

Biregular Elements in Radicals of Parabolic Subgroups in $GL(n)$

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Abstract. A biregular element in the radical \mathfrak{u} of the parabolic subgroup P is an element that is regular with respect to the adjoint actions of P and its maximal unipotent subgroup N simultaneously. We present a canonical biregular element in the radical of the parabolic subgroup of $GL(n)$. We construct a system of free generators of the field of Ad_N -invariants $K(\mathfrak{u})^N$.

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1. Introduction

Parabolic subgroups play an important role in the structure theory of reductive groups and their representations. Consider a reductive group G over an algebraically closed field K of zero characteristic and a parabolic subgroup P in G . The parabolic subgroup P acts on the nilpotent radical \mathfrak{u} of $\mathfrak{p} = \text{Lie}(P)$ by the adjoint representation. The problem of description of all orbits of this action is considered an extremely complicated, "wild" problem as it includes the problem of classification of all adjoint orbits of the triangular group.

It is well known that the action of P on \mathfrak{u} has an open dense orbit [11]. This orbit is referred to as the Richardson orbit. Respectively, its elements are called Richardson elements. There are different constructions of Richardson elements for simple Lie algebras of classical type (see [1, 3]). In the ongoing researches, the different authors study the complement of the Richardson orbit in \mathfrak{u} and its decomposition into adjoint orbits [2, 4, 5, 6].

From the other hand, the parabolic subgroup P contains the maximal unipotent subgroup N . The subgroup N also acts on the radical \mathfrak{u} by the adjoint action. According to K. Miyata's Theorem [8], the field of invariants of an arbitrary unitriangular transformation group action is rational (i.e. it is a pure transcendental extension of the main field). So, the field of invariants $K(\mathfrak{u})^N$ is rational. The problem is to present a system of free generators of $K(\mathfrak{u})^N$.

For $G = GL(n)$, the subgroup $N = UT(n)$ is a unitriangular group that consists of upper triangular matrices with ones on the diagonal. For parabolic subgroups of $GL(n)$, in the paper [9], A. N. Panov and V. V. Sevostyanova proposed the hypothetical system of free generators of the field $K(\mathfrak{u})^N$. In the next paper of

V. V. Sevostyanova [12], it was proved that the suggested system of invariants really freely generates the field $K(\mathfrak{u})^N$. However, the proof turned out to be quite complicated. It can't be generalized to the case of other root systems. Also, it doesn't contain any information on N -orbits of maximal dimension (call them N -regular orbits).

These shortcomings are eliminated in the present paper. We construct the element R_S in the radical \mathfrak{u} that is not only a Richardson element but also a N -regular element (see Theorem 2.13). We refer to these elements as biregular elements. As an application of this construction, we obtain a new, more simple proof of V. V. Sevostyanova's theorem (see Theorem 3.4).

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2. Construction of a biregular element

Let K be an algebraically closed field of zero characteristic. Consider the general linear group $\mathrm{GL}(n)$ over K . Fix the partition $[1, n] = I_1 \cup \dots \cup I_r$, where $r > 1$, of the integer segment $[1, n]$ into a system of consecutive segments I_1, \dots, I_r of lengths n_1, \dots, n_r respectively. Consider the upper parabolic subgroup P defined by the above partition of $[1, n]$. The subgroup P is a semidirect product $P = LU$ of the reductive part L and the unipotent radical U . The group P contains the upper triangular subgroup $N = \mathrm{UT}(n)$. The Lie algebra $\mathfrak{n} = \mathfrak{ut}(n)$ of the group N consists of strictly upper triangular matrices.

The Lie algebra \mathfrak{p} of the group P decomposes into a direct sum $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{u} = \mathrm{Lie}(U)$. As mentioned above, the adjoint action of P on \mathfrak{u} has the dense orbit (the Richardson orbit).

Definition 2.1. An element x in \mathfrak{u} is a *biregular element* if x is a Richardson element and the N -orbit $\mathrm{Ad}_N x$ has the maximal dimension among all N -orbits on \mathfrak{u} (i.e. x is a regular element with respect to the adjoint actions of P and N simultaneously).

Denote by Δ the root system for A_{n-1} and by Δ^+ the subsystem of positive roots. Each root α has the form $\alpha = \alpha_{ij} = \epsilon_i - \epsilon_j$. We identify α with the pair (i, j) , where $1 \leq i, j \leq n$, $i \neq j$. We write $i = \mathrm{row}(\alpha)$ and $j = \mathrm{col}(\alpha)$. The Lie algebra $\mathfrak{gl}(n)$ has the standard basis of matrix units $\{E_{ij} : 1 \leq i, j \leq n\}$. We denote by $\Delta_{\mathfrak{u}}^+$ and $\Delta_{\mathfrak{l}}^+$ the subsets of positive roots α_{ij} such that E_{ij} is contained in \mathfrak{u} and \mathfrak{l} relatively.

We define the binary relation on the set of positive roots $\alpha \succ \beta$ if $\alpha - \beta \in \Delta^+$ (there was a misprint $\Delta_{\mathfrak{l}}^+$ instead of Δ^+ in the papers [9, 12]). We say that α and β are incomparable if both $\alpha \succ \beta$ and $\beta \succ \alpha$ are not true.

Definition 2.2. Let $S \subset \Delta_{\mathfrak{u}}^+$. We say that S is a *base* of $\Delta_{\mathfrak{u}}^+$ if each two roots in S are incomparable and for each root α in $\Delta_{\mathfrak{u}}^+ \setminus S$ there exists a root $\xi \in S$ such that $\alpha \succ \xi$.

The set $\Delta_{\mathfrak{u}}^+$ has a unique base which one can construct as follows. Choose the system of minimal (in the sense of relation \succ) roots S_1 in $\Delta_{\mathfrak{u}}^+$. The subset S_1 consists of

simple roots from $\Delta_{\mathfrak{u}}^+$. Then we delete from $\Delta_{\mathfrak{u}}^+$ the system S_1 and such roots α that $\alpha \succ \xi$ for some $\xi \in S_1$. In the obtained set we choose a subset of minimal roots S_2 and continue the process. We obtain the subset S which is a union of all subsets $\{S_i\}$.

We say that the positive roots $\alpha_1, \alpha_2, \alpha_3$ form a *chain* if $\alpha_1 = (a_1, a_2)$, $\alpha_2 = (a_2, a_3)$, and $\alpha_3 = (a_3, a_4)$.

Below we define a VS-pair. This term was introduced in [6]. In the early papers [9, 12], we used the term admissible pair.

Definition 2.3. We call a pair $q = (\xi, \xi')$ of roots $\xi, \xi' \in S$ a *VS-pair* if there exists a root $\alpha_q \in \Delta_{\mathfrak{l}}^+$ such that the roots ξ, α_q, ξ' form a chain.

Observe that α_q is uniquely defined by q . We denote by Q the set of all VS pairs $\{q = (\xi, \xi') : \xi, \xi' \in S\}$. To each VS pair q , we attach the root $\varphi_q = \alpha_q + \xi'$. Denote $\Phi = \{\varphi_q : q \in Q\}$. Consider the set of matrices \mathcal{Y} of the form

$$y = \sum_{\xi \in S} y_{\xi} E_{\xi} + \sum_{q \in Q} y_{\varphi_q} E_{\varphi_q}, \quad (1)$$

where y_{ξ} and y_{φ_q} are from the field K .

Let w_L be the permutation matrix, which represents the element of the greatest length in the Weyl group for L . Recall that the Kostant cascade of L is the subset of roots in $\Delta_{\mathfrak{l}}^+$ of the form $(i, w_L(i))$ in $\Delta_{\mathfrak{l}}^+$, where $i < w_L(i)$.

Consider the subset $Q_1 \subset Q$ that consists of all VS-pairs $q = (\xi, \xi')$ such that α_q belongs to the Kostant cascade in $\Delta_{\mathfrak{l}}^+$ (i.e. $\alpha_q = (i, w_L(i)) \in \Delta_{\mathfrak{l}}^+$). Respectively, we take $\Phi_1 = \{\varphi_q : q \in Q_1\}$.

