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The Affine Closure of $T^*(\mathrm{SL}_n/U)$

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Abstract. We show that the affine closure $\overline{T^*(\mathrm{SL}_n/U)}$ has symplectic singularities, in the sense of Beauville. In the special case $n = 3$, we show that the affine closure $\overline{T^*(\mathrm{SL}_3/U)}$ is isomorphic to the closure $\overline{\mathcal{O}}_{\min}$ of the minimal nilpotent orbit $\mathcal{O}_{\min}$ in $\mathfrak{so}_8$. Moreover, the quasi-classical Gelfand-Graev action of the Weyl group W on $\overline{T^*(\mathrm{SL}_3/U)}$ can be identified with the restriction to $\overline{\mathcal{O}}_{\min}$ of E. Cartan's triality action on $\mathfrak{so}_8$.

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Key Words: Symplectic singularities, triality action.

1. Introduction

Let G be a complex connected semisimple algebraic group, B a Borel subgroup of G, and U the unipotent radical of B. For example, in the case $G = \mathrm{SL}_n$, we can take U to be the collection of all upper triangular matrices with all diagonal entries equal to 1. The homogeneous space G/U is called the "basic affine space". While G/B is projective, the basic affine space G/U is a quasi-affine variety. It turns out that many interesting problems in representation theory are related to the basic affine space. In particular, the algebra $\mathcal{D}(G/U)$ of algebraic differential operators on G/U is well-studied, for example in [3], [11]. In this paper, we study the total space of the cotangent bundle $T^*(G/U)$, of which the coordinate ring $\mathbb{C}[T^*(G/U)]$ is the quasi-classical counterpart of $\mathcal{D}(G/U)$. From a result by Ginzburg and Riche [12], the coordinate ring $\mathbb{C}[T^*(G/U)]$ is finitely generated, and the affine closure of the basic affine space is defined as

$$\overline{T^*(G/U)} := \operatorname{Spec} \mathbb{C}[T^*(G/U)].$$

Symplectic singularities, a notion of which was first introduced by Beauville in [1], play an important role in representation theory. For instance, conic symplectic singularities (affine symplectic singularities with a good $\mathbb{C}^*$-action) admit universal flat Poisson deformations and filtered quantizations. There are many examples of symplectic singularities, for example, finite quotient singularities [1], normalization of the closure of nilpotent coadjoint orbits [17], and Nakajima quiver varieties [2]. In [11], Ginzburg and Kazhdan conjectured that

Conjecture 1.1. The affine closure $\overline{T^*(G/U)}$ has symplectic singularities.

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And we prove the conjecture in section 2 for the special case $G = \mathrm{SL}_n$.

In the Lie algebra $\mathfrak{so}_8$, there is a unique nilpotent adjoint orbit $\mathcal{O}_{\min} \subset \mathfrak{so}_8$ of minimal (positive) dimension 10. The closure of the minimal orbit is $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$. In section 3, we show that there is an isomorphism of affine varieties

$$\overline{T^*(\mathrm{SL}_3/U)} \to \overline{\mathcal{O}}_{\min}. \tag{1}$$

In [11], Ginzburg and Kazhdan constructed an action on $\overline{T^*(G/U)}$ by the Weyl group W of G, called the "Gelfand-Graev action". So in the case $G = \mathrm{SL}_3$ we have an S_3-action on $\overline{T^*(\mathrm{SL}_3/U)}$. On the other hand, the Lie algebra $\mathfrak{so}_8$ has an S_3-symmetry called the triality action [4], [16], and the restriction of the triality action gives an S_3-action on $\overline{\mathcal{O}}_{\min}$. In section 4, we give a new interpretation of this triality action. In section 5, we show that the isomorphism (1) is S_3-equivariant. In section 6, we show the map (1), when restricted on smooth points, is a symplectic isomorphism.

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2. The affine closure $\overline{T^*(\mathrm{SL}_n/U)}$ has symplectic singularities

We recall the following quiver theoretic construction of $\overline{T^*(\mathrm{SL}_n/U)}$ in [7]. A *quiver* Q is a finite directed graph consisting of a vertex set I and an edge set E. Write $Q^{\mathrm{op}} = (I, \overline{E})$ for the opposite quiver obtained from Q by reversing the orientation of edges. The double quiver of Q is defined by $\overline{Q} = (I, E \sqcup \overline{E})$. A representation of Q assigns a vector space $V(i)$ to each vertex $i \in I$ and a linear map $V(e) : V(i) \to V(j)$ to each arrow $e \in E$ whose source and target are i and j respectively. Let Q be the following Dynkin quiver of type A_n.

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet$$

Let $V = \bigoplus_{k=1}^{n-1} \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{k+1})$, so each element of V defines a representation of the quiver Q. The cotangent space T^*V is identified with

$$\bigoplus_{k=1}^{n-1} \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{k+1}) \oplus \mathrm{Hom}(\mathbb{C}^{k+1}, \mathbb{C}^k)$$

via the trace pairing

$$\sum_{k=1}^{n-1} \mathrm{Tr}(\beta_k \circ \alpha_k),$$

for $\alpha_k \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{k+1})$ and $\beta_k \in \mathrm{Hom}(\mathbb{C}^{k+1}, \mathbb{C}^k)$. So each element of T^*V gives a representation of the double quiver $\overline{Q}$

$$\mathbb{C} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \mathbb{C}^2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} \mathbb{C}^{n-1} \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} \mathbb{C}^n \tag{2}$$

Throughout this paper, we use the expression (α,β) to denote an element

$$\bigoplus_{k=1}^{n-1}(\alpha_k,\beta_k)\in\bigoplus_{k=1}^{n-1}\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^{k+1})\oplus\mathrm{Hom}(\mathbb{C}^{k+1},\mathbb{C}^k)=T^*V.$$

There is a natural action of $H:=\prod_{k=2}^{n-1}\mathrm{SL}_i(\mathbb{C})$ on V defined as follows. Let $g=(g_2,\cdots,g_{n-1})\in H$, and $\alpha=(\alpha_1,\cdots,\alpha_{n-1})\in V$. Then we define

$$g.\alpha=(g_2\circ\alpha_1,g_3\circ\alpha_2\circ g_2^{-1},\cdots,g_{n-1}\circ\alpha_{n-2}\circ g_{n-2}^{-1},\alpha_{n-1}\circ g_{n-1}^{-1}). \tag{3}$$

This H-action on V induces a Hamiltonian H-action on T^*V, of which the moment map $\mu:T^*V\longrightarrow\mathrm{Lie}(H)^*$ is given by

$$\mu_H(\alpha,\beta)(X)=\sum_{k=2}^{n-1}\mathrm{Tr}\left((\alpha_{k-1}\beta_{k-1}-\beta_k\alpha_k)X_k\right),$$

where $(\alpha,\beta)=\bigoplus_{k=1}^{n-1}(\alpha_k,\beta_k)\in T^*V$, and

$$X=(X_2,X_3,\cdots,X_{n-1})\in\mathrm{Lie}(H)=\mathfrak{sl}_2\times\mathfrak{sl}_3\times\cdots\times\mathfrak{sl}_{n-1}.$$

We denote the zero fiber of the moment map by

$$N:=\mu_H^{-1}(0)\subset T^*V.$$

So N is the subvariety consists of all the (α,β) in T^*V such that $\alpha_{k-1}\beta_{k-1}-\beta_k\alpha_k$ is a k-by-k scalar matrix. Given $(\alpha,\beta)\in N$, and $k\in\{1,2,3,\cdots,n-1\}$, we define λ_k as follows:

$$\beta_k\alpha_k-\alpha_{k-1}\beta_{k-1}=\lambda_k\,\mathrm{Id}_{\mathbb{C}^k}. \tag{4}$$

Remark 2.1. A similar construction was applied in [15] to show the normality for the closure of every nilpotent orbit in $\mathfrak{gl}_n$. Note that the scalars λ_k depend on the choice of $(\alpha,\beta)\in N$, which is in contrast with the usual case of Nakajima quiver varieties. This difference is due to the definition of H as the product of SL$'s$ rather than a product of GL$'s$ as usual. The choice of such H also makes it much more difficult to construct an explicit resolution of singularities of $N//H$ using Geometric Invariant Theory.

