Journal of Lie Theory
Volume **35** (2025) 109–156

Hellmuth Kneser: A Noteworthy Mathematician of the 20th Century

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Communicated by K.-H. Neeb

Abstract. This text is a presentation of the mathematical biography of Hellmuth Kneser, 1898–1973. It contains a description of his substantial work in topology, including his pioneering work in low dimensional combinatorial theory of manifolds during the twenties, his influence on the work on the analysis of several complex variables, as well as his work on Lie groups and Lie algebras, and his later contributions to uncountable real analytic manifolds. It also gives an overview of diverse topics such as the teaching of mathematics in the years after 1945 as Kneser's efforts to compensate for losses suffered by German mathematics during 1933–1945.
One focus is Kneser's relation with Richard Courant whose activity at the university of Göttingen had contributed to its national and international reputation in mathematics during the nineteen-twenties, ending abruptly in 1933 with Courant's (and Emmy Noether's) dismissal in the course of the national socialist government takeover. Kneser was Courant's assistant in Göttingen from 1921 to 1924 and corresponded actively with him in 1933 from Greifswald, where he had held a professorship since 1925. He organized one of two direct appeals to the government protesting Courant's dismissal – both in vain.

Mathematics Subject Classification: 01-00, 22E15, 57-03.

Key Words: Biography, cohomology, David Hilbert, Dehn's Lemma, Emmy Noether, Erich Kamke, fundamental group, Göttingen, Greifswald, Hauptvermutung, Hellmuth Kneser, Helmut Wielandt, Hopf algebra, John Milnor, Kneser's Conjecture, Lie group, Lie algebra, long line, manifolds, P.S.Alexandroff, Richard Courant, topology, Tübingen, Walther von der Vogelweide.

Preface

Hellmuth Kneser is one of the significant mathematicians of the last century. In 2005 the collection of his works [28] was organized and presented by the current authors: A sizable book in one volume. Its "Bibliography" shows (on pages 917–923) that this collection rightfully can be called a complete collection of Hellmuth Kneser 's work. The book, however, yields more: It presents (on pages 803–914) a compact collection of commentaries on the impact of Kneser's work on the history of mathematics, written by experts in the various domains of the collection: I. N. Baker, D. Gabai, A. Huckleberry, W. M. Kazez, J. Kindler, C. McA. Gordan, G. Pickert, R. M. Range, R. Remmert, G. Thorbergsson and the authors [4, 22]. In 2015, J. Milnor emphasized Kneser's contributions again in the Bulletin of the American Mathematical Society [38]. In their totality these evaluations illuminate Kneser's fundamental contributions to the progress of mathematics in the 20th century.

ISSN 0949–5932 / \$2.50

Beginning in the year of 1952 the authors studied mathematics at the Eberhard-Karls-Universität in Tübingen, being taught by H. Kneser and by his colleagues E.Kamke, M.Müller, G.Pickert, and H.Wielandt, as well as by their assistants Bertram Huppert and Karl Zeller. Through the authors' own biographies they had close and personal experience of Hellmuth Kneser 's influence and his effect on the following generation of mathematicians. The amount of source material and pertinent facts they assembled for Kneser's collected work was considerable, and it was clear that not everything they had collected and recorded could find its way into the collection of Hellmuth Kneser 's work in 2005. In addition to this point, not every potential reader who is interested in Hellmuth Kneser 's biographical background will have the substantial volume with Hellmuth Kneser 's works collection readily at hand. And this fact justifies our presenting some of the biographical facts included in the works volume also in the present assembly of notes on Hellmuth Kneser 's biography. Finally, we would like to add one item. Various biographical contributions on Hellmuth Kneser 's life have pointed out his presence during the political development of mathematics at the German universities during the period 1933–1945 ([8, 9, 19, 38, 51, 52, 57, 58, 74]). His role had some prominence due to his presence at the department of mathematics of the university of Göttingen during the years of 1920 to 1925, in the last three years as Richard Courant's assistant. In 1925 Kneser was appointed professor at Greifswald but stayed in close contact with his colleagues in Göttingen, where Courant and Emmy Noether were tragically dismissed in 1933. In this context some isolated hints in prominent sources such as [38] or [74] did not always appear to be as helpful in clarifying the full facts. However, Segal's book [57] (2003) is an excellent source as a whole and in regard to the biography of "Hellmuth Kneser" prior to 1945.

The authors therefore feel that some clarifying points ought to contribute to a better understanding of Hellmuth Kneser 's biography – perhaps another worth-while afterthought to the works collection [28], during the preparation of which Segal's work had not yet been available to the authors. We shall feel free to quote from [28] as it appears appropriate.

In the 19th and 20th centuries mathematics progressed at a speed unprecedented in history. One has to throw a glance back to Hellenic Greece to observe mathematical developments of comparable historic impact. As early as 1900, the volume of accumulated mathematical information began to exceed the capacity of most individuals. If one looks for works or biographies of single mathematicians who successfully steered against this tide, retaining a global view over the wide horizons of their field, then one will find Hellmuth Kneser as eminent. This is evidenced by his works, as collected in [28], or as mentioned in prominent historical overviews such as the one of Milnor [39], as well as in the testimonials by colleagues about him such as in the excellent essays by Helmut Salzmann [55] and Helmut Wielandt [71, 72, 73] authored in his memory.

Observations on Kneser's early biography

On April 16, 1898, Hellmuth Kneser was born in Dorpat, a city of the Hanseatic League, now Tartu in Estland, and passed away on August 23, 1973 in Tübingen, in the south-west of Germany. His father was Julius Carl Christian Adolf

KNESER (1862–1930), a professor of mathematics at Dorpat University; his mother was Laura Kneser, née Booth. He was the second child of his parents. An older brother Lorenz was killed in the first world war; the younger brother HANS-OTTO KNESER (1901–1983) was professor of experimental physics at the University of Technology of Stuttgart. The father ADOLF KNESER was from 1905 to 1928 Professor at he University of Breslau (now Wrocław, Poland) and was 1929 President of the *Deutsche Mathematiker-Vereinigung* (= DMV, the German Union of Mathematicians). The most famous of his books was concerned with the calculus of variations and with integral equations.

At the turn from the 19th to the 20th century, at the international congress of mathematics in Paris in the year of 1900, DAVID HILBERT (1862–1943) gave his famous address in which he formulated 23 problems, which were to become a legend. He did not only describe the status of mathematical research at the time, but indeed determined its direction for half a century to come. His command of all the mathematics known at the turn of the century gave him the authority to take this groundbreaking step.

Another characteristic of HILBERT's universality is the almost incredible number of his PhD students, numbering, as it were, 69. HELLMUTH KNESER grew up in this climate of mathematical fermentation. He started his university studies at the age of 18 years at the University of Breslau where, at the side of his father, ERHARD SCHMIDT taught mathematics as well. Not untypically in those days, HELLMUTH also enrolled simultaneously at the University of Technology (TH) of Breslau. In the fall of 1918 he moved on to Göttingen to become a student of DAVID HILBERT's, under whose supervision he earned a doctorate in 1921 at the young age of 23.

As we indicated before, his father had made his reputation through his research on integral equations and notably his influential work on the calculus of variations, which set the standards in this area for a long time. Later the oldest of HELLMUTH KNESER 's three sons, MARTIN KNESER, born in 1928, was to continue the family tradition in mathematics and scholarship to have a career as a professor at Göttingen. In 1997 he was awarded the *Karl Georg Christian von Staudt Prize* at the University of Erlangen.

Still in his early youth, HELLMUTH KNESER must have made a daring resolution. HELMUT WIELANDT describes it in his address honoring KNESER's memory as follows [66]:

> *The width of the horizon which was characteristic for the mathematical life in the circle of* KLEIN *and* HILBERT *agreed with his manifold interests and gifts to such an extent, that he arrived at a remarkable decision: He renounced specialisation. He desired to gain an overview and a comprehension of all parts of his discipline.... He approached the realisation of his resolution, called reckless by himself at a later point in time, to such a degree that he amazed his colleagues.*[1]

[1] Die Weite des Gesichtsfeldes, die das mathematische Leben im Umkreis von KLEIN und HILBERT kennzeichnete, entsprach seinen vielseitigen Interessen und Fähigkeiten in einem solchen Maße, daß er einen bemerkenswerten Entschluß faßte: Er verzichtete darauf, sich zu spezialisieren. Er wünschte über alle Teile seiner Wissenschaft Übersicht und Urteil zu gewinnen ... Der Verwirklichung dieses, wie er selbst 30 Jahre später sagte, verwegenen Wunsches sollte er in einem Grade nahekommen, der seine Fachgenossen mit Staunen erfüllte ...

We have reports about the young HELLMUTH KNESER from the nineteen-twenties which confirm that he had succeeded to cultivate a body of mathematical knowledge, which was not only the cause of unrestrained admiration by his peers, but on occasions was apt to intimidate them.

B. L. VAN DER WAERDEN recalls how HELLMUTH KNESER on hikes and at lunch

> *used to start on a certain subject and make a few remarks which I couldn't understand at all. Then I would say to him that I would like to learn about that subject. Where could I find out about it? So he would give me the names of some books which I could find in the Lesezimmer. A day or so later I would be able to answer his questions and also make some significant remarks of my own, and then I learned much more.* ([51], p. 162, [52], p.85)[2]

In later years REINHOLD BAER told HELMUT SALZMANN, to whom we owe this reference, how he, a barely 23-year old PhD student, on these legendary "Göttingen hikes" with his "supervisor" (who was his senior only by four years) was treated with protracted lectures after which he was deeply depressed, needing a week to recover his demolished self-confidence.

HELLMUTH KNESER himself told his students later, not without a trace of pride, that a group of "disciples" of approximately equal age gathered around him to listen to the mathematical "oracle". Possibly, P. S. ALEXANDROFF belonged to this group; he was in Göttingen at this time ([51], pp.165, 166).

An episode from his oral doctoral exam (1921) was told by HELLMUTH KNESER in a lively fashion. The subject was ordinary differential equations $y'(x) = f(x, y(x))$. The examiner E. LANDAU asked about conditions for the existence of solutions. The candidate replied: 'The continuity of f is sufficient.' The examiner is satisfied. However, from the opposite side of the table, coexaminer HILBERT interjects:

'– but there is an additional condition, is there not!' (one should imagine this remark in HILBERT's East-Prussian idiom which KNESER imitated rather convincingly). HILBERT obviously referred to the stronger Lipschitz condition for f. It needed LANDAU's intervention referring to PEANO's 1890 paper [46] and thus confirming that continuity of f suffices. (Personal recollection by K. H. HOFMANN.) Two years later, KNESER was to take up the subject of ordinary differential equations ([7] 1923).

As soon as in the summer of 1922, his Habilitation thesis was submitted. Among KNESER's papers a handwritten note was found which was kindly shared with us by his son MARTIN.

> *When in the summer of 1922 my Habilitation Thesis* (Regular sets of curves on ring surfaces, Math. Ann.) [9-24b] *had been submitted to the faculty, Hilbert called me in and had me report about the content. After the conclusion of the report, he commented that the gist of the paper apparently was the Theorem about the existence of at least one closed curve in every set of curves that was free of singularities on Klein's bottle.*

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[2] This is the only place at which C. REID mentions KNESER in her biography of DAVID HILBERT.

I was a bit disappointed, because I thought some good things Hilbert could have mentioned in addition. In the course of the decades, however, my judgment had approached Hilbert's more closely.
When I had communicated this theorem in a letter to J. Nielsen, he praised it and raised the question, whether there did not always exist even two closed curves.
B.v.Kerékjártó, who was preparing his "Topology I" (1923) at that time let me know, that in the end the known theorems on sets of curves were presented and a really simple proof for my theorem was going to be shown.
In the course of my life I had the hope to prove a similar theorem in a dimension one unit higher, such as the existence of a a compact sheet in every 2-set of sheets of the 3-sphere. I have presented sketches of what I thought were proofs in various locations and wanted to do this also at a symposium on foliated manifolds in Grenoble (cf. correspondence with Reeb 1962/63). Fortunately I noticed, that my sketches did not add up to a proof. I renounced my participation at the symposium even though President G. Reeb tried to kindly persuade me, saying that this didn't really matter.[3]

In the twenties HELLMUTH KNESER was one of the most universal scholars among the German mathematical community. The subsequent explosion of mathematical knowledge exceeded even his capacity. Remarkably, in his later years he was still able to view mathematics from a high vantage point. The variety of subjects to which he contributed through decades is stunning: Theoretical physics, topology, the theory of functions of one and several variables, theory of Lie groups, ordinary and partial differential equations. After the devastating rule of National Socialism in Germany and after its total defeat in the Second World War, there was a massive demand in the German mathematical culture for reestablishing various fields such as stochastics and mathematical economy. A scholar of KNESER's caliber felt called upon not only to diagnose such deficits but to participate actively in their remedy. He was equally active in conveying to the general public the importance of mathematics for a modern community and to work for good teaching of mathematics in schools.

[3] Als im Sommer 1922 meine Habilitationsschrift (Kurvenscharen auf den Ringflächen, Math. Ann.) [9-24b] bei der Fakultät eingereicht war, bestellte mich Hilbert zu sich und ließ sich "uber den Inhalt berichten. Nach Beendigung dieses Berichtes meinte er, die "Poingte" der Arbeit sei der Satz von der Existenz mindestens einer geschlossenen Kurve in jeder singularitätenfreien Kurvenschar auf dem Kleinschen Schlauch.
Ich war etwas enttäuscht, denn ich dachte, einiges Gute hätte Hilbert auch [sonst noch] erwähnen können. Im Laufe der Jahrzehnte hat sich mein Urteil dem von Hilbert mehr angeglichen.
Als ich J. Nielsen diesen Satz brieflich mitgeteilt hatte, lobte er ihn, fragte aber, ob es nicht immer sogar zwei geschlossene Kurven gäbe.
B.v.Kerékjártó, der zu jener Zeit seine "Topologie I" (1923) vorbereitete, kündigte mir an, am Schluß würden die bekannten Sätze über Kurvenscharen dargestellt und für meinen Satz ein "ganz einfacher" Beweis gegeben werden.
Im weiteren Laufe meines Lebens hat mich die Hoffnung begleitet, einen ähnlichen Satz zu beweisen, der eine Dimension höher steht, etwa die Existenz eines kompakten Blattes in jeder 2-Blätterung der 3-Sphäre. Skizzen zu einem vermeintlichen Beweise habe ich an verschiedenen Orten vorgetragen und wollte es auch bei einem Symposium über geblätterte Mannigfaltigkeiten in Grenoble tun (cf. Briefwechsel Reeb 1962/3. Zu meinem Glück bemerkte ich, daß meine Skizzen keinen Beweis ergaben. Ich verzichtete auf die Teilnahme an dem Symposion, obwohl der Präsident G. Reeb mir freundlich zuredete, das mache ja nichts aus.

