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On the Rapidly Decreasing Property of Whittaker Functions for $Sp(2, \mathbb{R})$

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Abstract. The notion of Whittaker functions on quasi-split reductive groups over local fields is usually defined for non-degenerate characters of the maximal unipotent subgroups. In this paper the case of the real symplectic group of degree two is taken up. It is remarked for irreducible generic representations that if Whittaker functions in the usual sense are of moderate growth, they are always rapidly decreasing. The target of this paper is also Whittaker functions on the symplectic group of degree two for degenerate characters, which have been out of the targets in many studies. Motivated by the theory of the Fourier-Jacobi expansion of generic cusp forms, it is proved that there is no such Whittaker functions of rapidly decay for irreducible generic representations.

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1. Introduction

In the representation theory of quasi-split reductive groups G over local fields irreducible generic representations are recognized as a quite familiar class. Recall that an irreducible admissible representation π of G is called generic if, with the usual notation $\mathrm{Ind}_N^G(\psi)$ of the induced representation,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G(\psi)) \neq 0$$

holds for some non-degenerate character ψ of a maximal unipotent subgroup N, namely π admits a Whittaker model. For a definition of the non-degenerate characters see e.g. [28, Section 1], [30, Section 8.5] and [27, Section 3], in the latter two of which they are also called generic characters. This notion is naturally extended to irreducible admissible representations of adele groups for quasi-split group over a global field. In the theory of automorphic representations of adele groups irreducible generic automorphic representations have been studied in detail and drawn attention of experts since one can often discuss such representation in a general context e.g. by Langlands-Shahidi method (cf. [15], [27]), which provides a strong tool to investigate automorphic L-functions.

If such global Whittaker models are unique up to constant multiples, their studies are reduced to those of local models. Regarding this we review the multiplicity free

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property of Whittaker models for irreducible admissible representations of quasi-split real groups as follows (cf. [30, Theorem 8.8]):

$$\dim \mathrm{Hom}_G(\pi, \mathcal{A}_\psi(N\backslash G)) \le 1,$$

where $\mathcal{A}_\psi(N\backslash G)$ denotes the subspace of C^∞ sections of moderate growth in $\mathrm{Ind}_N^G(\psi)$.

However, we should note that there remain fundamental questions as follows:

- For what π and ψ, $\dim \mathrm{Hom}_G(\pi, \mathcal{A}_\psi(N\backslash G)) = 1$?
- When $\dim \mathrm{Hom}_G(\pi, \mathcal{A}_\psi(N\backslash G)) = 1$, is the Whittaker model rapidly decreasing?

As for the first question, we remark that even if π is generic, $\mathrm{Hom}_G(\pi, \mathcal{A}_\psi(N\backslash G))$ can be zero depending on ψ as known results (e.g. Oda [25], Miyazaki-Oda [17], Niwa [24] and Ishii [12] et al.) indicate (cf. Section 4). The second question naturally arises when one is interested in Whittaker models of cuspidal automorphic representations. In this paper we think of these problems for the real symplectic group $G = Sp(2,\mathbb{R})$ of degree two. For this case we provide a complete answer to the second question and point out that known results answer to the first question. In addition, we are interested in Whittaker models for degenerate characters. We formulate the notion of Whittaker models as intertwining operators for $(\mathfrak{g}, K)$-modules (cf. [31, Section 3.3.1]) with the Lie algebra $\mathfrak{g}$ of $Sp(2,\mathbb{R})$ and a maximal compact subgroup K. The statement of our theorem is as follows (cf. Theorem 4.9):

Theorem 1.1. (Main Theorem) *Let $G = Sp(2,\mathbb{R})$ and let us denote $\mathcal{S}_\psi(N\backslash G)$ by the subspace of C^∞-sections with rapid decay in $\mathrm{Ind}_N^G(\psi)$.*

(1) *Let π be any irreducible generic representation of G, a (limit of) holomorphic or a (limit of) anti-holomorphic discrete series representation. We have*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N\backslash G)) = \{0\}$$

for any degenerate character ψ.

(2) *Let π be any irreducible generic representation of G. Any moderate growth Whittaker models for π in the usual sense (i.e. when ψ is non-degenerate) are necessarily rapidly decreasing. We therefore see that if π admits a non-zero Whittaker model of moderate growth for a non-degenerate character ψ of N,*

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N\backslash G)) = 1.$$

This theorem is motivated by the Fourier-Jacobi expansion of automorphic forms on $Sp(2,\mathbb{R})$ (cf. [23]). Whittaker models contributes to the Fourier-Jacobi expansion by Whittaker functions, which are images of the restriction map of the models to a multiplicity one K-type (cf. Section 4.1). In fact, this study is based on a detailed analysis of explicit formulas for the Whittaker functions. By virture of the explicit formulas we know precisely when the dimension above is one or zero for many irreducible generic representations in Theorems 4.2, 4.3, 4.4 and 4.5 (see also Kostant [14, Theorems J, 6.6.2]). When an automorphic form is generic, we have to consider the contributions to the Fourier expansion by Whittaker functions, which can be neglected for holomorphic automorphic forms. However, we cannot avoid studying Whittaker functions for degenerate characters, which cannot be neglected for holomorphic automorphic forms as well as non-holomorphic ones. From the theorem above we know that we do not have to think of Whittaker functions for degenerate characters when we consider the Fourier-Jacobi expansion of generic cusp forms. We remark that Whittaker models for degenerate characters have been taken up quite rarely. We cite [6] for instance. Different from Whittaker models for non-degenerate characters, the multiplicity one property of moderate growth ones often collapses for the case of degenerate characters (cf. [7, Theorem 8.2], [10, Theorems 5.6 and 5.7]).

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2. Basic notation

2.1. Basic notation for real groups

Let $G = Sp(2,\mathbb{R})$ be the real symplectic group of degree two defined by

$$\{g \in GL_4(\mathbb{R}) \mid {}^t g J_4 g = J_4\},$$

where $J_4 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$. This has two maximal parabolic subgroups called the Jacobi (or Klingen) parabolic subgroup P_J and the Siegel parabolic subgroup P_S up to conjugation.

We first make a review on the group P_J and its subgroups. The parabolic subgroup P_J has the Langlands decomposition $P_J = N_J A_J M_J$ with

$$A_J := \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a_1 \in \mathbb{R}_{>0} \right\}, \quad M_J := \left\{ \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \epsilon & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \, \begin{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \\ \epsilon \in \{\pm 1\} \end{matrix} \right\}$$

and

$$N_J = \left\{ n(u_0, u_1, u_2) := \begin{pmatrix} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_0 & 1 \end{pmatrix} \middle| \, u_0, u_1, u_2 \in \mathbb{R} \right\}.$$

The group N_J is the unipotent radical of P_J, which is a Heisenberg group with the center $\{n(0, u_1, 0) \mid u_1 \in \mathbb{R}\}$.

We next review another maximal parabolic subgroup P_S. Its Langlands decomposition is given by $P_S = N_S A_S M_S$ with

$$A_S := \{\mathrm{diag}(a, a, a^{-1}, a^{-1}) \mid a \in \mathbb{R}_{>0}\},\ M_S := \left\{ \begin{pmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{pmatrix} \;\middle|\; A \in SL_2^{\pm}(\mathbb{R}) \right\}$$

and
$$N_S := \left\{ \begin{pmatrix} 1_2 & X \\ 0_2 & 1_2 \end{pmatrix} \;\middle|\; X = {}^tX \in M_2(\mathbb{R}) \right\}.$$

Here we use the notation $SL_2^{\pm}(\mathbb{R}) := \{A \in GL_2(\mathbb{R}) \mid \det(A) = \pm 1\}$. We obviously have $M_S \simeq SL_2^{\pm}(\mathbb{R})$.

We also need the minimal parabolic subgroup P_0 of G with the unipotent radical N_0, where N_0 is defined by

$$\left\{ n(u_0, u_1, u_2, u_3) = \begin{pmatrix} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & u_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_0 & 1 \end{pmatrix} \;\middle|\; u_i \in \mathbb{R}\ (0 \le i \le 3) \right\}.$$

We also review the Langlands decomposition $P_0 = N_0 A_0 M_0$ of the minimal parabolic subgroup P_0, where

$$A_0 := \{a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1,\ a_2 \in \mathbb{R}_{>0}\},$$
$$M_0 := \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2) \mid \epsilon_1,\ \epsilon_2 \in \{\pm 1\}\}.$$

Let us introduce the Cartan involution θ of G defined by $\theta(g) := {}^tg^{-1}$ for $g \in G$.

Then
$$K := \{g \in G \mid \theta(g) = g\} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \;\middle|\; A, B \in M_2(\mathbb{R}) \right\}$$

is a maximal compact subgroup of G. This is isomorphic to the unitary group $U(2)$ of degree two by the map

$$K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2).$$

We should remark that G has an Iwasawa decomposition $G = N_0 A_0 K$ with the notation above.

2.2. Lie algebras and root system

Following the standard manner of the notation, we denote the Lie algebras of real Lie groups by the corresponding German letters. For a real Lie algebra $\mathfrak{l}$ we denote its complexification by $\mathfrak{l}_{\mathbb{C}}$.

The Lie algebra $\mathfrak{g}$ of G is given by $\{X \in M_4(\mathbb{R}) \mid {}^tXJ_4 + J_4X = 0_4\}$. The Cartan involution of $\mathfrak{g}$, denoted also by θ, is defined by $\theta(X) = -{}^tX$ for $X \in \mathfrak{g}$. Then $\mathfrak{g}$ has the eigen-space decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; A,\ B \in M_2(\mathbb{R}),\ {}^tA = -A,\ {}^tB = B \right\},$$
$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \;\middle|\; A,\ B \in M_2(\mathbb{R}),\ {}^tA = A,\ {}^tB = B \right\}.$$

The former is nothing but the Lie algebra of K.

We consider the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to the complexification $\mathfrak{t}_{\mathbb{C}}$ of the compact Cartan subalgebra $\mathfrak{t} = \mathbb{R}T_1 \oplus \mathbb{R}T_2$ (in $\mathfrak{k}$), where

$$T_1 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The dual space $\mathfrak{t}_{\mathbb{C}}^*$ of $\mathfrak{t}_{\mathbb{C}}$ has a basis $\{\beta_1, \beta_2\}$ given by

$$\beta_i(T_j) = \sqrt{-1}\delta_{ij}.$$

We denote $\beta \in \mathfrak{t}_{\mathbb{C}}^*$ by (a, b) if $\beta = a\beta_1 + b\beta_2$. Hence the set of roots for the root space decomposition $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is given by $\Delta := \{\pm(2,0),\ \pm(0,2),\ \pm(1,1),\ \pm(1,-1)\}$. It has the standard choice of positive roots given by $\Delta^+ := \{(2,0), (0,2), (1,1), (1,-1)\}$. The roots $\{\pm(1,-1)\}$ forms the set of compact roots, whose roots vectors are in $\mathfrak{k}_{\mathbb{C}}$. Each root in $\{\pm(2,0),\ \pm(0,2),\ \pm(1,1)\}$ is called a non-compact root, whose root vector is in the complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$.

3. Representations of real groups

3.1. Discrete series representations of $SL_2(\mathbb{R})$ and $SL_2^{\pm}(\mathbb{R})$

We will need in the following discrete series representations of the groups $SL_2(\mathbb{R})$ and $SL_2^{\pm}(\mathbb{R}) := \{h \in GL_2(\mathbb{R}) \mid \det(h) = \pm 1\}$ to introduce parabolic inductions from maximal parabolic subgroups of $Sp(2, \mathbb{R})$. For $n \geq 1$, by $\mathcal{D}_n^+$ (respectively $\mathcal{D}_n^-$), we denote the discrete series representation with lowest weight n (respectively highest weight $-n$). A discrete series representation $\mathcal{D}_n^+$ (respectively $\mathcal{D}_n^-$) is called holomorphic (respectively anti-holomorphic). When $n = 1$, $\mathcal{D}_1^{\pm}$ is called a limit of holomorphic or anti-holomorphic discrete series representation. In addition to this, we recall that (limits) of discrete series representations of $SL_2^{\pm}(\mathbb{R})$ are of the form

$$\mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} \mathcal{D}_n^+ \simeq \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} \mathcal{D}_n^-,$$

which is isomorphic to $\mathcal{D}_n^+ \oplus \mathcal{D}_n^-$ as unitary representations of $SL_2(\mathbb{R})$. We denote this by $\mathcal{D}_n$.