Theorem 2.2. (Theorem 7.18 in [7]) *The affine closure $\overline{T^*(\mathrm{SL}_n/U)}$ is isomorphic to the categorical quotient $N//H$ as affine varieties.*

Lemma 2.3. *Let $0=m_0\le m_1\le m_2\le\cdots\le m_{n-1}\le m_n=n$. Suppose $m_k\le k$ for all k. Then*

$$\sum_{k=1}^{n-1}m_k(m_{k+1}-m_k)\le\sum_{k=1}^{n-1}k,$$

and the equality holds if and only if $m_k=k$ for all k.

Proof. Since $m_0=0$, we show that

$$\sum_{k=0}^{n-1}m_k(m_{k+1}-m_k)=\sum_{k=0}^{n-1}\sum_{j=m_k+1}^{m_{k+1}}m_k\le\sum_{k=0}^{n-1}\sum_{j=m_k+1}^{m_{k+1}}(j-1)=\sum_{k=1}^{n-1}k.$$ ∎

Remark 2.4. In fact, the LHS is the dimension of some (partial) flag variety of GL_n, so it obtains maximum if and only if the flag variety is a complete flag variety, which has dimension equals to the RHS.

Lemma 2.5. *The singular locus of* $\overline{T^*(\mathrm{SL}_n/U)}$ *has codimension at least* 4.

Proof. Following [7], for each $\underline{m} = (m_1, \cdots, m_{n-1})$ satisfying the condition of Lemma 2, we set

$$H(\underline{m}) := \prod_{k=1}^{n-1} \mathrm{SL}_{m_k}(\mathbb{C}), \quad \tilde{H}(\underline{m}) := \prod_{k=1}^{n-1} \mathrm{GL}_{m_k}(\mathbb{C}), \quad V(\underline{m}) := \bigoplus_{k=1}^{n-1} \mathrm{Hom}(\mathbb{C}^{m_k}, \mathbb{C}^{m_{k+1}}).$$

Then we have an exact sequence

$$1 \to H(\underline{m}) \to \tilde{H}(\underline{m}) \xrightarrow{\varphi} T(\underline{m}) \to 1,$$

where $T(\underline{m})$ is a complex torus of rank $n-1$, and φ is given by taking determinant of each product factor in $\tilde{H}(\underline{m})$. Let $\underline{n} = (1, 2, 3, \cdots, n-1)$. By Theorem 6.13 in [7], the affine variety X can be written as disjoint union

$$X = \bigsqcup_{S,\delta} Q_{(S,\delta)}$$

where each $Q_{(S,\delta)}$ is a smooth hyperkähler manifold that is a locally closed subset of X labeled by an injective subrelation S of $\leq$ on $\{1, 2, \cdots, n-1\}$ and a function $\delta : \mathrm{dom}S \to \mathbb{Z}_{>0}$ which tells us which dimension vector $\underline{m}$ to use in the hyperkähler reduction construction of the strata $Q_{(S,\delta)}$, more precisely $\underline{m} = \underline{n} - \sum_{(i,j)\in S} \delta(i) e_{ij}$, where $e_{ij} = (0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$ is the $(n-1)$-tuple with 1's in positions in-between i and j and 0's otherwise.

Let x be a singular point of $\overline{T^*(\mathrm{SL}_n/U)}$. Then x lies in some strata $Q_{(S,\delta)}$ other than the open dense generic strata $Q_{(\leq,0)}$. To the subrelation S, we associate the subtorus T_S of $T(\underline{m})$ whose Lie algebra is $\mathfrak{t}_S := \mathrm{span}\{e_{ij} \mid i \leq_S j\}$. Now by Proposition 6.9 of [7], the strata $Q_{(S,\delta)}$ has dimension the same as the Hamiltonian reduction of $T^*V(\underline{m})$ by the subgroup $H_S := \varphi^{-1}(T_S) \leq \tilde{H}(\underline{m})$. Then $\dim H_S \geq \dim H(\underline{m}) + 1$. Thus we have

$$\begin{aligned}\dim Q_{(S,\delta)} &= 2\left(\dim V(\underline{m}) - \dim H_S\right) \leq 2\left(\dim V(\underline{m}) - \dim H(\underline{m}) - 1\right) \\ &= 2\Big(\sum_{k=1}^{n-1} m_k m_{k+1} - (m_k^2 - 1)\Big) - 2 = 2\Big(n - 2 + \sum_{k=1}^{n-1} m_k(m_{k+1} - m_k)\Big).\end{aligned}$$

Since $\underline{m} \neq (1, 2, 3, \cdots, n-1)$, by the previous lemma, we have

$$\sum_{k=1}^{n-1} m_k(m_{k+1} - m_k) \leq \Big(\sum_{k=1}^{n-1} k\Big) - 1.$$

Thus

$$\dim Q_{(S,\delta)} \leq 2\Big((n-3) + \sum_{k=1}^{n-1} k\Big) = 2\Big(-1 + \sum_{k=1}^{n} k\Big) - 4 = \dim(\overline{T^*(\mathrm{SL}_n/U)}) - 4. \qquad \blacksquare$$

Proposition 2.6. *The smooth locus of* $\overline{T^*(\mathrm{SL}_n/U)}$ *admits a holomorphic symplectic form.*

Proof. Let $X = \overline{T^*(\mathrm{SL}_n/U)}$. Let $\pi : N \to N/\!/H = X$ be the categorical quotient map. Let X_{sm} be the smooth locus of X. We define the injective part N_{inj} of N as

$$N_{\mathrm{inj}} := \{(\alpha,\beta) \in N \mid \text{all the } \alpha_k \text{ are injective}\}. \tag{5}$$

Let $(\alpha,\beta) \in N_{\mathrm{inj}}$. Then the stabilizer H_α for the H-action on V defined in (2) at α is trivial. So is the stabilizer $H_{(\alpha,\beta)}$ for the induced Hamiltonian H-action on T^*V at (α,β). Now by Proposition 3.2 of [13], (α,β) is a smooth point of N.

Moreover, by Theorem 4.5 of [7], the H-orbit of α in V is closed. Now the H-orbit of (α,β) in N is the graph of the morphism between the orbits $H.\alpha \to \beta.H$ given by $g \circ \alpha \mapsto \beta \circ g^{-1}$, hence also a closed H-orbit. So, by a Corollary of Luna's étale slice theorem (cf. Proposition 5.7 in [8]), we have $\pi(N_{\mathrm{inj}}) \subset X_{\mathrm{sm}} \subset X$. From the proof of Lemma 7.17 in [7], we know $N \setminus N_{\mathrm{inj}}$ has codimension at least 2. So by the upper semicontinuity of the dimensions of the fibers of π we have

$$\operatorname{codim}(X_{\mathrm{sm}} \setminus \pi(N_{\mathrm{inj}})) \geq 2.$$

Now by the proof of Proposition 7.2 in [7], we know $\pi(N_{\mathrm{inj}})$ is identified with

$$T^*(\mathrm{SL}_n/U) \subset X_{sm} \subset X,$$

hence admitting a holomorphic symplectic form ω_0. Then by Hartogs' lemma, we can extend ω_0 to a closed holomorphic 2-form ω on X_{sm}. By taking the top wedge of ω, we have its points of degeneracy has codimension 1, so ω has to be non-degenerated on X_{sm}, hence symplectic. ∎

Definition 2.7. A normal variety X is said to have *symplectic singularities* if

1. its smooth locus X_{sm} carries a symplectic 2-form ω; and
2. if $\nu : \widetilde{X} \to X$ is any resolution of singularities, then the pull-back

   $$\nu^*\omega \in \Omega^2(\nu^{-1}(X_{\mathrm{sm}}))$$

   extends to a holomorphic 2-form on $\widetilde{X}$.

