are collected in the following proposition.

Proposition 5.2. (i) *The triangle inequality $\rho(a,c) \leq \rho(a,b) + \rho(b,c)$ holds for every a, b and c in $\overline{B(\mathbb{H}^n)}$.*

(ii) *ρ is a metric on $\partial B(\mathbb{H}^n)$.*

(iii) *For every $\omega \in \partial B(\mathbb{H}^n)$ and $0 < \delta \le \sqrt{2}$, the superficial measure of $B(\omega,\delta)$ satisfies the estimate*

$$\sigma(B(\omega,\delta)) \asymp \delta^{2(2n+1)}. \tag{22}$$

To prove this proposition, we will need the following standard calculus lemma (see [6]).

Lemma 5.3. *Let f be a $\mathbb{C}$-valued integrable function on $\partial B(\mathbb{H}^n)$ with the property $f(\theta_1,..,\theta_n) = g(\theta_1)$. Then*

$$\int_{\partial B(\mathbb{H}^n)} f(\omega)d\omega = c \int_{\{z\in\mathbb{H};\ |z|<1\}} g(z)(1-|z|^2)^{2n-3}dm(z),$$

for some positive constant c.

Now we come back to the proof of Proposition 5.2.

Proof. The properties (i) and (ii) being proved in [6], it remains to show iii).

Since both the metric $\rho(\theta,\omega)$ and the superficial measure σ are invariant under the action of $Sp(n)$, we reduce the problem to the calculation of $\sigma(B(e,\delta))$, where $e = (1,0,\dots,0)$. Hence, we write

$$\sigma(B(\omega,\delta)) = \sigma(B(e,\delta)) = \int_{\{\theta\in\partial B(\mathbb{H}^n);|1-\theta_1|<\delta^2\}} d\theta,$$

Using Lemma 5.3, we express the measure in integral form

$$\sigma(B(\omega,\delta)) = c \int_{\{z\in\mathbb{H};\ |z|<1\ \ and\ \ |1-z|<\delta^2\}} (1-|z|^2)^{2n-3}dm(z).$$

Next, we apply a coordinate transformation. Set $1-z = r\big(\cos(\theta)+\sin(\theta)y\big)$, where $r>0$, $\theta\in[0,\pi]$ and $y\in\mathbb{H}$ with $\Re(y)=0$, $|y|=1$. The measure $dm(z)$ becomes

$$dm(z) = r^3\sin^2(\theta)drd\theta.$$

Substituting this into the previous integral, we obtain

$$\sigma(B(\omega,\delta)) = c \int_{\{(r,\theta)\in]0,\delta^2[\times]0,\pi[;\ \ |1-re^{i\theta}|<1\}} (2\cos(\theta)-r)^{2n-3}r^{2n}\sin^2(\theta)drd\theta.$$

To simplify, we perform the change of variable $t = \frac{r}{\delta^2}$ we get $r = t\delta^2$ and $dr = \delta^2 dt$. This yields

$$\frac{\sigma(B(\omega,\delta))}{\delta^{2(2n+1)}} = c \int_{\{(t,\theta)\in]0,1[\times]0,\pi[;\ \ t\delta^2<2cos(\theta)\}} (2\cos(\theta)-t\delta^2)^{2n-3}\sin^2(\theta)dtd\theta.$$

As δ decreases from $\sqrt{2}$ to 0, both the integrand and the domain of integration increase. By applying Lebesgue theorem, we deduce that the normalized measure $\frac{\sigma(B(\omega,\delta))}{\delta^{2(2n+1)}}$ increases from A_2 to a finite limit A_1 as δ decreases from $\sqrt{2}$ to 0, where

$$A_2 = c\,\frac{2^{2n-4}\Gamma(3/2)\Gamma(\frac{2n-1}{2})}{(n-1)\Gamma(n+1)} \quad\text{and}\quad A_1 = c\,2^{2n-4}\frac{\Gamma(3/2)\Gamma(n-1)}{\Gamma(\frac{2n+1}{2})}. \qquad \blacksquare$$

From (22), it is easy to see that σ is doubling. As a consequence, it is well known that the Hardy-Littlewood maximal function satisfies the following L^p-boundedness estimate (see e.g. [22]).

$$\|Mf\|_{L^p(\partial B(\mathbb{H}^n))} \lesssim \|f\|_{L^p(\partial B(\mathbb{H}^n))}, \tag{23}$$

where Mf is the Hardy-Littlewood maximal function of a $\mathbb{C}$-valued function f from $L^1(\partial B(\mathbb{H}^n))$, defined on $\partial B(\mathbb{H}^n)$ by

$$Mf(\omega) = \sup_{\delta>0} \frac{1}{\sigma(B(\omega,\delta))} \int\limits_{B(\omega,\delta)} |f(\theta)| d\theta.$$

Next, we establish estimates for the generalized Poisson kernel $\mathcal{P}_{\lambda,l}(z,\omega)$.

