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Semisimple Algebras of Vector Fields on $\mathbb{C}^N$ of Maximal Rank

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Abstract. A local classification of semisimple Lie algebras of vector fields on $\mathbb{C}^N$ that have a Cartan subalgebra of dimension N is given. The proof uses basic representation theory and the local canonical form of semisimple Lie algebras of vector fields.

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Key Words: Vector field, semisimple Lie algebra, Levi decomposition.

1. Introduction

All vector fields considered in this paper are assumed to be complex analytic. By a Lie algebra of vector fields on $\mathbb{C}^N$ Lie meant a Lie algebra of analytic vector fields defined on some connected open subset $\mathcal{U}$ of $\mathbb{C}^N$. He considered two such Lie algebras L_1 and L_2, defined on open subsets $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively, to be equivalent if there is an open set $\mathcal{U} \subset \mathcal{U}_1$ and an analytic map $\phi : \mathcal{U} \longrightarrow \mathcal{U}_2$, which is a diffeomorphism between $\mathcal{U}$ and $\phi(\mathcal{U})$ and $\phi_*(L_1\big|_{\mathcal{U}}) = L_2\big|_{\phi(\mathcal{U})}$. This is the same as saying that all Lie algebras obtained from a given Lie algebra L of vector fields by a local change of coordinates are considered to be equivalent. Lie gave, in this sense, a classification of all finite dimensional Lie algebras of vector fields on $\mathbb{C}^2$ [6]. This was later extended to vector fields on the real plane in [4].

In this paper we give a local classification of all finite dimensional complex semisimple Lie algebras of vector fields on $\mathbb{C}^N$ that have a Cartan subalgebra of dimension N. The main result proved here is the following:

Theorem 1.1. *If $\mathcal{G}$ is a finite dimensional semisimple Lie algebra of vector fields on $\mathbb{C}^N$ which has a Cartan subalgebra of dimension N, then the simple factors of $\mathcal{G}$ must be of type A_{ℓ_i}, $1 \leq i \leq d$, such that $\sum_{i=1}^{d} \ell_i = N$.*

The proof of Theorem 1.1 uses the local canonical form of semisimple Lie algebras of vector fields on $\mathbb{R}^N$ given in [2, Theorem 2.1]. This result of [2] states that if a semisimple Lie algebra of vector fields on $\mathbb{R}^N$ has a split Cartan subalgebra then there are local coordinates $x_1, \cdots, x_N$ with respect to which the Cartan subalgebra is generated by ∂_{x_i}, $1 \leq i \leq r$, where r is the dimension of the Cartan subalgebra. If we have a complex semisimple Lie algebra of vector fields on $\mathbb{C}^N$ then all Cartan subalgebras of a complex semisimple Lie algebra are split.

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Thus we see, by repeating the arguments of [2, Theorem 2.1], that there are local analytic coordinates $x_1, \cdots, x_N$ with respect to which the Cartan subalgebra is generated by ∂_{x_i}, $1 \le i \le r$, where r is the dimension, over $\mathbb{C}$, of a given Cartan subalgebra. We refer the reader to [5] for results used from representation theory and root systems in the proof of Theorem 1.1.

After seeing a preprint of this paper, V. Popov sent us Vinberg's paper [8], where a similar result is proved for algebraic groups over an algebraically closed field of characteristic zero. In this paper we work with local analytic vector fields and do not assume that the vector fields are complete. The result of Vinberg is therefore a consequence of Theorem 1.1.

2. Structure of Lie algebras of vector fields

Lemma 2.1. *Let $\mathcal{L}$ and $\mathcal{L}_1$ be the Lie algebras of analytic vector fields on $\mathbb{C}^{k+m}$ of the form*

$$\sum_{i=1}^{k+m} f_i(x_1, \cdots, x_k)\partial_{x_i} \quad \textit{and} \quad \sum_{i=1}^{k} f_i(x_1, \cdots, x_k)\partial_{x_i}$$

respectively, where f_i are analytic functions of k variables. Then the linear map

$$\Pi : \mathcal{L} \longrightarrow \mathcal{L}_1, \quad \sum_{i=1}^{k+m} f_i(x_1, \cdots, x_k)\partial_{x_i} \longmapsto \sum_{i=1}^{k} f_i(x_1, \cdots, x_k)\partial_{x_i}$$

is a homomorphism of Lie algebras.