Theorem 2.8. *The affine closure* $\overline{T^*(\mathrm{SL}_n/U)}$ *has symplectic singularities.*

Proof. It is well-known that $X = \overline{T^*(\mathrm{SL}_n/U)}$ is normal (cf. the remark after Definition 4.3 in [19]). By Flenner's Theorem in [10], it suffices to show that the smooth locus X_{sm} is a symplectic variety and $\operatorname{codim}(X \setminus X_{\mathrm{sm}}) \geq 4$. So the statement follows from Proposition 2.6 and Lemma 2.5. ∎

3. Isomorphism between $\overline{T^*(\mathrm{SL}_3/U)}$ and $\overline{\mathcal{O}}_{\min} \subset \mathfrak{so}_8$

Let $m \geq 4$. Let $e_1, e_2, \ldots, e_{2m}$ be the natural basis of $\mathbb{C}^{2m}$. Define an Euclidean inner product $(\ ,\ )$ on $\mathbb{C}^{2m}$ by

$$(e_i, e_j) = \delta_{i,2m+1-j}. \tag{6}$$

For $v_1 \wedge v_2 \in \bigwedge^2 \mathbb{C}^{2m}$, we define

$$\begin{aligned} \varphi_{v_1 \wedge v_2}: \ \mathbb{C}^{2m} &\longrightarrow \mathbb{C}^{2m} \\ u &\longmapsto (v_1, u)\, v_2 - (v_2, u)\, v_1. \end{aligned} \tag{7}$$

Extend by linearity, we get the following isomorphism of $\mathfrak{so}_{2m}$-representations:

$$\bigwedge^2 \mathbb{C}^{2m} = \mathfrak{so}_{2m}. \tag{8}$$

Definition 3.1. We say that a subspace $W \subset \mathbb{C}^{2m}$ is *isotropic* if $(W, W) = 0$. An element $f \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m})$ is isotropic if its image is isotropic. An element $\alpha \in \bigwedge^2 \mathbb{C}^{2m}$ is isotropic if for all $v_1^*, v_2^* \in (\mathbb{C}^{2m})^*$,

$$(\iota_{v_1^*}\alpha, \iota_{v_2^*}\alpha) = 0.$$

Definition 3.2. We say that $\alpha \in \bigwedge^2 \mathbb{C}^{2m}$ is *decomposable* if there exist some $v_1, v_2 \in \mathbb{C}^{2m}$ such that $\alpha = v_1 \wedge v_2$.

Remark 3.3. By Plücker's Theorem we know that $\alpha \in \bigwedge^2 \mathbb{C}^{2m}$ is decomposable if and only if

$$\alpha \wedge \alpha = 0 \in \bigwedge^4 \mathbb{C}^{2m}.$$

Lemma 3.4. *Let $v_1 \wedge v_2 \in \bigwedge^2 \mathbb{C}^{2m}$ be a nonzero decomposable element. Then*

$$\mathrm{span}(v_1, v_2) \text{ is isotropic} \iff v_1 \wedge v_2 \text{ is isotropic.}$$

Proof. $(\Longrightarrow)$ Clear.

$(\Longleftarrow)$ It suffices to show that

$$(v_1, v_1) = (v_1, v_2) = (v_2, v_2) = 0.$$

Since $v_1 \wedge v_2 \neq 0$, we have v_1 and v_2 are linearly independent. So there exist $v_1^*, v_2^* \in (\mathbb{C}^{2m})^*$ such that

$$v_1^*(v_1) = v_2^*(v_2) = 1 \ \text{ and } \ v_1^*(v_2) = v_2^*(v_1) = 0.$$

Then

$$v_1 = -\iota_{v_2^*}(v_1 \wedge v_2) \ \text{ and } \ v_2 = \iota_{v_1^*}(v_1 \wedge v_2),$$

and our conclusion follows from the definition. ∎

Proposition 3.5. *We consider $\overline{\mathcal{O}}_{min}$, the closure of the minimal nilpotent orbit $\mathcal{O}_{min} \subset \mathfrak{so}_{2m}$. Then under the identification* (8),

$$\overline{\mathcal{O}}_{min} = \{\alpha \in \bigwedge^2 \mathbb{C}^{2m} \mid \alpha \text{ is decomposable and isotropic}\}. \tag{9}$$

Proof. Since $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$, it suffices to show that

$$\mathcal{O}_{\min} = \{\alpha \in \bigwedge^2 \mathbb{C}^{2m} \mid \alpha \text{ is a nonzero isotropic decomposable element}\}.$$

By Theorem 4.3.3 in [6], the minimal orbit $\mathcal{O}_{\min}$ is the adjoint orbit of the highest root vector $e_1 \wedge e_2 \in \bigwedge^2 \mathbb{C}^{2m} = \mathfrak{so}_{2m}$. Since both decomposable and isotropic properties are invariant under the SO_{2m} action, and $e_1 \wedge e_2$ is isotropic decomposable, so all elements in the minimal orbit are isotropic decomposable. But SO_{2m} acts transitively on the isotropic planes in $\mathbb{C}^{2m}$, and there is a scaling $\mathbb{C}^*$-action on $\mathcal{O}_{\min}$, so SO_{2m} acts transitively on the nonzero isotropic decomposable elements. ■

Fix basis $e_1, e_2 \in \mathbb{C}^2$, and define a symplectic bilinear form on $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m})$:

$$\omega_1(f_1, f_2) = (f_1(e_1), f_2(e_2)) - (f_1(e_2), f_2(e_1)). \tag{10}$$

The natural SL_2-action on $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m})$ is Hamiltonian with moment map

$$\mu_{SL_2} : \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m}) \longrightarrow \mathfrak{sl}_2^* \cong \mathfrak{sl}_2,$$
$$\mu_{SL_2}(f) = \begin{pmatrix} (f(e_1), f(e_2)) & (f(e_1), f(e_1)) \\ (f(e_2), f(e_2)) & -(f(e_1), f(e_2)) \end{pmatrix}.$$

So the zero fiber of the SL_2-moment map on $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m})$ is

$$N_1 := \mu_{SL_2}^{-1}(0) = \{f \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m}) \mid f \text{ is isotropic with respect to } (\ ,\ )\}. \tag{11}$$

The following result is probably known to experts, but we still include a proof here for completeness.

Proposition 3.6. *The Hamiltonian reduction $N_1 /\!/ SL_2$ is isomorphic to the closure $\overline{\mathcal{O}}_{min}$ of the minimal orbit in $\mathfrak{so}_{2m}$ as affine varieties.*

Proof. Recall from Proposition 3.5,

$$\overline{\mathcal{O}}_{\min} = \{\alpha \in \textstyle\bigwedge^2 \mathbb{C}^{2m} \mid \alpha \text{ is decomposable and isotropic}\}.$$

Define

$$\left(\textstyle\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec}} := \{\alpha \in \textstyle\bigwedge^2 \mathbb{C}^{2m} \mid \alpha \text{ is decomposable}\},$$
$$\left(\textstyle\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec,\,iso}} := \{\alpha \in \textstyle\bigwedge^2 \mathbb{C}^{2m} \mid \alpha \text{ is decomposable and isotropic}\}.$$

By the First Fundamental Theorem of Invariant Theory for SL_2 (Theorem 2 on p. 387 of [18]), we have

$$\mathbb{C}[(\mathbb{C}^2)^* \otimes \mathbb{C}^{2m}]^{SL_2} = \mathbb{C}\left[\left(\textstyle\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec}}\right].$$

Since SL_2 is reductive and $N_1 \subset \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{2m}) = (\mathbb{C}^2)^* \otimes \mathbb{C}^{2m}$ is an SL_2-invariant sub-variety, the restriction from $(\mathbb{C}^2)^* \otimes \mathbb{C}^{2m}$ to N_1 gives a surjective map

$$\begin{array}{ccc} \mathbb{C}\left[\left(\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec}}\right] & \longrightarrow & \mathbb{C}[N_1]^{SL_2} \\ f & \longmapsto & (e_1^* \otimes v_1 + e_2^* \otimes v_2 \mapsto f(v_1 \wedge v_2)) \end{array} \tag{12}$$