3.2. Representations of the maximal compact subgroup K

It is a fundamental fact that irreducible finite dimensional representations of a compact linear connected real reductive group are parametrized by dominant weights. For the maximal compact subgroup K of G the set of equivalence classes of irreducible finite dimensional representations of K are in bijection with the set of the dominant weights $\{(\Lambda_1, \Lambda_2) \in \mathbb{Z}^2 \mid \Lambda_1 \geq \Lambda_2\}$, where recall that (Λ_1, Λ_2) denotes the root $\Lambda_1\beta_1 + \Lambda_2\beta_2$ (cf. Section 2.2). The dominance respects the compact positive root $(1,-1)$. This is noting but the fact that the set of equivalence classes of irreducible finite dimensional representations of $U(2) \simeq K$ is parametrized by this set of dominant weights. For each dominant weight $\Lambda = (\Lambda_1, \Lambda_2)$, let τ_Λ be the irreducible representation of K parametrized by Λ, which is the pullback of the irreducible representation $\det^{\Lambda_2} \mathrm{Sym}^{\Lambda_1 - \Lambda_2}$ of $U(2)$ via the isomorphism $K \simeq U(2)$.

Here $\mathrm{Sym}^{\Lambda_1-\Lambda_2}$ denotes the $(\Lambda_1-\Lambda_2)$-th symmetric tensor representation of the 2-dimensional standard representation of $U(2)$. This representation is realized on

$$V_\Lambda := \{p \in \mathbb{C}[x_1,x_2] \mid p \text{ is homogeneous of degree } \Lambda_1-\Lambda_2\}$$

by $$\tau_\Lambda(u)p((x_1,x_2)) = \det(u)^{\Lambda_2} p((x_1,x_2)\,{}^t u) \quad (\forall u \in GL_2(\mathbb{C})).$$

In what follows, we put $d_\Lambda := \Lambda_1-\Lambda_2$. We have $\dim V_\Lambda = d_\Lambda+1$. We will use two bases of V_Λ as follows:

$$\{v_k := x_1^k x_2^{d_\Lambda-k} \mid 0 \le k \le d_\Lambda\},$$
$$\{u_k := (x_1+\sqrt{-1}x_2)^k (x_1-\sqrt{-1}x_2)^{d_\Lambda-k} \mid 0 \le k \le d_\Lambda\}.$$

In addition to these, we also need the following basis $\{u_k^{(l+n,l)} \mid 0 \le k \le n\}$ of $V_{(l+n,l)}$ for $l \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\ge 2}$: For $k=2j$ with $0 \le j \le [n/2]$,

$$u_{2j}^{(l+n,l)} := \begin{cases} (x_1^2+x_2^2)^{n/2-j}(x_1^2-x_2^2)^j & \text{(when } n \text{ is even)} \\ x_2(x_1^2+x_2^2)^{(n-1)/2-j}(x_1^2-x_2^2)^j & \text{(when } n \text{ is odd),} \end{cases}$$

and for $k=2j+1$ with $0 \le j \le [(n-1)/2]$,

$$u_{2j+1}^{(l+n,l)} := \begin{cases} x_1x_2(x_1^2+x_2^2)^{n/2-j-1}(x_1^2-x_2^2)^j & \text{(when } n \text{ is even)} \\ x_1(x_1^2+x_2^2)^{(n-1)/2-j}(x_1^2-x_2^2)^j & \text{(when } n \text{ is odd).} \end{cases}$$

This basis is used only to describe the Whittaker functions for P_S-principal series in Section 4.4. Though it is given in [12, Definition 1.3] for general dominant weights Λ we limit ourselves to the case of $\Lambda = (l+n,l)$ as above.

Let $(\tau_\Lambda^*, V_\Lambda^*)$ be the representation of K contragredient to $(\tau_\Lambda, V_\Lambda)$. The space V_Λ^* has a basis $\{v_k^*\}_{0\le k\le d_\Lambda}$ dual to $\{v_k\}_{0\le k \le d_\Lambda}$. Note that the highest weight of $(\tau_\Lambda^*, V_\Lambda^*)$ is $(-\Lambda_2,-\Lambda_1)$ and d_Λ remains the same under such replacement of the highest weights.

On the other hand, we introduce a basis $\{u_k^* \mid 0 \le k \le d_\Lambda\}$ of V_Λ^* dual to $\{u_k \mid 0 \le k \le d_\Lambda\}$. We write down the following formula

$$\tau_\Lambda(Z')u_k = \sqrt{-1}(2k-d_\Lambda)u_k,\ \tau_\Lambda^*(Z')u_k^* = \sqrt{-1}(2k-d_\Lambda)u_k^* \quad Z' := \begin{pmatrix} J_2 & 0_2 \\ 0_2 & J_2 \end{pmatrix} \in \mathfrak{k},$$

where $J_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This formula indicates that the two bases $\{u_k \mid 0 \le k \le d_\Lambda\}$ and $\{u_k^* \mid 0 \le k \le d_\Lambda\}$ are suitable to describe the infinitesimal action of τ_Λ and τ_Λ^* restricted to the Lie algebra of $P_S \cap K$ respectively. This is convenient to understand the infinitesimal actions by P_S-principal series representations taken up in Sections 3.5. For this see also (iii) in the proof of Proposition 4.7.

3.3. The (limits of) discrete series representations of G

We provide Harish-Chandra's parametrization of discrete series representations for G, namely irreducible unitary representations of G whose matrix coefficients belong to $L^2(G)$. More precisely we also include the limits of discrete series representations for our later discussion. Our discussion is based on [3, Theorem 16] and [13, Theorems 9.20, 12.21, 12.26, Corollary 12.27] with the help of [14] and [29].

To parametrize the discrete series representations we need regular dominant analytically integral weights, namely regular dominant weights coming from the derivatives of unitary characters of the compact Cartan subgroup $\exp(\mathfrak{t})$, where see Section 2.2 for the notation $\mathfrak{t}$. Now note that unitary characters of $\exp(\mathfrak{t})$ are parametrized by

$$\{a\beta_1 + b\beta_2 \mid (a,b) \in \mathbb{Z}^2\}.$$

We have identified this set with $\mathbb{Z}^2$. With the set Δ^+ of positive roots (cf. Section 2.2) we then see that the set of the regular analytically integral weights is in bijection with

$$\Xi'' := \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \mid \langle \lambda, \beta \rangle \neq 0 \ \forall \beta \in \Delta^+\}.$$

Here $\langle *, * \rangle$ denotes the inner product induced by the Killing form of $\mathfrak{g}_{\mathbb{C}}$, which can be regarded as the standard inner product of the two dimensional Euclidean space $\mathbb{R}^2$ in terms of the identification $\{a\beta_1 + b\beta_2 \mid (a,b) \in \mathbb{R}^2\} = \mathbb{R}^2$.

By Harish-Chandra's parametrization we mean the classification of discrete series representations in terms of the infinitesimal equivalence (which means the unitary equivalence of discrete series). Let π_λ be the discrete series representation of G parametrized by $\lambda \in \Xi''$. It is known that π_λ and $\pi_{\lambda'}$ are infinitesimally equivalent if and only if λ and λ' are conjugate by the Weyl group of $\mathfrak{k}$. As a result we see that the equivalence classes of discrete series representations of G are in bijection with

$$\Xi' := \{\lambda \in \Xi'' \mid \langle \lambda, (1,-1) \rangle > 0\} = \{(\lambda_1, \lambda_2) \in \Xi'' \mid \lambda_1 > \lambda_2\},$$

each of which is called a Harish Chandra parameter (cf. [13, Terminology after Theorem 9.20]).

Now we introduce the following four sets Δ_J^+ of positive roots system with capital roman letters $I \le J \le IV$, together with the set $\Delta_c^+ := \{(1,-1)\}$ of the compact positive root. We further need the sets $\Delta_{J,n}^+$ of non-compact positive roots with $I \le J \le IV$, specified as follows:

$$\begin{aligned}
\Delta_I^+ &:= \Delta_{I,n}^+ \cup \Delta_c^+ \quad \text{with } \Delta_{I,n}^+ := \{(2,0),\ (0,2),\ (1,1)\},\\
\Delta_{II}^+ &:= \Delta_{II,n}^+ \cup \Delta_c^+ \quad \text{with } \Delta_{II,n}^+ := \{(2,0),\ (0,-2),\ (1,1)\},\\
\Delta_{III}^+ &:= \Delta_{III,n}^+ \cup \Delta_c^+ \quad \text{with } \Delta_{III,n}^+ := \{(2,0),\ (0,-2),\ (-1,-1)\},\\
\Delta_{IV}^+ &:= \Delta_{IV,n}^+ \cup \Delta_c^+ \quad \text{with } \Delta_{IV,n}^+ := \{(-2,0),\ (0,-2),\ (-1,-1)\}.
\end{aligned}$$

For each J with $I \le J \le IV$ we put

$$\begin{aligned}
\Xi_J &:= \{\lambda \in \mathbb{Z}^2 \mid \langle \lambda, \alpha \rangle \ge 0 \ \forall \alpha \in \Delta_{J,n}^+,\ \langle \lambda, \beta \rangle > 0 \ \forall \beta \in \Delta_c^+ \text{s.t. } \beta \text{ is simple}\},\\
\Xi'_J &:= \{\lambda \in \Xi' \mid \langle \lambda, \alpha \rangle > 0 \ \forall \alpha \in \Delta_{J,n}^+\}.
\end{aligned}$$

These sets are explicitly given as follows:

$$\begin{aligned}
\Xi_I &= \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{\ge 0}^2 \mid \lambda_1 > \lambda_2\} \supset \Xi'_I = \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{>0}^2 \mid \lambda_1 > \lambda_2\},\\
\Xi_{II} &= \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\le 0}\} \mid \lambda_1 \ge -\lambda_2\} \supset \Xi'_{II} = \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{<0} \mid \lambda_1 > -\lambda_2\},\\
\Xi_{III} &= \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\le 0}\} \mid \lambda_1 \le -\lambda_2\} \supset \Xi'_{III} = \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{<0} \mid \lambda_1 < -\lambda_2\},\\
\Xi_{IV} &= \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{\le 0}^2 \mid \lambda_1 > \lambda_2\} \supset \Xi'_{IV} = \{(\lambda_1, \lambda_2) \in \mathbb{Z}_{<0}^2 \mid \lambda_1 > \lambda_2\}.
\end{aligned}$$

Discrete series representations (respectively The limits of discrete series representations) are parametrized by Ξ'_J (respectively $\Xi_J \setminus \Xi'_J$) for $I \leq J \leq IV$. Discrete series representations of G parametrized by Ξ'_I (respectively Ξ'_{IV}) are called holomorphic discrete series representations (respectively anti-holomorphic discrete series representations). The (limits) of discrete series representations of G parametrized by $\Xi_{II} \cup \Xi_{III}$ are called large in the sense of Vogan [29, Section 6]. The (limits of) large discrete series representations are characterized by the maximality of their Gelfand Kirillov dimensions (or by having the minimal primitive ideals) among the (limits of) discrete series representations. They are known to admit Whittaker models (cf. [14, Theorem 6.8.1]), and are therefore also called generic. In fact, we will see that they admits rapidly decreasing Whittaker models (cf. Theorem 4.2).

For $(\lambda_1, \lambda_2) \in \Xi'$ the discrete series representation parametrized by $(-\lambda_2, -\lambda_1)$ is contragredient to that parametrized by (λ_1, λ_2). We then see that the (limits of) discrete series representations parametrized by Ξ_{II} and Ξ_{III} are in bijection with contragredient representations of those parametrized by Ξ_{III} and Ξ_{II} respectively.

For each $\lambda \in \Xi'$ we recall that

$$\Lambda := \lambda + \rho_n - \rho_c$$

with the half sum ρ_n (respectively ρ_c) of non-compact positive roots (respectively compact positive roots) is called the Blattner parameter (cf. [13, Terminology after Theorem 9.20]), which provides the highest weight of the minimal K-type (cf. [13, p626]) of the discrete series representation π_λ. We should note that the minimal K type is the most important multiplicity one K-type for a discrete series representation. For $\lambda = (\lambda_1, \lambda_2) \in \Xi'_I$ or Ξ'_{IV} we have $\Lambda = (\lambda_1 + 1, \lambda_2 + 2)$ or $(\lambda_1 - 2, \lambda_2 - 1)$ respectively. When $\lambda = (\lambda_1, \lambda_2)$ is in Ξ'_{II} or Ξ'_{III}, we have $\Lambda = (\lambda_1 + 1, \lambda_2)$ or $(\lambda_1, \lambda_2 - 1)$ respectively. As for the limits of discrete series representations the similar explanation in terms of the minimal K-type is also valid.

3.4. P_J-principal series representations of G

We introduce the generalized principal series representations induced from the Jacobi parabolic subgroup P_J, which we call the P_J-principal series representations.

