On the Construction of the Embedding $D_4 < F_4 < E_6$ over Fields of Characteristic 2

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Abstract. The purpose of this paper is to give an elementary and explicit construction of the embedding $D_4 < F_4 < E_6$, over a field K of characteristic 2 and to show that the Chevalley group of type D_4 is isomorphic to $\Omega_8^+(K)$.

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1. Introduction

In this paper we give an elementary construction of the Lie algebras embedding $D_4 < F_4 < E_6$ over fields of characteristic 2. We say an elementary construction, on the account that we follow an elementary approach consists of little more than naive linear algebra notions. In [13, 14, 17], Jacobson studied the construction of F_4 in detail and generalized the construction of E_6 to arbitrary fields of characteristic not 2 or 3. It is remarkable to mention that fields of characteristic 2 or 3 are still problematic in Lie algebras and in particular in Jordan algebra context. In [12] Chevalley and Schafer showed that the algebra of derivations of the real Albert algebra is a Lie algebra of type F_4, and also showed how to extend this to E_6. Exceptional Lie algebras of type D_4 were first constructed by C. L. Caroll [11]. Jacobson in [20] showed that the algebra of derivations of an exceptional simple Jordan algebra which maps a cubic subfield K onto zero is an exceptional Lie algebra of type D_4, and this class of algebras contains all of those constructed by Caroll. In [24] R. Wilson proved the existence of a Lie algebra of type E_6 independently of the Chevalley construction and generalized it to arbitrary fields of characteristic not 3, and expressed the subalgebra of type F_4 in terms of the Hurwitz ring of quaternions. The importance of this construction is to provide descriptions which are sufficiently concrete to be used effectively in investigating subgroup structures of E_6, F_4 and D_4. For more information about these Lie algebras, one may refer Aschbacher [5, 6, 7, 8, 9], Vavilov et al. [21, 22], Wilson [23], Aldhafeeri and Bani-Ata [1, 2], Jacobson [19, 20], and Freudenthal [16].

2. Preliminaries and general setup

The construction starts with a 6-dimensional elliptic orthogonal space V over the field of two elements $\mathbb{F}_2$ and a non-degenerate quadratic form Q on V of minimal Witt-index. To Q belongs a bilinear form on V defined as:

$$(v \mid w) = Q(v + w) + Q(v) + Q(w).$$

Let $\mathbb{B} = \{0 \neq x \in V \mid Q(x) = 0\}$ and $\mathcal{L} = \{L \leqslant V \mid dim\, L = 2 \text{ and } Q(L) = 0\}$.

The elements of the quadric $\mathbb{B}$ are called points and the elements of $\mathcal{L}$ are called lines, and the vectors $s \in V$ with $Q(s) = 1$ are called exterior or anisotropic vectors. The incidence structure $(\mathbb{B}, \mathcal{L})$ is a generalized quadrangle with 27 isotropic points and 45 lines. Each line contains 3 points and each point lies on exactly 5 lines [2].

Let $W = \{g \in GL(V) \mid Q(x^g) = Q(x) \text{ for all } x \in V\}$ be the automorphism group of (V, Q), which is isomorphic to $\Omega_6^-(2).2 \cong U_4(2).2$. The group W is isomorphic to the Weyl group of type E_6 and has order 51840. W is a 3-transposition group with respect to the 36-reflections $\sigma_s : V \longrightarrow V$, $v^{\sigma_s} = v + (v \mid s)s$ for $v \in V$, where s is an anisotropic vector. It is obvious that the Weyl group W acts on the orthogonal space V (and hence on the quadrangle), where the reflections induce transvections with an anisotropic center.

Definition 2.1.   A subset B of $\mathbb{B}$ is a *root base* if
   1. B is a $\mathbb{F}_2$-base of V
   2. For any two distinct elements $x, y \in B$, $(x \mid y) = 1$.
The set of root bases contained in $\mathbb{B}$ is denoted by Φ.