By Lemma 3.4 we have the kernel of (12) is precisely the ideal $\mathcal{I}_{\mathrm{iso}}$ of functions on $\left(\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec}}$ vanishing identically on the isotropic elements. Hence

$$\mathbb{C}[N_1]^{SL_2} = \mathbb{C}\left[\left(\textstyle\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec}}\right] / \mathcal{I}_{\mathrm{iso}} = \mathbb{C}\left[\left(\textstyle\bigwedge^2 \mathbb{C}^{2m}\right)_{\mathrm{dec,\,iso}}\right].$$ ■

Let $\eta : \mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^* \longrightarrow \mathbb{C}^8$ be the isomorphism given by

$$\eta(z_1, z_2, z_3, a, b, w_1, w_2, w_3) = z_1e_2 + z_2e_3 - z_3e_8 + ae_4 + be_5 - w_3e_1 + w_2e_6 + w_1e_7. \quad (13)$$

Then η is an isometry with respect to the inner product on $\mathbb{C}^4 \oplus (\mathbb{C}^4)^*$ given by the natural pairing of $\mathbb{C}^4$ and $(\mathbb{C}^4)^*$ and the inner product $(\ ,\ )$ on $\mathbb{C}^8$ defined by taking $m = 4$ in (6). We define a linear isomorphism

$$\begin{aligned} F : T^*V &\longrightarrow \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^8) = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4 \oplus (\mathbb{C}^4)^*) \\ (\alpha, \beta) &\longmapsto \Big(v \mapsto (\alpha_2 \oplus -\beta_1)(v) \oplus \frac{(\beta_2 \oplus \alpha_1) \wedge v}{e_1 \wedge e_2}\Big) \end{aligned}$$

Proposition 3.7. *The map F is an* SL_2*-equivariant symplectic isomorphism between T^*V and* $(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^8), \omega_1)$, *where ω_1 is given by (10).*

Proof. First, we define $F_0 : T^*V \longrightarrow T^*(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4))$ which maps (α, β) to the following element in $\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4) \oplus \mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^2)$:

$$\mathbb{C}^2 \underset{\beta_2 \oplus \alpha_1}{\overset{\alpha_2 \oplus -\beta_1}{\rightleftarrows}} (\mathbb{C}^3 \oplus \mathbb{C}) = \mathbb{C}^4$$

Then F_0 is an SL_2-equivariant symplectic isomorphism. Next, under the identifications

$$T^*(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)) = T^*(\mathbb{C}^4 \oplus \mathbb{C}^4) = (\mathbb{C}^4 \oplus \mathbb{C}^4) \oplus ((\mathbb{C}^4)^* \oplus (\mathbb{C}^4)^*),$$
$$\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4 \oplus (\mathbb{C}^4)^*) = (\mathbb{C}^4 \oplus (\mathbb{C}^4)^*) \oplus (\mathbb{C}^4 \oplus (\mathbb{C}^4)^*),$$

we define the map

$$\begin{aligned} F_1 : \quad T^*(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)) &\longrightarrow \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4 \oplus (\mathbb{C}^4)^*) \\ (v_1 \oplus v_2) \oplus (v_1^* \oplus v_2^*) &\longmapsto (v_1 \oplus (-v_2^*)) \oplus (v_2 \oplus v_1^*). \end{aligned}$$

And we check that F_1 is a symplectic isomorphism:

$$\begin{aligned} &F_1^*\omega_1\big((v_1 \oplus v_2) \oplus (v_1^* \oplus v_2^*), (w_1 \oplus w_2) \oplus (w_1^* \oplus w_2^*)\big) \\ &= \omega_1\big((v_1 \oplus (-v_2^*)) \oplus (v_2 \oplus v_1^*), (w_1 \oplus (-w_2^*)) \oplus (w_2 \oplus w_1^*)\big) \\ &= \big(-v_2^*(w_2) + w_1^*(v_1)\big) - \big(v_1^*(w_1) - w_2^*(v_2)\big) \\ &= (w_1^* \oplus w_2^*)(v_1 \oplus v_2) - (v_1^* \oplus v_2^*)(w_1 \oplus w_2), \end{aligned}$$

which is the natural symplectic form on $T^*(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4))$. Notice that all three spaces

$$T^*V, \quad T^*(\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)), \quad \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4 \oplus (\mathbb{C}^4)^*)$$

have natural Hamiltonian SL_2-actions. Since both F_0 and F_1 are SL_2-equivariant, F is also an SL_2-equivariant symplectic isomorphism as

$$F = F_1 \circ F_0. \qquad \blacksquare$$

Corollary 3.8. *The map F induces an isomorphism between* $\overline{T^*(\mathrm{SL}_3/U)}$ *and* $\overline{\mathcal{O}}_{min}$ *as affine varieties.*

Proof. This is a direct corollary of Propositions 3.6 and 3.7. ■

4. The triality action on $\mathfrak{so}_8$

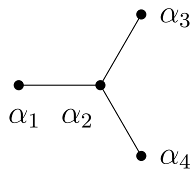
The triality action was first discovered by E. Cartan in his 1925 paper [4] in which he constructed a certain S_3-subgroup of the automorphism group of $\mathfrak{so}_8$, such that an order three element $\mathrm{Aut}(\mathfrak{so}_8)$ is constructed from the following matrix

$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Fix the following set of simple roots for the root system of type D_4.

$$\begin{aligned} \alpha_1 &= (1,-1,0,0) \\ \alpha_2 &= (0,1,-1,0) \\ \alpha_3 &= (0,0,1,-1) \\ \alpha_4 &= (0,0,1,1) \end{aligned}$$

So that the Dynkin Diagram D_4 is labeled as follows.



Choose a Chevalley basis $\{X_\alpha\}_{\alpha\in\Delta^+}, \{Y_{-\alpha}\}_{\alpha\in\Delta^+}, \{H_{\alpha_i}\}_{i=1,2,3,4}$ of $\mathfrak{so}_8 = \bigwedge^2(\mathbb{C}^8)$ as follows

$$\begin{aligned}
X_{\alpha_1} &= e_1 \wedge e_7 & Y_{-\alpha_1} &= e_2 \wedge e_8 \\
X_{\alpha_2} &= e_2 \wedge e_6 & Y_{-\alpha_2} &= e_3 \wedge e_7 \\
X_{\alpha_3} &= e_3 \wedge e_5 & Y_{-\alpha_3} &= e_4 \wedge e_6 \\
X_{\alpha_4} &= e_3 \wedge e_4 & Y_{-\alpha_4} &= e_5 \wedge e_6 \\
X_{\alpha_1+\alpha_2} &= e_1 \wedge e_6 & Y_{-\alpha_1-\alpha_2} &= e_3 \wedge e_8 \\
X_{\alpha_2+\alpha_3} &= e_2 \wedge e_5 & Y_{-\alpha_2-\alpha_3} &= e_4 \wedge e_7 \\
X_{\alpha_2+\alpha_4} &= e_2 \wedge e_4 & Y_{-\alpha_2-\alpha_4} &= e_5 \wedge e_7 \\
X_{\alpha_2+\alpha_3+\alpha_4} &= e_2 \wedge e_3 & Y_{-\alpha_2-\alpha_3-\alpha_4} &= e_6 \wedge e_7 \\
X_{\alpha_1+\alpha_2+\alpha_4} &= e_1 \wedge e_4 & Y_{-\alpha_1-\alpha_2-\alpha_4} &= e_5 \wedge e_8 \\
X_{\alpha_1+\alpha_2+\alpha_3} &= e_1 \wedge e_5 & Y_{-\alpha_1-\alpha_2-\alpha_3} &= e_4 \wedge e_8 \\
X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} &= e_1 \wedge e_3 & Y_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4} &= e_6 \wedge e_8 \\
X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4} &= e_1 \wedge e_2 & Y_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4} &= e_7 \wedge e_8 \\
H_{\alpha_1} &= e_1 \wedge e_8 - e_2 \wedge e_7 & H_{\alpha_3} &= e_3 \wedge e_6 - e_4 \wedge e_5 \\
H_{\alpha_2} &= e_2 \wedge e_7 - e_3 \wedge e_6 & H_{\alpha_4} &= e_3 \wedge e_6 + e_4 \wedge e_5
\end{aligned}$$

Since the Dynkin diagram D_4 has an S_3-symmetry and each automorphism of the Dynkin diagram does lift uniquely to a Lie algebra automorphism (cf. Theorem 2.108 in [14], and Lemma 2.6 in [9]), the existence of the triality action might seem easy at first glance. But, a priori, it is not clear that the these lifts gives an $S_3 \hookrightarrow \mathrm{Aut}(\mathfrak{so}_8)$. For example, not every lift of the cyclic permutation of the simple roots $\alpha_1, \alpha_3, \alpha_4$ has order three in $\mathrm{Aut}(\mathfrak{so}_8)$. In the work of [16], a lifted triality action on the Lie algebra $\mathfrak{so}(4,4)$ is constructed, but the authors proved their result by an explicit calculation in coordinates by computer. In this section, a new interpretation (see Definition 4.2) of the triality action on $\mathfrak{so}_8 = \mathfrak{so}(4,4)_{\mathbb{C}}$ is given by applying the decomposition of $\mathfrak{so}_8$ into irreducible $\mathfrak{sl}_3$-representations.