Let $\sigma := (D, \epsilon)$ be a representation of $M_J \simeq SL_2(\mathbb{R}) \times \{\pm 1_2\}$ with the sign character ϵ defined by $\epsilon(\pm 1_2) = \pm 1$ and a (limit of) discrete series representation $D = \mathcal{D}_n^{\pm}$ for $n \in \mathbb{Z}_{\geq 1}$ (cf. Section 3.1). Given $z \in \mathbb{C}$ we define the quasi character

$$\nu_z : A_J \ni \mathrm{diag}(a_1, 1, a_1^{-1}, 1) \mapsto a_1^z.$$

We then introduce the P_J-principal series representation $\mathrm{Ind}_{P_J}^G(1_{N_J} \otimes \nu_z \otimes \sigma)$ by the standard manner of the normalized parabolic induction.

By the Frobenius reciprocity of compact groups we can study the branching rule for the restriction of a P_J-principal series representation to K. Among the K-types τ_Λ of $\mathrm{Ind}_{P_J}^G(1_{N_J} \otimes \nu_z \otimes \sigma)$ the followings occur with multiplicity one (cf. [17, Proposition 2.1], [18]):

1. $\Lambda = (l, l)$ $(l \geq n,\ l \equiv n \mod 2)$ and $\Lambda = (n, l)$ $(l \leq n,\ l \equiv n \mod 2)$ for $\epsilon(-1_2) = (-1)^n$ and $D = \mathcal{D}_n^+$.
2. $\Lambda = (l, l-1)$ $(l \geq n)$ and $\Lambda = (n, l-1)$ $(l \leq n,\ l \equiv n \mod 2)$ for $\epsilon(-1_2) = -(-1)^n$ and $D = \mathcal{D}_n^+$.

3. $\Lambda = (l,l)$ $(l \le -n,\ l \equiv n \mod 2)$ and $\lambda = (l,-n)$ $(l \ge -n,\ l \equiv n \mod 2)$ for $\epsilon(-1_2) = (-1)^n$ and $D = \mathcal{D}_n^-$.
4. $\Lambda = (l+1,l)$ $(l \le -n)$ and $\lambda = (l+1,-n)$ $(l \ge -n,\ l \equiv n \mod 2)$ for $\epsilon(-1_2) = -(-1)^n$ and $\mathcal{D}_n^-$.

For these formulas we remark that the argument to use the Frobenius reciprocity works also for the case when $\mathcal{D}_n^{\pm}$ is a limit of discrete series $\mathcal{D}_1^{\pm}$ though [17, Proposition 2.1] and [18] state the formulas only when $\mathcal{D}_n^{\pm}$ is a usual discrete series. We call the K-type τ_Λ with $\Lambda = (n,n)$ (resp. $(n,n-1),(-n,-n)$ and $(-n+1,-n)$) the corner K-type of $\mathrm{Ind}_{P_J}^G(1_{N_J} \otimes \nu_z \otimes \sigma)$ for the case 1 (respectively 2, 3 and 4). For the cases 1 and 3 (respectively the cases 2 and 4) the dimension of the corner K-type is one (respectively two). Following [17] we call the P_J-principal series enumerated as 1 and 3 (respectively 2 and 4) even (respectively odd).

3.5. P_S-principal series representation of G

We also introduce another generalized principal series representations induced from the Siegel parabolic subgroup P_S, which we call the P_S-principal series representations.

Let us now recall that P_S has the Langlands decomposition $P_S = N_S A_S M_S$ (cf. Section 2.1). As representations of $M_S \simeq SL_2^{\pm}(\mathbb{R})$ we take discrete series representations $\mathcal{D}_n := \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} \mathcal{D}_n^+$ with a (limit of) holomorphic discrete series $\mathcal{D}_n^+$ (cf. Section 3.1), where $n \ge 1$. For $z \in \mathbb{C}$ we introduce the quasi character

$$\nu_z : A_S \ni \mathrm{diag}(a,a,a^{-1},a^{-1}) \mapsto a^z.$$

We then introduce the P_S-principal series representation $\mathrm{Ind}_{P_S}^G(1_{N_S} \otimes \nu_z \otimes \mathcal{D}_n)$ by the normalized parabolic induction. From [5, Chapter II, Proposition 1.2] we review the distribution of the K-type of $\mathrm{Ind}_{P_S}^G(1_{N_S} \otimes \nu_z \otimes \mathcal{D}_n)$ as follows:

Proposition 3.1. *The multiplicity of τ_Λ in the generalized principal series representation above is given by*

$$\begin{cases} \left[\dfrac{\Lambda_1 - \Lambda_2 - n}{2}\right] + 1 & (\Lambda_1 - \Lambda_2 - n \in 2\mathbb{Z}_{\ge 0}) \\ 0 & \textit{otherwise.} \end{cases}$$

We therefore see that the multiplicity of τ_Λ is one if and only if $(\Lambda_1, \Lambda_2) = (l+n, l)$ with $l \in \mathbb{Z}$. We call this multiplicity one K-type peripheral, following [5]. We remark that the explanation above is valid also for the limit $\mathcal{D}_1$ of discrete series representation.

3.6. Principal series representations of G (induced from the minimal parabolic subgroup)

Recall that P_0 denotes the minimal parabolic subgroup with the Langlands decomposition $P_0 = N_0 A_0 M_0$, where

$$A_0 := \{a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1,\ a_2 \in \mathbb{R}_{>0}\},$$
$$M_0 := \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2) \mid \epsilon_1,\ \epsilon_2 \in \{\pm 1\}\}$$

(cf. Section 2.1). Representations σ of M_0 are the sign characters determined by

$$\sigma_1 := \sigma(\mathrm{diag}(-1,1-1,1)) \in \{\pm 1\}, \quad \sigma_2 := \sigma(\mathrm{diag}(1,-1,1,-1)) \in \{\pm 1\}.$$

We introduce quasi characters of A_0 by

$$\nu_z : A_0 \ni a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mapsto a_1^{z_1} a_2^{z_2}$$

for $z = (z_1, z_2) \in \mathbb{C}^2$. With σ and ν_z above we define the principal series representation by the normalized parabolic induction $\mathrm{Ind}_{P_0}^G(1_{N_0} \otimes \nu_z \otimes \sigma)$.

From [16, Proposition 3.2] (see also [9, Proposition 2.1]) we have the list of the K-types of $\mathrm{Ind}_{P_0}^G(1_{N_0} \otimes \nu_z \otimes \sigma)$ with multiplicity one as follows:

1. $\Lambda = (l, l)$ with $l \in \mathbb{Z}$ such that $(-1)^l = \sigma_1 = \sigma_2$ for $(\sigma_1, \sigma_2) = \pm(1, 1)$.
2. $\Lambda = (l + 1, l)$ with $l \in \mathbb{Z}$ for $(\sigma_1, \sigma_2) = \pm(1, -1)$.

We call the principal series representations even (respectively odd) for the first case (respectively the second case). We note that the minimal K-type(s) of the principal series representation is/are given by $\tau_{(0,0)}$ (respectively $\{\tau_{(1,1)}, \tau_{(-1,-1)}\}$, and $\{\tau_{(1,0)}, \tau_{(0,-1)}\}$) if $(\sigma_1, \sigma_2) = (1, 1)$ (resp. $(-1, -1)$ and $(\pm 1, \mp 1)$). For the definition of the minimal K-type we cite [13, p. 626], as we also do in Section 3.3.

4. Whittaker functions for $Sp(2, \mathbb{R})$

4.1. A review on Whittaker functions

Let π be an admissible representation of $G = Sp(2, \mathbb{R})$ with a multiplicity one K-type τ and let ψ be a unitary character of the maximal unipotent subgroup N_0 of G. The Whittaker functions on G are defined as the elements in the image of the restriction

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, C_\psi^\infty(N_0 \backslash G)) \to \mathrm{Hom}_K(\tau, C_\psi^\infty(N_0 \backslash G))$$

to the K-type τ, where

$$C_\psi^\infty(N_0 \backslash G) := \{\phi \in C^\infty(G) \mid \phi(ng) = \psi(n)\phi(g) \quad \forall (n, g) \in N_0 \times G\}.$$

Now recall that (τ^*, V^*) is the notation for the contragredient representation of (τ, V). The image of the restriction map is contained in $C_{\psi,\tau^*}^\infty(N_0 \backslash G/K) :=$

$$\{C^\infty\text{-function } W : G \to V^* \mid W(ngk) = \psi(n)\tau^*(k)^{-1}W(g)\ \forall (n, g, k) \in N_0 \times G \times K\},$$

which is canonically identified with $\mathrm{Hom}_K(\tau, C_\psi^\infty(N_0 \backslash G))$. Now let us note that we have to impose the multiplicity one property on the K-type τ of π in order to ensure that the notion of the Whittaker functions is well-defined. By $W_{\psi,\pi}(\tau^*)$ we denote the image of the restriction map. We call an element in $W_{\psi,\pi}(\tau^*)$ a Whittaker function for π and K-type τ^*. We will need

$$W_{\psi,\pi}(\tau^*)^0 := \{w \in W_{\psi,\pi}(\tau^*) \mid w \text{ is rapidly decreasing when } \frac{a_1}{a_2} \to \infty,\ a_2^2 \to \infty\},$$
$$\mathcal{S}_\psi(N_0 \backslash G) := \{\phi \in C_\psi^\infty(N_0 \backslash G) \mid \phi \text{ is rapidly decreasing when } \frac{a_1}{a_2} \to \infty,\ a_2^2 \to \infty\}$$

since we are motivated by the Fourier expansion of cusp forms, which are rapidly decreasing in the sense above (cf. [4, Chapter 1, Section 4], [2]). For a definition of the rapidly decreasing condition, see [4, pp11, 12 and 13] for instance. For this we remark that another explanation of $a_1/a_2 \to \infty,\ a_2^2 \to \infty$ is that $a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1})$

sits inside a region in A_0 sufficiently regular with respect to the standard simple roots of $Sp(2,\mathbb{R})$, which is related to the reduction theory of arithmetic groups. Hereafter we call functions with the rapidly deceasing property in the above sense simply rapidly decreasing.

Let $\dim V^* = d+1$ and $\{v_k^*\}_{k=0}^d$ be a basis of V^* consisting of weight vectors with highest weight vector v_d^* as in Section 3.2. When the highest weight of τ is (Λ_1, Λ_2), we have $d = d_\Lambda = \Lambda_1 - \Lambda_2$ (cf. Section 3.2). We can express each Whittaker function $W \in W_{\psi,\pi}(\tau^*)$ as

$$W(g) = \sum_{k=0}^{d} c_k(g) v_k^*$$

with coefficient functions $c_k(g)$. For this we note that the notation for the coefficient functions will vary depending on the choices of bases of V^* etc., e.g. for the case of P_S-principal series representations (see Section 4.4 and (iii) in the proof of Proposition 4.7). Now recall that G admits the Iwasawa decomposition $G = N_0 A_0 K$ (cf. Section 2.1), where $A_0 := \left\{a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1,\ a_2 \in \mathbb{R}_{>0}\right\}$. We then see that the Whittaker function W is determined by the restriction to A_0.

Throughout Sections 4.1–4.4 we assume that the Whittaker functions are attached to a non-degenerate unitary character ψ given by

$$\psi(n(u_0, u_1, u_2, u_3)) = \exp(2\pi\sqrt{-1}(m_0 u_0 + m_3 u_3)) \in \mathbb{C}^\times \quad \text{with } m_0 m_3 \neq 0$$

for any standard admissible representations in our concern.

For the case of P_S-principal series we review the explicit formula only in a rough manner since the citation of [12, Theorems 3.1 and 3.2] is enough to know explicitly all the Whittaker functions for some specified peripheral K-types. However, we discuss the rapidly decreasing property of them, which is not taken up in [12, Theorems 3.1 and 3.2]. For the cases of other admissible representations we give a somewhat detailed account of the explicit formulas as well as their rapidly decreasing property. To discuss the rapidly decreasing property it is convenient to introduce the coordinate

$$y_1 := \frac{a_1}{a_2}, \quad y_2 := a^2.$$

With this we provide the following key lemma:

Lemma 4.1. *For $\nu, \nu_i, \mu, \mu_i, z, z_i \in \mathbb{C}$ with $i = 1, 2$ and $A_i \in \mathbb{R}_{>0}$ with $1 \le i \le 4$ we introduce functions on $\{(y_1, y_2) \in \mathbb{R}_{>0}^2\}$ by*

$$f(y_1, y_2) := y_1^{\nu_1} y_2^{\nu_2} e^{-A_1 y_2} \int_0^\infty t^\nu K_z(2\pi t) \exp(-A_2 t^2 y_2^{-1} - A_3 y_1^2 y_2 t^{-2}) \frac{dt}{t},$$

$$g(y_1, y_2) := y_1^{\nu_1} y_2^{\nu_2} \int_0^\infty \int_0^\infty t_1^{\mu_1} t_2^{\mu_2} K_{z_1}(2\pi t_1/t_2) K_{z_2}(2\pi t_1 t_2)$$
$$\times \exp(-A_1 y_1^2 y_2 t_1^{-2} - A_2 y_2^{-1} t_1^2 - A_3 y_2 t_2^{-2} - A_4 y_2 t_2^2) \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

These are rapidly decreasing when $y_1 \to \infty$ and $y_2 \to \infty$.