Proof. For vector fields X and Y, and a smooth function h, we have

$$[X, h \cdot Y] = h \cdot [X, Y] + X(h) \cdot Y.$$

Also, $\partial_{x_j} f(x_1, ..., x_k) = 0$ for $k+1 \le j \le k+m$, implying the lemma. ∎

Definition 2.2. Let L be a Lie algebra of vector fields defined on an open subset $\mathcal{U} \subset \mathbb{C}^N$. The rank of L is $\mathrm{Max}_{p \in \mathcal{U}} \dim\{X(p) \mid X \in L\}$. ∎

In general the rank and dimension are different: For example, the abelian Lie algebra $\langle \partial_x, y\partial_x, \cdots, y^n\partial_x \rangle$ has rank 1 and dimension $n+1$.
For a proof of the following basic fact see [3].

Theorem 2.3. *If $X_1, \cdots, X_r$ are commuting vector fields defined on an open subset $\mathcal{U} \subset \mathbb{C}^N$ and $p \in \mathcal{U}$ a point such that $X_1(p), \cdots, X_r(p)$ are linearly independent, then there are local coordinates $x_1, \cdots, x_N$ defined near p such that $X_i = \partial_{x_i}$ $(1 \le i \le r)$.*

Coordinates as in Theorem 2.3 are called *canonical coordinates* for the commuting fields $X_1, \cdots, X_r$. We will use Lemma 2.1 and Theorem 2.3 to derive the local canonical form of Lie algebras of type $A_1 \times A_1 \times \cdots \times A_1$. This is an essential ingredient for the proof of Theorem 1.1.

Notation: If a Lie algebra $\mathfrak{g}$ is generated as a Lie algebra by $X_1, \cdots, X_n$, we write $\mathfrak{g} = \langle X_1, \cdots, X_n \rangle$.

Proposition 2.4. *If* $L = \overbrace{A_1 \times A_1 \times \cdots \times A_1}^{N-times}$ *is a Lie algebra of vector fields on* $\mathbb{C}^N$ *then there are local coordinates* $x_1, \cdots, x_N$ *in which*

$$L = \langle \exp(x_1)\partial_{x_1}, \exp(-x_1)\partial_{x_1}\rangle \times \cdots \times \langle \exp(x_N)\partial_{x_N}, \exp(-x_N)\partial_{x_N}\rangle.$$

Proof. The Lie algebra L has generators X_i, Y_i $(1 \le i \le N)$ such that if we set $H_i = [X_i, Y_i]$, then

$$[H_i, X_i] = X_i, \ [H_i, Y_i] = -Y_i \tag{1}$$

for all $1 \le i \le N$ and $[X_i, X_j] = 0 = [Y_i, Y_j]$ for all $1 \le i, j \le N$. The subalgebra $\langle H_1, \cdots, H_N\rangle$ is a Cartan subalgebra of L. By Theorem 2.3 there are local coordinates $x_1, \cdots, x_N$ such that $H_i = \partial_{x_i}$ (see also [2, Theorem 2.1]). The eigenfields for ∂_{x_i} for eigenvalue λ are $\exp(\lambda x_i)(f_1\partial_{x_1} + \ldots + f_N\partial_{x_N})$, where $f_1, \cdots, f_N$ are functions independent of x_i. As $[H_j, X_i] = 0 = [H_j, Y_i]$ for $i \ne j$, we see that $X_i = \exp(x_i)(\lambda_{i1}\partial_{x_1} + \cdots + \lambda_{iN}\partial_{x_N})$, $Y_i = \exp(-x_i)(\mu_{i1}\partial_{x_1} + \cdots + \mu_{iN}\partial_{x_N})$, where λ_{ij} and μ_{ij} are constants.

We want to show that λ_{ii} and μ_{ii} are not zero for all $1 \le i \le N$ and $\lambda_{ij} = 0 = \mu_{ij}$ if $i \ne j$. To show that λ_{ii} and μ_{ii} are not zero for all $1 \le i \le N$, it suffices to show this for $i = 1$ — as this amounts to relabeling of variables.