A revitalisation of an open communication of the "two cultures"[4]is a pressing concern of both parties today. In a highly acclaimed address to the International Congress of Mathematicians in Berlin 1998 the prominent German writer HANS MAGNUS ENZENSBERGER described the image of mathematics in the eyes of the public in these terms[5]:

> *There is surely no other field in which the cultural time lag is so enormous. Popular consciousness trails research by centuries. Indeed, one can state dispassionately that great segments of the population have never progressed beyond the mathematical levels of the ancient Greeks. An equivalent backwardness in other fields – medicine, say, or physics – would arguably be perilous. Less directly this could be said of mathematics also, for never has a civilisation been so infused with mathematical methodology – right down to its everyday life – and so dependent on it as ours* ([15], p. 31).

In his own inimitable way, KNESER always contributed to bridging the gap between the two cultures.

Yet HELLMUTH KNESER 's contribution to 20th century mathematics as a whole continues to be underestimated or is, at worst, forgotten. It is the purpose of this text to put his work, published in its entirety in [28] in a more adequate perspective and to revive his memory in the mathematical community.

Short professional biography

Some bare facts are told expeditiously. After his 1921 doctorate, HELLMUTH KNESER passed his "Habilitation" in 1922, i.e. his official qualification for university teaching. In the years 1921 to 1924 he was RICHARD COURANT's Assistant in Göttingen ([52], p.85).

The following incident is typical for HELLMUTH KNESER 's period of being COURANT's assistant in Göttingen.

On October 21, 1924, L. E. J. BROUWER wrote a letter to HELLMUTH KNESER , in which he introduced BARTEL VAN DER WAERDEN who was on his way to join the Göttingen mathematics community that fall:

> *In a few days my student (or actually* WEITZENBÖCK*'s) will come to Göttingen for the winter semester. His name is* VAN DER WAERDEN*. He is very intelligent and has already published several papers (namely, on Invariant Theory). I do not know whether for a foreigner, who wants to register, there are difficult formalities to fulfill. Nevertheless, it would be very valuable for* VAN DER WAERDEN*, if he could find some assistance and guidance. May I ask if he could call on you in this regard? Thank you very much in advance.*

In his book [58], p. 29, SOIFER reports that he found BROUWER's original and a few copies in VAN DER WAERDEN's handwriting in the latter's archive in the ETH

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[4] C. P. Snow: The two cultures, Cambridge Univ. Press (1993).

[5] H. M. Enzensberger: Drawbridge up: Mathematics – A Cultural Anathema, German and English; translated from the original German by Tom Artin, A.K.Peters Ltd., Natick Massachusetts (1999) [15].

(= Eidgenössische Technische Hochschule Zürich). Soifer rightfully concludes that this indicates the significance that Van der Waerden himself attributed to this document.

As early as 1925, at the age of 27, Kneser was appointed professor of mathematics at the University of Greifswald, on the Baltic Sea coast, succeeding Johannes Radon on a chair which he would occupy for 12 years, until 1937. Before moving to Greifswald, he had received a fellowship of the Rockefeller Foundation to work with N. Nielsen in Kopenhagen. In 1932 and 1933 he occupied the position of the Dean of the “Philosophische Fakultät“ (School of Liberal Arts and Sciences) at the University of Greifswald.

A decisive turn in Kneser's career was brought about in 1937 by the offer to succeed the geometer Karl Kommerell at the University of Tübingen in southern Germany, which he accepted. He remained in Tübingen through his 1966 retirement until his death on August 23, 1973. In the post-war years 1951–52 he was again the Dean of the “Mathematisch-Naturwissenschaftliche Fakultät“ (School of Mathematics and Sciences), now at Tübingen. He declined a prestigious offer by the University of Munich in 1942 to succeed C. Carathéodory.

Kneser was married since 1927 to Hertha Adelheid Clara Kneser, née Scheurlen. They had three sons, Martin (born 1928), Hubert (born 1930), and Andreas (born 1933).

From 1949 through 1972 he was a member of the Advisory Board of Mathematische Zeitschrift (volumes 51–129). From 1952 on he was an editor of Archiv der Mathematik (Basel), and from 1968 an editor of Aequationes Mathematicae.

From 1953 to 1956 he was a member of the board of the Deutsche Mathematiker-Vereinigung (German Mathematical Union). Indeed in 1954, Hellmuth Kneser was President of this organisation like his father had been in 1929. In that year the *International Congress of Mathematicians* (ICM for short) was held in Amsterdam as reported in a comprehensive article by Ralf Bülow in the *Mitteilungen der Deutschen Mathematiker-Vereinigung* in 2024 [10]. An interesting photo in this article shows Queen Juliana of the Netherlands and 19 prominent mathematicians around the Fields medallists K. Kodaira and J.-P. Serre ([10], p. 129). Within this group is the President of the DMV, Hellmuth Kneser . (In the original version of the article he was misidentified as Arend Heyting of the Netherlands, but this was corrected on p. 204 in the same volume of the *Mitteilungen*.)

Hellmuth Kneser 's achievements received their due recognition in the form of memberships in prestigious academies. In 1957 he was elected ordinary member of the Heidelberg Academy of Sciences. He was corresponding member of the Academy of Sciences in Göttingen, and of the Academy of Sciences at Helsinki; furthermore he was honorary member of the Belgian Mathematical Society (Société Mathématique de Belgique).

Hellmuth Kneser's PhD Students and Candidates for “Habilitation”

The following is a listing of persons having acquired a Pd.D. degree or the “venia legendi” (Habilitation) under Hellmuth Kneser 's supervision.

Table 1

Who?	Where	Doctorate	Habilitation
Reinhold Baer (1902–1979)	Göttingen	1925	
Wilhelm Süß (1895–1958)	Greifswald		1928
Johannes Krzoska	Greifswald	1933	
Helmut Urban	Greifswald	1934	
Rudolf Witt	Greifswald	1935	
Günter Pickert (1917–2015)	Tübingen		1948
Karl Nickel (1924–2009)	Tübingen	1949	
Wilhelm Friedrich Stoll (1923–2010)	Tübingen	1953	1954
Walter Vogel (1923–2017)	Tübingen	1955	1960
Helmut Salzmann (1930–2022)	Tübingen	1957	
Irvine Noel Baker (1932–2001)	Tübingen	1957	
Karl Heinrich Hofmann (1932)	Tübingen	1958	1962
Erich Glock (1929)	Tübingern	1961	
Horst Dieter Ibisch (1932)	Tübingen	1961	1966
Manfred Reimer (1933)	Tübingen	1963	
Wilhelm Niethammer (1934–2023)	Tübingen	1964	
Peter Zahn (1930)	Tübingen	1965	
Frieder Schwenkel (1933)	Tübingen	1966	
Horst-Günter Zimmer (1937–2016)	Tübingen	1966	

It should be recalled in this context that HELLMUTH KNESER was merely 27 years old when REINHOLD BAER received his doctorate under his supervision.

If we wish to get a deeper appreciation of HELLMUTH KNESER 's Work and a better understanding of the influence it had in different fields we cannot but consider various of its facets individually.

It would be too daring to venture to a complete analysis of HELLMUTH KNESER 's œuvre in its entirety and in all of its details. The work represented here will have to speak for itself. In these paragraphs we shall attempt to guide the reader along some threads that weave through it; we shall necessarily restrict ourselves to what we deem to be essential aspects.

Topology

First we turn to Kneser's contributions to topology. Perhaps earlier surveys did not place enough emphasis on the significance of his influence in this field. JOHN MILNOR's surveys, notably the one presented in the Bulletin of the American Mathematical Society, Volume 52 (2015) corroberate our selection [38].

While young KNESER's dissertation was concerned with mathematical foundations of physics, it nevertheless led him into topology right away [2–21b][6]. In view of

[6] No statement in this paper shows that it is the author's dissertation or a part of it. However, a dissertation abstract [3–21c] in the "Jahrbuch der philosophischen Fakultät der Universität Göttingen" identifies it as KNESER's dissertation. In all citations of mathematians and physicists,

HILBERT's 6th Problem (s. [20]) it is not surpising that a dissertation subject in the mathematical foundations of physics was suggested, since in this problem Hilbert asked for an axiomatic foundation for physics similar to the one defining geometry – notably in the areas of probability and mechanics. Quantum mechanics in the strict sense of the word will be introduced as late as 1925–26, and the first sytematic and axiomatic description by VON NEUMANN follows in 1927 [43], nevertheless KNESER in his dissertation of 1921 studies evolutions in time of certain dynamical systems which in the phase space of n space coordinates $q_1, \ldots, q_n$ and n momentum coordinates $p_1, \ldots, p_n$ are subject to certain quantum conditions. The coordinates q_j are considered as angular variables that is, "the states in $(q_1, \ldots, q_j, \ldots, q_n, p_1, \ldots, p_n)$ and $(q_1, \ldots, q_j + 1, \ldots, q_n, p_1, \ldots, p_n)$ agree". Thus with respect to the space coordinates we operate on an n-torus or a manifold covered by an n-torus.

Frequently this sort of periodicity also entails the periodicity of the momentum; in this case the entire phase space is a torus or is a manifold with the torus as covering space. According to PLANCK a family of hyper surfaces decomposes this phase space. Certain phase trajectories which run through certain of these hypersurfaces, that are characterized by quantum numbers, are statistically distinguished. In the end KNESER restricts his attention to mechanical systems with two digrees of freedom, that is, to a 2×2-dimensional phase space. If two hypersurfaces are in general position, they intersect in a 2-dimensional analytic submanifold M^2, and all possible phase trajectories on it consitute a foliation of M^2 (see e.g. [64]). It is assumed that M^2 is compact. From the hypotheses on the physics of the system the existence of a compact leaf is deduced; such a leaf is a "closed curve".

HELLMUTH KNESER isolates all topological arguments and collects them in an appendix of barely two pages. But this appendix is to become the starting point of KNESER's next research program. He begins the appendix with the formulation of two theorems:

(1) *A compact foliated 2-manifold is a torus or a* KLEIN *bottle.*

(2) *Any compact manifold (possibly with a regular foliated boundary) contains at least one compact leaf.*

In the same year of 1921, KNESER publishes a note in the Jahresbericht der DMV (= Deutsche Mathematiker-Vereinigung) which he devotes to this topological portion of his dissertation.

These insights about possible foliations of compact surfaces KNESER expeditiously elaborates in the creation of his Habilitationsschrift (thesis required for admission to academic teaching at a university). The title of this thesis is "Bestimmung aller regulären Kurvenscharen auf geschlossenen Flächen" [5–22a] ("The determination of all regular families of curves on closed surfaces"), and it is ready for submission as early as 1922, that is, one year after its author graduated to the degree of Doctor of Philosophy. This is KNESER's first full fledged paper on topology. In it KNESER considers compact surfaces with foliation defined in purely topological terms. The appendix in the dissertation now unfolds and yields a thorough discussion. There is a full proof of the fact that the 2-torus and KLEIN's bottle are the only compact 2-manifolds with foliations.

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the author simply quotes them soberly by their names as usual. An exception is his doctoral adviser who is courteously referred to on p. 290 [2–21b] as "Herr Prof. HILBERT".

Let us pause for a moment and develop an intuition which may meet halfway a more contemporary observer whose training includes the rudiments of group actions. This will of course not eliminate the technical difficulties; we anticipate simply the actions of the POINCARÉ group. The concept of our departure is the euclidean plane $\mathbb{R}^2$ and the group $\mathbb{Z} \rtimes \mathbb{Z}$ with multiplication $(m,n)(m',n') = (m+(-1)^n m', n+n')$; it acts on $\mathbb{R}^2$ continuously via $(m,n)\cdot(x,y) = (m+(-1)^n x, n+y)$. The subgroup $\mathbb{Z} \rtimes 2\mathbb{Z}$ of index 2 of $\mathbb{Z} \rtimes \mathbb{Z}$ is abelian, and the action of its elements is simply by translation. Thus the orbit space of this subgroup is the factor group $\frac{\mathbb{R}\times\mathbb{R}}{\mathbb{Z}\times 2\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\mathbb{Z}$; it is obviously the 2-torus $\mathbb{T}^2$. HELLMUTH KNESER calls it the *zweiseitige Ringfläche* ("twosided ring surface"). The orbit space $\mathbb{K}_2 = \frac{\mathbb{R}^2}{\mathbb{Z}\rtimes\mathbb{Z}}$ is KLEIN's bottle; it is unlikely that at the time, during KLEIN's lifetime, it was labelled in this fashion. HELLMUTH KNESER calls it the *einseitigen Ringfläche* ("one-sided ring surface"). The natural function $p\colon \mathbb{T}^2 \to \mathbb{K}_2$ is the double covering which we may interpret as orbit space of the action of the two-element group $\mathbb{Z}(2) = \frac{\mathbb{Z}\rtimes\mathbb{Z}}{\mathbb{Z}\times 2\mathbb{Z}}$ on the torus. There are some very familiar foliations on the 2-torus, namely, those given by a one-dimensional subgroup $\{(ta+\mathbb{Z}, tb+2\mathbb{Z}) \in \mathbb{T}^2 : t \in \mathbb{R}\}$, $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ of the torus and its cosets. If a and b are two $\mathbb{Q}$-linearly independent real numbers then every coset is dense in the torus, otherwise they are circles. If one looks for those among these foliations which via p can be pushed down to the KLEIN bottle, one observes that a necessary and sufficient condition is the invariance of the foliation under the action of $\mathbb{Z}(2)$; this means that the foliation of $\mathbb{R}^2$ given by the cosets of the corresponding vector subspace $\mathbb{R}\cdot(a,b)$ of $\mathbb{R}^2$ is invariant under the action of $\mathbb{Z} \rtimes \mathbb{Z}$. But the image of a coset $\{(at+u+\mathbb{Z}, bt+v+2\mathbb{Z}) : t \in \mathbb{R}\}$ by the group element $(0,1) \in \mathbb{Z} \rtimes \mathbb{Z}$ is $\{(-at-u+\mathbb{Z}, bt+v+1+2\mathbb{Z}) : t \in \mathbb{R}\}$; this is again a coset if and only if $a = 0$. Such a foliation of $\mathbb{T}^2$ consists of circles. This is a plausibility argument for KNESER's Theorem 1 in his paper [9–24b], p. 153 which grew out of his habilitation thesis:

Satz 1. *Eine reguläre Kurvenschar auf einer einseitigen Ringfläche enthält mindestens eine geschlossenen Kurve.* (A regular foliation on KLEIN's bottle contains at least one compact leaf, i.e. one homeomorphic to a circle.)