Proof. We firstly verify the rapidly decreasing property of $f(y_1, y_2)$. In the integrand of $f(y_1, y_2)$, $W_{0,z/2}(t) = \sqrt{t} K_z(2\pi t)$ (cf. Theorem 4.2) is rapidly decreasing and bounded.

We therefore see that the integral in $f(y_1,y_2)$ is bounded by a constant multiple of

$$\int_0^\infty t^{\nu-1/2}\exp\Big(-A_2\frac{t^2}{y_2}-A_3\frac{y_1^2y_2}{t^2}\Big)\frac{dt}{t}.$$

It suffices to verify the rapidly decreasing property of this integral. By the change of variables $t\to x:=\sqrt{A_2/A_3}(\frac{t}{\sqrt{y_1y_2}})^2$ this can be understood by the integral expression of the K-Bessel function

$$K_\nu(y)=\int_0^\infty \exp\Big(-y\Big(x+\frac{1}{x}\Big)\Big)x^\nu\frac{dx}{x},$$

whose rapidly decreasing property for $y\to\infty$ is well known. This implies the rapidly decreasing property of $f(y_1,y_2)$.

We secondly take up $g(y_1,y_2)$. The absolute value of the integral in $g(y_1,y_2)$ is bounded by the product of the following two:

$$\Big|\int_0^\infty\int_0^\infty t_1^{\nu_1}t_2^{\nu_2}K_{z_1}(2\pi\sqrt{y_1y_2}\frac{t_1}{t_2})K_{z_2}(2\pi\sqrt{y_1y_2}t_1t_2)e^{-\frac{1}{2}(y_1(A_1t_1^{-2}+A_2t_1^2)+y_2(A_3t_2^{-2}+A_4t_2^2))}\frac{dt_1}{t_1}\frac{dt_2}{t_2}\Big|,$$

$$\exp(-\frac{1}{2}(\sqrt{A_1A_2}y_1+\sqrt{A_3A_4}y_2)).$$

In fact, to obtain the first one, we make the change of variables $t_1\to\sqrt{y_1y_2}t_1$, and for the second one we note $At_i^2+Bt_i^{-2}\ge 2\sqrt{AB}$ with $i=1,\ 2$ and $A,\ B\in\mathbb{R}_{>0}$.

Taking into account the rapidly decreasing property of the K-Bessel function $K_\nu(y)$ for $y\to\infty$ and the elementary formula $\exp(-b)<C_\alpha b^\alpha$ for $b>0$ with a constant C_α dependent on $\alpha\in\mathbb{R}$, the absolute value of the first one is estimated by a polynomial of y_1 and y_2.

As a result we verify that the absolute value of $g(y_1,y_2)$ is estimated by a product of a polynomial of $y_1,\ y_2$ and $\exp(-\frac{1}{2}(\sqrt{A_1A_2}y_1+\sqrt{A_3A_4}y_2))$ and have seen that $g(y_1,y_2)$ is rapidly decreasing. ∎

Furthermore, it is worthwhile to remark that, in Section 4.2, we often use an idea of reducing problems for the explicit formulas by some change of variables. This can be regarded as an analogue of "MVW-involution" (cf. [19], [26] et al.) or "real Chevalley involution" (cf. [1] et al.). For instance we can reduce the explicit formulas for (limits of) large discrete series π_λs with $\lambda\in\Xi_{III}$ to those with $\lambda\in\Xi_{II}$ etc.

4.2. The case of (limits of) large discrete series and P_J-principal series

These cases are settled by [25] and [17] (see also [18]). We first review the result by Oda [25] for the case of large discrete series representations. For this we note that π_λ is replaced by its contragredient π_λ^* with $\lambda\in\Xi_{II}$ in [25, Theorem 9.1] and further note that Moriyama [21, p913] pointed out that the result can be generalized to the case of limits of large discrete series.

Theorem 4.2. (Oda, Moriyama) (1) (*cf.* [25, Theorem 9.1]) *Let π_λ be a (limit of) discrete series representation with Harish Chandra parameter $\lambda\in\Xi'_{III}$ and τ_Λ be the minimal K-type of π_λ with highest weight $\Lambda=(\Lambda_1,\Lambda_2)$. We have*

$$\dim W_{\psi,\pi}(\tau^*)^0=\begin{cases}1 & (m_3<0),\\ 0 & (m_3>0).\end{cases}$$

For $\lambda \in \Xi_{III} \setminus \Xi'_{III}$ *we have*

$$\dim W_{\psi,\pi}(\tau^*)^0 \leq 1 \quad (m_3 < 0), \quad \dim W_{\psi,\pi}(\tau^*)^0 = 0 \quad (m_3 > 0).$$

The coefficient function $c_{d_\Lambda}(a_0)$ *of the restriction* $W|_{A_0}$ *of the rapidly decreasing Whittaker function* W *to* A_0 *is given explicitly by* $y_1^{1-\Lambda_2} y_2^{\frac{1-\Lambda_1-\Lambda_2}{2}} e^{2\pi m_3 y_2} h_{d_\Lambda}(a_0)$ *with*

$$h_{d_\Lambda}(a_0) = \int_0^\infty t^{-\Lambda_1 - 3/2} W_{0,\Lambda_1}(t) \exp\left(\frac{t^2}{64\pi m_3 y_2} + \frac{64\pi^3 m_0^2 m_3 y_1^2 y_2}{t^2}\right) \frac{dt}{t},$$

up to constant multiples. Here, for $y > 0$, $W_{\kappa,\mu}(y)$ *denotes a unique rapidly decreasing solution of the confluent hypergeometric equation by Whittaker (cf.* [34, Chapter XVI]):

$$\frac{d^2}{dy^2} W + \left\{-\frac{1}{4} + \frac{\kappa}{y} + \frac{1/4 - \mu^2}{y^2}\right\} W = 0.$$

The other coefficients c_k *for* $0 \leq k \leq d_\Lambda - 1$ *are obtained from* c_{d_Λ} *by the recurrence relation* $(\mathrm{E})_k$ *in [25, Section 8].*

(2) *For a discrete series representation* π_λ *with* $\lambda = (\lambda_1, \lambda_2) \in \Xi'_{II}$ *we have*

$$\dim W_{\psi,\pi}(\tau^*)^0 = \begin{cases} 1 & (m_3 > 0), \\ 0 & (m_3 < 0). \end{cases}$$

For $\lambda \in \Xi_{II} \setminus \Xi'_{II}$ *we have*

$$\dim W_{\psi,\pi}(\tau^*)^0 \leq 1 \quad (m_3 > 0), \quad \dim W_{\psi,\pi}(\tau^*)^0 = 0 \quad (m_3 < 0).$$

When $\dim W_{\psi,\pi}(\tau^*)^0 \neq 0$ *the rapidly decreasing Whittaker function* W^* *for this case is uniquely given by*

$$W^*(g) = W(\delta g \delta^{-1} \xi) \quad (g \in G)$$

with

$$\delta = \begin{pmatrix} -I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix}, \quad \xi = \begin{pmatrix} J'_2 & 0_2 \\ 0_2 & J'_2 \end{pmatrix} \quad \left(J'_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

up to constant multiples, where the function W *is the rapidly decreasing Whittaker function for a (limit of) discrete series representation with Harish Chandra parameter* $(-\lambda_2, -\lambda_1) \in \Xi_{III}$.

Proof. The idea to obtain the explicit formula in (1) is explained in [17, Section 8.1], for which note that the characterizing differential equations of the Whittaker functions for large discrete series (cf. [25, Lemma 8.1]) are similar to those for P_J-principal series as in [17]. From the calculation of the explicit formula we see that the condition $m_3 < 0$ is necessary for the Whittaker function to be rapidly decreasing. We mainly discuss the case of usual discrete series representations parametrized by Ξ'_{III}. The proof will end with a remark on the limits of discrete series.

We first check the rapidly decreasing property of the explicit Whittaker functions for $m_3 < 0$. Though this is already pointed out in the proof of [25, Theorem 9.1] we remark that this follows from such property of $f(a_1, a_2)$ in Lemma 4.1. More precisely, Lemma 4.1 is directly applicable for $c_{d_\Lambda}(a_0)$. As for the other c_k with $0 \leq k \leq d_\Lambda - 1$ the rapidly decreasing property also holds since the recurrence relation $(\mathrm{E})_k$ in [25, Section 8] preserves such property.

We next explain how to get $\dim W_{\psi,\pi}(\tau^*)^0$. In [25, Theorem 9.1] the result on $\dim W_{\psi,\pi}(\tau^*)^0$ is formulated as

$$\dim \mathrm{Hom}_{(g,K)}(\pi_\lambda, \mathcal{S}_\psi(N_0\backslash G)) = \begin{cases} 1 & (m_3 < 0) \\ 0 & (m_3 > 0) \end{cases}$$

for $\lambda \in \Xi_{III}$. Here note that the obtained Whittaker function is rapidly decreasing as we have seen. We can say that the formula for $\dim W_{\psi,\pi}(\tau^*)^0$ in the first assertion is due to the result on the explicit formula for the Whittaker functions and Yamashita's characterization theorem [35, Theorem 2.4] of some general intertwining operators for discrete series, the latter of which yields $\mathrm{Hom}_{(g,K)}(\pi_\lambda, \mathcal{S}_\psi(N_0\backslash G)) \simeq W_{\psi,\pi}(\tau^*)^0$. For [35, Theorem 2.4] we note that large discrete series representations satisfy the condition "far from the wall" (for the definition see [35, Definition 1.7]), which is necessary to use [35, Theorem 2.4].

To see the formula for $\dim W_{\psi,\pi}(\tau^*)^0$ in the second assertion we have two remarks. As the first remark we note that the Whittaker function W^* satisfies the left-equivariance with respect to $N_0 \ni n(u_0, u_1, u_2, u_3) \mapsto \psi(\delta n(u_0, u_1, u_2, u_3)\delta^{-1})$ and the right equivariance with respect to τ_Λ (not τ_Λ^*) when W is associated with the unitary character ψ of N_0 and τ_Λ is the minimal K-type of π_λ. In addition we note that $\delta g \delta^{-1}\xi \in G$ for $g \in G$ and that the unitary character

$$N_0 \ni n(u_0, u_1, u_2, u_3) \mapsto \psi(\delta n(u_0, u_1, u_2, u_3)\delta^{-1})$$

is parametrized by $(m_0, -m_3)$ instead of (m_0, m_3).

As the second remark let us note that the differential equations characterizing the Whittaker functions are induced by the infinitesimal action of the Schmid operator (cf. [25, Section 5]). We should further note that the Schmid operator for π_λ with $\lambda \in \Xi_{II}$ is conjugate to that for π_λ with $\lambda \in \Xi_{III}$ by $\delta^{-1}\xi$. In fact, the Schmidt operator for $\lambda \in \Xi_{II}$ is associated with non-compact roots $(1,1)$, $(2,0)$, $(0,-2)$ (for the notation of the roots see Section 2.2). These three roots and their root vectors are conjugate to $(-1,-1)$, $(0,-2)$, $(2,0)$ and their root vectors by $\delta^{-1}\xi$ respectively. The latter roots define the Schmid operator for $\lambda \in \Xi_{II}$. The Whittaker function W^* is therefore annihilated by the Schmid operator for $\lambda \in \Xi_{II}$. The Whittaker function W^* is a rapidly decreasing for $m_3 > 0$ as W is for $m_3 < 0$.

The change of the variables $G \ni g \mapsto \delta g \delta^{-1}\xi \in G$ therefore reduces the problem to the first assertion. We can thus say that the second assertion on $\dim W_{\psi,\pi}(\tau^*)^0$ follows from the first assertion.