Choose a system of non-zero constants $\{c_{\varphi} \in K : \varphi \in \Phi_1\}$. Consider the element $R_S \in \mathcal{Y}$ defined by the system of constants

$$R_S = R_{S,0} + R_{S,1}, \quad \text{where } R_{S,0} = \sum_{\xi \in S} E_{\xi} \quad \text{and} \quad R_{S,1} = \sum_{\varphi \in \Phi_1} c_{\varphi} E_{\varphi}.$$

Below we prove that R_S is a biregular element in \mathfrak{u} for some system of constants $\{c_{\varphi}\}$.

Example 2.4. Consider the parabolic subgroup of the type $(2, 4, 1, 3)$ in the group $\text{GL}(10)$. We draw the set \mathcal{Y} on the matrix by marking the roots from S by the symbol \otimes and the roots from Φ by the symbol \times ; see Figure 1.

There is a standard method to construct a Richardson element in \mathfrak{u} . First we construct a diagram D as follows. Let us place the columns of length n_1, n_2, \dots, n_r so that their upper levels coincide. Fill the columns sequentially from top to bottom with 1 to n in ascending order. Connect the adjacent numbers lying on the same level by lines. If $i < j$ are connected by a line, then we say that j is a *successor* of i and i is a *predecessor* of j .

If the column number of i in D is less than or equal to the column number of j , then the matrix unit E_{ij} belongs to the parabolic subalgebra \mathfrak{p} . If the column numbers are equal, then $E_{ij} \in \mathfrak{l}$. If the column number of i is less than the column number of j , then $E_{ij} \in \mathfrak{u}$.

We construct the element $R = \sum_{i \rightarrow j} E_{ij}$ in the radical \mathfrak{u} .

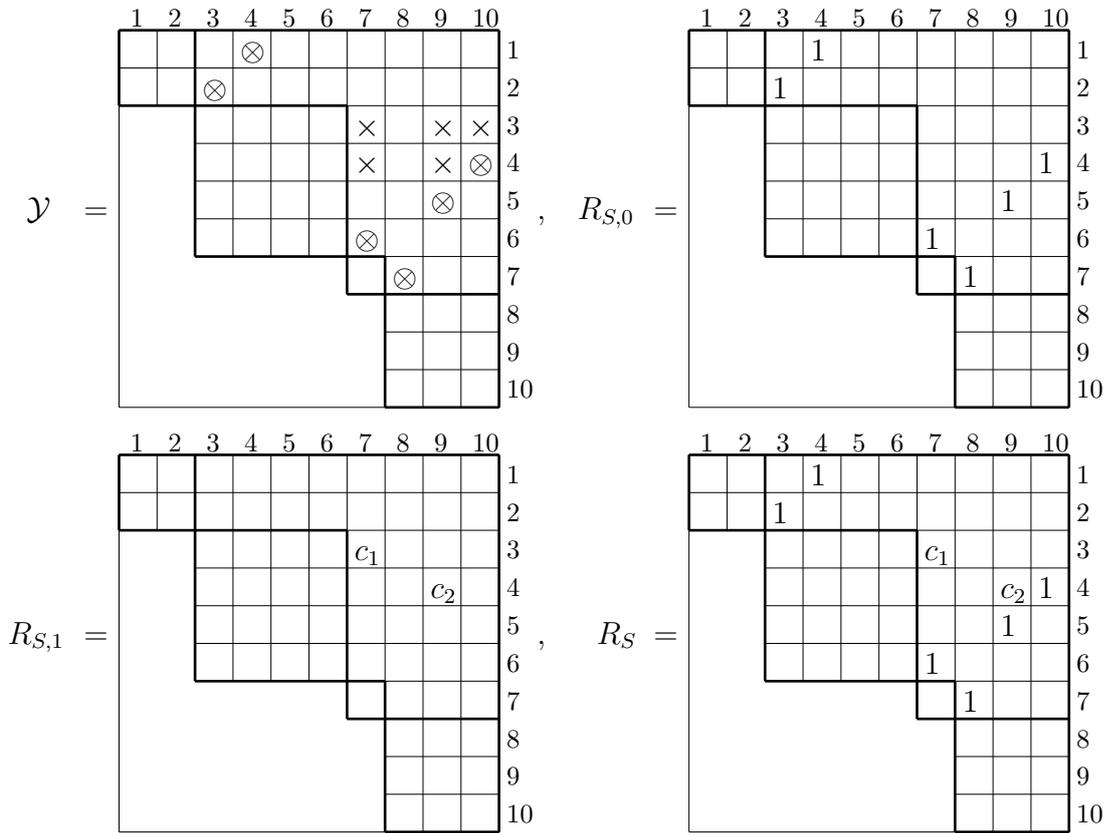


Figure 1

Proposition 2.5. [1, 3] *The matrix R is a Richardson element in \mathfrak{u} .*

We refer to R as a standard Richardson element in \mathfrak{u} .

Example 2.4 (continuation). For Example 2.4 we have the following diagram D and the standard Richardson element R :

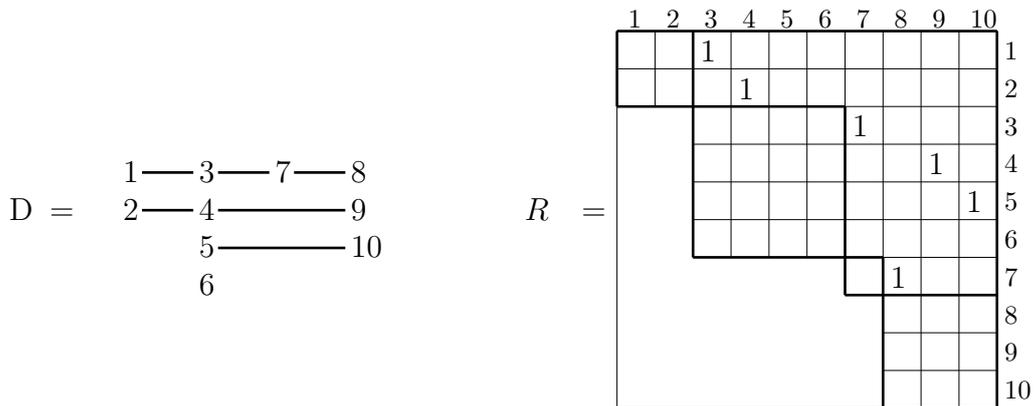


Figure 2

Consider the coordinate n -dimensional space $K^n = \{(x_1, \dots, x_n)\}$ and its standard basis $\{e_i : 1 \leq i \leq n\}$. The matrix R acts on K^n by the right multiplication. Then $e_i R = e_j$ if j is a successor of i , and $e_i R = 0$ if i has no successor.

Observe that the ones in each I_m block-row of matrices R and $R_{S,0}$ are placed symmetrically with respect to the center of I_m . It implies $R_{S,0} = w_L R$. The matrix

$R_{S,1}$ is a linear combination of the matrix units E_{ij} over all pairs $i—j$ such that $i < w_L(i)$ and i has a predecessor. For pair $i < w_L(i)$, we have

$$\begin{aligned} e_i R_S &= e_i(R_{S,0} + R_{S,1}) = e_i w_L R + e_i R_{S,1} = e_{w_L(i)} R + e_i R_{S,1}, \\ e_{w_L(i)} R_S &= e_{w_L(i)}(R_{S,0} + R_{S,1}) = e_{w_L(i)} w_L R + e_{w_L(i)} R_{S,1} = e_i R + e_{w_L(i)} R_{S,1}. \end{aligned}$$

Here $e_{w_L(i)} R_{S,1} = 0$. For instance in Example 2.4, if $i = 3$ (respectively, $i = 4$), then $w_L(i) = 6$ (respectively, $w_L(i) = 5$) and $e_6 R_{S,1} = 0$ (respectively, $e_5 R_{S,1} = 0$). We obtain

$$\begin{cases} e_i R_S = e_{w_L(i)} R + e_i R_{S,1}, \\ e_{w_L(i)} R_S = e_i R. \end{cases}$$

If $i < w_L(i)$ and i has a predecessor, then $e_i R_{S,1} = c_\varphi e_i R$, where φ corresponds to the cascade root $(i, w_L(i))$, and

$$\begin{cases} e_i R_S = c_\varphi e_i R + e_{w_L(i)} R, \\ e_{w_L(i)} R_S = e_i R. \end{cases} \quad (2)$$

Remark that here if $e_i R = e_j$ and $e_{w_L(i)} R = e_k$, then $j < k$. If $i < w_L(i)$ and i has no predecessor, then $e_i R_{S,1} = 0$ and

$$\begin{cases} e_i R_S = e_{w_L(i)} R, \\ e_{w_L(i)} R_S = e_i R. \end{cases} \quad (3)$$

If $i = w_L(i)$, then $e_i R_S = e_i R$.