Lemma 5.4. *Let $\lambda \in \pi^+$. The generalized Poisson kernel satisfies the following estimates:*

(1) *For every $z \in B(\mathbb{H}^n)$, we have*

$$|\mathcal{P}_{\lambda,l}(z,\omega)| \leq 2^{\frac{Re(i\lambda)+2n+1}{2}} (1-|z|)^{-\frac{Re(i\lambda)+2n+1)}{2}}. \tag{24}$$

(2) *For every $z \in D_\alpha(\theta)$, we have*

$$|\mathcal{P}_{\lambda,l}(z,\omega)| < \frac{\big(8\alpha(1-|z|)\big)^{\frac{Re(i\lambda)+2n+1}{2}}}{|1-<\theta,\omega>|^{Re(i\lambda)+2n+1}}. \tag{25}$$

Proof. (1) For $z \in B(\mathbb{H}^n)$, we use the standard estimates $1-|z|^2 \leq 2(1-|z|)$ and $1-|z| \leq |1-<z,\omega>|$, along with the fact that τ_l is unitary. So (24) follows.

(2) Let $z \in D_\alpha(\theta)$ and $\omega \in \partial B(\mathbb{H}^n)$. We have

$$|1-<z,\theta>| < \frac{\alpha}{2}(1-|z|^2) \leq \alpha(1-|z|) \leq \alpha|1-<z,\omega>|.$$

This gives us $\rho(z,\theta) \leq \sqrt{\alpha}\rho(z,\theta)$ and, since $\alpha > 1$, we obtain

$$\rho(\theta,\omega) \leq \rho(\theta,z) + \rho(z,\omega) < \sqrt{\alpha}\rho(z,\omega) + \rho(z,\omega) \leq 2\sqrt{\alpha}\rho(z,\omega).$$

Thus, we have
$$|1-<\theta,\omega>| < 4\alpha|1-<z,\omega>|. \tag{26}$$

Combining this with $1-|z|^2 \leq 2(1-|z|)$, we derive (25). ∎

Now, we proceed with the proof of Proposition 5.1.

Proof. Using the Poisson kernel estimates (24) and (25), we may follow the same method as in the proof of Theorem 5.4.5 in [17] and find that the modified admissible maximal function of the generalized Poisson transform $M_\alpha^\lambda[\mathcal{P}_{\lambda,l}f]$ is dominated by the Hardy-Littlewood maximal function Mf. Precisely, for every $\omega \in \partial B(\mathbb{H}^n)$, we have

$$M_\alpha^\lambda[\mathcal{P}_{\lambda,l}f](\omega) \lesssim Mf(\omega). \tag{27}$$

As a consequence of the L^p-bound of the maximal function Mf (see (23)), we get

$$\|M_\alpha^\lambda[\mathcal{P}_{\lambda,l}f]\|_{L^p(\partial B(\mathbb{H}^n))} \lesssim \|Mf\|_{L^p(\partial B(\mathbb{H}^n))} \lesssim \|f\|_{L^p(\partial B(\mathbb{H}^n))},$$

and the proof is finished. ∎

Now, we are able to give the proof of our first main result (Theorem 1.1).

Proof. (i)$\Rightarrow$(ii) This follows directly from Proposition 5.1.

(ii)$\Rightarrow$(iii) For $\Re(i\lambda)>0$, we have

$$\|\mathcal{P}_{\lambda,l}f\|_{\lambda,p}\leq\left(\int\limits_{\partial B(\mathbb{H}^n)}\left(\sup_{t\geq 0}(1-\operatorname{th}^2 t)^{\frac{Re(i\lambda)-(2n+1)}{2}}|\mathcal{P}_{\lambda,l}f(\operatorname{th}t\,\theta)|\right)^p d\theta\right)^{\frac{1}{p}}.$$

Since $\alpha>2$, for every $t\geq 0$ and $\omega\in\partial B(\mathbb{H}^n))$, we have $\operatorname{th}t\omega\in D_\alpha(\omega)$. This implies that

$$\sup_{t\geq 0}(1-\operatorname{th}^2 t)^{\frac{Re(i\lambda)-(2n+1)}{2}}|\mathcal{P}_{\lambda,l}f(\operatorname{th}t\,\theta)|\leq M_\alpha^\lambda[\mathcal{P}_{\lambda,l}f](\theta).$$

Consequently, we get $\|\mathcal{P}_{\lambda,l}f\|_{\lambda,p}\leq\|M_\alpha^\lambda[\mathcal{P}_{\lambda,l}f]\|_{L^p(\partial B(\mathbb{H}^n))}<+\infty$.

The proof for $\lambda=0$ follows similarly.