Lemma 4.1. *The map η defined in* (13) *induces an isomorphism*

$$\bigwedge^2\eta : \bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*) \longrightarrow \bigwedge^2(\mathbb{C}^8) = \mathfrak{so}_8$$

of $\mathfrak{sl}_3$-representations, where $\mathfrak{so}_8$ is viewed as an $\mathfrak{sl}_3$-representation with respect to the adjoint action under the embedding

$$\varphi_1 : \mathfrak{sl}_3 \longrightarrow \mathfrak{so}_8.$$

which restricts to an embedding $\varphi_1|_{\mathfrak{h}_{\mathfrak{sl}_3}} : \mathfrak{h}_{\mathfrak{sl}_3} \hookrightarrow \mathfrak{h}_{\mathfrak{so}_8}$ such that

$$\varphi_1^*(\alpha_2) = \alpha, \text{ and } \varphi_1^*(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \beta.$$

Proof. First, let $V_0 := \mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*$, and $\langle\, , \,\rangle$ denote the inner product on V_0 given by the natural pairing in between $\mathbb{C}^3 \oplus \mathbb{C}$ and $\mathbb{C}^* \oplus (\mathbb{C}^3)^*$. Then

$$\bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*) = \mathfrak{so}(V_0, \langle\, , \,\rangle).$$

Since η is an isometry between $(V_0, \langle\, , \,\rangle)$ and $(\mathbb{C}^8, (\, , \,))$, the map $\bigwedge^2\eta$ is a Lie algebra isomorphism between $\mathfrak{so}(V_0, \langle\, , \,\rangle)$ and $\mathfrak{so}_8$. We identify

$$\mathfrak{sl}_3 = \{A \in \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \mid \mathrm{tr}(A) = 0\} \subset \bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*).$$

Then $\mathfrak{sl}_3$ becomes an Lie subalgebra of $\mathfrak{so}(V_0, \langle\, , \,\rangle)$, and the $\mathfrak{sl}_3$-representation structure on $\bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*)$ is given by the adjoint action. Then we have the restriction of $\bigwedge^2\eta$ to $\mathfrak{sl}_3$ equals to the embedding φ_1. For example, for positive root vectors,

$$\begin{aligned}
\bigwedge^2\eta\,(e_1 \wedge e_2^*) &= e_2 \wedge e_6 = X_{\alpha_2},\\
\bigwedge^2\eta\,(e_2 \wedge e_3^*) &= e_3 \wedge (-e_1) = X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4},\\
\bigwedge^2\eta\,(e_1 \wedge e_3^*) &= e_2 \wedge (-e_1) = X_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}.
\end{aligned}$$

Since the $\mathfrak{sl}_3$-representation structure on $\mathfrak{so}_8$ is given by the Lie algebra embedding φ_1, our statement follows. ∎

Decompose $\mathfrak{so}_8 = \bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*)$ into irreducible $\mathfrak{sl}_3$-representations,

$$\mathfrak{so}_8 = \mathfrak{sl}_3 \oplus \mathbb{C}_{\mathrm{trace}} \oplus \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus (\mathbb{C}^3)^* \oplus (\mathbb{C}^3)^*. \tag{14}$$

Let $\mathfrak{h} = \{(c_1, c_2, c_3) \in \mathbb{C}^3 \mid c_1 + c_2 + c_3 = 0\}$. Define $\varphi_2 : \mathfrak{h} \to \mathfrak{so}_8$ by

$$\begin{aligned}\varphi_2(c_1, c_2, c_3) :&= c_1 H_{\alpha_1} + c_2 H_{\alpha_2} + c_3 H_{\alpha_3} \\ &= c_1(H_1 - H_2) + c_2(H_3 - H_4) + c_3(H_3 + H_4) \\ &= -c_1(H_2 + H_3 - H_1) + (c_3 - c_2)H_4 \in \mathbb{C}_{\text{trace}} \oplus \mathbb{C},\end{aligned}$$

where $H_i := e_i \wedge e_{9-i}$ for each i. So the image of φ_2 is precisely the subrepresentation $\mathbb{C}_{\text{trace}} \oplus \mathbb{C} \subset \mathfrak{so}_8$. Let V_1, V_2, V_3 (resp. V_1^*, V_2^*, V_3^*) be the three copies of $\mathbb{C}^3$ (resp. $(\mathbb{C}^3)^*$) in the decomposition (14) whose highest weight vectors are root vectors for $\mathfrak{so}_8$ corresponding to the roots $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4$ (resp. $\alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3$). Identify V_1, V_2, V_3 (resp. V_1^*, V_2^*, V_3^*) via the unique $\mathfrak{sl}_3$-equivariant linear isomorphisms which map the above choice of highest weight vectors to each other. Explicitly,

$$\begin{aligned}V_1 &= \mathbb{C}\langle -X_{\alpha_1+\alpha_2}, X_{\alpha_1}, Y_{-\alpha_2-\alpha_3-\alpha_4}\rangle = \mathbb{C}^3, \\ V_2 &= \mathbb{C}\langle X_{\alpha_2+\alpha_3}, X_{\alpha_3}, Y_{-\alpha_1-\alpha_2-\alpha_4}\rangle = \mathbb{C}^3, \\ V_3 &= \mathbb{C}\langle X_{\alpha_2+\alpha_4}, X_{\alpha_4}, Y_{-\alpha_1-\alpha_2-\alpha_3}\rangle = \mathbb{C}^3, \\ V_1^* &= \mathbb{C}\langle X_{\alpha_2+\alpha_3+\alpha_4}, Y_{-\alpha_1}, -Y_{-\alpha_1-\alpha_2}\rangle = (\mathbb{C}^3)^*, \\ V_2^* &= \mathbb{C}\langle X_{\alpha_1+\alpha_2+\alpha_4}, Y_{-\alpha_3}, Y_{-\alpha_2-\alpha_3}\rangle = (\mathbb{C}^3)^*, \\ V_3^* &= \mathbb{C}\langle X_{\alpha_1+\alpha_2+\alpha_3}, Y_{-\alpha_4}, Y_{-\alpha_2-\alpha_4}\rangle = (\mathbb{C}^3)^*.\end{aligned}$$

So we have an isomorphism of $\mathfrak{sl}_3$-representations

$$\varphi : \mathfrak{sl}_3 \oplus \mathfrak{h} \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_1^* \oplus V_2^* \oplus V_3^* \longrightarrow \mathfrak{so}_8 \tag{15}$$

Definition 4.2. We define an S_3-action on $\mathfrak{sl}_3 \oplus \mathfrak{h} \oplus V_1 \oplus V_2 \oplus V_3 \oplus V_1^* \oplus V_2^* \oplus V_3^*$ by fixing $\mathfrak{sl}_3$, acting on $\mathfrak{h}$ as the Weyl group S_3-action, and permuting subscripts of V_i and V_i^*. Then under the identification φ, we have an S_3-action act on $\mathfrak{so}_8$.