Finally, in Theorems 2 and 3, KNESER classifies, up to homeomorphism, all possible foliations on $\mathbb{K}$. HELLMUTH KNESER 's procedures rest on a persistent concatenation of geometric arguments which protract to considerable length. However, if one succeeds to visualize the variety of possible foliations on the torus one begins to comprehend why the proofs have to be so longwinded. Many years later, in the fifties or early sixties KNESER will tell his students about the KLEIN bottle and the existence of a closed leaf. The theme of these early studies emerges much later in various strong branches of topology and differential geometry: Topological dynamical systems were studied intensively in the second half of the 20th century. After being pioneered by POINCARÉ this line of topology became a seemingly inexhaustible source of fascinating problems down to the present time. On the other hand, the domain of foliations of manifolds – frequently endowed with a Riemannian metric – grew into a vital topic of research in the last quarter of the 20th century; this is evidenced by an inspection of pertinent textbooks (such as the one by TONDEUR [64]). Indeed HELLMUTH KNESER himself will return to foliations on manifolds: first in an article in the sequence "Topological Questions in Differential Geometry" ("Topologische Fragen der Differentialgeometrie") in the journal Hamburger Abhandlungen [31–32h], in which he contributes the Thomsen incidence condition in Blaschke's web geometry. In the foundations of geometry, webs and the associated binary structures will become well known through Kneser's Tübingen Habilitation candidate Günther Pickert who documented their use in his book [47] "Projektive Ebenen" [7]. The Russian school of differential web geometers around Akivis was active up to the early decades of the 21st century, documented e.g. by Akivis and Goldberg in [2].

While Hellmuth Kneser had treated foliations in his habilitation thesis, many years later, in 1962, Kneser comes back to foliations in modern formulation [78-62] as he turns to his last topological circle of ideas, namely, manifolds failing to satisfy countability conditions for their topology. Certain publications of his son Martin such as [31] indicate considerable interaction on this topic between the two mathematicians.

Manifolds. In the year of 1924 there appears in the Proceedings of the Royal Academy of Sciences in Amsterdam Hellmuth Kneser 's first paper on topology in the strict sense of the word; it was communicated by L. E. J. Brouwer who himself in 1911 had generalized the Jordan Curve Theorem to higher dimensions by showing that a compact $n-1$-dimensional submanifold in $\mathbb{R}^n$ disconnects $\mathbb{R}^n$. Of course, in place of $\mathbb{R}^n$ one might just as well consider $\mathbb{S}^n$. The fact that an $n-1$ dimensional compact submanifolds disconnected $\mathbb{S}^n$ is not true for all compact n manifolds M^n in place of $\mathbb{S}^n$. In his paper "Ein topologischer Zerlegungssatz" ("A topological decomposition theorem") Hellmuth Kneser proves that M^n is disconnected by an $n-1$-dimensional submanifold if and only if $H^1(M^n, \mathbb{Z})$ has no nontrivial 2-torsion. (In the paper this property is expressed, as was customary at the time, in terms of Betti numbers.) This is being proved in the **PL**-category; that is, it is assumed that M^n is triangulable. This remark opens up a theme that has become fundamental for topology throughout the century. Kneser's methods are those of combinatorial topology, that is, the geometry of cell complexes. Basically, integral homology of such complexes is used, but Kneser uses homological language sparingly.

In this paper as well as in the preceding papers on foliations of compact surfaces, Kneser assumes that manifolds were given in the **PL**-category; in other words, he assumed that the manifolds he considered were triangulable. While the triangulability of 2-manifolds belonged at the time to the firm ground of established topological knowledge, the question of the triangulability of manifolds of an arbitrary dimension occupied the attention of topologists for half a century, as we shall see presently.

Combinatorial Topology. In 1924, the 26 years old Hellmuth Kneser addresses the 88th Congress of the Society of German Naturalists and Physicians (Versammlung der Gesellschaft deutscher Naturforscher und Ärzte) with a groundbreaking lecture on "the Topology of Manifolds" [13–25b]. As a consequence of this lecture and one that is to follow, Hellmuth Kneser 's name will be mentioned in the literature on manifolds in the entire 20th century. The introduction to this lecture deserves to be cited verbally:

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[7] Günther Pickert is the author of one of the commentaries of Kneser's works collection. In [48] he writes that he owes much to Hellmuth Kneser, among other things his "becoming motivated to occupy himself with topology and geometry".

> *When I now set out to report to you about the topology of manifolds, then I shall focus primarily on that treatment, which so far has been the most successful one and which evidences the most prospects for future success: the combinatorial one. I would like to show you, how one progresses from a purely topological concept of manifolds to a combinatorial situation, and how, conversely, the combinatorial set-up yields topological results; I shall explain which difficulties oppose us on this path, difficulties that for the most part have not been overcome.*[8]

In restrospect we can only confirm how intricate these difficulties turned out to be. It took until 1960 to do away with them. The process of clarification was laborious and protracted.

At issue is the question whether for a topological manifold one can find a simplicial complex whose geometric realisation is homeomorphic to the given manifold. A simplicial complex is transformed into another one via simplicial subdivision. One called such a transformation or its inversion an *elementary transformation.* KNESER explains in his lecture, that DEHN and HEEGARD (s. e.g. Enzyklopädie der mathematischen Wissenschaften III A B 3) provided a foundation of combinatorial topology in such a way that for an application of combinatorially obtained results to the topological investigation of manifolds the following "Theorem" would have to be true:

Hauptvermutung (= Main Conjecture). *Sind* $\mathfrak{z}_1$ *and* $\mathfrak{z}_2$ *zwei Zerlegungen einer Mannigfaltigkeit in Elementarräume, so kann man von* $\mathfrak{z}_1$ *durch endliche viele elementare Transformationen zu* $\mathfrak{z}_2$ *oder einer mit* $\mathfrak{z}_2$ *isomorphen Zerlegung gelangen.*

In modern terminology:

Main Conjecture *Two homeomorphic locally finite simplicial complexes have isomorphic refinements in the category of simplicial complexes.*

The question of the validity of this statement according to KNESER was posed by STEINITZ in the year of 1908 (Sitzungsberichte der Berliner Mathematischen Gesellschaft **7** (1907), Fußnote auf S. 32). Indeed there the following statement is to be found about the homeomorphy of two complexes:

> DEHN*'s and* HEEGARD*'s axioms under (a), which characterize the conditions stated for homeomorphy as sufficient, might be accepted by everybody as conforming to intuition. The postulating of Axiom (b), that would imply the necessity of these conditions, seems daring to me, in particular, if one ascends to higher dimension, but indeed already in the three-dimensional domain. It is of course not sufficient to know that the axiom does not entail contradictions. I have not been able to*

[8] Wenn ich Ihnen über die Topologie der Mannigfaltigkeiten berichten will, so habe ich dabei besonders diejenige Behandlungsweise im Auge, die bisher die meisten Erfolge gehabt hat und auch für die Zukunft dazu berufen erscheint: die kombinatorische. Ich möchte Ihnen zeigen, wie man von einem rein topologisch gefaßten Begriff der Mannigfaltigkeit zu dem kombinatorischen Ansatz und von diesem wieder zu topologischen Ergebnissen kommt und welche Schwierigkeiten sich auf diesem Wege entgegenstellen, Schwierigkeiten, die zum erheblichen Teil noch nicht überwunden sind.

convince myself, whether one might not be compelled, to base for $n \geq 3$ the concept of homeomorphy on a more general cell subdivision. I shall return to this point, which requires a more detailed discourse, at another occasion.[9]

It appears that STEINITZ left us with this announcement. In view of the absence of a proof, in key publications at a later time authors speak of "STEINITZ' Hauptvermutung" (s. e.g. [41]). However, we do emphasize the fact that the word "Hauptvermutung" is KNESER's creation, and we may consider it as KNESER's remaining merit to point out the significance of the statement in such clarity at a time when he was still a very young man. KNESER assumed at that time that the "Hauptvermutung" had been established for triangulated manifolds of dimension 2 and 3. For dimension 2 he refers to V. KEREKJARTO (Vorlesungen über Topologie, 1923, S. 134–135) [25]. For dimension 3 he cites FURCH (Zur Grundlagen der kombinatorischen Topologie, Abh. Math. Sem. Hamburg **3** (1923), S. 69–88 and ibid., Zur kombinatorischen Topologie des 3-dimensionalen Raumes, 237–245). However these and other later attempts were eventually deemed faulty until MOISE [41] established the Hauptvermutung for manifolds of dimension 3. How clearly the 26 year old KNESER saw the significance of the "Hauptvermutung" is evident from his own formulation:

The combinatorial method, that yields so many results so simply is an outright invitation to create an independent combinatorial discipline, which immediately has topological applications, as soon as the requisite conditions are satisfied, e.g. the "Hauptvermutung". This is now indeed possible.[10]

KNESER defines the concept of what later is called *a simplicial set* and he develops the foundations of a combinatorial topology. Among HELLMUTH KNESER 's papers one found after his death the manuscript of a text book which contained all markings for the typesetter to be published; in this manuscript he does spread out all the details of such a theory. Certain indications make it plausible that the text was written after 1929 and before 1932. We do not know why this book was never published. When KNESER outlined his combinatorial topology in 1924, the algebraisation of a complex is not yet in full sight; it is true that homology groups appear implicitly, but they never are placed into the foreground. In his lecture and later in the book manuscript, HELLMUTH KNESER presents in convincing clarity to which extent one passes back and forth between the topological and combinatorial

[9] Die bei DEHN und HEGARD loc. cit. auf S. 168/169 unter (a) angeführten Axiome, durch welche die für den Homöomorphismus angegebenen Bedingungen als hinreichend charakterisiert werden, dürfte wohl jeder ohne weiteres als der Anschauung gemäß anerkennen. Die Aufstellung des Axioms (b) hingegen, aus welchem die Notwendigkeit der Bedingungen folgen würde, scheint mir doch – namentlich, wenn man zu höheren Dimensionen aufsteigt, aber auch schon im dreidimensionalen Gebiet – gewagt. Seine Widerspruchslosigkeit genügt natürlich nicht. Ich habe mich noch nicht davon überzeugen können, ob man nicht doch genötigt sein könnte, für $n \geq 3$ eine allgemeinere Zellteilung dem Begriff des Homöomorphismus zugrunde zu legen. Ich komme bei anderer Gelegenheit auf diesen Punkt zurück, welcher eine ausführlichere Erörterung erfordert.

[10] Die kombinatorische Methode, die viele Ergebnisse so einfach liefert, lädt dazu ein, eine selbständige rein kombinatorische Disziplin zu schaffen, die sofort topologische Anwendung findet, sobald die dazu nötigen Voraussetzungen, etwa die Hauptvermutung, erfüllt sind. Das ist jetzt in der Tat möglich.

theories, and that the "Hauptvermutung" is a basic prerequisite for this free transition. Thus the problem of the relationship of the topological and the piecewise linear categories is presented for the first time in full clarity. To have even recognized this problem belongs to the great achievements of the topologist HELLMUTH KNESER.

Three-Dimensional Manifolds. On September 18, 1928, HELLMUTH KNESER, at the age of 30, again addresses the Congress of the Society of German Naturalists and Physicians (s. [19–29]). The publication of this lecture is to have an even greater impact than his preceding publications. Now KNESER treats the embedding of compact 2-manifolds into 3-manifolds and concentrates on the removability of singularities. The first important lemma states the following: *if a compact 2-manifold M^2 inside a compact 3-manifold M^3 contains the continuous image of a circle that is contractible in M^3 but not in M^2, then M^3 contains a compact 2-cell which intersects M^2 in an $\mathbb{S}^1$ which is free of singularities except for possibly a finite set of boundary points and which is not contractible in M^2.* A continuous image of a cell or a circle is then also called a singular cell, respectively circle. For a proof, KNESER uses a Lemma of DEHN's from the year of 1910 ([13], S. 147).

Dehn's Lemma. *If a singular 2-cell $'E^2$ in a 3-manifold has a singularity free boundary, then there is a singularity free 2-cell E^2 with the same boundary which in addition may be chosen to be arbitrarily close to $'E^2$.*

Among many other things, DEHN's Lemma has the consequence that in an $\mathbb{S}^3$ every compact connected 2-manifold arises by inserting *handles* (A handle arises if we consider in any manifold M two closed cells A and B, remove the interior, and glues the respective boundaries ∂A and ∂B together via an orientation reversing homeomorphism $f\colon \partial A \to \partial B$.) The following construction KNESER uses will perhaps become even more important in the course of time, namely, the *connected sum* $M\#N$ of two connected disjoint manifolds M and N: If M and N have the same dimension, assume that a closed cell A in M, a closed cell B in N and a homeomorphism $f\colon \partial A \to \partial B$ are given. Identifying the two boundaries via f yield a new manifold $(M \setminus \text{interior}\, A) \cup_f (N \setminus \text{interior}\, B)$, the connected sum.

A compact 3-manifold is called *irreducible*, if every 2-sphere in it bounds a 3-cell, and if the only way how it can be represented as a connected sum $M\#N$ is that one of M or N is a sphere. KNESER's paper [19–29] of 1929 which we discuss here contains a result that would become extraordinarily fruitful in the classification of 3-manifolds: MILNOR reports that in his investigations he discovered KNESER's result after he had independently found a proof (see e.g. [39]). The final version looks like this:

The Kneser-Milnor Decomposition Theorem. *For each compact 3-manifold there exists a natural number k such that it is the connected sum of k prime manifolds; the decomposition is unique up to permutations of the summands.*

The theorem is treated in textbooks today (s. e.g. [14], S. 69). KNESER proved the existence of the decomposition at age 31 in 1929, and MILNOR provides the proof of uniqueness in 1962 aged 31.[11] About KNESER's proof of the existence of the number

[11] In the year of 1953 JOHN MILNOR (1931–) visited the Mathematisches Institut der Universität Tübingen. It is difficult to imagine that he would not have talked with HELLMUTH KNESER about manifolds, but the subject matter of his colloquium lecture on Monday, June 8, 1953, 5:15 pm.

k, MILNOR writes: *"This* KNESER *proves by an ingenious argument."* In a personal memo [40] MILNOR recalls a passage in his collected papers, Volume 2, where he writes:

> *After working for some time on the problem of decomposing a 3-manifold into irreducible pieces, I was rather chagrined to discover that Kneser had obtained sharper results more than thirty years earlier. The paper "*A unique decomposition theorem for 3-manifolds*" shows that Kneser's work leads quite directly to a unique factorization theorem: Every compact, oriented, triangulated* 3*-manifold is a connected sum of "prime"* 3*-manifolds which are uniquely determined up to order and up to isomorphism.*

See also MILNOR's fascinating overview article [39] in the Bulletin of the American Mathematical Society in 2015.

Every irreducible manifold is prime, and $\mathbb{S}^3$ is irreducible by a theorem of ALEXANDER's (cf. [36], p. 2). Apart from $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^3$, a manifold is prime if it is irreducible [36]. In his survey article on the structure of 3-manifolds of 1982 [63], THURSTON features the KNESER-MILNOR Decomposition Theorem on page 1: "KNESER proved that this process terminates after a finite number of steps… " – namely the process of continuing the decomposition; that was the proof which MILNOR termed "ingenious".