We first show that $\lambda_{11} \ne 0$. Suppose $\lambda_{11} = 0$. Then

$$X_1 = \exp(x_1)(\lambda_{12}\partial_{x_2} + \cdots + \lambda_{1N}\partial_{x_N}) \quad \text{and} \quad Y_1 = \exp(-x_1)(\mu_{11}\partial_{x_1} + \cdots + \mu_{1N}\partial_{x_N}).$$

Using the formula

$$[\exp(\chi)U, \exp(\psi)V] = \exp(\chi + \psi)(U(\psi)V - V(\chi)U + [U, V]), \tag{2}$$

where χ and ψ are analytic functions, we see, using that λ_{ij}, μ_{ij} are constants, that $H_1 = [X_1, Y_1] = -\mu_{11}(\lambda_{12}\partial_{x_2} + \cdots + \lambda_{1N}\partial_{x_N})$. Thus $[H_1, X_1] = 0$; but this contradicts (1). Therefore $\lambda_{11} \ne 0$. Similarly $\mu_{11} \ne 0$. As mentioned above, this means that $\lambda_{ii} \ne 0$ and $\mu_{ii} \ne 0$ for all i.

Now we want to show that $\lambda_{ij} = 0 = \mu_{ij}$ for all $i \ne j$. Without loss of generality, we may suppose that $i = 1$ and $j = 2$. The Lie algebra

$$\begin{aligned} S_{12} = &\langle \exp(x_1)(\lambda_{11}\partial_{x_1} + \cdots + \lambda_{1N}\partial_{x_N}), \exp(-x_1)(\mu_{11}\partial_{x_1} + \cdots + \mu_{1N}\partial_{x_N})\rangle \\ &\times \langle \exp(x_2)(\lambda_{21}\partial_{x_1} + \cdots + \lambda_{2N}\partial_{x_N}), \exp(-x_2)(\mu_{21}\partial_{x_1} + \cdots + \mu_{2N}\partial_{x_N})\rangle \end{aligned}$$

is a Lie subalgebra of vector fields of the type

$$f_1(x_1, x_2)\partial_{x_1} + f_2(x_1, x_2)\partial_{x_2} + \cdots + f_N(x_1, x_2)\partial_{x_N}.$$

Note that S_{12} is isomorphic to $A_1 \times A_1$. By Lemma 2.1, the linear map of vector fields defined by

$$\textstyle\sum_{i-1}^N f_i(x_1, x_2)\partial_{x_i} \longmapsto f_1(x_1, x_2)\partial_{x_1} + f_2(x_1, x_2)\partial_{x_2} \tag{3}$$

is a homomorphism of Lie algebras. As λ_{11}, μ_{11}, λ_{22}, μ_{22} are all not zero, the Lie algebra S_{12} is mapped isomorphically, by the map in (3), onto its image

$$\begin{aligned} &\big\langle \exp(x_1)(\lambda_{11}\partial_{x_1} + \lambda_{12}\partial_{x_2}), \ \exp(-x_1)(\mu_{11}\partial_{x_1} + \mu_{12}\partial_{x_2})\big\rangle \\ \times &\big\langle \exp(x_2)(\lambda_{21}\partial_{x_1} + \lambda_{22}\partial_{x_2}), \ \exp(-x_2)(\mu_{21}\partial_{x_1} + \mu_{22}\partial_{x_2})\big\rangle, \end{aligned}$$

which is of type $A_1 \times A_1$.

To prove the proposition, it therefore suffices to show that if

$$\mathfrak{g} = \langle \exp(x)(\partial_x + \lambda\partial_y),\, \exp(-x)(\partial_x + \mu\partial_y)\rangle \times \langle \exp(y)(\partial_y + \widetilde{\lambda}\partial_x),\, \exp(-y)(\partial_y + \widetilde{\mu}\partial_x)\rangle,$$

where λ, μ, $\widetilde{\lambda}$ and $\widetilde{\mu}$ are constants, is a Lie algebra of type $A_1 \times A_1$ on $\mathbb{C}^2$ with Cartan subalgebra $\langle \partial_x, \partial_y \rangle$, then

$$\lambda = \mu = \widetilde{\lambda} = \widetilde{\mu} = 0. \tag{4}$$

Now $[\exp(x)(\partial_x + \lambda\partial_y), \exp(-x)(\partial_x + \mu\partial_y)]$ should be a non zero multiple of ∂_x. Using the formula (2), we see that $\lambda + \mu = 0$. Similarly, we have $\widetilde{\lambda} + \widetilde{\mu} = 0$. Consequently, we have

$$\mathfrak{g} = \langle \exp(x)(\partial_x + \lambda\partial_y),\, \exp(-x)(\partial_x - \lambda\partial_y)\rangle \times \langle \exp(y)(\partial_y + \widetilde{\lambda}\partial_x),\, \exp(-y)(\partial_y - \widetilde{\lambda}\partial_x)\rangle.$$