We finally remark that almost all the argument so far goes similarly for limits of discrete series representations. The only difference is the formula for $\dim W_{\psi,\pi}(\tau^*)^0$. This is due to the fact that Yamashita's formula [35, Theorem 2.4] is not useful for limits of discrete series. We can only verify by the irreducibility of a limit of discrete series π that there is an injection $\mathrm{Hom}_{(g,K)}(\pi_\lambda, \mathcal{S}_\psi(N_0\backslash G)) \hookrightarrow W_{\psi,\pi}(\tau^*)^0$. ∎

We next review the results by Miyazaki-Oda [17, Theorems 8.1, 8.2] (see also [18]) on the Whittaker functions for P_J-principal series representations. Their results are given only for the P_J-principal series representations $\mathrm{Ind}_{P_J}^G(1_{N_J} \otimes \nu_z \otimes \sigma)$ with a discrete series $\sigma = (\mathcal{D}_n^-, \epsilon)$. However, the Whittaker functions for these P_J-principal series are related to those for P_J-principal series with $\sigma = (\mathcal{D}_n^+, \epsilon)$ by the change of variables $G \ni g \mapsto \delta g \delta^{-1}\xi \in G$. The results can be therefore stated for P_J-principal series with both choices of $\mathcal{D}_n^\pm$. According to [13, Theorem 14.15] we know that the P_J-principal series just mentioned are irreducible when the parameters z are non-zero and purely imaginary. However, such irreducibility is not assumed for the coming theorem. Recall that the P_J-principal series enumerated as 1 and 3 (respectively 2 and 4) are called even (respectively odd) (cf. Section 3.4).

Theorem 4.3. (Miyazaki-Oda) (1) *Let π be a P_J-principal series representation with a discrete series $\mathcal{D}_n^{\pm}$ in σ.*

(a) *Let π be even and associated with $\sigma = (\mathcal{D}_n^-, \epsilon)$, and let W be the rapidly decreasing Whittaker function for π and τ_Λ^*, where τ_Λ is the corner K-type of π with $\Lambda = (-n, -n)$. When $m_3 < 0$ the restriction of $W|_{A_0}$ to A_0 of the Whittaker function W is uniquely given by*

$$y_1^{n+1} y_2^{n+\frac{1}{2}} \exp(2\pi m_3 y_2) \int_0^\infty t^{-n+(1/2)} W_{0,z}(t) \exp\left(\frac{t^2}{64\pi m_3 y_2} + \frac{64\pi^3 m_0^2 m_3 y_1^2 y_2}{t^2}\right) \frac{dt}{t}$$

up to constant multiples. If $m_3 > 0$ there is no such Whittaker function.

When π is associated with $(\mathcal{D}_n^+, \epsilon)$ the corner K-type is replaced by τ_Λ with $\Lambda = (n, n)$ and the explicit formula is obtained by replacing (m_0, m_3) with $(m_0, -m_3)$.

(b) *Let π be odd and associated with $\sigma = (\mathcal{D}_n^-, \epsilon)$, and let $W(g) = c_0(g)v_0^* + c_1(g)v_1^*$ be the the rapidly decreasing Whittaker function for π and τ_Λ^*, where τ_Λ is the corner K-type of π with $\Lambda = (-n+1, -n)$. When $m_3 < 0$ the restriction $W|_{A_0}$ of W to A_0 is uniquely given by*

$$c_0(a_0) = y_1^{n+2} y_2^{n+1} \exp(2\pi m_3 y_2) \int_0^\infty t^{-1/2-n} W_{0,z}(t) \exp\left(\frac{t^2}{64\pi m_3 y_2} + \frac{64\pi^3 m_0^2 m_3 y_1^2 y_2}{t^2}\right) \frac{dt}{t},$$

$$c_1(a_0) = y_1^{n+1} y_2^{n} \exp(2\pi m_3 y_2) \int_0^\infty t^{3/2-n} W_{0,z}(t) \exp\left(\frac{t^2}{64\pi m_3 y_2} + \frac{64\pi^3 m_0^2 m_3 y_1^2 y_2}{t^2}\right) \frac{dt}{t},$$

up to constant multiples. If $m_3 > 0$ there is no such Whittaker function.

When π is associated with $(\mathcal{D}_n^+, \epsilon)$ the corner K-type is replaced by τ_Λ with $\Lambda = (n, n-1)$ and the explicit formula is obtained by replacing (m_0, m_3) with $(m_0, -m_3)$.

(2) *Let π be a P_J-principal series associated with $\sigma = (\mathcal{D}_n^-, \epsilon)$ and the K-type τ_Λ above. We have*

$$\dim W_{\psi,\pi}(\tau_\Lambda^*)^0 = \begin{cases} 1 & (m_3 < 0) \\ 0 & (m_3 > 0) \end{cases}.$$

On the other hand, for π with $\sigma = (\mathcal{D}_n^+, \epsilon)$ and the K-type τ_Λ as above,

$$\dim W_{\psi,\pi}(\tau_\Lambda^*)^0 = \begin{cases} 1 & (m_3 > 0) \\ 0 & (m_3 < 0) \end{cases}.$$

Proof. First of all, we remark that how to check the rapidly decreasing property of the explicit Whittaker functions is quite similar to the case of large discrete series representations. We do not thus go into detail for this case.

The assertions for the case of $\sigma = (\mathcal{D}_n^-, \epsilon)$ are stated as [17, Theorems 8.1 and 8.2] (see also [18]). We can deduce the dimension formula for $W_{\psi,\pi}(\tau^*)^0$ also from the fact that the multiplicity of the moderate growth Whittaker model for π coincides with that for $\mathcal{D}_n^-$, which is due to [33, Theorem 40] (see also [32, Theorem 15.6.7]). In fact we have now known that both Whittaker functions are rapidly decreasing, where note that the Whittaker function for $\mathcal{D}_n^-$ with respect to the additive character indexed by $m_3 < 0$ is given uniquely up to scalars in terms of the exponential function and proved to be rapidly decreasing.

A Whittaker function for the case of $\sigma = (\mathcal{D}_n^+, \epsilon)$ is written as $W^*(g) = W(\delta g \delta^{-1}\xi)$, where $g \in G$, with the Whittaker function W for the case of $\sigma = (\mathcal{D}^-, \epsilon)$. This reduction of the problem is quite similar to the case of large discrete series representations. To see this we now note that the Whitaker functions for P_J-principal series are characterized by the differential equations arising from the Casimir operators and the shift operators (for the definition of the shift operators see [16, Section 8, Definition (8.1), Section 9]). The shift operators for P_J-principal series with $\sigma = (\mathcal{D}_n^+, \epsilon)$ and $(\mathcal{D}_n^-, \epsilon)$ are related to each other by $\delta^{-1}\xi$-conjugate. The Casimir operator remains unchanged by G-conjugate. We thereby see that W^* satisfies the characterizing differential equations for the case of P_J-principal series with $\sigma = (\mathcal{D}_n^+, \epsilon)$. ■

4.3. The case of the principal series representations (induced from the minimal parabolic subgroup)

Recall that $\mathrm{Ind}_{P_0}^G(1_{N_0} \otimes \nu_z \otimes \sigma)$ has denoted a principal series representation. The following theorem is essentially due to Ishii [11, Theorems 3.2, 3.3, 3.4] and Niwa [24].

Theorem 4.4. *Let π be a principal series representation and let $(m_0, m_3) \in \mathbb{R}^2$ with $m_0 m_3 \neq 0$.*

(1) *Let π be an even principal series representation. Up to constant multiples, the rapidly decreasing Whittaker functions with minimal K-type τ_Λ restricted to A_0 can be expressed uniquely as*

$$(|m_0|y_1)^2(|m_3|y_2)^{3/2}\int_0^\infty\int_0^\infty K_{(z_1-z_2)/2}(2\pi t_1/t_2)K_{(z_1+z_2)/2}(2\pi t_1 t_2)$$
$$\times \exp\Big(-\pi\Big(\frac{m_0^2|m_3|y_1^2y_2}{t_1^2} + \frac{t_1^2}{|m_3|y_2} + \frac{|m_3|y_2}{t_2^2} + |m_3|y_2t_2^2\Big)\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

for $\Lambda = (0,0)$, and

$$(|m_0|y_1)^{5/2}(|m_3|y_2)^2\int_0^\infty\int_0^\infty K_{(z_1-z_2)/2}(2\pi t_1/t_2)K_{(z_1+z_2)/2}(2\pi t_1 t_2)$$
$$\times \Big(\frac{1}{t_1t_2} - l\frac{t_2}{t_1}\Big)\exp\Big(-\pi\Big(\frac{m_0^2|m_3|y_1^2y_2}{t_1^2} + \frac{t_1^2}{|m_3|y_2} + \frac{|m_3|y_2}{t_2^2} + |m_3|y_2t_2^2\Big)\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

for $\Lambda = (l,l) = \pm(1,1)$.

(2) *Let π be an odd principal series representations with $(\sigma_1, \sigma_2) = (1, -1)$. The explicit formula for the rapidly decreasing Whittaker functions $c_0(g)v_0^* + c_1(g)v_1^*$ with minimal K-type τ_Λ is uniquely given as*

$$c_0(a_0) = \begin{cases} P_1^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2)) + P_2^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2)) & (\Lambda = (1,0), \\ Q_1^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2)) - Q_2^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2)) & (\Lambda = (0,-1)), \end{cases}$$

$$c_1(a_0) = \begin{cases} Q_1^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2)) + Q_2^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2) & (\Lambda = (1,0)), \\ -P_1^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2)) + P_2^{(z_1,z_2)}((|m_0|y_1,|m_3|y_2) & (\Lambda = (0,-1)). \end{cases}$$

up to constant multiples. As for odd principal series representations with $(\sigma_1,\sigma_2) = (-1,1)$ *the explicit formula for the Whittaker functions with minimal* K*-type* τ_Λ *is given similarly by exchanging* z_1 *and* z_2*. Here*

$$\begin{aligned} P_1^{(z_1,z_2)}(y) &= (y_1y_2)^2 \int_0^\infty \int_0^\infty \Big(\frac{z_2+1}{2} - \frac{\pi y_1^2 y_2}{t_1^2}\Big) K_{(z_1-z_2-1)/2}(2\pi t_1/t_2) K_{(z_1+z_2+1)/2}(2\pi t_1 t_2) \\ &\quad \times \exp\Big(-\pi\Big(\frac{y_1^2y_2}{t_1^2} + \frac{t_1^2}{y_2} + \frac{y_2}{t_2^2} + y_2t_2^2\Big)\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}, \\ P_2^{(z_1,z_2)}(y) &= -P_1^{(z_1,-z_2)}(y), \\ Q_1^{(z_1,z_2)}(y) &= y_1^3y_2^2 \int_0^\infty \int_0^\infty K_{(z_1-z_2-1)/2}(2\pi t_1/t_2) K_{(z_1+z_2+1)/2}(2\pi t_1 t_2) \\ &\quad \times \exp\Big(-\pi\Big(\frac{y_1^2y_2}{t_1^2} + \frac{t_1^2}{y_2} + \frac{y_2}{t_2^2} + y_2t_2^2\Big)\Big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}, \\ Q_2^{(z_1,z_2)}(y) &= -Q_1^{(z_1,-z_2)}(y). \end{aligned}$$

(3) *We have* $\dim W_{\psi,\pi}(\tau_\Lambda^*)^0 = 1$ *for* ψ *and* (π,τ_Λ)*s above.*

Proof. We put $a_{m_0,m_3} := \mathrm{diag}(m_0\sqrt{|m_3|}, \sqrt{|m_3|}, (m_0\sqrt{|m_3|})^{-1}, \sqrt{|m_3|}^{-1})$, and recall that

$$\delta := \begin{pmatrix} -I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} \quad \text{and} \quad \xi := \begin{pmatrix} J_2' & 0_2 \\ 0_2 & J_2' \end{pmatrix} \quad \text{with} \quad J_2' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(cf. Theorem 4.2 (2)). The explicit formulas for the Whittaker functions in the first and second assertions are obtained by considering

$$\begin{cases} W(a_{m_0,m_3}g) & (m_3 > 0), \\ W(\delta a_{m_0,m_3} g \delta^{-1}\xi) & (m_3 < 0), \end{cases}$$

for $g \in G$, where W denotes the Whittaker functions attached to the character of N_0 with $(m_0,m_3) = (1,1)$, whose explicit formula is given by Ishii [11, Theorems 3.2, 3.3, 3.4] and Niwa [24]. As we have remarked in the proof of Theorem 4.2 (2) we note that $\delta a_{m_0,m_3} g \delta^{-1}\xi \in G$ for $g \in G$. The differential equations characterizing the Whittaker functions are induced by the infinitesimal actions of the two generators of the center of the universal enveloping algebra for $\mathfrak{g}$, one of which is the Casimir operator and another of which coincides with some composite of the shift operators (cf. [11, Remark 3]). For the definition of the shift operators see [16, Section 8, Definition (8.1), Section 9].