Aside from many other achievements, in KNESER's article of 1929, an "addition in proof" ("Zusatz bei der Korrektur") printed in small letters is possibly one of the most significant observations:

> *During the printing I noticed, that the proof of Dehn's Lemma contains a gap. The argument of the 2nd paragraph on page 150–151* [13] *is not valid; counterexamples show that it cannot be corrected. Until this gap is filled, my results remain unsecured as well, but they nevertheless cast a new light on the significance of this Lemma for this entire circle of ideas.*[12]

The fact that the proof of Dehn's Lemma was recognized as having a serious gap caused circumspection regarding KNESER's results. Let us first remark, that the fundamental group $\pi_1(X)$ of a topological space had been known since POINCARÉ. KNESER called it the path group ("Wegegruppe")[13]. Throughout his life as a

in Lecture Hall 11 in the "Neue Aula" was "Games against nature."MILNOR obtained his PhD in 1954 in Princeton. In [39] MILNOR remarks that he coauthored in 1953 an article about n-person games jointly with JOHN FORBES NASH, Jr. and others [24].

[12] Während der Drucklegung bemerkte ich, daß der Beweis des DEHNschen Lemmas eine Lücke enthält. Die Überlegung des Absatzes 2 auf Seite 150–151 [13] ist nicht stichhaltig und läßt sich, wie Gegenbeispiele zeigen, auch nicht in Ordnung bringen. Bis zur Ausfüllung dieser Lücke sind auch meine Ergebnisse nicht sichergestellt, werfen aber immerhin von neuem Licht auf die Bedeutung des Lemmas für diesen Fragenkreis.

[13] KNESER bemerkt zur Bezeichnung: "Wenn man im allgemeinen von einer eingeführten Benennung nicht ohne besonderen Grund abgehen soll, so glaube ich Grund genug zu haben, für die seit POINCARÉ so genannte Fundamentalgruppe diese Bezeichnung vorzuschlagen. Sie ist kürzer, sagt mehr und läßt sich leicht in andere Sprachen übertragen. Mehr oder weniger fundamental sind auch die Homologiegruppen; unter ‚Wegegruppe' kann man kaum etwas anderes verstehen,

scholar, HELLMUTH KNESER made very conscious and conscientious use of the German language. Only one publication in a foreign language is known, namely, the relatively late article [71–58b] in the Bulletin de la Société Mathématique de Belgique in 1958. In his effort to create and use terminology in the German language he was consistent throughout his scholarly life. He wrote articles concerned with terminology (s. e.g. [49–40c], [51—43]). The comments on terminology we cited above verbatim antedate the beginning of the National Socialist rule; this deserves mentioning as it was part of its propaganda themes to find German equivalents for technical terminology that had a non-German etymology.

If $X = A \cup B$ denotes a space with base point in the intersection of the subspaces A and B, then a theorem due to VAN KAMPEN yields

$$\pi_1(X) = \pi(A) *_{\pi_1(A\cap B)} \pi_1(B),$$

where $*_U$ denotes the free product with amalgamated subgroup U. Since

$$\pi_1(\mathbb{S}^{n-1}) = \{1\} \text{ for } n > 2,$$

we may conclude that *if a manifold* M^n *of dimension* $n > 2$ *is decomposed into a connected sum* $M^n = M_1^n \# M_2^n$, *then*

$$(\dagger) \qquad \pi_1(M^n) = \pi_1(M_1^n) * \pi_1(M_2^n).$$

The results in KNESER's paper [18–28b] which were put into question by the gap in the proof of DEHN's Lemma contained the proposition that the converse is true for 3-manifolds. This entered the topological tradition of the 20th century as

Kneser's Conjecture. *If the fundamental group* $\pi_1(M^3)$ *of a three manifold is a free product* $A*B$ *of two subgroups, then* M^3 *can be represented as a connected sum* $M^3 = M_A^3 * M_B^3$ *in such a fashion that the relations* $\pi_1(M_A^3) = A$ *and* $\pi_1(M_B^3) = B$ *hold.*

Of course that is a central result in the spirit of *algebraic topology*, through which the KNESER–MILNOR Decomposition Theorem for 3-manifolds shines in resplendence. If, for the present purpose alone, we call a group G *prime* whenever the relationship $G = A * B$ entails $A = G$ or $B = G$, then *a closed* 3*-manifold is prime if and only if its fundamental group is prime, and the* KNESER–MILNOR *connected sum decomposition of a* 3*-manifold correspondes exactly to the prime decomposition of its fundamental group.*

We shall have more on the impact of the KNESER-conjecture presently!

It was still the year of 1928 in which KNESER's article [18–28b] on the smoothing of maps between surfaces appears. It was apparently conceived while its author was in contact with HEINZ HOPF, who had just developed the concept of the *degree of a map*. KNESER shows that a continuous function of degree n from one compact orientable surface to another can be continuously deformed into a covering map.

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als das Wort bedeuten soll."[19–29], S. 256. (While it is true that one should not deviate from an established notation without serious reason, I believe to have reason enough to propose this notation for the fundamental group, so called since POINCARÉ. It is shorter, tells more, and can be easily translated into other languages. More or less fundamental are the homology groups as well; but 'path group' cannot be understood in any other way than the word itself suggests.)

Compact surfaces had been recognized as triangulable, and thus KNESER was able to employ his exceptional intuition for combinatorial topology for a proof of this fact.

If we now glance back on HELLMUTH KNESER 's achievements in the twenties in topology alone, then we see that out of his work on compact manifolds, the following insights were probably the most influential ones for the development of topology in the 20th century.

(1) The significance of triangulability of manifolds: **TOP** versus **PL** [13–25b], [18–28b].
(2) The formulation of the *Hauptvermutung* and the insight into its central significance for combinatorial topology [13–25b].
(3) The KNESER-Decomposition Theorem for 3-manifolds [17–28a], [19–29].
(4) The KNESER-Conjecture on the free decomposition of the homotopy group of a 3-manifold [19–29].
(5) The detection of a gap in the proof of DEHN's Lemma [19–29].

What effect did these insights have on the further development of topology? In the year of 1952, the triangulability of 3-manifolds and the Hauptvermutung for 3-manifolds was proved by E. MOISE [41]. From that moment on it was known that every compact 3-manifold is triangulable and, in fact, that it carries a C^∞-structure. For 3-manifolds, this completes KNESER's program. A second proof was given by R. H. BING in the year of 1959 [5].

In 1961, for simplicial complexes MILNOR gave counterexamples [35] to the Hauptvermutung from dimension 5 on up; STEINITZ' suspicion therefore had been more than justified.

DEHN's Lemma (and the so-called Sphere Theorem), belonging to the KNESER Conjecture were proved as late as 1957 by C. PAPAKYRIAKOPOULOS [45]. In his Annals paper, in which he proves DEHN's Lemma, he expressly mentions KNESER:

> *As far as the sphere theorem is concerned we have to remark that, to the best knowledge of this author, the first one to attempt a theorem of this kind was* HELLMUTH KNESER *in* 1928 [17–28a], p. 257 ["KNESER's Conjecture"]*; however, his proof does not seem to be conclusive.*

At that point in time, after having been in abeyance for more than three decades, HELLMUTH KNESER 's results on 3-manifolds have finally been confirmed as being correct. The theory of 3-manifolds is now complete, including the confirmation of the classical "POINCARÉ-Conjecture" of 1904, which is verified as a portion of GRIGORI PERELMAN's work (2006) (see [33]) settling WILLIAM THURSTON's "Geometrization Conjecture" (1982). The excellent shape in which the topology of 3-manifolds is at the beginning of the 21st century is largely owed to HELLMUTH KNESER 's pioneering work on which the ingenious achievements of the topologists of the second half of the last century was based.

Indeed, C. PAPAKYRIAKOPOULOS' contributions were followed by pertinent results of HAKEN and WALDHAUSEN, and finally the famous insights of THURSTON of 1977

[63] showed that certain "simple" types of 3-manifolds are geometric in the sense that they allow the introduction of RIEMANNian metrics with negative sectional curvature. The prime summands in the KNESER-MILNOR Sum Decomposition thus could be more explicitly classified. The Decomposition Theorem, after KNESER's Conjecture has been verified, has thus become a fundamental tool in the description of 3-manifolds. The KNESER-Conjecture, having become a theorem again after PAPAKYRIAKOPOULOS' proof of DEHN's Lemma[14] was made the topic of J. STALLINGS' 1959 Princeton dissertation [59] and of his subsequent work. He provided a topological proof of a purely group theoretical theorem due to GRUSHKO (1940 [18]) and WAGNER (1957 [65]), saying indeed the following:

If φ is a homomorphism of a group G onto the free product of a family of subgroups H_j of the image, then G is a free product of a family of subgroups G_j such that $\varphi(G_j) \subseteq H_j$.

In this context STALLINGS remarks (loc.cit. p. 6):

> *My original version of this proof involved 3-dimensional manifolds and has been expanded into one of the proofs of* KNESER*'s conjecture. In fact, with hindsight we can see that a proof of* GRUSHKO*'s Theorem could have been derived from the discussion of* KNESER*'s paper.*

The ties between topology and geometry have been tightening up progressively in the theory of three and four dimensional topology; in some sense topology is returning to its home. HELLMUTH KNESER surely considered topology as a branch of geometry. Even if his mathematical legacy contained nothing but his papers of 1924 and 1928, his name would have been permanently engraved into the history of mathematics of the 20^{th} century.

The Tables 2–5 permit an overview of the after-effects of KNESER's results in topology prior to the contributions of THURSTON and PERELMAN [33, 42]:

Table 2

M^n	Who	Circumstances	when	Bibliogr.
$n=2$	T. Rado	always	1924–26	[49]
$n=3$	E. E. Moise	always	1953	[41]
$n=3$	R. H. Bing	2^{nd} proof	1959	[5]
smooth manifold all n	J. H. C. Whitehead	smooth always	1941	[66]
$n=4$		not always		
$n\geq 5$	R. C. Kirby & L. C. Siebenmann	sometimes obstruction $k(M)\in H^4(M,\mathbb{Z}(2))$ known	1969	[26]

Personal encounters. The records seem not to inform us precisely which personal encounters HELLMUTH KNESER had with prominent persons in the history of topology. Even the records of his later life are somewhat clouded on the issue whether the protagonists of the ongoing developments of topology met personally.

[14] Further proofs and generalisations of DEHN's Lemma were to follow, e.g. [69].

Table 3: The Hauptvermutung

dim	who	manifolds	loc. finite complexes	when	Bibliogr.
$n=2$	v.Kerékjártó	always	—	1923	[25]
$n=2$	Ch. Papakyriakopoulos	always	always	1943	[44]
$n=3$	E. E. Moise	always	—	1953	[41]
$n=3$	E. M. Brown	always	always	1964	[7]
$n \geq 5$	D. Sullivan	sometimes: obstructions known	—	1968	[62]
$n \geq 5$	J. Milnor	—	counter-examples for all $n \geq 5$	1961	[35]

Table 4: Dehn's Lemma

dim	who	manifolds	when	Bibliogr.
$n=3$	Papakyriakopoulos	always	1957	[45]

Table 5: The Kneser Conjecture

dim	who	manifolds	when	Bibliogr.
$n=3$	Papakyriakopoulos	Sphere Theorem	1957	[45]
$n=3$	J. H. C. Whitehead	Sphere Theorem	1959	[68]
$n=3$	Milnor	Decomposition Theorem	1962	[36]
$n=3$	Stallings	Grushko's Theorem	1965	[59]

It is certain that MILNOR and KNESER met in Tübingen in the year of 1953 on the occasion of MILNOR's visit to the University of Tübingen. During HELLMUTH KNESER 's 1961 visit to Princeton, which was very probably his first excursion to the US, he did meet PAPAKYRIAKOPULOS as he reports in a letter to HELMUT WIELANDT in Tübingen about his first excursion into the USA:

> *Since there is space left I might as well relate that I begin to lead a loose life. Thursday of last week in Philadelphia* (RADEMACHER, BESIKOVITSCH)*, Saturday tour of Manhattan with* DOLD*, then on to New Rochelle to* COURANT, FRIEDRICHS, MAGNUS*, today without any scholarly pretext visiting the Institue out there, simply to stroll around in the wood for an hour with* E. THOMA*, yesterday with* FELLER *for lunch on a somewhat superior level out to the country which is rather handsome at the Delaware River, notably during this beautiful season which under the sun still causes short shirt sleeves to be the most agreeable attire. My*

> *lecture today was a high point: The structure of* PRÜFER- *and* ALEXANDROFF-*manifolds in front of an audience of six, among whom* THOM *and* PAPA...[15]

Topological groups and Lie groups

We should be far amiss had we created the impression that we had now touched all of HELLMUTH KNESER 's early contributions to topology. As soon as 1926 there appeared in Mathematische Zeitschrift his article on "Die Deformationssätze der einfach zusammenhängenden Flächen" [15–26b][16], in which it is shown by means of classical complex function theory that the orthogonal groups $\mathrm{O}(2)$ and $\mathrm{O}(3)$ are (in modern parlance) strong deformation retracts of the groups of self-homeomorphisms, equipped with the compact-open topology, of the spaces $\mathbb{R}^3$, respectively, $\mathbb{S}^2$. If $\mathrm{TOP}(3)$ denotes the topological group of autohomeomorphisms of $\mathbb{R}^3$ with the compact-open topology, then KNESER's results show that the inclusion $\mathrm{O}(3) \to \mathrm{TOP}(3)$ is a homotopy equivalence by showing that $\mathrm{TOP}(3)/\mathrm{O}(3)$ has the homotopy type of a point. For the modern point of view of these issues we refer to the book of KIRBY and SIEBENMANN [27]. (We owe LINUS KRAMER the more contemporary aspects and references of these results.)

At the University of California at Los Angeles in the early seventies D. S. GILLMAN and R. C. KIRBY posed a thesis topic to a PhD candidate to find a proof of KNESER's result accessible to topologists directly and without detour through complex function theory. The candidate was BJORN FRIBERG; a friend of his mentioned to the authors that KIRBY had told FRIBERG, KNESER's proof was "unelementary, untopological, and un-American(!)." In the introduction of FRIBERG's paper [16] "A topological proof of a theorem of Kneser" the first two adjectives are on record; the last one therefore has to be classified as hearsay.

Thus KNESER's paper of the deformation theorems had its after-effects 45 years later, just like the others. However, this story does not reveal one remarkable aspect of this paper of KNESER's: namely, the fact that KNESER addressed explicitly and with great clarity the issue of topologies on function spaces. He emphasized, in particular, that homeomorphism groups should always be considered as equipped with function space topologies:

> *In the following I outline a proof of deformation theorems, which has one advantage over other proofs, namely, that the desired deformation depends uniquely and continuously of the given map. This property permits us to assert more precise statements on the structure of the group of all deformations of the surface we consider. As a preparation I collect in §1 definitions and remarks on general convergence spaces which are*

[15] Da noch Platz ist, kann ich erzählen, daß ich ein liederliches Leben zu führen beginne. Donnerstag der letzten Woche in Philadelphia (RADEMACHER, BESIKOVITSCH), Sonnabend Rundfahrt um Manhattan mit DOLD, dann nach New Rochelle zu COURANT, FRIEDRICHS, MAGNUS, heute ohne jeden wissenschaftlichen Vorwand hinaus zum Institut, einfach um mit E. THOMA eine Stunde im Wald herumzustrolchen, gestern mit FELLER zu einem ein bischen [sic] besseren Lunch hinaus aufs Land, das gerade am Delaware River recht hübsch ist, zumal in dieser schönen Jahreszeit, die immer noch in der Sonne kurze Hemdsärmel zur angenehmsten Kleidung macht. In meiner Vorlesung war heute ein Höhepunkt: Strukturen der PRÜFER- und ALEXANDROFF-Mannigfaltigkeiten vor 6 Zuhörern, darunter THOM und PAPA...