We introduce a linear order on Φ_1 as follows: $\varphi_1 < \varphi_2$ if $\text{row}(\varphi_1) < \text{row}(\varphi_2)$. The field K contains the subfield of rational numbers \mathbb{Q} . For each $\varphi \in \Phi_1$, let K_φ be the subfield in K generated over \mathbb{Q} by the elements c_ψ , $\varphi < \psi$.

Proposition 2.6. *Suppose that the field K contains a system of elements $\{c_\varphi : \varphi \in \Phi_1\}$ such that each c_φ is transcendental over the subfield K_φ (for example, $K = \mathbb{C}$). Then the matrix R_S defined by $\{c_\varphi\}$ is a Richardson element in \mathfrak{u} .*

Proof. For each $1 \leq m \leq r - 1$, we denote by \mathcal{M}_m the submatrix of R_S with the system of rows I_m and the system of columns $\text{col}(\xi)$, where ξ runs through the elements of S belonging to the m th block-row. Observe that the number of rows n_m of the submatrix \mathcal{M}_m is greater or equal to its number of columns.

For each $2 \leq m \leq r$, we consider the submatrix \mathcal{N}_m of R_S with the system of columns I_m and the system of rows $\text{row}(\xi)$, where ξ runs through the elements of S belonging to the m th block-column. The submatrix \mathcal{N}_m is an upper block-triangular matrix with respect to the anti-diagonal. Its blocks are matrices that look like $\{\mathcal{M}_k\}$ (i.e. they have ones on the anti-diagonal, some $\{c_i\}$ on the diagonal, and zeros on the other places, see examples below).

For instance, for Example 2.4, we have $\mathcal{M}_2 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathcal{N}_4 = \begin{pmatrix} 0 & c_2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Example 2.7. In Example 2.4, we replace the length of the first block $n_1 = 2$ by $n_1 = 1$. Then the submatrices will have the form

$$M'_2 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N'_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Item 1. Let us show that there exists a permutation matrix $w \in L$ such that for the submatrices \mathcal{M}_m , $1 \leq m \leq r-1$, of $R'_S = \text{Ad}_w R_S$ all left upper corner minors $\{\delta_i\}$ are non-zero.

Observe that if the element $g \in L$ has the m th component g_m and all other components are the identity matrices, then the transformation $R_S \rightarrow \text{Ad}_g R_S$ implies the transformations of the submatrices $\mathcal{M}_m \rightarrow g_m \mathcal{M}_m$ and $\mathcal{N}_m \rightarrow \mathcal{N}_m g_m^{-1}$.

Let s_m be the number of elements from S that belong to the m th block-column of R_S , ϕ_m is the number of roots $\alpha_q \in \Delta^+(\mathfrak{n}_m)$ that are defined by the VS-pairs (it is equal to the number of φ_q in the m th block-row).

If $s_m \geq \lceil \frac{n_m}{2} \rceil$, then each left upper corner minor δ_i of \mathcal{M}_m is non-zero. For instance, in the Example 2.4 and the submatrix \mathcal{M}_2 , we have

$$\delta_1 = c_1 \neq 0, \quad \delta_2 = \begin{vmatrix} c_1 & 0 \\ 0 & c_2 \end{vmatrix} \neq 0, \quad \delta_3 = \begin{vmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{vmatrix} \neq 0.$$

If $s_m < \lceil \frac{n_m}{2} \rceil$, then $n_m > 2\phi_m + 1$. Decompose the segment I_m into three consecutive segments $J_1 \sqcup J_2 \sqcup J_3$ of lengths $|J_1| = |J_3| = \phi_m$. Observe that there are no elements of Φ_1 in the J_2 -rows and there are no elements of S in the J_2 -columns. Let w_m^0 be the permutation matrix of the greatest length of the block $J_2 \times J_2$. We put ones on the diagonal in all other rows (and columns) and obtain the element $w_m^0 \in L$. The transformation $R_S \rightarrow w_m^0 R_S (w_m^0)^{-1}$ implies the transformations of the submatrices $\mathcal{M}_m \rightarrow w_m^0 \mathcal{M}_m$ and $\mathcal{N}_m \rightarrow \mathcal{N}_m (w_m^0)^{-1}$. Since there are no elements of S in the J_2 -columns, the matrix \mathcal{N}_m has zeros in J_2 -columns. Then $\mathcal{N}_m (w_m^0)^{-1} = \mathcal{N}_m$.

Observe that each left upper corner minor δ_i of $w_m^0 \mathcal{M}_m$ is non-zero. For instance, in Example 2.7, we have

$$w_2^0 M'_2 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \delta_1 = c_1 \neq 0, \quad \delta_2 = \begin{vmatrix} c_1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0, \quad \delta_3 = \begin{vmatrix} c_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0.$$

Let w be a product of all w_m^0 over all m such that $s_m < \lceil \frac{n_m}{2} \rceil$. Then $R'_S = \text{Ad}_w R_S$ satisfies the conditions of Item 1.

Item 2. Decompose the linear space $V = K^n$ into a direct sum of subspaces

$$V = \bigoplus_{m=1}^r V_m,$$

where each V_m is a subspace spanned by the vectors of the standard basis e_j , $j \in I_m$. Observe that the numbers of I_m fill the m th column in the diagram.

For each $0 \leq k < m$, we consider the subspace $V_{m,k}$ of V_m defined as follows. The subspace $V_{m,0}$ is spanned by all vectors e_j such that j has no predecessor. For $k \neq 0$, the subspace $V_{m,k}$ is spanned by all e_j such that j has a predecessor in I_k .

We denote by $V_k^{(m)}$ the subspace spanned by all $\{e_i\}$, where $i \in I_k$ is a predecessor of some $j \in I_m$. We have $R : V_k^{(m)} \rightarrow V_{m,k}$. Define the subspace

$$W_{m,k} = (\oplus_{s \geq k} V_{m,s}) \oplus (\oplus_{t > m} V_t).$$

In Item 2, we show that there exists $g \in L$ such that the element $R'_S = \text{Ad}_g(R'_S)$ sends $V_k^{(m)} \rightarrow V_{m,k} \bmod W_{m,k+1}$ by an upper triangular transformation for each $1 \leq k < m \leq r$.

We start a construction of g from the block $r-1$. Consider the field K_{r-1} generated over \mathbb{Q} by the elements c_φ , where $\varphi \in \Phi_1$ and it lies in the $(r-1)$ th block-row. The submatrix \mathcal{M}_{r-1} has size $n_{r-1} \times \ell$ with $n_{r-1} \geq \ell$. Since the left upper corner minors $\{\delta_i\}$ of \mathcal{M}_{r-1} are non-zero, the submatrix \mathcal{M}_{r-1} decomposes over the field K_{r-1} into a product $V_{r-1}^- B_{r-1}^+$ of the lower triangular $(n_{r-1} \times n_{r-1})$ -matrix V_{r-1}^- and the upper triangular $(n_{r-1} \times \ell)$ -matrix B_{r-1}^+ . Replacing R'_S to $\text{Ad}_{V_{r-1}^-}^{-1}(R'_S)$, the submatrix \mathcal{M}_{r-1} changes to the upper triangular matrix B_{r-1}^+ .

At the same time, in the submatrix \mathcal{N}_{r-1} the anti-diagonal blocks \mathcal{M}_k change to $\mathcal{M}_k U^-$, where U^- is the lower triangular matrix. Since each c_φ , $\varphi \in \Phi_1$, is transcendental over K_φ , the left upper corner minors $\{\delta_i\}$ of $\mathcal{M}_k U^-$ are non-zero. We decompose the submatrix $\mathcal{M}_{r-2} U^-$ into a product $V_{r-2}^- B_{r-2}^+$ and act by $\text{Ad}_{V_{r-2}^-}$. Continuing the process until $m=2$, we obtain the matrix R''_S satisfying the condition of Item 2.