As $[\exp(x)(\partial_x + \lambda\partial_y), \exp(y)(\partial_y + \widetilde{\lambda}\partial_x)] = 0$ we see using the formula (2) that $\lambda(\partial_y + \widetilde{\lambda}\partial_x) - \widetilde{\lambda}(\partial_x + \lambda\partial_y) = 0$. Therefore,

$$\lambda = \lambda\widetilde{\lambda} = \widetilde{\lambda}. \tag{5}$$

On the other hand, as $[\exp(x)(\partial_x + \lambda\partial_y), \exp(-y)(\partial_y - \widetilde{\lambda}\partial_x)] = 0$, from (2) it follows that

$$-\lambda(\partial_y - \widetilde{\lambda}\partial_x) + \widetilde{\lambda}(\partial_x + \lambda\partial_y) = 0.$$

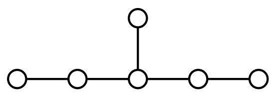
So considering the coefficient of ∂_x we conclude that $\lambda\widetilde{\lambda} + \widetilde{\lambda} = 0$. This and (5) together imply that $\lambda = 0$ and $\widetilde{\lambda} = 0$. This proves (4). As noted before, (4) completes the proof of the proposition. ■

Lemma 2.5. *Any simple Lie algebra which is not of type A_ℓ for some ℓ contains a subalgebra of type B_2, G_2 or D_4.*

Proof. If the Dynkin diagram of a given simple Lie algebra $\mathfrak{g}$ contains a multiple bond, then $\mathfrak{g}$ contains a subalgebra of type B_2 or G_2. Assume that there is no multiple bond in the Dynkin diagram of $\mathfrak{g}$.

If $\mathfrak{g}$ is of type D_n, with $n \geq 4$, then it must contain a subalgebra of type D_4.

If $\mathfrak{g}$ is exceptional, it contains a subalgebra of type E_6 whose Dynkin diagram is



Removing the two extreme vertices of the row we get the Dynkin diagram of D_4. So $\mathfrak{g}$ contains a subalgebra of type D_4. ■

Part (1) of the following proposition is due originally to Lie in the sense that it is a consequence of Lie's classification of finite dimensional subalgebras of vector fields on $\mathbb{C}^2$ [6]. This classification is also listed in [1, p. 369–372], [4, p. 3] and [7, pp. 472–475].

Proposition 2.6. (1) *There is no faithful representation of a Lie algebra of type B_2 or G_2 as analytic vector fields on $\mathbb{C}^2$.*

(2) *There is no faithful representation of a Lie algebra of type D_4 as analytic vector fields on $\mathbb{C}^4$.*

Proof. We will first show that there is no faithful representation of a Lie algebra of type B_2 and G_2 in vector fields on $\mathbb{C}^2$. To see this, if the positive roots of B_2 are α, β, $\alpha+\beta$ and $\alpha+2\beta$, then subalgebra of B_2 with roots

$$\langle \alpha,\, \alpha+2\beta,\, -\alpha,\, -\alpha-2\beta \rangle$$

is of type $A_1 \times A_1$. Similarly, if G_2 is a Lie subalgebra of Lie algebra of vector fields on $\mathbb{C}^2$ and the positive roots are α, β, $\alpha+\beta$, $\alpha+2\beta$, $\alpha+3\beta$ and $2\alpha+3\beta$, then the Lie subalgebra of G_2 with roots

$$\langle \alpha,\, \alpha+2\beta,\, -\alpha,\, -\alpha-2\beta \rangle$$

is of type $A_1 \times A_1$. By Proposition 2.4, any Lie algebra of vector fields on $\mathbb{C}^2$ of type $A_1 \times A_1$ is equivalent to

$$S = \langle \exp(x)\partial_x,\, \exp(-x)\partial_x,\, \partial_x \rangle \times \langle \exp(y)\partial_y,\, \exp(-y)\partial_y,\, \partial_y \rangle.$$