[16] "The deformation theorems of simply connected surfaces"

essentially known; my motivation for this is not in the first line because the appropriate concepts can be introduced and the relevant questions can be posed in this context, but rather because the structure of the deformation group expresses itself in the property of being a convergence space and because the topological properties of this convergence space are immediately properties of this group.[17]

Lie groups

HILBERT's programmatic lecture in Paris in the year of 1900 had enormous influence on the development of research in mathematics. Among the famouse 23 problems it was the fifth one which would assign a central role to the theory of Lie groups in the 20th century. So it may not be a surprise that his student KNESER makes Lie group theory the topic of a paper in 1930 [21–30b], which is a piece of evidence that KNESER taught courses on Lie groups in Greifswald[18]. According to the author's own description he is dealing with a new presentation of known facts. However to the expert who knows the development of the pedagogy of Lie group theory since Chevalley's treatise of 1946 [12] it is abundantly clear that KNESER's approach is astonishingly modern. With a surprising lucidity the exponential function is placed in the focus in a way that could be immediately generalized to apply to the exponential function of a Banach algebra which is the context of the contemporary approach (see e.g. [6, 23]). The role of the entire function defined by

$$f(z) = (\exp z - 1)z^{-1} \quad \text{for} \quad z \neq 0 \quad \text{and} \quad f(0) = 1$$

with the power series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

is clearly recognized. The application of the representation as an infinite product

$$f(z) = \exp \frac{z}{2} \prod_{n=1}^{\infty} (1 + \frac{z}{2\pi i n})(1 - \frac{z}{2\pi i n}),$$

known very well in complex function theory, remaining valid for elements z from a Banach algebra, seems to be largely forgotten among modern authors, including BOURBAKI [6]. This approach yields instantly an explicit formula for the derivative

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[17] Im folgenden gebe ich einen Beweisansatz für Deformationssätze, der vor anderen die Eigenschaft voraus hat, daß die gesuchte Deformation eindeutig und stetig von der gegebenen Abbildung abhängt. Diese Eigenschaft erlaubt es, genauere Aussagen über die Struktur der Gruppe aller Deformationen der betrachteten Fläche zu machen. Zur Vorbereitung stelle ich in §1 im wesentlichen bekannte Definitionen und Bemerkungen über allgemeine Konvergenzräume zusammen, nicht so sehr weil sich schon bei diesen die hierher gehörigen Begriffe einführen, die Fragen stellen lassen, vielmehr weil sich die Struktur der Deformationsgruppe besonders in der Eigenschaft äußert, ein Konvergenzraum zu sein und die topologischen Eigenschaften dieses Konvergenzraumes ohne weiteres Eigenschaften jener Gruppe sind.

[18] The first sentence reads: "Die folgenden Bemerkungen, entstanden anläßlich einer Vorlesung über Gruppentheorie… (The following remarks arose on the occasion of a course on group theory…)

of the exponential function $\exp\colon \mathfrak{g} \to G$ for a Lie group G in a given point $X \in \mathfrak{g}$ of the Lie algebra of the group. Indeed if G is a Lie group with Lie algebra $\mathfrak{g}$, then the analytic exponential function $\exp\colon \mathfrak{g} \to G$ induces at any point $X \in \mathfrak{g}$ a linear map $T_X(\exp)\colon T_X(\mathfrak{g}) \to T_{\exp X}(G)$. By the translation with X in $\mathfrak{g}$ we get an isomorphism $T_0(\mathfrak{g}) \to T_X(\mathfrak{g})$ and by multiplication with $g = \exp X$ in G we obtain an isomorphism $\mathfrak{g} = T_1(G) \to T_g(G)$. Therefore we may define the linear map $\exp'(X)\colon \mathfrak{g} \to \mathfrak{g}$ by the commutative diagram

$$\begin{array}{ccc} T_X(\mathfrak{g}) & \xrightarrow{T_x(\exp)} & T_{\exp X}(G) \\ \cong\downarrow & & \downarrow\cong \\ \mathfrak{g} & \xrightarrow{\exp'(X)} & \mathfrak{g}. \end{array}$$

For $x \in \mathfrak{g}$ one defines the linear self-maps $\operatorname{ad} X$ of $\mathfrak{g}$ by $(\operatorname{ad} X)(Y) = [X, Y]$ and knows that $\exp'(X) = f(\operatorname{ad} X)$ with the entire function f mentioned above. (One has to choose the group translations on the appropriate side.) Now KNESER's deliberations yield at once the little known but very interesting formula

$$\begin{aligned} \exp'(X) &= (\exp \tfrac{1}{-} \operatorname{ad} X) \cdot \prod_{n=1}^{\infty} (1 + \frac{1}{2\pi i n} \cdot \operatorname{ad} X)(1 - \frac{1}{2\pi i n} \cdot \operatorname{ad} X) \\ &= (\exp \frac{1}{2} \operatorname{ad} X) \cdot \prod_{n=1}^{\infty} (1 + \frac{1}{(2\pi n)^2} \cdot (\operatorname{ad} X)^2). \end{aligned}$$

Since his Tübingen lecture course on Lie groups in 1961–62, KNESER's student K.H.HOFMANN endeavored to find an approach to the foundations of Lie group theory which would be suitable for presentation in the classroom (cf. [23], Chap. 5), but at that time in the early sixties he was unaware of KNESER's paper of 1930, more than thirty years earlier. His Tübingen Lecture Notes on Lie Theory were circulated relatively widely in the sixties. The lecture notes in the end appeared smoother than the lectures and did (of course) not show that at one point, in the use of real analytic in place of smooth functions, the lecturer ran into a dead end. HELLMUTH KNESER , who was in the audience at the time, courteously remarked to the lecturer in private: "Herr Hofmann I don't think you can pull *that* off."[19]

In personal correspondence KNESER writes to HOFMANN with the overtones of mild irony of the person in the know, which were characteristic for him [30]:

> *As I don't have your letters before me – another indication of my derelict condition – I can remember a controversy only vaguely. A remark of mine made you vituperate me saying in fact that in dealing with Lie groups one had "to dirty one's hands" from time to time. Should I have caused offense? I jump upon your Lecture Notes on Lie Groups – and have to realize that, alas, I had not studied them –. . . . Yet I promise you, that under your guidance I shall dirty my hands however much you shall demand it.*[20]

[19] "Herr Hofmann, ich glaube, Sie kommen damit nicht durch."

[20] Da ich Ihre Briefe nicht zur Hand habe – auch dies ein Zeichen meines heruntergekommen Zustandes –, kann ich mich an eine Kontroverse nur ungenau erinnern. Auf irgend eine Bemerkung von mir hielten Sie mir vor, man müsse sich bei den Lie-Gruppen auch gelegentlich "die Hände

He might have said: First read my article of 34 years ago [21–30b], then we shall see. But Hellmuth Kneser 's comportance was forever immaculately gentlemanlike, in his courses, in conversation, in his letters as well.

One thing is certain: The central role given the exponential function in Lie theory which Kneser had emphasized as early as 1930 emerged in textbooks which appear in increasing rapidity in the seventies. Sophus Lie and his immediate students had not yet clearly recognized the significance of the exponential function. However, the encyclopedic treatment of Bourbaki does do full justice to it (s. e.g. [6], p. 51ff.) and became a model. Recent texts place the exponential function into the center of their developments (see [23] as an example).

Teaching Topology

In the roughly 15 years following the Second World War Hellmuth Kneser 's contributions to topology manifest themselves primarily in his courses on the subject. In those years students learned topology in Tübingen mainly with Kneser and Günther Pickert (1917-2015). In a letter [48] of October 25, 1998, Pickert emphasizes the positive influence that Kneser had on his professional career:

> *– his positive vote when I was appointed to an assistant professorship pro tem [at the University of Tübingen] in the winter term of 1946-47 [about one year after the universities resumed a rocky restart after the war the latter parts of which Pickert had spent as a POW in the US, the Eds.],*
> *– his motivating me to orient my mathematical activities towards topology and geometry,*
> *– and, notably, with questions concerning the teaching of mathematics and the continuing education of teachers.*
> *– The title of the course "The Scholarly Foundations of Mathematics Education in Schools, I–IV" that I introduced at the University of Giessen is to be traced back to* Hellmuth Kneser*. He rejected the [then current] designation of "school mathematics" as offense against the unity of mathematics.*[21]

In retrospect it appears quite natural that from Kneser one would learn in those years a topology with a strong combinatorial and geometric flavor. This is reflected in his lecture notes of the summer term of 1952 [87–52].

Of course, Kneser knew of the success story of algebraic topology. In his courses, he taught simplicial and singular homology. It is here where the bridge from combinatorial topology to the complete algebraisation of topology is shortest. In

schmutzig machen". Sollte ich Sie da gekränkt haben? Ich stürze auf die Ausarbeitung Ihrer Lie-Gruppen los – und muß feststellen, daß ich sie noch gar nicht studiert hatte – was übrigens bis heute nur in unzulänglichem Maße geschehen ist. Ich verspreche Ihnen aber, mir unter Ihrer Anleitung die Hände so schmutzig zu machen, wie Sie es irgend verlangen.

[21] . . . daß ich Hellmuth Kneser viel zu verdanken habe:
— Sein positives Votum zu meiner Einstellung als "Vertreter einer Assistentenstelle" WS 46/47;
— Motivation zu meiner Beschäftigung mit *Topologie* und Geometrie
— sowie mit Fragen des mathematischen Schulunterrichts und der Lehrer(fort)bildung.
Der Titel der Vorlesung "Wissenschaftliche Grundlagen des mathematischen Schulunterrichts I-IV", die ich hier in Giessen eingeführt habe, geht auf Kneser zurück; die Bezeichnung "Schulmathematik" lehnte er ab als Verstoß gegen die Einheit der Mathematik.

a conversation in the late fifties KNESER spoke of H. HOPF with whom he must have had a lively exchange in the twenties. The evidence is his paper [18–28b] of 1928 which appears in Mathematische Annalen immediately after that paper of HOPF's, in which the latter defines the degree of a map. KNESER's article solves a problem suggested by HOPF about the degree of continuous maps between compact surfaces.

When the conversation turned to cohomology as a tool of algebraic topology, HELLMUTH KNESER made the comment that people had passed to (graded) cohomology groups primarily because these carried a ring structure. (Personal recollection by KARL H. HOFMANN.) From the position of a certain level of homological algebra and functorial practice, this is not so mysterious: If $H(X)$ denotes the graded cohomology module of a topological space over some commutative ring, then by one part of the Künneth formula there is a natural injection $\kappa\colon H(X)\otimes H(X)\to H(X\times X)$. The diagonal embedding $\delta\colon X\to X\times X$ induces a morphism of graded modules $H(\delta)\colon H(X\times X)\to H(X)$. The composition $H(\delta)\circ\kappa\colon H(X)\otimes H(X)\to H(X)$ is none other than the ring multiplication in question, called the *"cup product"*. If G is a compact connected Lie group, then we also have the multiplication $m\colon G{\times}G\to G$. If the ground ring for cohomology is taken to be a field of characteristic 0, then κ is an isomorphism, and we have, in addition to the cup product, a morphism of graded vector spaces in the reverse direction

$$H(G)\xrightarrow{\ m\ }H(G)\times H(G)\longrightarrow H(G)\otimes H(G);$$

this morphism is called a *comultiplication*, yielding the *cohomology* HOPF-*algebra of G* which permitted HOPF and SAMELSON to arrive at important conclusions regarding the cohomology of Lie groups.

When a letter from HOFMANN to his teacher KNESER expressed enthusiasm for what the former called the "functorial thinking" (still rather novel at the time), HELLMUTH KNESER wrote him on May 3, 1964:

> *In an effort to come to an end by presenting something astute, I link up with your impression of the "categorical thinking". This sounds almost as though you copied* F. KLEIN *with his "functional thinking" which my father [the mathematician J. C. Ch. Adolf Kneser, 1862–1930, the Eds.] already ridiculed. But seriously: You might have a point there. Regrettably, I don't find the place in a letter by* FROBENIUS *to* H. WEBER *where he writes how happy he was about the continuation of the latter's "Algebra";* FROBENIUS *continues to write that he was also glad that it was* WEBER *who wrote the book and not the all too abstract (or words to that effect)...* DEDEKIND*! That was all said in high esteem and friendship for D.; for both of them were well aware of what the latter meant [to mathematics].– This is the way it is likely to have been at all times. Let therefore everybody grasp however much he may be able to assimilate and leave to the senior the apprehensive query, how mathematics should be taught in schools (high and low [meaning university, secondary and primary level, the Eds.]).*[22]

[22] Um noch etwas Gescheites vorzubringen, knüpfe ich an Ihren Eindruck vom "kategoriellen

Contributions to Topology in the Sixties: Uncountable manifolds

After Hellmuth Kneser 's enormously influentional contributions to topology in the decade of the twenties, his publication activity in topology appears to take a pause from 1930 on; however, this period is richly filled with creativity of a similiar caliber in other mathematical fields of which complex function theory is perhaps the most prominent. We shall return to this point; but here we complete our discourse on topology by turning to Kneser's late and lesser known work on "large" manifolds which do not satisfy the second axiom of countability. It is characteristic of Hellmuth Kneser 's mathematical profile, that this new direction of research in topology is firmly rooted in the history of manifold theory and its basis in analysis. In the introduction to his paper [70–58a] on the analytical structure and countability, Kneser expounds on the fact that in his book of 1914 "the Idea of the Riemann Surface" ("die Idee der Riemannschen Fläche"), Hermann Weyl among other things formulated a postulate, which, expressed topologically, said that the topology of a manifold should have a countable basis. Kneser calls topological spaces with this property *countable*, all other spaces *uncountable* (überabzählbar). It was shown by Rado that this postulate was superfluous in the case of Riemann surfaces: it is in fact a consequence of the other axioms defining a Riemann surface. However, this is no longer the case for complex connected analytical manifolds of higher dimension as Calabi and Rosenlicht showed in 1953 [11]. In his paper, Kneser illustrated this with a complex analytical manifold of complex dimension 2 which he also used to present in his lectures. Thus: *for complex connected analytical manifolds, the second axiom of countability is automatic in the case of dimension one; in higher dimensions there exist examples of connected analytical manifolds failing to satisfy it.* From the Calabi-Rosenlicht construction, by restricting the scalar domain to that of real numbers, one obtains at once examples of uncountable connected real analytic manifolds of dimensions greater than one. Now Kneser's paper exhibits the surprising fact that there are uncountable connected real analytic manifolds of dimension 1. His proof uses an example first pointed out by P. S. Alexandroff from 1924 ([3], p. 295); already G. Cantor and L. Vietoris had observed Alexandroff's space as early as 1883, respectively 1921. Let $[0,\Omega[$ denote the space of all countable ordinals and $[0,1[$ the interval of all real numbers r satisfying $0 \le r < 1$. On the product $[0,\Omega[\, \times [0,1[$ one can consider the lexicographic order for which one has $(\alpha, r) < (\beta, s)$ iff $\alpha < \beta$, or $\alpha = \beta$ and $r < s$. The order topology makes this totally ordered set, and if one drops the smallest element $(0,0)$, an uncountable connected locally euclidean space arises, called Alexandroff's open half-line, "Halbgerade" according to Hellmuth Kneser . With an ingenious argument using complex uniformisation, Kneser succeeds in endowing this topological 1-manifold with a real analytic structure. In completing this study, Kneser observes that the literature exhibits no classification of connected topological 1-manifolds. He fills this

Denken" an. Das klingt beinahe, wie wenn Sie F. Klein mit seinem "funktionalen Denken" kopierten, über das sich schon mein Vater lustig machte. Im Ernst: Sie werden schon recht haben. Leider finde ich die Brief-Stelle nicht, wo Frobenius an H. Weber schreibt, wie er sich über das Fortschreiben von dessen "Algebra" freue; er sei auch froh, daß Weber das Buch schreibe und nicht der allzu abstrakte (o.ä)... Dedekind! Das in aller Hochachtung und Freundschaft für D.; denn beide wußten sehr wohl, was dieser bedeutete.