We now justify the above remark on the explicit formula. To be precise, the formula in the assertion is $W(a_{|m_0|,m_3}a_0)$ (or $W(a_{|m_0|,m_3}a_0\xi)$) for $a_0 \in A_0$. Now let us note that $a_{|m_0|,m_3} = a_{m_0,m_3}\mathrm{diag}(\epsilon,1,\epsilon,1)$ with some $\epsilon \in \{\pm 1\}$. Taking into account the right equivariance of the Whittaker functions with respect to τ_Λ^*, we can then verify that the difference between $W(a_{|m_0|,|m_3|}a_0)$ and $W(a_{m_0,m_3}a_0)$ (or $W(a_{|m_0|,|m_3|}a_0\xi)$ and $W(a_{m_0,m_3}a_0\xi)$) is given at most as the multiples of the K-type vectors by $\{\pm 1\}$. We remark that the change of variables $g \mapsto \delta g \delta^{-1}\xi$ leads to switching between the Whittaker function for an even principal series with minimal K-types $\tau_{(1,1)}$ and that for an even principal series with minimal K-type $\tau_{(-1,-1)}$, and also to switching between the Whittaker functions for odd principal series with the different minimal K-types $\tau_{(1,0)}$ and $\tau_{(0,-1)}$. For this note further that the change of variables $g \mapsto (\delta^{-1}\xi)^{-1}g(\delta^{-1}\xi)$ induces the conjugation of the shift operators and the Casimir operator as in the case of P_J-principal series.

Regarding the third assertion we remark that the rapidly decreasing property of $g(y_1, y_2)$ in Lemma 4.1 is useful to see such property for the explicit given Whittaker functions. The result of $\dim W_{\psi,\pi}(\tau^*)^0$ follows from [33, Theorem 40] (also from [32, Theorem 15.6.7]), together with this remark. ■

4.4. The case of P_S-principal series representations

For the case of $\pi = \mathrm{Ind}_{P_S}^G(1_{N_S} \otimes \nu_z \otimes \mathcal{D}_n)$ we make a rather rough review on the explicit formulas for Whittaker functions by Ishii [12, Theorems 3.1 and 3.2], which were deduced from studying characterizing differential equations by Hasegawa [5, Chapter II, Theorem 3.1]. To know the Whittaker functions explicitly we choose the peripheral K-type $\tau_{(l+n,l)}$ (cf. Section 3.5, [5, p. 105]) such that

- n is even, and $l + n/2 = 0$,
- n is odd, and $l + (n+1)/2 = 0$ or $l + (n-1)/2 = 0$.

To illustrate the explicit formulas Ishii chooses the basis $\{u_i^{(l+n,l)} \mid 0 \le i \le n\}$ of $V_{(l+n,l)}$ in Section 3.2 (cf. [12, Definition 1.3]) or $\{\mathbf{g}_i^{(l+n,l)} \mid 0 \le i \le n\}$ in [12, Definition 1.9]). By $\{u_i^{(l+n,l),*}\}_{0\le i\le n}$ we denote the basis of $V_{(l+n,l)}^*$ dual to $\{u_i^{(l+n,l)}\}_{0\le i\le n}$. In addition, for $\sigma \in \mathbb{R}$, we introduce the vertical path $L(\sigma)$ of integration in the complex plane from $\sigma - \sqrt{-1}\infty$ to $\sigma + \sqrt{-1}\infty$.

We then state the explicit formula for the Whitaker functions attached to the P_S-principal series in [12, Theorems 3.1 and 3.2]. More precisely, the formulas we need are obtained by putting $c = 0$ in the formula of [12, Theorems 3.1 and 3.2].

Theorem 4.5. *Suppose $n > 1$, which means that $\mathcal{D}_n$ is a discrete series representation.*

When n is even, namely $l = -n/2$, let W_i be the coefficient function of $u_i^{(l+n,l),*}$ for the rapidly decreasing Whittaker function attached to the contragredient of the peripheral K-type $\tau_{(l+n,l)}$. For an odd n, $l = -(n+1)/2$ or $l = -(n-1)/2$ by assumption. When $l = -(n+1)/2$ (respectively $l = -(n-1)/2$) let W_i (respectively W_i') be the coefficient function of $u_i^{(l+n,l),*}$ for the rapidly deceasing Whittaker function as above.

(1) *Suppose that n is even. The restriction of the function W_i to A_0 is explicitly described as*

$$\frac{y_1^2 y_2^{\frac{3}{2}}}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_1)}\int_{L(\sigma_2)} V_i(s_1,s_2)(\pi|m_0|y_1)^{-s_1}(\pi|m_3|y_2)^{-s_2}ds_1ds_2,$$

where $$V_0(s_1,s_2) := \frac{1}{(2\pi\sqrt{-1})^2}\int_{L(\tau_1)}\int_{L(\tau_2)}\Gamma(\frac{s_1}{2}+\frac{n}{4})\Gamma(\frac{s_1-t_1-t_2}{2}+\frac{n}{4})$$
$$\times\Gamma(\frac{s_2-t_1}{2}+\frac{n}{4})\Gamma(\frac{s_2-t_2}{2}+\frac{n}{4})$$
$$\times\Gamma(\frac{t_1}{2}+\frac{z}{4})\Gamma(\frac{t_1}{2}-\frac{z}{4})\Gamma(\frac{t_1}{2}+\frac{1}{4})\Gamma(\frac{t_1}{2}-\frac{1}{4})dt_1dt_2,$$
$$V_1(s_1,s_2) := \frac{1}{(2\pi\sqrt{-1})^2}\int_{L(\tau_1)}\int_{L(\tau_2)}\Gamma(\frac{s_1+1}{2}+\frac{n}{4})\Gamma(\frac{s_1-t_1-t_2}{2}+\frac{n}{4})$$
$$\times\Gamma(\frac{s_2-t_1-1}{2}+\frac{n}{4})\Gamma(\frac{s_2-t_2}{2}+\frac{n}{4})$$
$$\times\Gamma(\frac{t_1}{2}+\frac{z}{4})\Gamma(\frac{t_1}{2}-\frac{z}{4})\Gamma(\frac{t_1}{2}+\frac{1}{4})\Gamma(\frac{t_1}{2}-\frac{1}{4})dt_1dt_2,$$

and $V_j(s_1,s_2)$ with $j\geq 2$ are defined explicitly in terms of $V_0(s_1,s_2)$ and j for even j (respectively $V_1(s_1,s_2)$ and j for odd j). For the further detail on $V_i(s_1,s_2)$, see the statement of [12, Theorem 3.1]. *Here $\sigma_i,\tau_i\in\mathbb{R}$ are taken so that*

$$\sigma_1>\tau_1+\tau_2-n/2,\ \sigma_2>\max\{\tau_1,\tau_2\},\ \tau_1>|\mathrm{Re}(z)/2|,\ \tau_2>1/2.$$

(2) *Suppose that n is odd. The restrictions of the functions W_i and W_i' to A_0 are explicitly described as*

$$\begin{cases}\frac{y_1^2y_2^{\frac{3}{2}}}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_2)}\int_{L(\sigma_1)}V_i(s_1,s_2)(\pi|m_0|y_1)^{-s_1}(\pi|m_3|y_2)^{-s_2}ds_1ds_2 & (l=-(n+1)/2),\\ \frac{y_1^2y_2^{\frac{3}{2}}}{(2\pi\sqrt{-1})^2}\int_{L(\sigma_2)}\int_{L(\sigma_1)}V_i'(s_1,s_2)(\pi|m_0|y_1)^{-s_1}(\pi|m_3|y_2)^{-s_2}ds_1ds_2 & (l=-(n-1)/2),\end{cases}$$

where $V_0(s_1,s_2)$ and $V_1(s_1,s_2)$ denotes the functions defined by explicit linear combinations of the integrals

$$U^{\pm}(s_1,s_2) := \frac{1}{(2\pi\sqrt{-1})^2}\int_{L(\tau_1)}\int_{L(\tau_2)}\Gamma(\frac{s_1}{2}+\frac{n+1}{4})\Gamma(\frac{s_1-t_1-t_2}{2}+\frac{n-1}{4})$$
$$\times\Gamma(\frac{s_2-t_1}{2}+\frac{n}{4})\Gamma(\frac{s_2-t_2}{2}+\frac{n-1}{4})$$
$$\times\Gamma(\frac{t_1}{2}\pm\frac{z}{4})\Gamma(\frac{t_1}{2}\mp\frac{z}{4}+\frac{1}{2})\Gamma(\frac{t_1}{2})\Gamma(\frac{t_1}{2}+\frac{1}{2})\,dt_1dt_2.$$

over $L(\tau_1)\times L(\tau_2)$ ($V_0'(s_1,s_2)$ and $V_1'(s_1,s_2)$) are defined similarly), and $V_j(s_1,s_2)$ (respectively $V_j'(s_1,s_2)$) with $j\geq 2$ are defined explicitly in terms of $V_0(s_1,s_2)$ or $V_1(s_1,s_2)$, and j (respectively $V_0'(s_1,s_2)$ or $V_1'(s_1,s_2)$, and j). For the notations $V_i(s_1,s_2)$, $V_i'(s_1,s_2)$, $U^{\pm}(s_1,s_2)$ and unexplained recurrence relations, see the statement of [12, Theorem 3.2]. Here σ_i, τ_i are taken so that

$$\sigma_1>\tau_1+\tau_2-(n-3)/2,\quad \sigma_2>\max\{\tau_1,\tau_2\},\ \tau_1>|\mathrm{Re}(z)/2|,\ \tau_2>0.$$

(3) *We have $\dim W_{\psi,\pi}(\tau_\Lambda^*)^0=1$ for ψ and $(\pi,\tau_\Lambda=\tau_{(n+l,n)})$s above. In other words, the Whittaker functions in the first and second assertion are unique, up to scalars.*

Proof. We can say that the explicit formulas and the assertion on the uniqueness are essentially included in [12, Theorems 3,1 and 3.2]. To be precise about the uniqueness, the formula $\dim W_{\psi,\pi}(\tau_\Lambda^*)^0 = 1$ is verified by the rapidly decreasing property of the Whittaker function and the argument using [33, Theorem 40] (see also [32, Theorem 15.6.7]) as in the proof of Theorem 4.3. What remains is thus to prove that the Whittaker functions in the first and second assertions are rapidly decreasing. We see the essence of the proof by showing the rapidly decreasing property of W_0 for an even n.

We now review the well known definition of the K-Bessel function together with its Mellin-Barnes type integral as follows:

$$\begin{aligned} K_r(y) &= \frac{1}{2}\int_0^\infty x^{r-1}\exp(-\frac{y}{2}(x+\frac{1}{x}))dx \\ &= \frac{1}{2^3\pi\sqrt{-1}}\int_{L(c)}(y/2)^{-s}\Gamma(\frac{s+r}{2})\Gamma(\frac{s-r}{2})ds \quad (y>0), \end{aligned}$$

the latter of which converges for $\mathrm{Re}(s) > |\mathrm{Re}(r)|$. Here c denotes a positive number satisfying $c > |\mathrm{Re}(r)|$. By this formula we see that the absolute value of $W_0|_{A_0}$ is estimated by that of

$$\begin{aligned} \int_{L(\tau_1)}\int_{L(\tau_2)} & K_{\frac{t_1+t_2}{2}}(2\pi|m_0|y_1)K_{\frac{t_2-t_1}{2}}(2\pi|m_3|y_2) \\ & \times\Gamma(\frac{t_1}{2}+\frac{z}{4})\Gamma(\frac{t_1}{2}-\frac{z}{4})\Gamma(\frac{t_2}{2}+\frac{1}{4})\Gamma(\frac{t_2}{2}-\frac{1}{4})dt_1dt_2, \end{aligned}$$

up to multiplication by a polynomial of a_1, a_2. This integral is calculated as follows:

$$\begin{aligned} & 2^{-2}(\int_{L(\tau_1)}\int_{L(\tau_2)}\Big(\int_0^\infty\int_0^\infty x_1^{\frac{t_1+t_2}{2}-1}x_2^{\frac{t_2-t_1}{2}-1} \\ &\qquad \exp(-\pi(|m_0|y_1(x_1+\frac{1}{x_1})+|m_3|y_2(x_2+\frac{1}{x_2})))dx_1dx_2\Big) \\ &\qquad \times\Gamma(\frac{t_1}{2}+\frac{z}{4})\Gamma(\frac{t_1}{2}-\frac{z}{4})\Gamma(\frac{t_2}{2}+\frac{1}{4})\Gamma(\frac{t_2}{2}-\frac{1}{4})dt_1dt_2 \\ &= 2^{-2}(\int_0^\infty\int_0^\infty\Big(\int_{L(\tau_1)}\int_{L(\tau_2)}(x_1/x_2)^{\frac{t_1}{2}}(x_1x_2)^{\frac{t_2}{2}-1} \\ &\qquad \Gamma(\frac{t_1}{2}+\frac{z}{4})\Gamma(\frac{t_1}{2}-\frac{z}{4})\Gamma(\frac{t_2}{2}+\frac{1}{4})\Gamma(\frac{t_2}{2}-\frac{1}{4})dt_1dt_2\Big) \\ &\qquad \times\exp(-\pi(|m_0|y_1(x_1+\frac{1}{x_1})+|m_3|y_2(x_2+\frac{1}{x_2})))dx_1dx_2 \\ &= -2^4\pi^2\int_0^\infty\int_0^\infty(x_1x_2)^{-1}K_{z/2}(2\sqrt{x_2/x_1})K_{1/2}(\frac{2}{\sqrt{x_1x_2}}) \\ &\qquad \exp(-\pi(|m_0|y_1(x_1+\frac{1}{x_1})+|m_3|y_2(x_2+\frac{1}{x_2})))dx_1dx_2. \end{aligned}$$

Here we take τ_1 and τ_2 so that $\tau_1 > \max\{n/2,\ |\mathrm{Re}(z)/2|\}$ and $\tau_2 > 1/2$.