Therefore, $\langle \exp(x)\partial_x,\, \exp(y)\partial_y \rangle$ is a rank two abelian Lie subalgebra in the derived algebra of a unique Borel subalgebra of S, which is denoted by $\mathfrak{b}$. Thus if S is contained in a Lie subalgebra $\mathfrak{g}$ of vector fields on $\mathbb{C}^2$, then $\mathfrak{g}$ must have a highest weight vector v in an S-invariant complement of S in $\mathfrak{g}$.

The derived algebra of the above Borel subalgebra $\mathfrak{b}$ of S is

$$\langle \exp(x)\partial_x,\; \exp(y)\partial_y \rangle.$$

In the coordinates $\widetilde{x}$, $\widetilde{y}$ in which

$$\exp(x)\partial_x = \partial_{\widetilde{x}} \quad \text{and} \quad \exp(y)\partial_y = \partial_{\widetilde{y}},$$

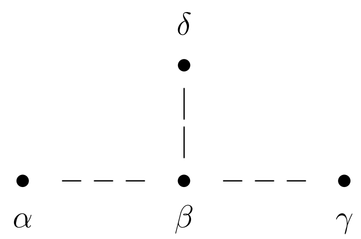
we see that $[\partial_{\widetilde{x}}, v] = 0 = [\partial_{\widetilde{y}}, v]$, as v is the above highest weight vector and a highest weight vector is, by definition, in the null space of the derived algebra of a Borel subalgebra $\mathfrak{b}$. Now v is of the form

$$v = f(\widetilde{x},\widetilde{y})\partial_{\widetilde{x}} + g(\widetilde{x},\widetilde{y})\partial_{\widetilde{y}}.$$

Therefore, $[\partial_{\widetilde{x}}, v] = 0 = [\partial_{\widetilde{y}}, v]$ implies that f and g are constant functions. Hence $v \in \langle \partial_{\widetilde{x}}, \partial_{\widetilde{y}} \rangle$, which is a contradiction. Thus the only semisimple Lie algebras of vector fields on $\mathbb{C}^2$ are of the type A_1, $A_1 \times A_1$ and A_2.

Now we will prove that there is no faithful representation of a Lie algebra of type D_4 as vector fields on $\mathbb{C}^4$.

We note that if α, β, γ, δ are the simple roots of D_4



with α, γ, δ orthogonal and $\langle \alpha, \beta \rangle = \langle \beta, \gamma \rangle = \langle \beta, \delta \rangle = -1$, the highest root is $\alpha+2\beta+\gamma+\delta =: \alpha_0$ and α_0 is orthogonal to α, γ, δ, and thus the collection

$$\{\alpha,\, \gamma,\, \delta,\, \alpha_0,\, -\alpha,\, -\gamma,\, -\delta,\, -\alpha_0\}$$

corresponds to a Lie subalgebra of type $A_1 \times A_1 \times A_1 \times A_1$.

Its realization as vector fields on $\mathbb{C}^4$ is

$$S = \prod_{i=1}^{4} \langle \exp(x_i)\partial_{x_i},\, \exp(-x_i)\partial_{x_i},\, \partial_{x_i} \rangle,$$

with respect to some coordinates x_1, x_2, x_3, x_4, and

$$\langle \exp(x_1)\partial_{x_1},\, \exp(x_2)\partial_{x_2},\, \exp(x_3)\partial_{x_3},\, \exp(x_4)\partial_{x_4} \rangle$$

is a rank 4 abelian Lie subalgebra in the derived algebra of a Borel subalgebra of S. By the same argument as before, there is no highest weight vector in its complement in D_4. This proves that there is no faithful representation of D_4 in vector fields on $\mathbb{C}^4$. ■

The following proposition is also due to Lie in the sense that it is a consequence of Lie's classification of finite dimensional subalgebras of vector fields on $\mathbb{C}^2$ [6]; see the lists in [1, p. 369–372], [4, p. 3] and [7, pp. 472–475].