Es wird wohl immer so gewesen sein. Also greife jeder zu, ders assimilieren kann und lasse den Alten das ängstliche Fragen, wie man Mathematik auf den Schulen (hoch und niedrig) lehren soll.

gap in the literature in his paper [71–58b] by showing that up to homeomorphism there are only four topological 1-manifolds, namely, the 1-sphere $\mathbb{S}^1$, the real line, the Alexandroff long line $\mathbb{A}$ and the open Alexandroff half-line $\mathbb{A}^+$. Both papers taken together show that every topological 1-manifold can be endowed with a real analytical structure. HELLMUTH KNESER discussed this topic with his son MARTIN KNESER. In a joint paper [74–60c] they show that on the long line there exist indeed uncountably many nonisomorphic real analytic structures.

With the aid of uncountable 1-manifolds one can of course construct various uncountable 2-manifolds such as e.g. the manifold $M = \mathbb{A} \times \mathbb{S}^1$. This provides in particular an $\mathbb{S}^1$-bundle $\mathbb{S}^1 \to M \xrightarrow{\text{pr}} \mathbb{A}$; here "pr" simply denotes the projection onto the factor $\mathbb{A}$. One day, perhaps in 1962, HELLMUTH KNESER came to the department and waved a postcard: "Look what my son sent me!" On the back of the postcard MARTIN KNESER had written the following elegant argument: Let $\mathbb{A}$ be the Alexandroff long line with a real analytical structure. In the cotangent bundle $\mathbb{R} \to \mathrm{T}'(\mathbb{A}) \xrightarrow{p} \mathbb{A}$ let $A \subseteq T'(\mathbb{A})$ be the zero section. Since $\mathbb{A}$ is orientable, $T'(\mathbb{A}) \setminus A$ decomposes into two components; let us say that $M \overset{\text{def}}{=} T'(\mathbb{A})_+$ is one of them and let us call it the "positive" one. Now let $\mathbb{P} =]0, \infty[$ be the positive real half-line.

Then P is a multiplicative group and $\mathbb{P} \to M \xrightarrow{p|M} \mathbb{A}$ is a bundle.

MARTIN KNESER argued, that this bundle cannot have a global cross section as follows: A cross section $\omega\colon \mathbb{A} \to M$ for $p|M$ is a nowhere vanishing differential 1-form; by setting $f(a) = \int_{a_0}^{a} \omega$ we obtain a continuous and strictly increasing continuous function $f\colon \mathbb{A} \to \mathbb{P} \subseteq \mathbb{R}$. But such a function cannot exist, as every continuous function $f\colon \mathbb{A} \to \mathbb{R}$ is eventually constant in either direction. Thus every differential 1-form is eventually zero and thus cannot take all of its values in P. Now the subgroup $e^{\mathbb{Z}} \cong \mathbb{Z}$ acts on M discretely via multiplication on the fiber $\mathbb{P}$. Then $\mathbb{S}^1 \to M/e^{\mathbb{Z}} \to \mathbb{A}$ is a fiber bundle which is locally indistinguishable from $\mathbb{S}^1 \times \mathbb{A}$. However, it does not allow a global cross section. As KNESER would say: "There is no flight path ("Fluchtweg") leading out of this manifold."

Functions of Several Complex Variables: An overview

In regard to the aftereffects, HELLMUTH KNESER 's contributions to complex function theory are comparable to the influence of his contributions to topology. His papers impacted upon the entire area; but his work also contains wonderful gems scintillating in their own brilliance.

For the present purpose, however, we shall overview KNESER's work in the area of complex analysis more compactly.

Kneser's Papers in Complex Function Theory. Beside his only book "Funktionentheorie" von 1958 [91–58], which appeared in 1966 in a second edition, and the jointly authored survey with EGON ULLRICH [85–48] of 1948, the following papers appear to be relevant as we collect them from KNESER's cumulative bibliography for the convenience of the reader: [22–30c, 23–30d, 25–32b, 27–32d, 28-32e, 29-32f, 37–35a, 40–36b, 41–36c, 43–38, 50-42, 52-48a, 58-50b, 61-51a]. WILHELM STOLL has called [41–36] and [43–38] the "most significant and valuable" papers; they contain HELLMUTH KNESER's foundation of the value distribution theory of meromorphic functions of several variables into complex projective space. In his obituary honoring

Hellmuth Kneser [71], Helmut Wielandt (1910–2001) cites Kneser's own description of his work in complex function theory [99–57b]:

Among my function theoretical papers I would like to emphasize the theory of entire and meromorphic functions of several variables. After the foundational achievements of Poincaré, Cousin, Hahn *and* Gronwall *it was* S. Bergmann, H. Cartan *and others who showed among others that portions of the highly developed theory of one variable can be extended to several variables. However, the results remained unsatisfactory, as they did not yield an explicit expression of an entire function in terms of its zeros, as in the case of one variable it was possible in terms of* Weierstrass' *product formula as completed by* Hadamard. *My idea was simple, indeed naïve: If in* Weierstrass*' formula the logarithm of the function was expressible in terms of of a sum over the isolated zeros of the function, then in the case of several variables a suitable integral extended over the continuously distributed manifold of zeros should accomplish the purpose. My stubborn search for such an integral eventually was successful after I utilized the tools of the* Kähler *metric. An appropriate elaboration of the methods that I developed for the purpose furthermore provided the first level of the construction of a theory that can duly be called the natural extension of* Rolf Nevanlinna*'s theory of meromorphic functions to the domain of several variables. My student* W. Stoll *elaborated the theory and vastly expanded its domain of validity.*[23]

Reinhold Remmert views Hellmuth Kneser as the first mathematician since Hartogs to seek a thorough understanding of the phenomena of meromorphic continuation [53]. He points out (ibid.) that in his paper on "Analytische Mannigfaltigkeiten im komplexen Raum" [61–51a], Kneser provides a proof of the important Theorem of W.L.Chow which asserts that every *analytical* set in complex projective space $\mathbb{P}^n$ is in fact *algebraic.* As Remmert further states, Kneser was next to K.Stein the only German mathematician recognizing the significance of Chow's Theorem. As is observed in Remmert's Commentary on Chow's Theorem in Kneser's collected works, Kneser's proof was still too complicated.

In the early ninteen thirties, through his paper [29–32f], Hellmuth Kneser came into contact with Hodge-theory. As Remmert points out [53], Kneser entered a correspondence with Hodge in this context. Later Hodge is to make use of an idea of Hellmuth Kneser emerging from these contacts.

[23] Unter meinen funktionentheoretischen Arbeiten möchte ich die Theorie der ganzen und meromorphen Funktionen mehrerer Veränderlicher herausheben. Nach den grundlegenden Leistungen von Poincaré, Cousin, Hahn, und Gronwall hatten S. Bergmann, H. Cartan und andere gezeigt, daß man Teile der bei einer Veränderlichen hoch ausgebildeten Theorie auf mehrere Veränderliche übertragen kann. Die Ergebnisse befriedigten unter anderem deshalb noch nicht, weil sie keinen expliziten Ausdruck einer ganzen Funktion durch ihre Nullstellen lieferten, so wie es bei einer Veränderlichen die durch Hadamard vervollständigte Produktformel von Weierstraß tat. Meine Idee war einfach, ja naïv: bei Weierstraß wird der Logarithmus der Funktion ausgedrückt durch eine Summe über die isolierten Nullstellen der Funktion; bei mehreren Veränderlichen muß ein geeignetes Integral über die stetig ausgedehnte Nullstellenmannigfaltigkeit dasselbe leisten. Mein hartnäckiges Suchen nach einem solchen Integral führte zum Ziel, nachdem ich die Hilfsmittel der Kähler-Metrik herangezogen hatte. Der gehörige Ausbau der dabei ausgebildeten Methoden ergab ferner die erste Stufe im Aufbau einer Theorie, die als die natürliche Übertragung von Rolf Nevanlinnas Theorie der meromorphen Funktionen auf das Gebiet mehrerer Veränderlicher angesprochen werden darf. Mein Schüler W. Stoll baute die Theorie aus und erweiterte ihren Gültigkeitsbereich in hohem Maße.

Even this preliminary summary of Hellmuth Kneser 's contributions to complex function theory indicate, that his far-sighted contributions to this area measure up to his papers in topology in the nineteen twenties.

At the University of Tübingen, Hellmuth Kneser educated many generations of students in the theory of complex variables on all levels. His pedagogical efforts in this direction are elegantly documented in his textbook [91–58] which very aptly renders the spirit of his lectures. While, it is true, this text included, wherever possible also functions of several complex variables, it is basically a book on the function theory of one variable. However, Kneser taught special courses on the theory of several complex variables which remained alive in the memory of his erstwhile students.

Hellmuth Kneser – The Person

Whoever has met Hellmuth Kneser as a teacher knows that an extraordinary personality has crossed one's biography. None of Hellmuth Kneser 's students could know the wealth of his mathematical achievements in so many areas, let alone be familiar with them. But the far reaching scope of his intelligence convinced anybody following his presentation. Whoever may recall the reports of those mathematicians who knew Hellmuth Kneser in his Göttingen period will not be surprised that the members of the audiences of his lectures will have occasionally been aware of the "difference of height" of his and their own intellectual levels. The style of the noble modesty of his presentation, which was very rarely interspersed with personal comments, may have contributed to this feeling of a personal distance which, on the level of an interpersonal relationship, diminished in the measure in which one was able to approach him personally.

In the present text, personal testimonies will be related to Hellmuth Kneser 's Tübingen period, and even for this segment of time there remains for most witnesses alive merely the time past 1945, the year in which the Second World War came to an end. A future generation may possibly arrive at the conclusion that Hellmuth Kneser's most grandiose and persistent achievements as a mathematician are to be found prior to his Tübingen time. Yet as a *teacher and pedagogue* he impressed witnesses in a way that they recall at this point in time.

Traces of doctoral students from Hellmuth Kneser 's Greifswald period are barely detectable.

However, his first doctoral student in Göttingen, Reinhold Baer, himself left a gigantic impression on many genenerations of mathematicians in the 20th century. Such a historic impact also requires a thorough culture of vivid traditions. Yet Baer's mathematics is detectable mainly in the domain of algebra and geometry, often close to the foundations – at any rate in domains which we do not connect immediately with Kneser's mathematical heritage. Those historical witnesses who are close to both mathematicians by biographical circumstances or for mathematical constellations tend to see this relationship as a fact of complementarity rather than one of competition.

In Tübingen there existed opportunities of personal approach to HELLMUTH KNESER through, say, a student's lecture in one of his seminars, or better still, through a thesis written for a German State Board Examination (for subsequent service in higher education) or a "Diploma Thesis" (comparable to a Master's Thesis) supervised by him. Extraneously, such an approach manifested itself by a progression, beginning in a first step perhaps by a brief conversation in the hallway of the "New Auditorium " (called the "Neue Aula", referring to the building complex of the university dating back to 1845) between the "Auditorium 11" ("Hörsaal" 11) and the entrance to the Department of Mathematics, and leading possibly to a longer conversation in the rooms of the departmental library; from there it was no longer far to his personal office in the corner of the Department. There he kept on the side of mathematical material also literature from the classics. Occasionally he would resort to this part of his personal library in order to verify some classical citation, which he anyhow knew by heart. In his office important official procedures took place, for instance, an official oath of office a future employee in the department would have to take, or, much more significantly, the instruction for scientific work on the highest level. Here it would happen that the professor would inform an author of a State Board Examination thesis, or, perhaps the writer of dissertation in this fashion:

"Herr Hofmann, I liked your text. But you will still have to learn the writing of mathematics. For example, one does not begin a sentence with a mathematical formula or, likewise, with a letter from a formula – and most certainly not if the preceding sentence ends with a formula! Also you do not write '... *Thus we have an implication* (in German: "somit folgt aus $x < y$ $y < z$ $x < z$."); there are enough words to fill in, like e.g. '... *Thus* $x < y$ *and* $y < z$ *imply* $x < z$.' And so on." Then for the following 40 years, the person so instructed will teach all students and candidates, but also, say, as an editor of journals or an author of articles or books, KNESER's rules of acceptable writing.

The person who not only had found the way into KNESER's office, but was invited to join his family at home, upon entering, could not fail to notice opposite the entrance a small framed sentence on the wall:

La cortesia e una chiave
d'oro che apre tutte porte

One might surmise that the citation originated from Libro del Cortegiano (1528) by Baldassare Castiglioni (1478–1529), but we are not sure. In any case that source is held to be a codex of the courteous style of the period. What counts here is the attitude in KNESER's family which definitively dominated his personal style: in appearance and behavior a gentleman, correct from top to the bottom.

From these forms of KNESER's *cortesia* there was a straight path to his warm and caretaking heart. Once when his assistant had been hopitalized for imprecisely diagnosed liver complications in a clinic high up on one of the hills surrounding Tübingen, he never failed a weekly arduous excursion on foot uphill to explore the patient's condition. *"Herr Hofmann, don't take the liver as a joke!"* he declared to his young colleague and former student, *"You have only one. Take good care of yourself!"*. Unforgettably, he even returned to the advice in his correspondence across the Atlantic years later.