Item 3. We show that there exists $g \in L$ such that the element $\tilde{R}_S = \text{Ad}_g(R''_S)$ obeys the property: if $i \dashv j$ and $e_j \in V_{m,k}$, then

$$e_i \tilde{R}_S = e_j \bmod W_{m,k+1}.$$

The matrix R''_S obeys the property of Item 3 for $m=1$ and $m=2$. For $m=3$ acting by the upper triangular matrix B_3^+ , we change B_3^+ to the matrix with ones in the diagonal. Continuing the process for $m=4, \dots, r$, we obtain the matrix \tilde{R}_S .

Item 4. For each $1 \leq j \leq n$, we denote by $p(j)$ the greatest number such that $e_j \in \text{Im} R^{p(j)}$. The number $p(j)$ coincides with the length of the chain of predecessors of j in the diagram. Denote

$$W_m = W_{m,0} = \oplus_{t \geq m} V_t.$$

In Item 4 let us show that for each $1 \leq m \leq r$ and each $j \in I_m$ there exists a vector $\tilde{e}_j = e_j + a$, where $a \in W_{m+1}$, belonging to $\text{Im} \tilde{R}_S^{p(j)}$.

We shall prove by the induction method on j . In the case $j=1$, the vector e_1 does not lie in $\text{Im} R$ (and does not lie in $\text{Im} \tilde{R}_S$). We take $\tilde{e}_1 = e_1$.

Suppose that the statement is proved for all numbers $< j$. We shall prove for j . Let $j \in I_m$. We denote $p = p(j)$. Suppose that i is a predecessor of j in the diagram and $i \in I_k$, where $k < m$. Then $e_j \in V_{m,k}$, $e_i \in V_k^{(m)}$ and $e_i \in \text{Im} R^{p-1}$. According to the induction assumption, there exists $\tilde{e}_i = e_i + a$, where $a \in W_{k+1}$, belonging to $\text{Im} \tilde{R}_S^{p-1}$. Therefore $\tilde{e}_i \tilde{R}_S \in \text{Im} \tilde{R}_S^p$.

Since i is connected by a line with j in the diagram, the lengths of the columns between I_k and I_m are strictly less than the lengths n_k and n_m of columns I_k and I_m . Let $t \in I_{k+1} \sqcup \dots \sqcup I_{m-1}$. According to the induction assumption, there

exists a vector $\tilde{e}_t \in \text{Im}\tilde{R}_S^{p(t)}$. The number $p(t)$ is greater than $p - 1$. Therefore $\tilde{e}_t \in \text{Im}\tilde{R}_S^{p-1}$. Observe that the vectors $\{\tilde{e}_t\}$, where $t \in I_{k+1} \sqcup \dots \sqcup I_{m-1}$, form a basis in $W_{k+1} \bmod W_m$.

Recall that $\tilde{e}_i = e_i + a$, where $a \in W_{k+1}$. Write the vector $a \in W_{k+1}$ in the form $a = \tilde{a} + a_1$, where $\tilde{a} \in \text{Im}\tilde{R}_S^{p-1}$ and $a_1 \in W_m$. Then $a\tilde{R}_S = \tilde{a}\tilde{R}_S + a_1\tilde{R}_S$, where $\tilde{a}\tilde{R}_S \in \text{Im}\tilde{R}_S^p$ and $a_1\tilde{R}_S \in W_{m+1}$.

Item 3 implies that $e_i\tilde{R}_S = e_j + v$, where $v \in W_{m,k+1}$. According to the induction assumption, for each $e_s \in \bigoplus_{s \geq k+1} V_{m,s}$ there exists $\tilde{e}_s \in \text{Im}\tilde{R}_S^p$. There exists the vector $\tilde{v} \in \text{Im}\tilde{R}_S^p$ such that $v = \tilde{v} + b$, where $b \in W_{m+1}$. We obtain

$$\tilde{e}_i\tilde{R}_S = (e_i + a)\tilde{R}_S = e_i\tilde{R}_S + a\tilde{R}_S = e_j + \tilde{v} + b + \tilde{a}\tilde{R}_S + a_1\tilde{R}_S.$$

Then the vector $\tilde{e}_j = e_j + b + a_1\tilde{R}_S = \tilde{e}_i\tilde{R}_S - \tilde{a}\tilde{R}_S - \tilde{v}$ lies in $\text{Im}\tilde{R}_S^p$ and $b + a_1\tilde{R}_S$ lies in W_{m+1} . The statement of Item 4 is proved.

Item 5. Let us show that R_S is similar to R . It is sufficient to prove that \tilde{R}_S is similar to R . The system of vectors $\{e_j\}$, where j has the length p of the chain of predecessors, form a basis in $\text{Im}R^p$. It follows from Item 4 that the system of vectors \tilde{e}_j , where $e_j \in \text{Im}R^p$, is linear independent. Therefore $\dim \text{Im}\tilde{R}_S^p \geq \dim \text{Im}R^p$. On the other hand, $\dim \text{Im}\tilde{R}_S^p \leq \dim \text{Im}R^p$ because R is a Richardson element and \tilde{R}_S lies in the closure of P -orbit of R . Finally, $\dim \text{Im}\tilde{R}_S^p = \dim \text{Im}R^p$ for each p . The matrix \tilde{R}_S similar to R , and R_S is a Richardson element. \blacksquare

Corollary 2.8. *There exists a nonempty open subset $\{c_\varphi \in K : \varphi \in \Phi_1\}$ in $K^{|\Phi_1|}$ such that R_S is a Richardson element.*

The proof is similar to the proof of Proposition 2.6. The nonempty open subset is defined by the conditions $\{\delta_i \neq 0\}$, where $\{\delta_i\}$ are the left upper corner minors encountered in Item 2.

Example 2.9. If $n = 6$ and the parabolic subgroup of the type $(1, 2, 2, 1)$, then

$$R_S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a Richardson element iff $c_1c_2 + 1 \neq 0$.

Conjecture 2.10. If $c_\varphi = 1$ for each $\varphi \in \Phi_1$, then R_S is a Richardson element.

In what follows the matrix R_S is constructed by the system of constants $\{c_\varphi\}$ that obeys the condition of Proposition 2.6 or belongs to the open subset from Corollary 2.8. So, R_S is a Richardson element in \mathfrak{u} .

For every element $A \in \mathfrak{u}$ and every subalgebra $\mathfrak{p} \supseteq \mathfrak{m} \supseteq \mathfrak{u}$, we introduce the following notations: $C_{\mathfrak{m}}(A) = \{x \in \mathfrak{m} : [x, A] = 0\}$ is a stabilizer of A in \mathfrak{m} , $\overline{C}_{\mathfrak{m}}(A)$ is the factor algebra of $C_{\mathfrak{m}}(A)$ modulo $C_{\mathfrak{u}}(A)$, $\bar{c}_{\mathfrak{m}}(A) = \dim \overline{C}_{\mathfrak{m}}(A)$, for each $1 \leq m \leq r$ let $n_{m,\text{left}} = \max\{n_i : 1 \leq i < m\}$, and $n_{m,\text{right}} = \max\{n_i : m < i \leq r\}$.

In the following we calculate the dimensions $\bar{c}_n(R_S)$ and $\bar{c}_p(R)$. We begin with $\bar{c}_n(R_S)$. Let $\text{col}(S)$ (respectively, $\text{row}(S)$) stand for the set of columns (respectively, rows) of S . Let \mathbb{V}_S be the subset of roots $\alpha = (i, j) \in \Delta_l^+$, where $i \notin \text{col}(S)$ and $j \notin \text{row}(S)$. Decompose $\mathbb{V}_S = \sqcup_{m=1}^r \mathbb{V}_{S,m}$, where $\mathbb{V}_{S,m}$ is the subset of all roots from the m th block, which belong to \mathbb{V}_S .