Proposition 2.7. *All simple Lie algebras of vector fields on $\mathbb{C}^2$ must be of type A_1 or A_2.*

Proof. By the main result of [2], the rank of such a Lie algebra can be at most two. Now Lemma 2.5 and Proposition 2.6 together complete the proof. ■

3. Proof of Theorem 1.1

By [2, Theorem 2.1], if $\mathcal{G}$ is a semisimple Lie algebra of vector fields on $\mathbb{C}^N$, and $C \subset \mathcal{G}$ is a Cartan subalgebra of dimension n of $\mathcal{G}$, then there are coordinates $x_1, \cdots, x_N$ so that the root spaces corresponding to a simple set of roots and their negatives are of the form $\exp(x_i)V_i$, $\exp(-x_i)W_i$, $i = 1, \cdots, n$, where V_i and W_i are vector fields whose coefficients with respect to the basis $\partial_{x_1}, \cdots, \partial_{x_N}$ are independent of $x_1, \cdots, x_n$ and the linear span of the vector fields $[\exp(x_i)V_i, \exp(-x_i)W_i]$, $1 \le i \le n$ is that of $\partial_{x_1}, \cdots, \partial_{x_n}$, and the Lie algebra generated by $\exp(x_i)V_i$, $\exp(-x_i)W_i$ is a copy of $sl(2, \mathbb{C})$ for every $i = 1, \cdots, n$.

Thus in case the Cartan subalgebra is of dimension N, the vector fields V_i, W_i are constant ones. Consequently, if $\alpha_1, \cdots, \alpha_N$ are the simple roots, the corresponding root vectors are vector fields of the form

$$X_{\alpha_i} = \exp(x_i)\Big(\sum_{j=1}^{N} \lambda_{i,j}\partial_{x_j}\Big), \quad X_{-\alpha_i} = \exp(-x_i)\Big(\sum_{j=1}^{N} \mu_{i,j}\partial_{x_j}\Big), \tag{6}$$

where $\lambda_{i,j}$ and $\mu_{i,j}$, $1 \le i, j \le N$, are constants with $\lambda_{i,i} \ne 0 \ne \mu_{i,i}$ for every i as shown in the proof of Proposition 2.4.

Now suppose J is a set of simple roots. By permuting $\alpha_1, \cdots, \alpha_N$ we may assume that $J = \{\alpha_1, \cdots, \alpha_k\}$. Note that $\{X_{\alpha_i}, X_{-\alpha_i}\}_{i=1}^{k}$ (see (6)) generate a semisimple Lie algebra

$$\mathcal{G}_J \subset \mathcal{G}. \tag{7}$$

Moreover, $\mathfrak{g}_J = \mathfrak{b}_J \oplus \mathfrak{c}_J \oplus \mathfrak{b}_{-J}$, where $\mathfrak{c}_J$ is a Cartan subalgebra of $\mathfrak{g}_J$ and $\mathfrak{b}_J$, $\mathfrak{b}_{-J}$ are Lie subalgebras obtained by taking commutators of all possible order starting from the root vectors corresponding to the simple roots and their negatives.

The formula

$$[\exp(\chi)V,\ \exp(\psi)W] \;=\; \exp(\chi+\psi)([V,\ W] + V(\psi)W - W(\chi)V)$$

implies $$[\exp(\chi)V,\ \exp(\psi)W] \;=\; \exp(\chi+\psi)(V(\psi)\cdot W - W(\chi)\cdot V)$$

if V and W are constant vector fields. Consequently, $\mathcal{G}_J$ in (7) is a Lie subalgebra of the Lie algebra of all vector fields of the type

$$\sum_{i=1}^{N} f_i(x_1,\, \cdots\,,\, x_k)\partial_{x_i}\,,$$

and the projection $$\sum_{i=1}^{N} f_i(x_1,\, \cdots\,,\, x_k)\partial_{x_i} \;\longmapsto\; \sum_{i=1}^{k} f_i(x_1,\, \cdots\,,\, x_k)\partial_{x_i}\,,$$

is a homomorphism of Lie algebras by Lemma 2.1. Consequently, $\mathcal{G}_J$ in (7) is isomorphic to a Lie subalgebra of Lie algebra of vector fields on $\mathbb{C}^k$.

Suppose $\mathcal{G}_J$ is not of type A_ℓ.