We now see that the rapidly decreasing property of $g(a_1,a_2)$ in Lemma 4.1 is useful for $W_0|_{A_0}$. We can reduce the problem to the estimates of the integrals like the above one, applying the fundamental formula $\Gamma(s+1) = s\Gamma(s)$ to V_i and V_i' if necessary. In fact, for both of even n and odd n, W_i and W_i' can be written as linear combinations of integrals similar to $g(a_1,a_2)$ for a general i. ■

Remark 4.6. The explicit integral expressions of the Whittaker functions in Theorem 4.5 are referred to as "Mellin-Barnes type integrals", which are more suitable to the archimdean local theory of automorphic L-functions. For the cases of large discrete series, P_J-principal series and the principal series, the Mellin-Barnes type integrals are also obtained. For this we cite Moriyama [20] and Ishii [11], [12].

4.5. Whittaker functions attached to degenerate characters of N_0 and a general consequence

The Whittaker functions we have studied are proved to be rapidly decreasing. They are attached to non-degenerate characters of N_0. However, towards the Fourier Jacobi expansion of automorphic forms on $Sp(2,\mathbb{R})$, we have to also study the Whittaker functions attached to degenerate characters of N_0, i.e. unitary characters ψ of N_0 parametrized by (m_0, m_3) with $m_0 m_3 = 0$. For the following proposition we remark that the case of $m_3 = 0$ for principal series and the case of the P_S-principal series are due to Taku Ishii.

Proposition 4.7. *Let π be a (limit of) large discrete series representation, a P_J-principal series representation, a P_S-principal series representation or a principal series representation, where $\mathcal{D}_n^{\pm}$ and $\mathcal{D}_n$ are taken as usual discrete series (i.e. $n > 1$) for P_J- and P_S-principal series. Let τ be the minimal K-type of π when π is a large discrete series representation or a principal series representation. When π is a P_J-principal series representation (respectively a P_S principal series representation) let τ be the corner K-type of π in the sense of Section* 3.4 *(respectively the peripheral K-type in the sense of Section* 3.5*).*

For degenerate characters ψ of N_0 the Whittaker functions for π above with respect to τ is not rapidly decreasing.

Proof. We divide the proof into three cases.

(i) The case of (limits of) large discrete series and P_J-principal series.

For this we omit the case of P_J-principal series representations. In fact, following the coming argument for the case of large discrete series representations, the case of P_J-principal series representations is settled similarly by the differential equations in [17, Propositions 7.1, 7.3] (see also [18]).

Let π be a (limit of) large discrete series representation and $\Lambda = (\Lambda_1, \Lambda_2)$ be the highest weight of the minimal K-type τ_Λ of π. We think only of π with Harish-Chandra parameter in Ξ_{III} since the argument for the case of Ξ_{II} is reduced to this by the reasoning to prove Theorem 4.2 (2). We put $d_\Lambda := \Lambda_1 - \Lambda_2$ as in Section 3.2. Recall that the Whittaker function W is written as $W(g) = \sum_{k=0}^{d_\Lambda} c_k(g) v_k^*$. Here note that W satisfies the right K-equivariance with respect to τ_Λ^*, whose highest weight is $(-\Lambda_2, -\Lambda_1)$. Following [25, Section 8] (see also Theorem 4.2) we put

$$c_k(a_0) = a_1^{-\Lambda_2+1-d_\Lambda} a_2^{-\Lambda_2} \left(\frac{a_1}{a_2}\right)^k e^{2\pi m_3 a_2^2} h_k(a_0)$$

for $a_0 \in A_0$. According to [25, Section 8, (G-1), Lemma 8.1] $h_{d_\Lambda}(a_0)$ satisfies the following differential equations:

$$\partial_1 h_{d_\Lambda}(a_0) + 2\pi\sqrt{-1}\frac{a_1}{a_2} m_0 h_{d_\Lambda - 1}(a_0) = 0, \tag{1}$$

$$(\partial_1\partial_2 + 4\pi^2\frac{a_1^2}{a_2^2}m_0^2)h_{d_\Lambda}(a_0) = 0, \tag{2}$$

$$((\partial_1+\partial_2)^2 + (-2\Lambda_1 - 2)(\partial_1+\partial_2) + (2\Lambda_1+1) - 4\pi\sqrt{-1}a_2^2m_3\partial_2)h_{d_\Lambda}(a_0) = 0, \tag{3}$$

where $\partial_i = a_i\frac{\partial}{\partial a_i}$ for $i = 1, 2$. Here we note that the condition $m_0 \neq 0$ is necessary to obtain (4.2) and (4.3). For this differential equations we remark that Moriyama [21, p913] pointed out that they hold also for the case of the limits of large discrete series representations.

Suppose first that $m_0 = 0$. Then, due to (4.1), $h_{d_\Lambda}(a_0)$ is constant with respect to a_1, which implies that $c_{d_\Lambda}(a_0)$ is not rapidly decreasing. We next assume that $m_3 = 0$, for which we may assume that $m_0 \neq 0$. Then the equation (4.3) admits the following decomposition:

$$(\partial_1+\partial_2-1)(\partial_1+\partial_2-2\Lambda_1-1)h_{d_\Lambda}(a_0) = 0.$$

We now note that $y_2\frac{\partial}{\partial y_2} = \frac{1}{2}(\partial_1+\partial_2)$, and then see that

$$(y_2\frac{\partial}{\partial y_2} - \frac{1}{2})(y_2\frac{\partial}{\partial y_2} - (\Lambda_1+\frac{1}{2}))h_{d_\Lambda}(a_0) = 0.$$

That is, $h_{d_\Lambda}(a_0)$ is the eigen-function with respect to $y_2\frac{\partial}{\partial y_2}$ with the eigenvalues $\frac{1}{2}$ or $\Lambda_1 + \frac{1}{2}$. We therefore see that $c_{d_\Lambda}(a_0)$ is not rapidly decreasing with respect to y_2 since it differs from $h_{d_\Lambda}(a_0)$ only by a polynomial of y_1 and y_2.

(ii) The case of principal series.

For this case recall that the differential equations characterizing the Whittaker function W are given in [16, Theorems 10.1, 11.3] (see also [11, Theorems 1.5, 1.6]). Furthermore recall that we have used the coordinate $(y_1, y_2) := (a_1/a_2, a_2^2)$ for the case of principal series. With the notation $\partial_i = y_i\frac{\partial}{\partial y_i}$ for $i = 1,\ 2$ the characterizing differential equations are written for

$$\varphi(a_0) = y_1^{-3/2}y_2^{-2}W(a_0) \quad (a_0 \in A_0).$$

The differential equations just mentioned in [16] and [12] are those for the case of $(m_0, m_3) = (1, 1)$. The characterizing differential equation for a general (m_0, m_3) is obtained by replacing $2\pi y_1$ and $2\pi y_2$ with $2\pi m_0 y_1$ and $2\pi m_3 y_2$ respectively. This has led to Theorem 4.4 essentially by Ishii [12] and Niwa [24]. The characterizing differential equation for a degenerate (m_0, m_3) is obtained by specializing (m_0, m_3) with $m_0 = 0$ or $m_3 = 0$ for that of the general (m_0, m_3).

We first explain only the consequences. For the case of an even principal series representation we deduce

$$\begin{cases} (\partial_1^2 - z_1^2)(\partial_1^2 - z_2^2)\varphi = 0 & (m_0 = 0), \\ (\partial_2^2 - \frac{(z_1+z_2)^2}{4})(\partial_2^2 - \frac{(z_1-z_2)^2}{4})\varphi = 0 & (m_3 = 0), \end{cases}$$

from [11, Theorem 1.5] or [16, Theorem 10.1]. For the case of an odd principal series representation we write $\varphi_0(a_0)v_0^* + \varphi_1(a_0)v_1^*$ for $\varphi(a_0)$. We deduce from [11, Theorem 1.6] or [16, Theorem 11.3] that

$$\begin{cases} (\partial_1^2 - z_0^2)\varphi_0 = 0,\ (\partial_1^2 - (z_1^2 + z_2^2 - z_0^2))\varphi_1 = 0 & (m_0 = 0), \\ (\partial_2^2 - \frac{(z_1+z_2)^2}{4})(\partial_2^2 - \frac{(z_1-z_2)^2}{4})\varphi_i = 0 \text{ for } i = 0,\ 1 & (m_3 = 0). \end{cases} \tag{4}$$

Here we put $z_0 := z_1$ (respectively $z_0 := z_2$) if the pair of σ and the minimal K type is $(\sigma_1, \sigma_2) = (1, -1)$ and $\tau_{(0,-1)}$ or $(\sigma_1, \sigma_2) = (-1, 1)$ and $\tau_{(1,0)}$ (respectively $(\sigma_1, \sigma_2) = (1, -1)$ and $\tau_{(1,0)}$ or $(\sigma_1, \sigma_2) = (-1, 1)$ and $\tau_{(0,-1)}$).

As a result we see that W is an eigenfunction of ∂_1 or ∂_2 for both of even and odd principal series representations. From this we deduce that W is of polynomial order but not of rapid decay with respect to y_1 or y_2 when $m_0 = 0$ or $m_3 = 0$ respectively.

We next explain how to verify this. The case of $m_0 = 0$ is not difficult to prove. We therefore write down the outline of the proof for the case of an odd principal series representation with $m_3 = 0$, which is more difficult to verify than the case of an even principal series with $m_3 = 0$. We put

$$f := \varphi_0 + \sqrt{-1}\varphi_1, \ g := \varphi_0 - \sqrt{-1}\varphi_1.$$

With the notation we have

$$D_1 = \partial_1^2 + 2\partial_2^2 - 2\partial_1\partial_2 - \frac{z_1^2+z_2^2}{2}, \ \ D_2 = \partial_1^2 - z_0^2, \ \ D_3 = (\partial_1 - 2\partial_2)^2 - z_0^2, \ \ Y_1 = 2\pi m_0 y_1,$$

so that the characterizing differential equations in [11, Theorem 1.6] or [16, Theorem 11.3] can be rewritten as

$$\begin{aligned}
&(D_1 - Y_1^2 - Y_1)f = 0,\\
&(D_1 - Y_1^2 + Y_1)g = 0,\\
&(D_2 + D_3 - 2Y_1^2 - 2Y_1)f + (D_2 - D_3 + 4Y_1\partial_2)g = 0,\\
&(D_2 - D_3 - 4Y_1\partial_2)f + (D_2 + D_3 - 2Y_1^2 + 2Y_1)g = 0.
\end{aligned}$$

From these differential equations we see that

$$\begin{aligned}
&(D_2 + D_3 - 2D_1)f + (D_2 - D_3 + 4Y_1\partial_2)g = 0,\\
&(D_2 - D_3 - 4Y_1\partial_2)f + (D_2 + D_3 - 2D_1)g = 0,
\end{aligned}$$

which leads to

$$\begin{aligned}
&(z_1^2 + z_2^2 - 2z_0^2)f + 4(\partial_1\partial_2 - \partial_2^2 + Y_1\partial_2)g = 0,\\
&4(\partial_1\partial_2 - \partial_1^2 - Y_1\partial_2)f + (z_1^2 + z_2^2 - 2z_0^2)g = 0.
\end{aligned}$$

From this and the relations $\partial_1\partial_2(Y_1\partial_2) = Y_1(\partial_1 + 1)\partial_2^2$, $\partial_2^2 D_1 f = (Y_1^2 + Y_1)\partial_2^2 f$, $\partial_2^2 D_1 g = (Y_1^2 - Y_1)g$ we then deduce

$$\{(\partial_1\partial_2 - \partial_2^2)^2 - \partial_2^2 D_1 - \frac{1}{16}(z_1^2 + z_2^2 - 2z_0^2)^2\}h = 0$$

for both of $h = f$ and $h = g$. From this we see that f and g satisfy the same differential equation as (4). As a result we have the desired differential equations for φ_0 and φ_1.