Remark 2.2.   Any set of six pairwise non-orthogonal points is a root base. If B is a root base, then $s_B = \sum_{x \in B} x$ is an anisotropic, and $B^* = B + s_B$ is also a root base. We call B and B^* corresponding root bases. $B_0 = \{v \in \mathbb{B} \mid (v \mid s_B) = 0\}$ then $\mathbb{B} = B_0 \cup B \cup B^*$. A line is either contained in B_0 or it is a transversal line i.e it intersects each of B, B^*, and B_0. ∎

Proposition 2.3.   (1) $\mathbb{B}$ *contains exactly 72 root bases, i.e* $|\Phi| = 72$.

(2)   *The group W acts transitively on Φ with stabilizer S_6 and imprimitively on Φ with 36 blocks $\{B, B^*\}$, $B \in \Phi$.*

Proof.   See [2]. ∎

Remark 2.4.   Root bases can be constructed as follows. For two points x and y with $(x \mid y) = 1$, set $B_{x,y} = \{x\} \cup \{v \in \mathbb{B} \mid (x \mid v) = 1, (y \mid v) = 0\}$. For a root base B and $x \in B$ one has $\mathbb{B} = B_{x,x+s_B}$. It is obvious that for a point x, there exist 16 points y with $(x \mid y) = 1$. Hence again $|\Phi| = \frac{27 \cdot 16}{6} = 72$. Any root base corresponds to an anisotropic vector $s_B = \sum_{b \in B} b$ and every anisotropic vector s or a reflection σ_s corresponds to two root bases $B_1 = B_2$ such that $s = s_B$. Moreover, $B_2 = B_1^{\sigma_s} = B_1 + s$ and $B_1 \cup B_2 = \{v \in \mathbb{B} \mid (s, v) = 1\}$. We denote σ_{s_B} by σ_B and $B_1^{\sigma_s}$ by B_1^*. ∎

Proposition 2.5. [1] *For root bases B and C with $s_B \neq s_C$ it holds*
(1) *If $(s_B \mid s_C) = 0$, then $|B \cap C| = |B \cap C^*| = 1$ and $B^{\sigma_C} = B$.*
(2) *If $(s_B \mid s_C) = 1$, then:*

(i) $|B \cap C| = |B^* \cap C^*| = 3$ *and* $B \cap C^* = B^* \cap C = \phi$ *or*

(ii) $|B \cap C^*| = |B^* \cap C| = 3$ *and* $B \cap C = B^* \cap C^* = \phi$.

(iii) *If* $B \cap C = \phi$, *then* $(B^* \cap C)^{\sigma_B} \cup (C^* \cap B)^{\sigma_C} = B^{\sigma_C} = C^{\sigma_B}$ *is a root base corresponding to* $s_B + s_C$.

3. Lie algebras of type E_6 and of characteristic 2

Let K be a field of characteristic 2 and $(\mathbb{B}, \mathcal{L})$ a generalized quadrangle of type $O_6^-(2)$ as defined above, and let A be a 27-dimensional vector space over K with base $\{e_x \mid x \in \mathbb{B}\}$. Let $\mathrm{End}_K(A)$ denote the Lie algebra consisting of all elements of $GL(A)$ with Lie product $[X, Y] = XY - YX$. For $v \in V$ define $H_v \in \mathrm{End}(A)$ by $e_x^{H_v} = (x \mid v)e_x$, $x \in \mathbb{B}$, and for a root base B define the Lie root R_B on A by

$$e_x^{R_B} = \begin{cases} e_x^{\sigma_B} = e_x + s_B, & x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that R_B has rank 6 and R_B, R_{B^*} are transposed to each other with respect to the base $\{e_x \mid x \in \mathbb{B}\}$ and $R_B^2 = 0$.

Proposition 3.1.   *For root bases B and C and for $v, w \in V$ it holds:*

(1) $[H_v, H_w] = 0$ *and* $[H_v, R_B] = (s_B \mid v) R_B$.