Hellmuth Kneser 's students indeed recall from their daily life certain characteristic events which soon turned into legends, although one cannot claim that there were Kneser-anecdotes in the line of Hilbert-anecdotes. The Kneser-stories represent his spontaneity. Helmut Salzmann recalls the following event:

> *I recall the following experience with* Herr Kneser *towards the end of the summer term of 1951 from a course in differential geometry. He got stuck in the proof of a theorem, pondered about it, pulled a small sheet of paper torn off a calendar out of his pocket, compared the notes on it with what he had written on the chalk board a couple of times, and concluded elegantly the proof. Upon leaving the room, he carlessly left the slip of paper on the desk. Being a polite young man I picked it up and took it for him into his office, yet there was nothing on it that related in any way to the content of the lecture.* Herr Kneser *received the slip of paper with a trace of a smile and touched his lips with his finger. – As long as he lived indeed I never told the story.*

Hellmuth Kneser 's cultured relationship for classical literature was manifest through his distinct scholarly interest for languages. We commented on it earlier in the context of his steady effort to deal with the German language in the context of the formulation of mathematical statements (see e.g. [49–40c]). Indeed there are two published articles by him in French ([64–52b], [71–58b]), but none in English. His appointment in Tübingen offered the north-German Hellmuth Kneser the opportunity to study the dialect of his Swabian social environment. The postman who in the morning greeted him with the words

"Herr Professer, i hân an Brieaf fir Siea!"

caused him to consider a linguistic consideration on the immediate derivation of the Swabian dialect from the Central High-German environment. Indeed from Walther von der Vogelweide (1170–1230) one observes rather directly, e.g. in his poem *"Nû wachet… "* the lines:

"wir hân der zeichen vil gesehen,
daran wir sîne kunft wol spehen"

or in the poem:

"Ich hân gemerket von der Seine unz an die Muore"

Kneser assured us Swabians, that our pronunciation of the German word *"Haus"* clearly reflects the central high German word *"hûs"*. And, by the way, he said, the Swabians are the only one of the German tribes for whom the difference between a *"Seite"* (= page) of a book and a *"Saite"* (= string) of a violin is audible. It did require the linguistic sensitivity of a Hellmuth Kneser to classify us Swabians as the linguistic offspring of Walter von der Vogelweide and his likes.

Hellmuth Kneser and the period 1933–1945

The period from January 1933, when the National Socialist rule began until the total collapse of Germany in 1945, comprises the darkest segment of German history. In the conscience and self-reflection of the German universities and notably their departments of mathematics it left wounds that may never heal completely[24].

[24] How intellectual life would fare from here on became evident through the infamous and equally

An exhibition at the University of Technology of Berlin organized on the occasion of the International Congress of Mathematicians from the 18th to the 27th of August 1998 in Berlin under the title "Persecution and Expulsion of Mathematicians from Berlin between 1933 and 1945" commemorated the terror to which mathematicians merely of the University of Berlin were exposed (see [8, 9]). The fate of the Department of Mathematics of Göttingen is known by numerous individual presentations, so e.g. by the biographies of DAVID HILBERT and RICHARD COURANT by CONSTANCE REID [51, 52], the biography of BARTHEL LEENDERT VAN DER WAERDEN by ALEXANDER SOIFER, and the comprehensive history of German mathematics in the Nazi period by SANFORD S. SEGAL [57] in which HELLMUTH KNESER is widely mentioned. Another significant source is the rather recent collection of biographies of Austrian mathematicians from this period by W.A.F. RUPPERT and P.W. MICHOR [54]. Its relevance concerns the general background, but contains only one reference to HELLMUTH KNESER , namely, that he is the inventor of the name "Hauptvermutung" (see above).

HELLMUTH KNESER published a total of three articles ([39–36a], [44–39a], and [xlc–40a]) in the journal "Deutsche Mathematik" (see [50], p. 387 ff.) which was founded by the early national socialist THEODOR VAHLEN (1869-1945) (see e.g. [9], p. 8, also [57, 52]) with the cooperation of LUDWIG BIEBERBACH. VAHLEN was professor in Greifswald since 1911; in the years from 1924 to 1927 he was "Gauleiter" (= state leader) of the substitute organisation of the then illegal national socialist party "NSDAP" in the state (or province) of Pomerania. In the year of 1927 he was dismissed from civil service without pay on account of "actions hostile to the republic": as the "Prorektor" (= vice rector) of the University of Greifswald he had ordered the removal of all flags of the republic from the university buildings which had been hoisted on the occasion of the celebration of the republic. He returned to university service at the time of the seizure of power of the national socialists in the year of 1933 and was charged with the authority over all universitities inside the Prussian Ministery for University Education.

HELLMUTH KNESER, who in the years of 1925 to 1937 was a member of the faculty of Greifswald, clearly knew VAHLEN personally and probably considered himself VAHLEN's junior colleague. In any case, KNESER's elegant contribution [xxxix–39a] in the issue which was expressly published for VAHLEN on the occasion of his 70th birthday is dedicated to him "als Zeichen der Verehrung" (= as a token of reverence). Yet in 1933 KNESER had witnessed the dismissal from office of his Göttingen colleagues COURANT and NOETHER each with an alleged conflict with the new authorities in charge. The journal "Deutsche Mathematik" was reprinted in the year of 1966 by Swets & Zeitlinger in Amsterdam with the omission of ideological or political material. This indicates that in this journal among other things legitimate science of high level had been published. Surely HELLMUTH KNESER 's contributions [39–36a] and [44–39a] were purely scientific articles, while [49–40c] was a wittily ingenious article in linguistics. None of KNESER's contributions in the journal "Deutsche Mathematik" shows any evidence that their author had any

ruthless burning of books on May 10, 1933 on what is today Berlin Bebelplatz facing the front of *Humboldt University* by perpetrators who were students. The event is now commemorated by an underground memorial created by MICHAEL ULLMANN visible through a plate of glass inserted into the floor of the square, showing a large room surrounded by empty bookshelves.

inclination towards the ideology of the founders of "Deutsche Mathematik". In view of other material published in that journal, this deserves to be emphasized. Hellmuth Kneser 's linguistic purism, which manifests itself in the tendency to express, if possible, all concepts of the technological language of the mathematical environment in the words and the verbiage of the German language is manifest in his writings already in the early twenties, and it accompanies him throughout his entire life as a researcher and a teacher. This sort of purism, however, was later excessively exaggerated in the "Germanism" practiced by the National Socialists.

The intellectual mood in Germany in the late twenties and the early thirties exhibited a degree of disdain for parliamentary democracy while sympathizing with the concept of a strong government and crediting it with substantial moral values. The economic crisis of 1929 was erroneously considered as a weakness of the "Weimar Republic". It was believed that the power attainment (labelled the "Machtergreifung") of Adolf Hitler and his party would lead to a powerful German government which were to solve the social and economic problems haunting the community at that time. There are clear indications that Kneser was subject to these contemporaneous inclinations. On October 30, 1933 he joined the SA (= *"Sturmabteilung"*) and was promoted on the 23rd of August 1934 to the rank of *"Sturmmann"*. In 1937 he also joined the *NSDAP* (= *national-sozialistische deutsche Arbeiterpartei*), the core organisation of the contemporary system.

In the years of 1932 and 1933 Hellmuth Kneser was the Dean of the School of Liberal Arts and Sciences ("Philosophische Fakultät") of the University of Greifswald at the age of 34 and 35 years. As a former member of the department of mathematics at the University of Göttingen and as Richard Courant's former assistant, Hellmuth Kneser was greatly puzzled and perturbed by the dismissal of Emmy Noether and the forced time of leave imposed on Courant in the spring of 1933, and equally so about similar actions concerning scholars of Jewish background. The reasons given, however, were termed "unreliability in the political domain" or such terms, but never "Jewish descent". The authorities carefully avoided any impression of anti-semitism, while rather invoking dangers for the community "Volk und Staat" from which the state ("das Reich") had to be protected by all means. The invoked law, called "Ermächtigungsgesetz", was described as the "Law for the Protection of the People and the State" (= "Gesetz zum Schutz von Volk and Staat"). The Nobel laureate James Franck voluntarily renounced his professorial position in protest of this discrimination of Jewish colleagues. However, his action was critizied by a majority in the academic community who trusted the "Prussian State of Law and Order", which was believed to be completely free of antisemitism. Accordingly, friends of the dismissed colleagues were led to hope they might be able to influence the official actions by officially presenting objections to the reasons alleged for the dismissals. Actions of this sort were attempted in the case of Emmy Noether by submitting testimonies for her personal integrity and her faithfulness to the German Nation. Notably, the fact that a person like Richard Courant, who was a highly decorated officer in the army of the First World War, now had been put on leave from his ordinary position in the civil service appeared totally incomprehensible. In Kneser's opinion and that of many colleagues this was only explicable as a faulty judgment of Courant's situation on the part of the authorities, and a failure of knowing and understanding of his person. The entire situation appeared as a challenge for COURANT's colleagues and students to testify for him. For instance, this explains the petition submitted to the authorities by FRIEDRICHS, KNESER and PRANDTL which we are about to describe in the following. KNESER's position in this action is represented dramatically in the correspondence between Greifswald und Göttingen leading to a petition to the government which is significant enough for us to be reproduced here. (The translation of the correspondence between KNESER and COURANT originates from sources out of the archives of the Department of Mathematics at the University of Tübingen.)

- On April 26, 1933, KNESER to COURANT:

Dear Herr COURANT:

Just now I hear about your predicament of being put on leave and that of other colleagues in Göttingen, and I want to tell you one word in all haste. I am totally horrified and fail to understand the whole thing unless it is merely a precaution against active disturbances. If you or NEUGEBAUER *have the time to write me details, then I would be very grateful.*

Cordial greeting

your H. Kneser.

- On April 28, 1933, COURANT to KNESER:

Dear Herr KNESER:

Many thanks for your kind letter. I have to confess, that the news of my forced retirement has hit me less prepared and much more shaken as it should have under reasonable circumstances. I feel myself too deeply connected with everything I built up here, with my activity and with Germany, in order to be able to consider this event with equanimity, even though it is clear to me, that to many persons in very different positions and professions the same thing will happen.

You want to know more details. I myself know nothing. Up to now I have no written declaration. Of course the question occupies myself strongly, what truly might be the cause for the procedure of the Office of the Minister; however on this point I can only offer conjectures. Firstly: I was an active participant in the war, officer of the infantery, seriously wounded, and I have otherwise, as I believe, accumulated some merits through my contributions to terrestrial telegraphy. According to the law of civil service I should not have to expect my dismissal from duty. The reasons therefore must be be found in the personal domain. My friends assume, that these reasons are of a political nature. They could then merely be connected with the fact that in the year of 1919 in passing I was a member of the S.P.D. and thereby atttained–incidentally upon FELIX KLEIN*'s initiative, who regarded this to be in the interest of the university, a seat in the Göttingen City Parliament. In this position I was practically never active, except that on one occasion I gave a really academic public address on the events at and after the end of the war. On a factual basis, my entry into the S.P.D. happened on the basis of the idea, widely present at the time, to fortify the only stronghold extant against bolshevism in the face of a necessary reconstruction, and thereby to initiate such a rebuilding. Yet I recognized very soon, that the development I had hoped for did not materialize in the S.P.D., and I quit my mandate and left the party. Since that time I was politically inactive in every aspect.*

I cannot imagine, that these facts represent a sufficient reason for my dismissal. Equally insufficient is the totally nonpolitical fact that, at the time of the termination of the war, I stayed, together with a group of about 70 men, for certain experimental purposes in the Harz[25]*, where the group elected me and my subordinate officer-in-charge (Wachtmeister) as military councillors-in-charge (Soldatenräte) and thus endowed me with the authority to keep the order and the smooth dissolution of the group.*

Yet during the last months it became evident around here that those old facts in diverse circles became the origins of completely distorting and sometimes very fanciful rumors about my political past and my political position. Our faculty and notably our department of mathematics is not considered agreeable instinctively by many groups, where without doubt a certain jealousy is unconsciously involved. The fact, that I was personally and primarily involved in the build-up of the institute and the development of the institutions in mathematics and natural sciences, notably through the Rockefeller-negotiations, and that I frequently pursued this fact-oriented line without deep concern about personal sensitivities, is no doubt the reason that a large component of these sensitivities inspired by suspicions is concentrated to a large part against my person.

Thus there arose an atmosphere of rumors. Assertions, which never clearly penetrated to us and which culminated in the slogan "Stronghold of Marxism" ("Hochburg des Marxismus")*. The currently significant group of young colleagues at the university, which is in no way connected with us, is doubtlessly the center of this attitude, and I could imagine that the considerations of this group about myself and other colleagues were not only carried into the community of students outside of the department of mathematics, but also penetrated all the way to the government.*

In addition, we have the case of FRANCK*. Before I returned from holidays,* FRANCK *had confirmed more and more his decision for a voluntary petition to resign from his civil service duties.* NEUGEBAUER *and I, but also other friends, had discouraged him time and again from this decision and urged him to delay it. One day, however, I believe it was Easter Sunday, Franck's decision attained a definitive form, contrary to our original wishes. We, that is, Born and I, contemplated for a short time, whether we were not morally obligated to follow Franck. We decided however not to do that, but rather to try to stay and with all of our forces to preserve our present institutions in this place. With the aid of rumors and disfigured sayings, however, a completely distorted conception of this story was distributed, namely, that Franck's withdrawal had been the result of a common decision of sabotage and that, for purely tactical reasons, we had dispatched* FRANCK *ahead of ourselves. This grotesque idea motivated a comparatively small group of Göttingen colleagues to issue a public declaration opposing* FRANCK*'s letter of resignation and demanding a quick cleanup of the University. On the day after the publication of this declaration we were dismissed from our duties.*

I can well imagine that the situation I described caused the action of the Office of the Minister. Of couse we are urgently interested to obtain clear information about the full connection of this matter. After what I explained you can imagine very well, that I would experience a final dismissal as a profound injustice. At any rate I hope,

[25] a mountainous region about 30 km north-east of Göttingen

that a final step will not be taken without offering me the opportunity of an open discussion with the authorities unvolved, respectively, an investigation of potential accusations.

It is my personal intention not to let myself be dominated by a feeling of bitterness, but rather stay in place as long as possible and to pursue with the highest concentration my scientific work (completion of the second volume of the "Methods of Mathematical Physics"). I cannot see in any way what might happen in the long run when the events attain a definitive character.

I have essentially only written about myself. As concerns Born *things may develop similarly. Incidentally,* Born *is less protected than myself by the Conditions of Exceptions to the Law of Civil Service* (= "Ausnahmebedingungen des Beamtengesetzes" possibly meaning the "Gesetz zur Wiederherstellung des Berufsbeamtentums"). *This applies even more to* Emmy Noether.

I should be highly interested in knowing your personal opinion about this situation. You know me and the local conditions for such a long time that you are in a position to pass an unbiased judgment.

In the meantime I send you cordial greetings
your ...

- On 29th of April 1933, Courant to Kneser:

Dear Herr Kneser:
Permit me to complement a little my letter of yesterday, now as the situation continues to harden.
...
It is truly a pity which values are being destroyed after constructive work of more than ten years in this way. What hurts me most is to see which senseless damage is done to Germany. All you have to do is to think of the Americans and other persons of foreign nationality here all of whom now consider to terminate their studies and prepare for their departure. At any rate the spirit of our institute already now is destroyed; the appearence of ugly competition may be noticed now.
...