By Lemma 2.5, either $\mathcal{G}_J$ is of type G_2 or it has a Lie subalgebra of type B_2 or D_4. Now applying Lemma 2.1 we see that either we have a faithful representation of a Lie algebra of type G_2 or B_2 as a Lie algebra of vector fields on $\mathbb{C}^2$ of rank two, or we have a faithful representation of a Lie algebra of type D_4 as a Lie algebra of vector fields on $\mathbb{C}^4$ of rank four. But by Proposition 2.6 none of these representations exist. Thus every simple components of $\mathcal{G}$ must of type A_ℓ.

The faithful representations of A_k in the Lie algebra of vector fields on $\mathbb{C}^k$ are already given in Corollary 3.2 of [2]. This describes all faithful representations of maximal rank in the Lie algebra of vector fields on $\mathbb{C}^N$.

4. An application

A corollary of Theorem 1.1 is the following; it is due originally to Lie in the sense that it can be obtained by inspection of the lists in [1, p. 369–372] and [4, p. 3].

Recall that every finite dimensional Lie algebra $\mathfrak{g}$ has a Levi decomposition $\mathfrak{g} = S \ltimes R$, where S – the Levi complement – is semisimple and R – the radical – is a solvable ideal.

Corollary 4.1. *If $\mathfrak{g}$ is a Lie algebra of vector fields on $\mathbb{C}^2$ with a proper Levi decomposition, then the Levi complement must be $sl(2,\mathbb{C})$.*

Proof. Let $\mathfrak{g} \subset V(\mathbb{C}^2)$ be a finite dimensional Lie algebra of vector fields on $\mathbb{C}^2$ with Levi decomposition $\mathfrak{g} = S \ltimes R$. Assume that both S and R are not zero. By adopting the arguments given in the proof of [2, Theorem 2.1], as all Cartan subalgebras of a complex semisimple Lie algebra are split, we see that for any complex semisimple Lie algebra of vector fields on $\mathbb{C}^N$, its Cartan subalgebra can be of dimension at most N.

Thus S is of rank 1 or 2. If S is of rank 2, then it is of type $A_1 \times A_1$ or A_2 – by Proposition 2.7. We have already seen that any Lie algebra of type $A_1 \times A_1$ in $\mathbb{C}^2$ contains the abelian Lie algebra $\langle \exp(x)\partial_x, \exp(y)\partial_y\rangle$ – for suitable local coordinates x, y – in the derived algebra of a Borel subalgebra.

A Lie algebra of type A_2 is isomorphic to $sl(3,\mathbb{C})$ and the Lie subalgebra of upper triangular matrices is a Borel subalgebra of $sl(3,\mathbb{C})$. The derived algebra of this Borel subalgebra of $sl(3,\mathbb{C})$ has generators X, Y, Z such that $[X, Y] = Z$ with Z commuting with both X and Y. Choose coordinates in which $Z = \partial_x$. Thus $X = f_1(y)\partial_x + g_1(y)\partial_y$ and $Y = f_2(y)\partial_x + g_2(y)\partial_y$. If both the Lie algebras $\langle Z, X\rangle$ and $\langle Z, Y\rangle$ are of rank 1, say $X = f_1(y)\partial_x$, $Y = f_2(y)\partial_x$, then we have $[X, Y] = 0$. Thus, one of $\langle Z, X\rangle$ or $\langle Z, Y\rangle$ must be of rank 2. Consequently, the derived algebra of a Borel subalgebra of both the Lie algebras of type $A_1 \times A_1$ and A_2 contains an abelian Lie algebra of rank 2. In the canonical coordinates x, y of this abelian Lie algebra, any highest weight vector in $V(\mathbb{C}^2)$ must be in the Lie algebra $\langle \partial_x, \partial_y\rangle$.

Thus the Levi complement S must be of type A_1. Therefore, the only Lie algebras on $\mathbb{C}^2$ with a proper Levi decomposition must be of the form $sl(2,\mathbb{C}) \rtimes R$, where R is the radical of the Lie algebra. ■

Theorem 1.1 reduces the classification of semisimple Lie algebras of vector fields on $\mathbb{C}^3$ and those Lie algebras with a proper Levi decomposition to the representations of ranks one and two in the vector fields on $\mathbb{C}^3$; proofs of this classification will appear elsewhere. (Here by rank we mean the dimension of a Cartan subalgebra of a given semisimple Lie algebra.)

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