Consider the subspace \mathcal{V}_S spanned by all E_{ij} , where $(i, j) \in \mathbb{V}_S$. We have

$$\mathcal{V}_S = \bigoplus_{m=1}^r \mathcal{V}_{S,m}, \quad \text{where } \mathcal{V}_{S,m} = \text{span}_{(i,j) \in \mathbb{V}_{S,m}} E_{i,j}.$$

Lemma 2.11. (1) *The subspace \mathcal{V}_S is a subalgebra in $C_n(R_S)$ and*

$$C_n(R_S) = \mathcal{V}_S \oplus C_u(R_S).$$

(2) $\bar{c}_n(R_S) = \dim \mathcal{V}_S = |\mathbb{V}_S|$.

(3) *If maximum of $\{n_m : 1 \leq m \leq r\}$ is not unique, then $\mathbb{V}_S = \emptyset$ (respectively, $\mathcal{V}_S = \{0\}$). Suppose that n_{m_0} is greater than each n_m , $m \neq m_0$. Then $\mathcal{V}_{S,m} = \{0\}$ for each $m \neq m_0$ and*

$$\dim \mathcal{V}_{S,m_0} = \begin{cases} 0, & \text{if } n_{m_0} \leq n_{m_0,\text{left}} + n_{m_0,\text{right}}, \\ \dim \mathfrak{ut}(n_{m_0} - n_{m_0,\text{left}} - n_{m_0,\text{right}}), & \text{if } n_{m_0} > n_{m_0,\text{left}} + n_{m_0,\text{right}}. \end{cases}$$

Proof. Easy to see that if $x \in \mathcal{V}_S$, then $[x, R_S] = 0$ and therefore $x \in C_n(R_S)$. From the other side, let $x \in \mathfrak{n}$ such that $[x, R_S] = 0$ and $x = x_l + x_u$, where $x_l \in \mathfrak{l}$ and $x_u \in \mathfrak{u}$. Let us show that $x_l \in \mathcal{V}_S$.

The element x_l decomposes into a sum of representatives of blocks in the form $x_l = (x_l)_1 + \dots + (x_l)_r$. Let us prove that each component belongs to \mathcal{V}_S beginning with the first one. If the first block-row of the matrix R_S contains d roots from S , then $(x_l)_1 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$, where $B \in \mathfrak{ut}(n_1 - d)$. Therefore $(x_l)_1 \in \mathcal{V}_S$.

Suppose that for all $k < i$ the components $(x_l)_k$ belong to \mathcal{V}_S . Let us prove that $(x_l)_i \in \mathcal{V}_S$. Indeed, if the i th block-row of R_S contains d roots from S , and the i th block-column contains b roots from S , then the i th component has the form $(x_l)_i = \begin{pmatrix} 0 & B & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $B \in \mathfrak{ut}(n_i - b - d)$ (in the case $n_i \leq b + d$ we have $B = 0$). Then $(x_l)_i \in \mathcal{V}_S$. This proves (1) and (2).

For each $1 \leq m \leq r$, the number of roots from S in the m th block-column of R_S equals to $\min\{n_m, n_{m,\text{left}}\}$, and the number of roots from S in the m th block-row equals to $\min\{n_m, n_{m,\text{right}}\}$. Let m_0 be a number from $[1, r]$ such that $n_{m_0} = \max\{n_m : 1 \leq m \leq r\}$. For each $m \neq m_0$, the number of roots from S from the m th block-column or the number of roots from S in the m th block-row equals to m_0 . Then $\mathbb{V}_{S,m} = \emptyset$ (respectively, $\mathcal{V}_{S,m} = \{0\}$). For $m = m_0$, we have $\dim \mathcal{V}_{S,m_0} = |\mathbb{V}_{S,m_0}| = \dim \mathfrak{ut}(n_{m_0} - b - d)$, where $b = n_{m_0,\text{left}}$ and $d = n_{m_0,\text{right}}$. This proves (3). ■

We turn to a description of $C_p(R)$ and calculation of $\bar{c}_p(R)$. Let (\cdot, \cdot) be the Killing form of $\mathfrak{gl}(n)$. The parabolic subalgebra \mathfrak{p} is a polarization for R in $\mathfrak{gl}(n)$ (i.e. it is a subalgebra and maximal totally isotropic subspace with respect to the skew-symmetric bilinear form $(R, [x, y])$ on $\mathfrak{gl}(n)$). Indeed, if \mathfrak{p} is extended to some totally isotropic subspace \mathfrak{m} , then $(R, [\mathfrak{p}, \mathfrak{m}]) = 0$. Since R is a Richardson element, $[\mathfrak{p}, R] = \mathfrak{u}$. Then $0 = (R, [\mathfrak{p}, \mathfrak{m}]) = ([\mathfrak{p}, R], \mathfrak{m}) = (\mathfrak{u}, \mathfrak{m})$ and therefore $\mathfrak{m} \subset \mathfrak{p}$.

Since \mathfrak{p} is a polarization for R , then stabiliser $C(R) = \{x \in \mathfrak{gl}(n) : [x, R] = 0\}$ of R in $\mathfrak{gl}(n)$ is contained in \mathfrak{p} and coincides with $C_{\mathfrak{p}}(R)$.

The subalgebra $C(R) = C_{\mathfrak{p}}(R)$ is spanned by the elements of the form described below (see [7, 8.2]). Let us agree that we count the rows in the diagram D from top down. For the rows \mathbf{i} and \mathbf{j} in the diagram D , let $\ell(\mathbf{i})$ and $\ell(\mathbf{j})$ be their lengths respectively. Let $k \leq \min\{\ell(\mathbf{i}), \ell(\mathbf{j})\}$. The subalgebra $C_{\mathfrak{p}}(R)$ has the basis \mathcal{E} that consists of the elements

$$E_{\mathbf{i}, \mathbf{j}}^{(k)} = \sum_{a=1}^k E_{i_a, j_a}, \quad (4)$$

where $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_k$ is the system of the first k numbers in the row \mathbf{i} , and $j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_k$ is the system of the last k numbers in the row \mathbf{j} .

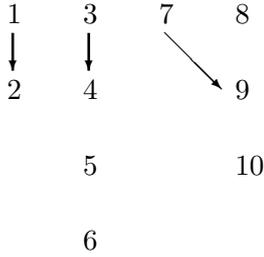
We depict the element $E_{\mathbf{i}, \mathbf{j}}^{(k)}$ on the diagram connecting the numbers i_a, j_a by arrows $i_a \rightarrow j_a$. If $\mathbf{i} \leq \mathbf{j}$, the arrows will be from top down or along the rows of the diagram D . Denote $\mathcal{E}_{\downarrow} = \{E_{\mathbf{i}, \mathbf{j}}^{(k)}, \mathbf{i} \leq \mathbf{j}\}$.

If $\mathbf{i} > \mathbf{j}$, then the arrows will be from bottom up. Denote $\mathcal{E}_{\uparrow} = \{E_{\mathbf{i}, \mathbf{j}}^{(k)}, \mathbf{i} > \mathbf{j}\}$. We have

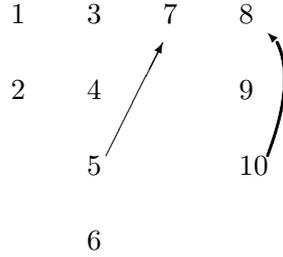
$$\mathcal{E} = \mathcal{E}_{\downarrow} \sqcup \mathcal{E}_{\uparrow}.$$

For instance, the diagram D of the Example (2.4) has the rows $\mathbf{1}, \mathbf{2}, \mathbf{3}$. Depict the basic elements $E_{\mathbf{1}, \mathbf{2}}^{(3)}$ and $E_{\mathbf{3}, \mathbf{1}}^{(2)}$ on the diagram:

$$E_{\mathbf{1}, \mathbf{2}}^{(3)} = E_{12} + E_{34} + E_{79}$$



$$E_{\mathbf{3}, \mathbf{1}}^{(2)} = E_{57} + E_{10,8}$$



Let $\bar{\mathcal{E}}$ be the subset of \mathcal{E} that consists of all elements $E_{\mathbf{i}, \mathbf{j}}^{(k)}$ such that there exists a pair $\{i_a, j_a\}$ (see (4)) from a common column in the diagram D (or in other words from a common segment I_b , $1 \leq b \leq r$). Analogously to the previous, we define $\bar{\mathcal{E}}_{\downarrow}$ and $\bar{\mathcal{E}}_{\uparrow}$. We obtain

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}_{\downarrow} \sqcup \bar{\mathcal{E}}_{\uparrow}.$$

Then $\bar{C}_{\mathfrak{p}}(R) = \text{span}\{E_{\mathbf{i}, \mathbf{j}}^{(k)} \in \bar{\mathcal{E}}\}$ and $\bar{c}_{\mathfrak{p}}(R) = |\bar{\mathcal{E}}|$.