- No date – Kneser to Courant:

Dear Herr Courant:

For both of your letters many thanks! For these I obtained the information I asked for, but it is unpleasant. Right now I have to travel away as agreed; tomorrow I am occupied by the holiday; therefore only briefly:

From Franck*'s resignation I fear consequences of the kind he appears to have had; in this context you do not tell me new aspects on the motivation. To begin with, I should like to confirm you in the attitude you have assumed up to now. If you are willing to accept my advice, you will read the last chapters in* Plato*'s* Kriton.

I agree with your view on the situation and the position of the government. Regarding Vahlen *I know for sure that he thinks like you hope he would. Indeed I myself have to remind myself that no doubt we have a revolution behind us, even though it happened (despite everything) with a totally unrevolutionary quiet and order, and that rather great achievements were attained (national unification =* Reichseinigung*), which*

Bismarck did not achieve nor did the 1918s (= 1918er*), that on many domains now something happens where all former governments could not get past their intentions (a small example: settlement of university access according to quality), and that one has to have the patience to see how things develop in the end–this happened not before 1920. Of course meanwhile everybody acts as is expected in his place. Your job this summer is* Hilbert-Courant Vol. 2*. Therefore again my cordial request: try to recognize, as well as possible, the positive aspect in our political development: not so much in the interest of your own self but rather of the whole. (Excuse the common platitudes: this is what is being told to many people and by many people.)*

...

With cordial greetings

your H. Kneser

Subsequent to this correspondence, a petition was formulated by Hellmuth Kneser together with K. O. Friedrichs and Ludwig Prandtl as follows:

An den Herrn Minister für Wissenschaft Kunst und Volksbildung
Berlin
Hochschulabteilung

Prof. Dr. R. Courant *in Göttingen ist bis zur Entscheidung im Sinne des Gesetzes zur Wiederherstellung des Berufsbeamtentums beurlaubt worden. Von uns Unterzeichneten kennt jeder Prof.* Courant *aus mehrjähriger enger Zusammenarbeit. Nach unserer Kenntnis hat sich Prof.* Courant *in seinem Wirken als deutscher Staatsbürger gefühlt und bewusst als solcher gehandelt. Er hat sich der ihm anvertrauten Studenten und jungen Forscher nicht nur als Lehrer sondern auch als Fürsorger und Berater weit mehr angenommen, als die Amtspflicht erforderte und als öffentlich bekannt ist. Die seit 1921 wesentlich von ihm geschaffenen mathematischen Unterrichtseinrichtungen in Göttingen stellen ein Werk dar, das für die Wissenschaftspflege in Deutschland und für Deutschlands wissenschaftliche Weltgeltung von hoher Bedeutung ist und von seiner Person nicht ohne wesentliche Schädigung zu trennen ist.*

Aus diesen Gründen glauben wir, zu der Klärung der Unterlagen für die Anwendung des Beamtengesetzes auf Prof. Courant *wesentlich beitragen zu können, und bitten darum, in dieser Frage mündlich gehört zu werden, oder uns schriftlich äussern zu dürfen*[26].

[26] Prof. Dr. R. Courant in Göttingen has been relieved from his duties until a finite descision in the sense of the "Law for the Restitution of the Professional Civil Service". Each of the signers of this letter knows Prof.Courant from a close cooperation through several years. According to our knowledge Prof.Courant has been deeply conscious of his German citizenship in his activities and has consciously acted accordingly. Towards the students and junior researchers who were entrusted to him he was devoted not only as a teacher but also as a caretaker and advisor to an extent that exceeded his official duties and that is known publicly. The institutions for the instruction of mathematics in Göttingen that were created since 1921 essentially by him represent a work which for the cultivation of science in Germany and for Germany's scientific relevance in the world is of high significance; it cannot be separated from his person without damaging it severely.
For these reasons we believe to be able to contribute significantly to the clarification of the fundamentals of the application of the Law of the Professional Civil Service to Prof.Courant and request to be heard orally about this question, or else be permitted to submit a written statement about it.

(gez.) K. Friedrichs	*(gez.) H. Kneser*	*(gez.) L. Prandtl*
Dr. K. Friedrichs	*Dr. H. Kneser*	*Dr. L. Prandtl*
o. Prof. f. Mathematik	*o. Prof. f. Mathematik*	*o. Prof. f. angew. Mechanik*
an der TH Braunschweig	*an der Univ. Greifswald*	*an der Univ. Göttingen*
		Dir. d. Kaiser-Wilhelm-
		Inst. f. Strömungsforschung

The petition remained unanswered by the Office of the Minister (German *„Ministerium“*) as did another petition prepared by Friedrichs and Neugebauer which had been signed by the likes of Artin, Blaschke, Carathéodory, Hasse, Heisenberg, Herglotz, Hilbert, von Laue, Mie, Planck, Prandtl, Schrödinger, Sommerfeld, van der Waerden, and Weyl (see [52], pp. 151, 152). Kneser believed that the shorter letter reproduced above would be more effective. He also talked with Vahlen ([52, pp. 148 and 149]); after all Kneser was Professor in Greifswald and Vahlen still had connections there ([52], p. 148).

These events happened in 1933. The upheaval hitting German mathematics happened in 1934 in the events around the Bad Pyrmont Meeting of the DMV (Deutsche Mathematiker-Vereinigung = German Mathematical Union) when Ludwig Bieberbach (1886–1982) vigorously attempted to reorganize it in the structure and spirit of the National Socialist government. This struggle is amply and accurately documented in [57], pp. 263–288. This documentation also indicates on pp. 276, 280–287 the involvement of the now 36 years old Hellmuth Kneser . At first he appeared to be rather supportive of Bieberbach as is indicated in Kneser's letter of July 14th, 1934 to him, which concluded with the sentences ([57], p. 276).

"In your feud in matters of personality-structure, composition, etc., I take an active (lebhaften) part. May God grant German science a unitary, powerful and continued political position."

The last sentence of this quote is indeed cited in the English Wikipedia article on "Hellmuth Kneser" [74]. What has not yet been mentioned in our present text is the fact, aptly described in [57] on pp. 286, 289, namely, that Kneser had distanced himself toward the end of the thirties internally from his original supportive attitude of the 1933 government takeover expressed in the preceding quote.

For the final course of Courant's biography we refer to Constance Reid's colorful description [52]. Courant's and Kneser's professional lives were close in Göttingen and continued until the point in time when Courant had to leave Germany. This fact is the basis of Courant's opinion of Kneser in general, and in particular through the critical years of the political crisis in Göttingen. It is expressed in a statement on the relationship of Kneser and Vahlen stated in Reid's book [52] (p. 148), quoting Courant at an unspecified later point in time:

> *"*Kneser *was young, and he got some ideas under the influence of this dean* (Vahlen)*. . . . I don't think he ever did anything really bad. He is a good man. A very good mathematician."*

In the year of 1961 Hellmuth Kneser and Courant met again in New Rochelle in the USA as we reported earlier (see the footnote [15] above).

All sources agree that the developments of 1933 had devastating consequences for mathematics in Göttingen. Hellmuth Kneser who had joined Greifswald in 1925, left Greifswald and joined the department of mathematics in Tübingen in 1937.

By comparison with the situation in Göttingen, mathematics departments such as the one in Tübingen may have escaped in a less impaired fashion. But even there the Nazis caused the dismissal of Professor Erich Kamke on November 1, 1937, who had never concealed his opposition to the national socialist rule (cf. [1, p. 36 and 37]) and whose wife was Jewish. In those days Kamke was *"Außerordentlicher Professor."*

Helmut Wielandt, like all applicants for a career in the civil service, was confronted with a barrier as long as he did not identify himself sufficiently with the national socialist movement. He obtained the position of an assistantship at the University of Tübingen. There are indications that Hellmuth Kneser recommended his employment to the *"Dozentenführer"* (the person representing the national socialist structure of the university)[27].

The documents do not conclusively explain, which reactions had been caused in 1937 by Kamke's dismissal in the department. After the war, in 1945, Kamke's decisively formulated application to be re-instated, and the intensive engagement of the mathematicians Knopp and Kneser as well as of the Physicist Back, resulted in his complete rehabilitation and reinstallation, now as a full professor, on July 26, 1945. Immediately, Kamke found himself confronted with the preliminary dismissal of almost all of his colleagues in the department of mathematics when the so-called "epuration" [German: "Entnazifizierung"] (i.e. the official procedure of investigation and possible reinstallation of former members of any of the national socialist organisations) began in the German provinces under French administration. We have credible information that Kamke himself intervened with considerable energy for Kneser and Wielandt to achieve their reinstallation from the épuration. This may allow the interpretation that Kamke considered these colleagues' comportment during the period of his own dismissal as correct–in any case under the conditions prevailing at that time. In fact, Segal wrote in [57], p. 280 that Kneser

> *... seems not to have been able to follow Vahlen deep into the Nazi thicket. Indeed, on the contrary, Kneser and Erich Kamke became good friends.*

Kneser was also connected through a close personal friendship with his coauthor Wilhelm Süss, senior to Kneser by three years. In fact, Kneser was the advisor of Süss' habilitation (i.e. the German qualification test and thesis for a subsequent professorship) in 1929. In contrast to Kneser his friend Süss appears to have approached the centers of power of the Third Reich if that appeared useful in the interest of mathematics. Indeed, as Grüttner described in [19], pp. 476–478, in an address to the conference of the university rectors in 1943 he openly criticized the national administration of the universities for its insufficient observation of the significance of science and mathematics for the German war efforts. Wilhelm Süss is also remembered as the founder of the "Mathematisches Forschungsinstitut Oberwolfach" in the Blackforest.

[27] Letter from Wielandt to Hofmann 06/15/74.

The period 1945–1966

In the years after 1945 Hellmuth Kneser himself rarely spoke about political or societal concepts. In the postwar period of that time it became evident, that the mathematical culture in the academic and pedagogical environment at the universities had suffered substantially during the years of the national socialist rule from 1933 through the devastating war it had caused. This war lasted from 1939 to 1945. Germany had become isolated from progress in sciences and mathematics.

In mathematics the setback was evident significantly in areas for which the following three are exemplary: firstly, probability and statistics, secondly, the mathematics of economy, and thirdly, the instruction of mathematics at schools on an advanced level. While, in the postwar area, Hellmuth Kneser continued to contribute significantly to the theory of real analytical spaces, indicated by publications like [59-50c, 61-51a, 70-58a, 71-58b, 74-60c, 79-63a] (to a good deal in collaboration with his son Martin Kneser, as we indicated earlier, and as may be observed by publications like [74–60b] and [31]), the records show that Hellmuth Kneser invested considerable energy in his teaching efforts to systematic improvements in each of the three defects we mentioned. His publications in these decades demonstrate the direction of his efforts during that period:

(i) in probability theory and statistics: [77–61b], [100–61];

(ii) in the mathematics of economy: [64–52b] (one of the few of his publications in French), [66–53], [68–54b] (representing a lecture he gave at the International Congress of Mathematicians 1954 at which he appeared in the photo with Queen Juliana, see [10], 2024), [88–53] (the Lecture Notes of a course in the Winter term 1952/53);

(iii) the pedagogy and teaching of mathematics on a scholarly level [63–52a], [69–56], [76–61a], [89–54].

The last entry represents a set of Lecture Notes of a course of the winter term 1954/55. It became the project of a publication in the form of a book, that unfortunately did not materialize. However, we remind the reader that all of the items we list here are accessible completely in the volume of Kneser's collected works [28].

The authors themselves had the opportunity of getting close to Hellmuth Kneser at the University of Tübingen after the mid-fifties. Hofmann took the the State Board Examination for the Instruction at Highschools (Erstes Staatsexamen für das Lehramt an Höheren Schulen) in 1957, obtained a PhD (Dr. rer. nat.) under the direction of Hellmuth Kneser in 1958, and acquired later the "Habilitation" in 1962 also under his direction. During the academic year of 1959–60 he was a member of the Research Group of Mathematical Statistics headed by Hellmuth Kneser and Walter Vogel. Similarly, Betsch followed Wielandt and his research environment and received his PhD in 1963 at the University of Tübingen under the direction of Helmut Wielandt.

Complete Listing of Hellmuth Kneser's Publications

(The listing follows the works collection [28] of HELLMUTH KNESER .)

[1–21a] *Eine Erweiterung des Begriffs "konvexer Körper"*, Math. Ann. 82 (1921) 287–296.

[2–21b] *Untersuchungen zur Quantentheorie*[28], Math. Ann. 84 (1921) 277–302.

[3–21c] *Untersuchungen zur Quantentheorie*, Jahrbuch d. Phil. Fakultät d. Univ. Göttingen, 4s.

[4–21d] *Kurvenscharen auf geschlossenen Flächen*, Jahresber. d. Dt. Math. Verein 30 (1921) 83–85.

[5–22a] Habilitationsschrift Göttingen 1922: *Bestimmung aller regulären Kurvenscharen auf geschlossenen Flächen*; die Habilitationsschrift ging offenbar in verschiedene Publikationen ein.

[6–22b] *Neuer Beweis des Vierscheitelsatzes*, Christiaan Huygens 2 (1922-23) 315–318.

[7—23] *Über die Lösungen eines Systems gewöhnlicher Differentialgleichungen, das der Lipschitzschen Bedingung nicht genügt*, Sitz.ber. Preuß. Akad. Wiss., Phys.-Math. Klasse (1923) 171–174.

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[10–24c] *Die adiabatische Invarianz des Phasenintegrals bei einem Freiheitsgrad*, Math. Ann. 91 (1924) 155–160; (Nachtrag F. d. M. 50, 677).

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[17–28a] *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Naturwissenschaften 16 (1928) 973.[31] Vgl. [19–29].

[18–28b] *Glättung von Flächenabbildungen*, Math. Ann. 100 (1928) 609–617.

[28] Laut F.d.M.: Göttinger Dissertation. Martin Kneser, mündlich: "Mein Vater erzählte, Hilbert habe seine Dissertation nicht gelesen, sondern ihn nur zu einem Bericht einbestellt.

[29] Vortrag auf der 88. Versammlung der Ges. deutscher Naturforscher und Ärzte in Innsbruck, 21.–27. Sept. 1924; Hauptvermutung auf Seite 6.

[30] Aus einem Brief an Herrn H. Behnke.

[31] Auszug aus einem Vortrag am 18.9.1928 bei der 90. Versammlung der Gesellschaft Deutscher Naturforscher und Ärzte in Hamburg.

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Acknowledgments

We thank everybody who wrote us in the context the preparation of HELLMUTH KNESER 's Collected Works and of the present text, notably Ralf Bülow, Jürgen Flachsmeyer, Erich Glock, Norbert Heldermann, Isolde Hofmann, Linus Kramer, Matthias Kreck, John Martin, John Milnor, Michael W. Mislove, Sidney A. Morris, Karl-Hermann Neeb, Günther Pickert, Reinhold Remmert, Helmut Salzmann, Peter Schreiber, and Wilhelm Stoll.

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Received December 1, 2024
and in final form January 6, 2025