(iii)$\Rightarrow$(i) The result for $\lambda=0$ follows from (11). Similarly, for $\Re(i\lambda)>0$, the result follows from Theorem 2.1 in [8]. ∎

In order to prove Theorem 1.3, we need the following Lebesgue differentiation type result (see e.g. [22]).

Lemma 5.5. *Let $f\in L^1(\partial B(\mathbb{H}^n))$, then for almost every $\omega\in\partial B(\mathbb{H}^n)$, we have*

$$\lim_{\delta\to 0}\frac{1}{\sigma(B(\omega,\delta))}\int\limits_{B(\omega,\delta)}|f(\theta)-f(\omega)|d\theta=0. \tag{28}$$

Now, we proceed with the proof of Theorem 1.3.

Proof. We will prove the theorem for the case $\Re(i\lambda)>0$, while the case $\lambda=0$ follows similarly.

Step 1. Let $f\in L^1(\partial B(\mathbb{H}^n))$ be such that

$$\lim_{\delta\to 0}\frac{1}{\sigma(B(\omega,\delta))}\int\limits_{B(\omega,\delta)}|f(\theta)|d\theta=0,$$

for some point $\omega\in\partial B(\mathbb{H}^n)$. Because of the estimate (27), we may follow the same method as in the proof of (Theorem 5.4.7 in [17]) and get

$$\lim_{\substack{z\to\omega\\ z\in D_\alpha(\omega)}}(1-|z|^2)^{\frac{i\lambda-2n-1}{2}}\mathcal{P}_{\lambda,l}f(z)=0.$$

Step 2. Now, let $f\in L^1(\partial B(\mathbb{H}^n))$ and define the function h_λ on $B(\mathbb{H}^n)$ by $h_\lambda(z)=(1-|z|^2)^{\frac{i\lambda-2n-1}{2}}$. We can write $h_\lambda(z)\mathcal{P}_{\lambda,l}f(z)-\mathbf{c}_l(\lambda)f(\omega)$ as

$$h_\lambda(z)\mathcal{P}_{\lambda,l}f_\omega(z)+h_\lambda(z)\int\limits_{\partial B(\mathbb{H}^n)}\mathcal{P}_{\lambda,l}(z,\theta)d\theta f(\omega)-\mathbf{c}_l(\lambda)f(\omega),$$

where $f_\omega(\theta)=f(\theta)-f(\omega)$.

Using the triangle inequality, we obtain

$$\begin{aligned}&|h_\lambda(z)\mathcal{P}_{\lambda,l}f(z)-\mathbf{c}_l(\lambda)f(\omega)|\\&\leq |h_\lambda(z)\mathcal{P}_{\lambda,l}f_\omega(z)|+|f(\omega)|\Big|h_\lambda(z)\int\limits_{\partial B(\mathbb{H}^n)}\mathcal{P}_{\lambda,l}(z,\theta)d\theta-\mathbf{c}_l(\lambda)\Big|.\end{aligned}$$

Thus, it suffices to show that the right-hand side above tends to 0 as z tends to ω with $z\in D_\alpha(\omega)$. According to the Lemma 5.5, we have for almost every $\omega\in\partial B(\mathbb{H}^n)$

$$\lim_{\delta\to 0}\frac{1}{\sigma(B(\omega,\delta))}\int\limits_{B(\omega,\delta)}|f_\omega(\theta)|d\theta=0.$$

From Step 1, we conclude that

$$\lim_{\substack{z\to\omega\\ z\in D_\alpha(\omega)}} h_\lambda(z)\mathcal{P}_{\lambda,l}f_\omega(z)=0.$$

On the other hand, by (5) and (4), we find

$$\begin{aligned}&h_\lambda(z)\int\limits_{\partial B(\mathbb{H}^n)}\mathcal{P}_{\lambda,l}(z,\theta)d\theta=h_\lambda(z)\Phi^{(0,0)}_{\lambda,l}(|\ z\ |)\\&=\frac{\pi(2l+1)\Gamma(2n-2)}{4\Gamma(2n)}{}_2F_1\Big(\frac{-i\lambda+2n+1}{2}-l-1,\frac{-i\lambda+2n+1}{2}+l;2n;|z|^2\Big).\end{aligned}$$

By (2), we get

$$\lim_{|z|\to 1}h_\lambda(z)\int\limits_{\partial B(\mathbb{H}^n)}\mathcal{P}_{\lambda,l}(z,\theta)d\theta=\mathbf{c}_l(\lambda).$$

Therefore

$$\lim_{\substack{z\to\omega\\ z\in D_\alpha(\omega)}}\Big|h_\lambda(z)\int\limits_{\partial B(\mathbb{H}^n)}\mathcal{P}_{\lambda,l}(z,\theta)d\theta-\mathbf{c}_l(\lambda)\Big|=0,$$

and the proof is finished. ■

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