(iii) The case of P_S-principal series.

Different from the case of non-degenerate characters we use $\{u_i^* \mid 0 \le i \le n\}$ as basis of $V_{(l+n,l)}^*$ as in Section 3.2, instead of $\{u_i^{(l+n,l),*} \mid 0 \le i \le n\}$ for the case of non-degenerate characters as in Section 4.4. The basis $\{u_i^* \mid 0 \le i \le n\}$ or $\{u_i \mid 0 \le i \le n\}$ corresponds to $\{\mathbf{f}_i^{(l+n,l)}\}_{0 \le i \le n}$ given in [12, p. 294]). For this case let W_i be the coefficient function of u_i^* for the Whittaker function of this case and put $\partial_i := a_i \frac{\partial}{\partial a_i}$ for $i = 1, 2$.

We first consider the case of $m_0 = 0$. For this case we review the characterizing differential equations in [12, (2.4), (2.5), (2.6)] as follows:

$$\begin{aligned}
&(\partial_1 + l - i - 2)W_i + (\partial_2 - 4\pi m_3 a_2^2 + l + n - i - 2)W_{i+2} = 0,\ (0 \le i \le n-2),\\
&(\partial_2 + 4\pi m_3 a_2^2 - l - n + i)W_i + (\partial_1 - l - 2n + i)W_{i+2} = 0,\ (0 \le i \le n-2),\\
&\{\partial_1^2 + \partial_2^2 - 4\partial_1 - 2\partial_2 - 16(\pi m_3 a_2^2)^2 + 8(l+n-i)\pi m_3 a_2^2\}W_i = \frac{1}{2}(z^2 + (n-1)^2 - 10)W_i,
\end{aligned}$$

with $0 \le i \le n$ in the last equation. From the first and second equation we deduce

$$\{(\partial_1 - \partial_2 - n)(\partial_1 + \partial_2 - n - 2) + 16(\pi m_3 a_2^2)^2 + 8\pi m_3 a_2^2(-l - n + i)\}W_i = 0$$

for $0 \le i \le n$. Adding this equation to the third equation above we have

$$(\partial_1 - \frac{n+3+z}{2})(\partial_1 - \frac{n+3-z}{2})W_i = 0 \quad (0 \le i \le n),$$

which implies that W_i is of moderate growth but not rapidly decreasing with respect to a_1 or $y_1^2 y_2$ $(= a_1^2)$.
Next let $m_3 = 0$ and $m_0 \ne 0$. We put $y_1 := 4\pi|m_0|\frac{a_1}{a_2}$ and $y_2 := a_1 a_2$. By abuse of notation we denote $\mathrm{sgn}(m_0)W_i$ also by W_i with the signature $\mathrm{sgn}(m_0)$ of m_0. In this setting we mean ∂_i by $y_i\frac{\partial}{\partial y_i}$ for $i = 1,\ 2$. The differential equations [12, (2.4), (2.5), (2.6)] are then written as

(A_i) $\quad (\partial_1 + \partial_2 + l - i - 2)W_i - \sqrt{-1}y_1 W_{i+1} + (-\partial_1 + \partial_2 + l + n - i - 2)W_{i+2} = 0$ $(0 \le i \le n-2)$,

(B_i) $\quad (-\partial_1 + \partial_2 - l - n + i)W_i + \sqrt{-1}y_1 W_{i+1} + (\partial_1 + \partial_2 - l - 2n + i)W_{i+2} = 0$ $(0 \le i \le n-2)$,

(C_i) $\quad \{2\partial_1^2 + 2\partial_2^2 - 2\partial_1 - 6\partial_2 - \frac{1}{2}y_1^2 - \frac{1}{2}(z^2 + (n-1)^2 - 10)\}W_i - \sqrt{-1}(n-i)y_1 W_{i+1} + \sqrt{-1}iy_1 W_{i-1} = 0$ $(0 \le i \le n)$.

Consider $(A_i) + (B_i)$, which leads to

(D_i) $\quad (2\partial_2 - n - 2)(W_i + W_{i+2}) = 0 \quad (0 \le i \le n-2)$.

We can thus replace the system $(A_i), (B_i), (C_i)$ by $(A_i), (C_i), (D_i)$.
To go further we define $\beta_{i,j}^{(n)} \in \mathbb{C}$ by

$$(x_1 + \sqrt{-1}x_2)^i (x_1 - \sqrt{-1}x_2)^{n-i} = \sum_{j=0}^{n} \beta_{i,j}^{(n)} x_1^j x_2^{n-j} \tag{5}$$

for $0 \le i, j \le n$ and put $h_i := \sum_{j=0}^{n} \beta_{i,j}^{(n)} W_i \quad (0 \le i \le n)$.
In addition, applying the differential operator $x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}$ to (5), we get

$$\sum_{j=0}^{n} \beta_{i,j}^{(n)}\{-(n-j)W_{j+1} + jW_{j-1}\} = \sqrt{-1}(n-2i)h_i.$$

Consider $\sum_{j=0}^{n} \beta_{i,j}^{(n)}(C_j)$, and we can then rewrite (C_i) as

(C_i') $\quad \{\partial_1^2 + \partial_2^2 - \partial_1 - 3\partial_2 - \frac{1}{4}y_1^2 - \frac{1}{4}(z^2 + (n-1)^2 - 10) - \frac{1}{2}(n-2i)y_1\}h_i = 0 \quad (0 \le i \le n)$.

By a direct calculation we see $\sum_{j=0}^{n-2}\beta_{i,j}^{(n-2)}(W_j+W_{j+2})=h_{i+1}$, whence

(D_i') $\quad (2\partial_2-n-2)h_{i+1}=0, \quad (0\le i\le n-2)$.

As a result we see that h_i is of moderate growth but not rapidly decreasing with respect to y_2 for $1\le i\le n-1$. We are done. ■

Remark 4.8. (1) According to [7, Theorem 8.2 (2)] and [8, Theorem 7.2] the multiplicities of the moderate growth Whittaker functions (or models) for non-trivial degenerate characters with $m_3=0$ are two or at most two for the cases of large discrete series or P_J-principal series respectively. For the case of large discrete series representations the moderate growth Whittaker functions for all degenerate characters are written down explicitly in [10, Theorems 5.6, 5.7 and 5.9].

(2) As for the case of P_S-principal series with $m_3=0$ and $m_0\neq 0$, some further effort enables us to obtain

$$h_i(y_1,y_2)=C_i y_2^{\frac{n+2}{2}}W_{i-\frac{n}{2},\frac{z}{2}}(y_1)$$

for $0\le i\le n+1$, where C_i is a constant determined by some recurrence relation from C_0 and C_1. ■

As a result of this proposition and the explicit formulas in Theorems 4.2, 4.3, 4.4, 4.5 we obtain a general consequence as follows:

Theorem 4.9. (1) *Let π be any irreducible generic representation of G, a (limit of) holomorphic or a (limit of) anti-holomorphic discrete series representation. We have*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi,\mathcal{S}_\psi(N_0\backslash G))=\{0\}$$

for any degenerate character ψ of N_0. In particular, π chosen as above satisfies $\dim W_{\psi,\pi}(\tau^*)^0=0$ for any multiplicity one K-type τ.

(2) *Let π be any irreducible generic representation of G. Any moderate growth Whittaker functions for π in the usual sense (i.e. when ψ is non-degenerate) are necessarily rapidly decreasing. We have*

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi,\mathcal{S}_\psi(N_0\backslash G))\le 1$$

for any non-degenerate character ψ of N_0. This implies that π as above satisfies $\dim W_{\psi,\pi}(\tau^*)^0\le 1$ for any multiplicity one K-type τ. In particular, for π admitting a non-zero moderate growth Whittaker model,

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi,\mathcal{S}_\psi(N_0\backslash G))=1.$$

Proof. We mainly give a proof for the first assertion. This proof ends with a brief explanation of the second assertion, which is settled in a manner quite similar to the first one.

When π is a holomorphic discrete series the assertion is nothing but [22, Proposition 7.1(1), (2), Theorems 7.2, 7.3]. The case of an anti-holomorphic discrete series is settled by a similar calculation since the characterizing differential equations of this case are deduced from the complex conjugate of the Cauchy Riemann condition. Or the idea to prove Theorem 4.2 (2) reduces the problem of the anti-holomorphic discrete series to that of holomorphic ones. The argument for holomorphic or anti-holomorphic discrete series can be extended to limits of them since their minimal K-types are characterized by the Cauchy Riemann condition or its complex conjugate.

Let π be generic. By [14, Theorem 6.8.1] and [29, Theorem 6.2 f)], such π is a (limit of) large discrete series representation or a parabolic induction induced from a cuspidal parabolic subgroup. All the parabolic subgroups P_0, P_J and P_S are cuspidal. Regarding the parabolic induction from P_J and P_S the representations of M_J and M_S (modulo centers of M_J and M_S) are taken as limits of discrete series as well as usual discrete series.

Recall that, to define the Whittaker functions for π, we consider the restriction map

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N_0\backslash G)) \to \mathrm{Hom}_K(\tau, \mathcal{S}_\psi(N_0\backslash G)) \simeq W_{\psi,\pi}(\tau^*)^0$$

to a multiplicity one K-type τ (see the beginning of Section 4.1). Proposition 4.7 implies that $\dim W_{\psi,\pi}(\tau^*)^0 = 0$ when π is a (limit of) large discrete series or a parabolic induction whose representation of M_0, M_J or M_S is a discrete series. Here, as τ, we take the minimal K-type, the corner K-type or a peripheral K-type as in Proposition 4.7. By the irreducibility of π the restriction map is injective, and we thus have $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N_0\backslash G)) = \{0\}$, which yields $W_{\psi,\pi}(\tau^*)^0 = \{0\}$ for any multiplicity one K-type τ.

We are left with the case where π is a parabolic induction from P_J or P_S whose representation of M_J or M_S is a limit of discrete series (modulo center). It is known that a limit of discrete series of $SL_2(\mathbb{R})$ or $SL_2^{\pm}(\mathbb{R})$ is a summand of a principal series representation. By the double induction (cf. [13, p. 170–171]) we see that π is embedded into a principal series of G. When π is a P_S-principal series representation and its representation of M_S is a limit of discrete series, the set of the K-types of π includes all the minimal K-types $\{\tau_{(1,0)},\ \tau_{(0,-1)}\}$ of an odd principal series (see Proposition 3.1 and Section 3.6). For this we note that there is no possibility that even principal series includes the K-types $\{\tau_{(1,0)},\ \tau_{(0,-1)}\}$, which is verified by the distribution of K-types of even principal series (see e.g. [16, Proposition 3.2] and [9, Proposition 2.1]). This and the multiplicity free property of minimal K-types verify that π occurs in an odd principal series with multiplicity one. The restriction map of $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N_0\backslash G))$ to such a minimal K-type gives rise to the Whittaker functions for an odd principal series with respect to the minimal K-type since such restriction factors through the odd principal series. Namely the problem is reduced to that for the case of a principal series. When π is a P_J-principal series induced from a limit of discrete series we see that the corner K-types of π coincides with a minimal K-type of an even or odd principal series (for this see Sections 3.4 and 3.6). We can then do a similar reduction of the problem to the case of a principal series. We have proved the first assertion.

Finally we explain the proof of the second assertion briefly. Let ψ be non-degenerate and suppose first that π is an irreducible generic representation except for P_J-principal series and P_S-principal series induced from limits of discrete series of M_J and M_S. For these representations Theorems 4.2, 4.3, 4.4 and 4.5 tell us the rapidly decreasing property of the moderate growth Whittaker functions, for which note that all the Whittaker functions we have reviewed are referred to as moderate growth ones in all the papers cited in Sections 4.1–4.4. Now note the injectivity of $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N_0\backslash G)) \hookrightarrow W_{\psi,\pi}(\tau^*)^0$ remarked above. The four theorems mentioned above also show that

$$\dim \mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \mathcal{S}_\psi(N_0\backslash G)) \leq 1$$

since $\dim W_{\psi,\pi}(\tau^*)^0 \leq 1$ is verified for the similar specific choices of multiplicity one K-types. As for the aforementioned remaining two parabolic inductions for P_J and P_S induced from limits of discrete series of M_J and M_S we can reduce the problem to the case of principal series representations in a manner similar to the previous discussion for degenerate ψ. As a result we are done for all the assertions. ∎

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