Theorem 4.3. *The S_3-action* act *gives an embedding $S_3 \hookrightarrow \operatorname{Aut}(\mathfrak{so}_8)$ which coincides with the triality action.*

Proof. Let (M, c, u, u^*) be an element in $\mathfrak{sl}_3 \oplus \mathfrak{h} \oplus (\mathbb{C}^3)^3 \oplus ((\mathbb{C}^3)^*)^3$. Express each components in terms of the Chevalley basis, we see the S_3-action act does permutes the root vectors accordingly (i.e. fixing α_2, and permutes $\alpha_1, \alpha_3, \alpha_4$). So it suffices to show that this action preserves the Lie bracket of $\mathfrak{so}_8$. There is a linear action of (M, c, u, u^*) on elements $(v, a, b, v^*) \in \mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*$ coming from the linear action of $\mathfrak{so}_8$ on $\mathbb{C}^8$ given by

$$(M, c, u, u^*).\begin{pmatrix} v \\ a \\ b \\ v^* \end{pmatrix} = \begin{pmatrix} Mv + c_1 v + a u_2 + b u_3 + v^* \wedge u_1^* \\ u_2^*(v) + (c_3 - c_2)a - v^*(u_3) \\ u_3^*(v) - (c_3 - c_2)b - v^*(u_2) \\ -v^* M - c_1 v^* - a u_3^* - b u_2^* + v \wedge u_1 \end{pmatrix},$$

which can be applied to compute the Lie bracket structure on

$$\mathfrak{sl}_3 \oplus \mathfrak{h} \oplus (\mathbb{C}^3)^3 \oplus ((\mathbb{C}^3)^*)^3.$$

Let $A = (M_A, c_A, u_A, u_A^*), B = (M_B, c_B, u_B, u_B^*) \in \mathfrak{sl}_3 \oplus \mathfrak{h} \oplus (\mathbb{C}^3)^3 \oplus ((\mathbb{C}^*)^3)^3$. Then the Lie bracket can be written as $[A, B] = (M_C, c_C, u_C, u_C^*)$ where

$$M_C = [M_A, M_B] + \sum_{i=1}^{3}(u_{A,i} \otimes u_{B,i}^* - u_{B,i} \otimes u_{A,i}^*) + \frac{1}{3}\sum_{i=1}^{3}(u_{A,i}^*(u_{B,i}) - u_{B,i}^*(u_{A,i})) \operatorname{Id}_{\mathbb{C}^3}$$

$$c_C = \begin{pmatrix} -2/3 & 1/3 & 1/3 \\ 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} u_{A,1}^*(u_{B,1}) - u_{B,1}^*(u_{A,1}) \\ u_{A,2}^*(u_{B,2}) - u_{B,2}^*(u_{A,2}) \\ u_{A,3}^*(u_{B,3}) - u_{B,3}^*(u_{A,3}) \end{pmatrix}$$

$$u_C = \begin{pmatrix} M_A\, u_{A,1} - M_B\, u_{B,1} + u_{A,2}^* \wedge u_{B,3}^* - u_{B,2}^* \wedge u_{A,3}^* + 2c_{A,1}\, u_{B,1} - 2c_{B,1}\, u_{A,1} \\ M_A\, u_{A,2} - M_B\, u_{B,2} + u_{A,3}^* \wedge u_{B,1}^* - u_{B,3}^* \wedge u_{A,1}^* + 2c_{A,2}\, u_{B,2} - 2c_{B,2}\, u_{A,2} \\ M_A\, u_{A,3} - M_B\, u_{B,3} + u_{A,1}^* \wedge u_{B,2}^* - u_{B,1}^* \wedge u_{A,2}^* + 2c_{A,3}\, u_{B,3} - 2c_{B,3}\, u_{A,3} \end{pmatrix}$$

$$u_C^* = -\begin{pmatrix} u_{A,1}^* M_A - u_{B,1}^* M_A + u_{A,2} \wedge u_{B,3} - u_{B,2} \wedge u_{A,3} + 2c_{A,1}\, u_{B,1}^* - 2c_{B,1}\, u_{A,1}^* \\ u_{A,2}^* M_A - u_{B,2}^* M_A + u_{A,3} \wedge u_{B,1} - u_{B,3} \wedge u_{A,1} + 2c_{A,2}\, u_{B,2}^* - 2c_{B,2}\, u_{A,2}^* \\ u_{A,3}^* M_A - u_{B,3}^* M_A + u_{A,1} \wedge u_{B,2} - u_{B,1} \wedge u_{A,2} + 2c_{A,3}\, u_{B,3}^* - 2c_{B,3}\, u_{A,3}^* \end{pmatrix}$$

Then we see that the Lie bracket is manifestly equivariant under the lifted triality action. ∎

In fact, the embedding of $T^*(\mathrm{SL}_3/U)$ into the LHS of (15) is a special case of the following conjecture due to Ginzburg and Kazhdan.

Conjecture 4.4. There is an W-equivariant embedding of the affine variety

$$\overline{T^*(G/U)} \hookrightarrow \mathfrak{g} \oplus \mathfrak{h} \oplus \bigoplus_{\varpi \in \text{Fund. Weights}} V_{\varpi}^{|\text{the } W\text{-orbit of } \varpi|},$$

such that it restricts to the embedding $\overline{G/U} \hookrightarrow \bigoplus_{\varpi \in \text{Fund. Weights}} V_{\varpi}$, where the Weyl group W acts on $\overline{T^*(G/U)}$ via the Gelfand-Graev action, acts on $\mathfrak{h}$ as the Weyl group, and permutes the copies of the fundamental representation V_{ϖ}.

Remark 4.5. Fix the standard basis e_1, e_2, e_3 of $\mathbb{C}^3$, and identify

$$\wedge^3\mathbb{C}^3 = \wedge^3(\mathbb{C}^3)^* = \mathbb{C},\ \wedge^2\mathbb{C}^3 = (\mathbb{C}^3)^*,\ \wedge^2(\mathbb{C}^3)^* = \mathbb{C}^3.$$

For any given $A \in \operatorname{End}(\mathbb{C}^3)$ and $v \in \mathbb{C}^3$ we define $A \wedge v \in \operatorname{Sym}^2(\mathbb{C}^3)^*$ by

$$(A \wedge v)(w_1, w_2) = (Aw_1) \wedge v \wedge w_2 + (Aw_2) \wedge v \wedge w_1.$$

Similarly we can define $A \wedge v^* \in \operatorname{Sym}^2\mathbb{C}^3$ for $v^* \in (\mathbb{C}^3)^*$. Let (M, c, u, u^*) be an element in $\mathfrak{sl}_3 \oplus \mathfrak{h} \oplus (\mathbb{C}^3)^3 \oplus ((\mathbb{C}^3)^*)^3$.

Then $\varphi(M, c, u, u^*) \in \overline{\mathcal{O}}_{\min}$ if and only if:

$u_i^*(u_j) = 0$ for all distinct $i, j \in \{1, 2, 3\}$,

$u_1^*(u_1) = (c_1 - c_2)(c_1 - c_3),$ and its cyclic permutations ("c.p." in short),

$u_1 \wedge u_2 = (c_1 - c_2)u_3^*,$ and its c.p.,

$u_1^* \wedge u_2^* = (c_1 - c_2)u_3,$ and its c.p.,

$(M - c_3 \operatorname{Id}_{\mathbb{C}^3}) \wedge u_3 + u_1^* \cdot u_2^* = 0,$ and its c.p.,

$(M - c_3 \operatorname{Id}_{\mathbb{C}^3}) \wedge u_3^* + u_1 \cdot u_2 = 0,$ and its c.p.,

$(M - c_3 \operatorname{Id}_{\mathbb{C}^3})^2 + u_1 \otimes u_1^* + u_2 \otimes u_2^* - u_3 \otimes u_3^* + u_3^*(u_3) \operatorname{Id}_{\mathbb{C}^3} = 0,$ and its c.p.

It would be interesting to find all the analogous relations for $\overline{T^*(\mathrm{SL}_n/U)}$ if the conjecture holds.