Remark 2.12. (1) Let $E_{\mathbf{i}, \mathbf{j}}^{(k)} \in \bar{\mathcal{E}}_{\downarrow}$ (i.e. $\mathbf{i} \leq \mathbf{j}$). Then in the sum (4) each of the several first pairs $(i_1, j_1), \dots, (i_b, j_b)$, where $b \leq \ell(\mathbf{i})$, is lying in a common column, and the next arrows $i_a \rightarrow j_a$ are inclined. The basic element $E_{\mathbf{i}, \mathbf{j}}^{(k)}$ is uniquely determined by the first pair (i_1, j_1) such that both i_1 and j_1 has no predecessors (see the above diagram for $E_{\mathbf{1}, \mathbf{2}}^{(3)}$).

Observe that if $\mathbf{i} = \mathbf{j}$, the basic element $E_{\mathbf{i}, \mathbf{i}}^{(k)}$ belongs to $\bar{\mathcal{E}}$ whenever $k = \ell(\mathbf{i})$.

(2) Let $E_{\mathbf{i}, \mathbf{j}}^{(k)} \in \bar{\mathcal{E}}_{\uparrow}$ (i.e. $\mathbf{i} > \mathbf{j}$). Then in the sum (4) each of the several last pairs $(i_b, j_b), \dots, (i_k, j_k)$, where $k = \ell(\mathbf{i})$, is lying in a common column, and the previous arrows $i_a \rightarrow j_a$ are inclined. The basic element $E_{\mathbf{i}, \mathbf{j}}^{(k)}$ is uniquely determined by the

last pair (i_k, j_k) , such that both i_k and j_k has no successors (see the above diagram for $E_{3,1}^{(2)}$).

Theorem 2.13. *Suppose that the system of constants $\{c_\varphi \in K : \varphi \in \Phi_1\}$ obeys the condition of Proposition 2.6 or belongs to the open subset from Corollary 2.8. Then R_S is a biregular element in \mathfrak{u} .*

Proof. Since R_S is a Richardson element (see Proposition 2.6 and Corollary 2.8), it is sufficient to prove that R_S is a N -regular element in \mathfrak{u} . For every $x \in \mathfrak{u}$, we have

$$\dim \operatorname{ad}_{\mathfrak{n}} x = \dim \operatorname{Ad}_N x \leq \dim \mathfrak{u} - \dim \mathcal{Y}$$

(see Proposition 3.2, which do not depend on the present theorem). In particular, $\dim \operatorname{ad}_{\mathfrak{n}} R_S \leq \dim \mathfrak{u} - \dim \mathcal{Y}$. Then our goal is to prove that

$$\dim \operatorname{ad}_{\mathfrak{n}} R_S = \dim \mathfrak{u} - \dim \mathcal{Y}, \tag{5}$$

which implies the statement of the present theorem.

Since R_S is a Richardson element, we have $\mathfrak{u} = \operatorname{ad}_{\mathfrak{p}} R_S$. Then

$$\dim \operatorname{ad}_{\mathfrak{n}} R_S = \dim \operatorname{ad}_{\mathfrak{p}} R_S - \dim \mathcal{Y}.$$

Substituting the formulas

$$\begin{cases} \dim \operatorname{ad}_{\mathfrak{p}} R_S = \dim \mathfrak{p} - \dim C_{\mathfrak{p}}(R_S), \\ \dim \operatorname{ad}_{\mathfrak{n}} R_S = \dim \mathfrak{n} - \dim C_{\mathfrak{n}}(R_S), \end{cases}$$

we obtain that the equality (5) is equivalent to

$$\dim \mathfrak{n} - \dim C_{\mathfrak{n}}(R_S) = \dim \mathfrak{p} - \dim C_{\mathfrak{p}}(R_S) - \dim \mathcal{Y}.$$

The equality (5) has the form

$$\dim \mathcal{Y} + \dim C_{\mathfrak{p}}(R_S) - \dim C_{\mathfrak{n}}(R_S) = \dim \mathfrak{p}/\mathfrak{n}.$$

The subalgebras $C_{\mathfrak{p}}(R_S)$ and $C_{\mathfrak{n}}(R_S)$ have common intersection with \mathfrak{u} , which coincides with the stabilizer $C_{\mathfrak{u}}(R_S)$ of R_S in \mathfrak{u} . Recall the notations

$$\begin{cases} \overline{C}_{\mathfrak{p}}(R_S) = C_{\mathfrak{p}}(R_S) \bmod C_{\mathfrak{u}}(R_S), \\ \overline{C}_{\mathfrak{n}}(R_S) = C_{\mathfrak{n}}(R_S) \bmod C_{\mathfrak{u}}(R_S), \end{cases}$$

$$\overline{c}_{\mathfrak{p}}(R_S) = \dim \overline{C}_{\mathfrak{p}}(R_S), \quad \overline{c}_{\mathfrak{n}}(R_S) = \dim \overline{C}_{\mathfrak{n}}(R_S).$$

We get

$$\dim \mathcal{Y} + \overline{c}_{\mathfrak{p}}(R_S) - \overline{c}_{\mathfrak{n}}(R_S) = \dim \mathfrak{p}/\mathfrak{n}.$$

Since R_S is a Richardson element, the element R_S is Ad_P -conjugate to the standard Richardson element R . Their stabilizers $C_{\mathfrak{p}}(R_S)$ and $C_{\mathfrak{p}}(R)$ are Ad_P -conjugate. Their intersections with the radical \mathfrak{u} are also Ad_P -conjugate. The dimension $\overline{c}_{\mathfrak{p}}(R_S)$ coincides with the dimension $\overline{c}_{\mathfrak{p}}(R)$ of the subalgebra $C_{\mathfrak{p}}(R)$ modulo \mathfrak{u} .

Denote $y := \dim \mathcal{Y}$. We obtain the equality that is equivalent to (5):

$$y + \overline{c}_{\mathfrak{p}}(R) - \overline{c}_{\mathfrak{n}}(R_S) = \dim \mathfrak{p}/\mathfrak{n}. \tag{6}$$

Item 2. Consider the case $m = r$ (the number of the last column of diagram D). The number $|\overline{\mathcal{E}}_{\uparrow,r}|$ equals to the number of arrows from bottom up $i_r \rightarrow j_r$ of the last column. We get $|\overline{\mathcal{E}}_{\uparrow,r}| = \dim \mathfrak{ut}(n_r)$.

If $n_r \leq n_{r,\text{left}}$, then $y_r = n_r$ and $|\overline{\mathcal{E}}_{\downarrow,r}| = |\mathbb{V}_{S,r}| = 0$. It implies equality (7).

Suppose that $n_r > b$, where $b = n_{r,\text{left}}$. The number b coincides with the number of lines in the diagram D passing the last column. Then

$$y_r = b, \quad |\overline{\mathcal{E}}_{\downarrow,r}| = \dim \mathfrak{t}(n_r - b) = n_r - b + \dim \mathfrak{ut}(n_r - b),$$

$$|\overline{\mathcal{E}}_{\uparrow,r}| = \dim \mathfrak{ut}(n_r), \quad |\mathbb{V}_{S,r}| = \dim \mathfrak{ut}(n_r - b).$$

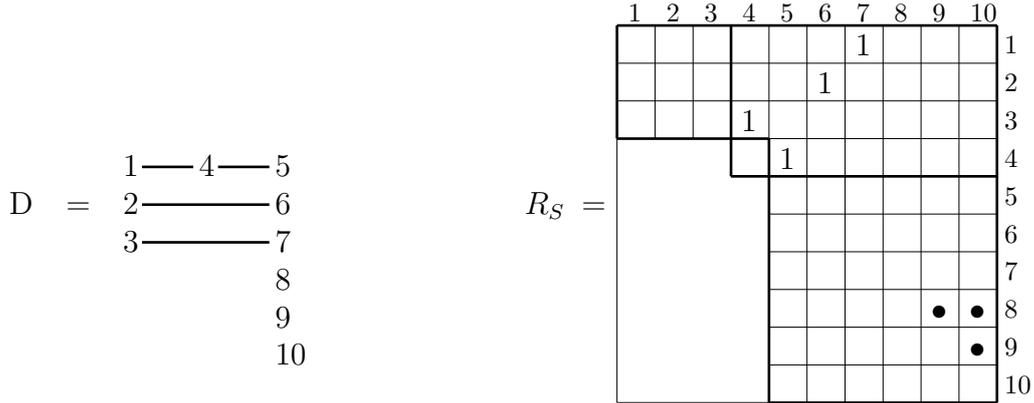
We obtain

$$y_r + |\overline{\mathcal{E}}_{\downarrow,r}| + |\overline{\mathcal{E}}_{\uparrow,r}| - |\mathbb{V}_{S,r}|$$

$$= b + (n_r - b) + \dim \mathfrak{ut}(n_r - b) + \dim \mathfrak{ut}(n_r) - \dim \mathfrak{ut}(n_r - b) = \dim \mathfrak{t}(n_r).$$

This proves (7) for $m = r$.