5. The Gelfand-Graev action

We first recall the reconstruction in [19] of the Gelfand-Graev action in Corollary 1.3.4 of [11]. Let $B = (\alpha, \beta) \in N := \mu_H^{-1}(0)$. Let $k \in \{1, 2, 3, \cdots, n-1\}$. Define

$$\begin{aligned} \operatorname{out}_k(B) &:= \alpha_k \oplus \beta_{k-1} \in \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^{k+1} \oplus \mathbb{C}^{k-1}), \\ \operatorname{in}_k(B) &:= \beta_k \oplus (-\alpha_{k-1}) \in \operatorname{Hom}(\mathbb{C}^{k+1} \oplus \mathbb{C}^{k-1}, \mathbb{C}^k). \end{aligned}$$

Then
$$\operatorname{in}_k(B) \operatorname{out}_k(B) = \lambda_k \operatorname{Id}_{\mathbb{C}^k}.$$

We also define Z_k to be a subvariety of $N \times N$ consisting of pairs

$$(B, B') = ((\alpha, \beta), (\alpha', \beta')),$$

such that

(1) $\forall j \notin \{k-1, k\},\ \alpha_j = \alpha_j',$ and $\beta_j = \beta_j'$.

(2) The following short sequence is exact

$$0 \longrightarrow \mathbb{C}^k \xrightarrow{\operatorname{out}_k(B')} \mathbb{C}^{k+1} \oplus \mathbb{C}^{k-1} \xrightarrow{\operatorname{in}_k(B)} \mathbb{C}^k \longrightarrow 0\,.$$

Moreover, for each j we fix some volume form $\mathsf{vol}_j \in \bigwedge^j \mathbb{C}^j$, and require

$$\operatorname{out}_k(B')(\mathsf{vol}_k) \wedge \operatorname{in}_k(B)^{-1}(\mathsf{vol}_k) = \mathsf{vol}_{k-1} \wedge \mathsf{vol}_{k+1} \in \textstyle\bigwedge^{2k}(\mathbb{C}^{k+1} \oplus \mathbb{C}^{k-1}).$$

(Since the short sequence is exact, we may take any element of $\operatorname{in}_k(B)^{-1}(\mathsf{vol}_k)$.)

(3)
$$\operatorname{out}_k(B') \operatorname{in}_k(B') = \operatorname{out}_k(B) \operatorname{in}_k(B) - \lambda_k \operatorname{Id}_{\mathbb{C}^{k+1} \oplus \mathbb{C}^{k-1}}.$$

Theorem 5.1. [19] *For any simple reflection $s_k \in W$, we can construct an automorphism S_k on $N/\!/H$, such that for all $(B, B') \in Z_k$,*

$$S_k(\pi(B)) = \pi(B').$$

And the automorphisms S_k satisfy the braid relations, hence we constructed a W-action on $N/\!/H$. This W-action coincides with the quasi-classical Gelfand-Graev action for $\overline{T^*(\mathrm{SL}_n/U)}$.

Theorem 5.2. *In the case of $n = 3$, and under the identification*

$$\overline{T^*(\mathrm{SL}_3/U)} = \overline{\mathcal{O}}_{min} \subset \mathfrak{so}_8,$$

the Gelfand-Graev action coincides with the triality S_3-action act *on $\mathfrak{so}_8$ restricted on $\overline{\mathcal{O}}_{min}$.*

Proof. Since Lie algebra automorphism preserves the minimal orbit, so the triality action can be restricted to an S_3 action on $\overline{\mathcal{O}}_{\min}$. It suffices to check for two of the simple transpositions $s_1 = (23), s_2 = (13) \in S_3$, the triality action satisfies the condition that for each $k \in \{1, 2\}$, and $(B, B') \in Z_k$, we have

$$S_k(\pi(B)) = \pi(B').$$

Since the Gelfand-Graev action is uniquely determined by its restriction on the regular semisimple open part, we take (M, c, u, u^*) to be an element in

$$\mathcal{O}_{\min} \subset \mathfrak{sl}_3 \oplus \mathfrak{h} \oplus (\mathbb{C}^3)^3 \oplus ((\mathbb{C}^3)^*)^3$$

with distinct c_1, c_2, c_3. Then the corresponding isotropic decomposable element in $\bigwedge^2(\mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}^* \oplus (\mathbb{C}^3)^*)$ is

$$(u_3 \oplus 0 \oplus -(c_3 - c_2) \oplus -u_2^*) \wedge \left(\frac{u_2}{c_3 - c_2} \oplus 1 \oplus 0 \oplus -\frac{u_3^*}{c_3 - c_2} \right).$$

So with this choice of representative, we have a lift $B = (\alpha, \beta) \in N$:

$$\alpha_1 = \begin{pmatrix} 0 \\ c_3 - c_2 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & 1 \end{pmatrix},$$

$$\alpha_2 = \left(\begin{array}{|c|c|} \hline u_3 & \dfrac{u_2}{c_3 - c_2} \\ \hline \end{array} \right), \qquad \beta_2 = \left(\begin{array}{|c|} \hline -\dfrac{u_3^*}{c_3 - c_2} \\ \hline u_2^* \\ \hline \end{array} \right).$$

First, for $k = 1$, we have

$$\mathrm{out}_1(B) = \begin{pmatrix} 0 \\ c_3 - c_2 \end{pmatrix}, \quad \mathrm{in}_1(B) = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \lambda_1 = c_3 - c_2.$$

Apply the action by $s_1 = (23)$, and rewrite the bivector so that the first defining property of Z_1 holds.

$$\begin{aligned} &(u_2 \oplus 0 \oplus -(c_2 - c_3) \oplus -u_3^*) \wedge \left(\frac{u_3}{c_2 - c_3} \oplus 1 \oplus 0 \oplus -\frac{u_2^*}{c_2 - c_3} \right) \\ &= (u_3 \oplus (c_2 - c_3) \oplus 0 \oplus -u_2^*) \wedge \left(\frac{u_2}{c_3 - c_2} \oplus 0 \oplus 1 \oplus -\frac{u_3^*}{c_3 - c_2} \right). \end{aligned}$$

So we have $$\mathrm{out}_1(B') = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathrm{in}_1(B') = \begin{pmatrix} c_2 - c_3 & 0 \end{pmatrix}.$$

Hence, $$\mathrm{out}_1(B')\,\mathrm{in}_1(B') = \mathrm{out}_1(B)\,\mathrm{in}_1(B) - \lambda_1\,\mathrm{Id}_{\mathbb{C}^2}.$$

So $(B, B') \in Z_1$, and the action of s_1 on $\overline{\mathcal{O}}_{\min}$ coincides with the action of S_1 on $\overline{T^*(\mathrm{SL}_3/U)}$.

Next, for $k=2$, we have $\lambda_2 = c_1 - c_3$, and

$$\mathrm{out}_2(B) = \left(\begin{array}{c|c} u_3 & \dfrac{u_2}{c_3 - c_2} \\ \hline 0 & 1 \end{array}\right), \quad \mathrm{in}_2(B) = \left(\begin{array}{c|c} -\dfrac{u_3^*}{c_3 - c_2} & 0 \\ \hline u_2^* & c_2 - c_3 \end{array}\right).$$

Applying the triality action by element $s_2 = (13) \in S_3$, we get B'' such that

$$\mathrm{out}_2(B'') = \left(\begin{array}{c|c} u_1 & \dfrac{u_2}{c_1 - c_2} \\ \hline 0 & 1 \end{array}\right), \quad \mathrm{in}_2(B'') = \left(\begin{array}{c|c} -\dfrac{u_1^*}{c_1 - c_2} & 0 \\ \hline u_2^* & c_2 - c_1 \end{array}\right).$$

Then we can check

$$\mathrm{out}_2(B)\,\mathrm{in}_2(B) = \left(\begin{array}{c|c} -M + c_1 \mathrm{Id}_{\mathbb{C}^3} & u_2 \\ \hline u_2^* & c_2 - c_3 \end{array}\right),$$

$$\mathrm{out}_2(B'')\,\mathrm{in}_2(B'') = \left(\begin{array}{c|c} -M + c_3 \mathrm{Id}_{\mathbb{C}^3} & u_2 \\ \hline u_2^* & c_2 - c_1 \end{array}\right),$$

where we have used the relations:

$$\frac{u_2 \wedge u_2^* - u_3 \wedge u_3^*}{c_3 - c_2} = -M + c_1 \mathrm{Id}_{\mathbb{C}^3}, \qquad \frac{u_2 \wedge u_2^* - u_1 \wedge u_1^*}{c_1 - c_2} = -M + c_3 \mathrm{Id}_{\mathbb{C}^3}.$$