Example 2. Let the parabolic subalgebra \mathfrak{p} be of the type $(3, 1, 6)$ and $m = 3$. On the matrix R_S , we fill the squares from $\mathbb{V}_S = \mathbb{V}_{S,3}$ by the symbol \bullet . In this case $y_3 = 3$, $|\overline{\mathcal{E}}_{\downarrow,3}| = |\{(i, j) : 8 \leq i \leq j \leq 10\}| = 6$, $|\overline{\mathcal{E}}_{\uparrow,3}| = |\{(i, j) : 5 \leq i < j \leq 10\}| = 15$, $|\mathbb{V}_{S,3}| = 3$. Equality (7) has the form $3 + 6 + 15 - 3 = 21 = \dim \mathfrak{t}(6)$.



Item 3. Let $1 < m < r$. In this item, we suppose that the m th column is not the strongly greatest column in D that is, $n_m \leq n_{m,\text{left}}$ or $n_m \leq n_{m,\text{right}}$.

Case 3a. Suppose that $n_m \leq n_{m,\text{left}}$ and $n_{m,\text{right}} < n_m$. Denote $d = n_{m,\text{right}}$. The number of roots of S in the m th block-row equals $\min(n_m, n_{m,\text{right}}) = \min(n_m, d) = d$ (we use this fact in the proof of Lemma 2.11). The positive roots $\alpha_q \in \Delta^+(\mathfrak{n}_m)$ fill all squares in $\Delta^+(\mathfrak{n}_m)$ apart from the left upper $(n_m - d, n_m - d)$ -block. Then

$$\phi_m = \dim \mathfrak{ut}(n_m) - \dim \mathfrak{ut}(n_m - d).$$

We have $s_m = n_m$, $y_m = s_m + \phi_m = n_m + \dim \mathfrak{ut}(n_m) - \dim \mathfrak{ut}(n_m - d)$, $|\overline{\mathcal{E}}_{\downarrow,m}| = 0$, $|\overline{\mathcal{E}}_{\uparrow,m}| = \dim \mathfrak{ut}(n_m - d)$, $|\mathbb{V}_{S,m}| = 0$. We obtain

$$y_m + |\overline{\mathcal{E}}_{\downarrow,m}| + |\overline{\mathcal{E}}_{\uparrow,m}| - |\mathbb{V}_{S,m}|$$

$$= n_m + \dim \mathfrak{ut}(n_m) - \dim \mathfrak{ut}(n_m - d) + \dim \mathfrak{ut}(n_m - d) = n_m + \dim \mathfrak{ut}(n_m) = \dim \mathfrak{t}(n_m).$$

This proves (7).

the left upper $(n_m - d, n_m - d)$ -block and the right lower $(n_m - b, n_m - b)$ -block. Since $n_m - (n_m - b) - (n_m - d) = -n_m + b + d > 0$, these two blocks do not intersect. We have

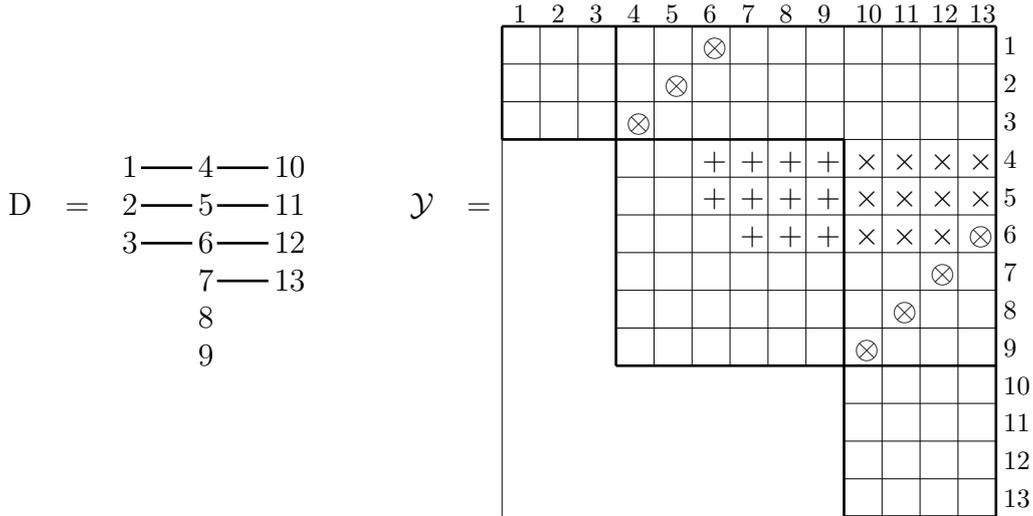
$$\begin{aligned} s_m &= b, & \phi_m &= \dim \mathfrak{ut}(n_m) - \dim \mathfrak{ut}(n_m - b) - \dim \mathfrak{ut}(n_m - d), \\ y_m &= b + \dim \mathfrak{ut}(n_m) - \dim \mathfrak{ut}(n_m - b) - \dim \mathfrak{ut}(n_m - d), \\ |\bar{\mathcal{E}}_{\downarrow, m}| &= \dim \mathfrak{t}(n_m - b), & |\bar{\mathcal{E}}_{\uparrow, m}| &= \dim \mathfrak{ut}(n_m - d), & |\mathbb{V}_{S, m}| &= 0. \end{aligned}$$

Finally we obtain

$$\begin{aligned} &y_m + |\bar{\mathcal{E}}_{\downarrow, m}| + |\bar{\mathcal{E}}_{\uparrow, m}| - |\mathbb{V}_{S, m}| \\ &= b + \dim \mathfrak{ut}(n_m) - \dim \mathfrak{ut}(n_m - b) - \dim \mathfrak{ut}(n_m - d) + \dim \mathfrak{t}(n_m - b) + \dim \mathfrak{ut}(n_m - d) \\ &= n_m + \dim \mathfrak{ut}(n_m) = \dim \mathfrak{t}(n_m). \end{aligned}$$

This proves (7).

Example 4b. Let the parabolic subalgebra \mathfrak{p} be of the type $(3, 6, 4)$ and $m = 2$. On the diagram \mathcal{Y} , we depict elements from S by the symbol \otimes , the elements from Φ by the symbol \times , the elements $\{\alpha_q\}$ for VS-pairs by the symbol $+$. Recall that $|\{\alpha_q\}| = |\Phi|$. Here $s_2 = 3$, $\phi_2 = 11$, $y_2 = 14$, $|\bar{\mathcal{E}}_{\downarrow, 2}| = |\{(i, j) : 7 \leq i \leq j \leq 9\}| = 6$, $|\bar{\mathcal{E}}_{\uparrow, 2}| = |\{(i, j) : 8 \leq i < j \leq 9\}| = 1$, $|\mathbb{V}_{S, 2}| = 0$, $\dim \mathfrak{t}(n_2) = 21$. The equality (7) has the form $14 + 6 + 1 - 0 = 21 = \dim \mathfrak{t}(6)$. \blacksquare



3. Field of N -invariants

The unitriangular group $UT(n)$ acts on the radical \mathfrak{u} by the adjoint representation. In this section, we present the system of free generators for the field of Ad_N -invariants $K(\mathfrak{u})^N$. Following the papers [9, 12], we construct two systems of Ad_N -invariants: $\{M_\xi\}$, where ξ runs through S , and $\{L_q\}$, where q runs through the set of VS-pairs Q .