So
$$\mathrm{out}_2(B'')\,\mathrm{in}_2(B'') = \mathrm{out}_2(B)\,\mathrm{in}_2(B) - \lambda_2\,\mathrm{Id}_{\mathbb{C}^4}.$$

Hence $(B, B'') \in Z_2$. This shows that the Gelfand-Graev action on $\overline{T^*(\mathrm{SL}_3/U)}$ coincides with the restriction of the triality action on $\overline{\mathcal{O}}_{\min}$. ■

6. Symplectic form on the smooth locus

Let us recall the following known results from [5]. Let $\mathcal{O}_{\mathrm{nilp}}$ be a nilpotent adjoint orbit in $\mathfrak{g}$. Since $\mathcal{O}_{\mathrm{nilp}}$ is a nilpotent orbit, there is a $\mathbb{C}^*$-action on it such that $\omega^{\mathrm{KKS}} = \mathcal{L}_{Eu}\omega^{\mathrm{KKS}}$, where Eu is the Euler vector field on $\mathcal{O}_{\mathrm{nilp}}$. Define

$$\lambda^{\mathrm{KKS}} := \iota_{Eu}\omega^{\mathrm{KKS}}.$$

So we have $$d\lambda^{\mathrm{KKS}} = d\iota_{Eu}\omega^{\mathrm{KKS}} = \mathcal{L}_{Eu}\omega^{\mathrm{KKS}} = \omega^{\mathrm{KKS}}.$$

Proposition 6.1. *Let $X \in \mathcal{O}_{nilp}$ and $Y \in \mathfrak{g}$. Then*

$$\lambda_X^{KKS}(\xi_Y) = \kappa(X, Y).$$

Proof. Since X is nilpotent, we have $Y \in \mathfrak{g}_X$ if and only if $\kappa(X, Y) = 0$. So the RHS of the formula only depends on ξ_Y (i.e. independent of the choice of Y). Let $E \in \mathcal{O}_{\mathrm{nilp}}$. Fix some $\mathfrak{sl}_2$-triple (E, H, F) in $\mathfrak{g}$. Then we have

$$(\xi_H)_E = 2\,\mathrm{Eu}_E \in T_E\mathcal{O}_{\mathrm{nilp}}.$$

Since λ^{KKS} is invariant under the G-action, it suffices to prove the statement for $X = E$. Let $Y \in \mathfrak{g}$. We compute

$$(\iota_{\mathrm{Eu}}\omega_E^{\mathrm{KKS}})(\xi_Y) = \frac{1}{2}\omega_E^{\mathrm{KKS}}(\xi_Y, \xi_H) = \frac{1}{2}\kappa(E, [Y, H]) = \frac{1}{2}\kappa([H, E], Y) = \kappa(E, Y). \quad \blacksquare$$

Theorem 6.2. *The symplectic form on $\overline{T^*(\mathrm{SL}_3/U)}_{sm}$ coincides (up to a scalar multiple) with ω^{KKS} on $\mathcal{O}_{min} \subset \mathfrak{so}_8$.*

Proof. Since the symplectic form on $\overline{T^*(\mathrm{SL}_3/U)}_{\mathrm{sm}}$ constructed in Proposition 2.6 is the extension of the symplectic form given by the Hamiltonian reduction of N_{inj} by H, it coincides with the pull-back of the symplectic form on $(N_1//\mathrm{SL}_2)_{\mathrm{sm}}$ by the restriction of the isomorphism in Corollary 3.8 to the smooth points:

$$F : \left(\overline{T^*(\mathrm{SL}_3/U)}\right)_{\mathrm{sm}} \to \mathcal{O}_{\mathrm{min}}.$$

So it suffices to show that the symplectic form on $(N_1//\mathrm{SL}_2)_{\mathrm{sm}}$ coming from the Hamiltonian reduction coincides (up to a scalar multiple) with ω^{KKS}. It suffices to check for one forms.

Let $v_1 \wedge v_2 \in \mathcal{O}_{\mathrm{min}} \subset \bigwedge^2\mathbb{C}^8$. Recall the element $\varphi_{v_1\wedge v_2} \in \mathfrak{so}_8$ is defined in (7) by the formula

$$\varphi_{v_1\wedge v_2}(u) = (v_1, u)v_2 - (v_2, u)v_1.$$

Observe that the tangent space $T_{v_1\wedge v_2}\mathcal{O}_{\mathrm{min}}$ is spanned by vectors given by infinitesimal actions of some $w_1 \wedge w_2 \in \mathfrak{so}_8$. Let

$$\xi_{w_1\wedge w_2} = [w_1 \wedge w_2, v_1 \wedge v_2]$$

be such a tangent vector.

To calculate $\xi_{w_1\wedge w_2}$ we first we compute

$$\begin{aligned}(\varphi_{v_1\wedge v_2}\circ\varphi_{w_1\wedge w_2})(u) &= [(w_2,u)(v_2,w_1) - (w_1,u)(v_2,w_2)]v_1\\ &\quad + [(w_1,u)(v_1,w_2) - (w_2,u)(v_1,w_1)]v_2,\\ (\varphi_{w_1\wedge w_2}\circ\varphi_{v_1\wedge v_2})(u) &= [(v_2,u)(w_2,v_1) - (v_1,u)(w_2,v_2)]w_1\\ &\quad + [(v_1,u)(w_1,v_2) - (v_2,u)(w_1,v_1)]w_2.\end{aligned}$$

Then we compute the Lie bracket $[\varphi_{v_1\wedge v_2},\varphi_{w_1\wedge w_2}]$ and get

$$\begin{aligned}\xi_{w_1\wedge w_2} &= (v_2,w_2)\,w_1\wedge v_1 - (v_1,w_2)\,w_1\wedge v_2 - (v_2,w_1)\,w_2\wedge v_1 + (v_1,w_1)\,w_2\wedge v_2\\ &= \varphi_{w_1\wedge w_2}(v_1)\wedge v_2 + v_1\wedge\varphi_{w_1\wedge w_2}(v_2).\end{aligned}$$

The value of the one form λ^{KKS} on $\mathcal{O}_{\min}$ at $v_1\wedge v_2$ is

$$\lambda^{\mathrm{KKS}}_{v_1\wedge v_2}(\xi_{w_1\wedge w_2}) = \mathrm{Tr}(\varphi_{v_1\wedge v_2}\circ\varphi_{w_1\wedge w_2}) = 2((v_2,w_1)(v_1,w_2) - (v_1,w_1)(v_2,w_2)).$$

Recall from (10), we have $\omega_1 = (1/2)d\lambda'$, where λ' is the SL_2-invariant one form on $\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^8) = \mathbb{C}^8\oplus\mathbb{C}^8$ defined by

$$\lambda'_{v_1\oplus v_2}(x_1\oplus x_2) = (v_1,x_2) - (v_2,x_1).$$

Lift the tangent vector $\xi_{w_1\wedge w_2}$ to $(x_1\oplus x_2)\in T_{v_1\oplus v_2}(\mathbb{C}^8\oplus\mathbb{C}^8)$, where

$$\begin{aligned}x_1 &= \varphi_{w_1\wedge w_2}(v_1) = (w_1,v_1)w_2 - (w_2,v_1)w_1,\\ x_2 &= \varphi_{w_1\wedge w_2}(v_2) = (w_1,v_2)w_2 - (w_2,v_2)w_1.\end{aligned}$$

So we have

$$\lambda'_{v_1\oplus v_2}(x_1\oplus x_2) = 2((w_1,v_2)(v_1,w_2) - (w_2,v_2)(v_1,w_1)) = \lambda^{\mathrm{KKS}}_{v_1\wedge v_2}(\xi_{w_1\wedge w_2}). \qquad\blacksquare$$

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