For each $\gamma \in \Delta_{\mathfrak{u}}^+$, we denote by S_γ the subset of all roots $\alpha \in S$ such that $\text{row}(\alpha) > \text{row}(\gamma)$ and $\text{col}(\alpha) < \text{col}(\gamma)$. Roughly speaking S_γ is the subset of all roots of S that are lying below and on the left side from the root γ .

Let $x \in \mathfrak{u}$. We denote by M_γ the minor of x with the system of rows $\text{row}(S_\gamma \cup \{\gamma\})$ and columns $\text{col}(S_\gamma \cup \{\gamma\})$.

For each VS-pair $q = (\xi, \xi')$, we define the polynomial

$$L_q = \sum_{\substack{\alpha_1, \alpha_2 \in \Delta_1^+ \sqcup \{0\} \\ \alpha_1 + \alpha_2 = \alpha_q}} M_{\xi + \alpha_1} M_{\alpha_2 + \xi'}.$$

The polynomials $\Psi = \{M_\xi : \xi \in S\} \sqcup \{L_q : q \in Q\}$ (8)

are Ad_N -invariants [9, Theorem 1]. Observe that the restriction of M_ξ and L_q on the element $y \in \mathcal{Y}$ from (1) are calculated by the formulas (see formulas (3) and (4) from [9]):

$$M_\xi(y) = \pm y_\xi \cdot \prod_{\alpha \in S_\xi} y_\alpha, \quad L_q(y) = \pm y_{\varphi_q} \cdot y_\xi \left(\prod_{\alpha \in S_\xi} y_\alpha \right) \left(\prod_{\beta \in S_{\xi'}} y_\beta \right). \quad (9)$$

Consider the open subset $\mathfrak{u}_0 = \{x \in \mathfrak{u} : M_\xi(x) \neq 0 \text{ for all } \xi \in S\}$ and its intersection $\mathcal{Y}_0 = \mathfrak{u}_0 \cap \mathcal{Y}$, which is a nonempty open subset in \mathcal{Y} .

Proposition 3.1. (1) *Every element $y_0 \in \mathcal{Y}_0$ is uniquely determined by the system of the values $\Psi(y_0)$ at y_0 .*

(2) *For every $y_0 \in \mathcal{Y}_0$ the intersection $\text{Ad}_N(y_0) \cap \mathcal{Y}$ consists of the only y_0 .*

Proof. Recall that the system of roots S is constructed consecutively by the induction process: first we define S_1 , then S_2 and so on. If $\xi_1 \in S_1$, then $M_{\xi_1}(y_0) = y_{\xi_1}$. If $\xi_2 \in S_2$, then $M_{\xi_2}(y_0) = \pm y_{\xi_2} \prod_{\xi_1 \in S'_1} M_{\xi_1}(y_0)$ for some subset $S'_1 \subset S_1$. We have

$$y_{\xi_2} = \pm M_{\xi_2}(y_0) \left(\prod_{\xi_1 \in S'_1} M_{\xi_1}(y_0) \right)^{-1}.$$

Given the system of numbers $\Psi(y_0) = \{M_\xi(y_0), L_q(y_0)\}$, we obtain the system of coordinates $\{y_\xi \neq 0, y_{\varphi_q}\}$ applying formulas (9). This proves 1) and 2). ■

Define the regular map $\rho : \mathfrak{u}_0 \rightarrow \mathcal{Y}_0$ such that $\rho(x_0) = y_0$ if $M_\xi(x_0) = M_\xi(y_0)$ and $L_q(x_0) = L_q(y_0)$ for each $\xi \in S$ and $q \in Q$. The regular map ρ is surjective. Since M_ξ and L_q are Ad_N -invariants, the fibers of ρ are invariant with respect to Ad_N . It follows from the fiber dimension theorem that for every $y_0 \in \mathcal{Y}_0$ the dimension of $\rho^{-1}(y_0)$ is greater than or equal to $\dim \mathfrak{u} - \dim \mathcal{Y}$. There exists a nonempty open subset $\mathcal{Y}_1 \in \mathcal{Y}_0$ such that $\dim \rho^{-1}(y_1) = \dim \mathfrak{u} - \dim \mathcal{Y}$ for every $y_1 \in \mathcal{Y}_1$. ■

Proposition 3.2. *For every $x \in \mathfrak{u}$ the dimension $\text{Ad}_N(x)$ is less than or equal to $\dim \mathfrak{u} - \dim \mathcal{Y}$.*

Proof. The set of Ad_N -orbits of maximal dimension $\mathfrak{u}_{\text{reg}}$ is open in \mathfrak{u} . The intersection of two nonempty open subsets $\mathfrak{u}_{\text{reg}} \cap \rho^{-1}(\mathcal{Y}_1)$ is also open and nonempty. There exists $x_{\text{reg}} \in \rho^{-1}(y_1) \cap \mathfrak{u}_{\text{reg}}$ for some $y_1 \in \mathcal{Y}_1$. Then $\text{Ad}_N x_{\text{reg}} \subset \rho^{-1}(y_1)$ and

$$\dim \text{Ad}_N x_{\text{reg}} \leq \dim \rho^{-1}(y_1) = \dim \mathfrak{u} - \dim \mathcal{Y}.$$

For every $x \in \mathfrak{u}$, we have $\dim \text{Ad}_N x \leq \dim \text{Ad}_N x_{\text{reg}} \leq \dim \mathfrak{u} - \dim \mathcal{Y}$. ■

Proposition 3.3. *The subset $\text{Ad}_N \mathcal{Y}$ is dense in \mathfrak{u} .*

Proof. According to Theorem 2.13, the element R_S is Ad_N -regular, i.e. $R_S \in \mathfrak{u}_{\text{reg}}$. The subset $\mathfrak{u}_{\text{reg}}$ is open in \mathfrak{u} . The intersection $\mathfrak{u}_{\text{reg}} \cap \mathcal{Y}$ is open in \mathcal{Y} and

nonempty (it contains R_S). The subset $\mathfrak{u}_{\text{reg}} \cap \mathcal{Y}_1$ is an intersection of two nonempty open subsets \mathcal{Y}_1 and $\mathfrak{u}_{\text{reg}} \cap \mathcal{Y}$. Therefore, $\mathfrak{u}_{\text{reg}} \cap \mathcal{Y}_1$ is open and nonempty. For every $y_1 \in \mathfrak{u}_{\text{reg}} \cap \mathcal{Y}_1$, we have $\dim \text{Ad}_N y_1 = \dim \rho^{-1}(y_1) = \dim \mathfrak{u} - \dim \mathcal{Y}$. An orbit of every unipotent group action on an affine variety is closed (see [10, §1.3]). So, the orbit $\text{Ad}_N y_1$ is closed. Two closed subsets $\text{Ad}_N y_1 \subseteq \rho^{-1}(y_1)$ have a common dimension. Hence $\text{Ad}_N y_1 = \rho^{-1}(y_1)$. The subset $\text{Ad}_N \mathcal{Y}_1$ coincides with $\rho^{-1}(\mathcal{Y}_1)$, and therefore it is open in \mathfrak{u} . This implies the statement. ■

Theorem 3.4. *The system (8) freely generates the field of N -invariants $K(\mathfrak{u})^N$.*

Proof. Consider the restriction map $\pi : K[\mathfrak{u}]^N \rightarrow K[\mathcal{Y}]$, which corresponds for every polynomial from $K[\mathfrak{u}]^N$ its restriction on \mathcal{Y} . Since $\text{Ad}_N \mathcal{Y}$ is dense in \mathfrak{u} , we have $\text{Ker } \pi = 0$. The map π is extended to an embedding of fields $\pi : K(\mathfrak{u})^N \hookrightarrow K(\mathcal{Y})$. By formulas (9), the polynomials $\pi(\Psi)$ freely generate the field $K(\mathcal{Y})$. The map π is an isomorphism of fields and the system of polynomials Ψ freely generate the field of Ad_N -invariants. ■

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