(2) $[R_B, R_{B^*}] = H_{S_B}$.

(3) *If s_B and s_C are distinct and orthogonal, then* $[R_B, R_C] = 0$.

(4) *If $(s_B \mid s_C) = 1$ and $B \cap C \neq \phi$, then* $[R_B, R_C] = 0$.

(5) *If $(s_B \mid s_C) = 1$ and $B \cap C = \phi$, then* $[R_B, R_C] = R_{C^{\sigma_B}} = R_{B^{\sigma_C}}$.

(6) *If $B \neq C$, then* $R_B R_C R_B = R_B [R_B, R_C] = 0$.

Proof.   For the proof see [1] and [3].   ■

Definition 3.2.   The Lie algebra $\mathbb{E}$ generated by the elements H_v, $v \in V$, and the 72 Lie roots R_B, $B \in \Phi$ is a subalgebra of $\mathrm{End}_K(A)$ of dimension 78 over K. Proposition 3.1 implies that the Lie algebra $\mathbb{E}$ is of type E_6. The group W acts on A by $e_x^g = e_{x^g}$, W permutes the roots R_B and induces a Weyl group on the Lie algebra $\mathbb{E}$, normalizing the Cartan subalgebra $\mathbb{H} = \langle H_v \mid v \in V \rangle$.

4. The Chevalley group E of type E_6

Definition 4.1.   Let B be a root base. For $k \in K$ define the root elements $r_B(k) = I + kR_B$ on A and define the root subgroups $U_B = \langle r_B(k) \mid k \in K \rangle$. The following holds for root bases B and root elements $r_B(k)$.

Proposition 4.2.   (1) *The root group U_B is isomorphic to $(K, +)$.*

(2) *If $k \neq 0$, then* $C_A(r_B(k)) = \langle e_x \mid x \in B^* \cup B_0 \rangle$.

(3) $[A, r_B(k)] := \langle a + a^{r_B(k)} \mid a \in A \rangle$ *has dimension 6.*

(4) *For root bases B and $C \neq B$ and $k, m \in K$, it holds:*

   (i) $[r_B(k), r_C(m)] = I + km[R_B, R_C]$.

   (ii) *If $B \cap C \neq \phi$, then* $[r_B(k), r_C(m)] = I$.

   (iii) *If $B \cap C = \phi$, then* $[r_B(k), r_C(m)] = r_D(km)$ *for the root base* $D = B^{\sigma_C} = C^{\sigma_B}$.

   (iv) *The group $\langle U_B, U_{B^*} \rangle$ is isomorphic to $SL_2(K)$.*

Proof.   (1) Let $r_B(k_1)$ and $r_B(k_2)$ be two root elements in U_B. Then

$$r_B(k_1) \cdot r_B(k_2) = (I + k_1 R_B)(I + k_2 R_B) = I + k_1 R_B + k_2 R_B + k_1 k_2 R_B^2$$
$$= I + (k_1 + k_2) R_B, \text{ as } R_B^2 = 0.$$

(2) This directly follows from the definition of $r_B(k)$.

(3) The commutator $[A, r_B(k)] := \langle a + a^{r_B(k)} \mid a \in A \rangle = \langle e_x \mid x \in B^* \rangle$ is of dimension 6.

(4)(i) Consider
$$r_B(k)\, r_C(m)\, r_B(k) = (I + kR_B)(I + mR_C)(I + kR_B)$$
$$= (I + mR_C + kR_B + kmR_B R_C)(I + kR_B)$$
$$= I + mR_C + kR_B + kmR_B R_C + kR_B + mkR_C R_B + k^2 R_B^2 + k^2 m R_B R_C R_B$$
$$= I + mR_C + km\,[R_B, R_C] \text{ as } R_B^2 = R_B R_C R_B = 0,$$

by Proposition 3.1. As root elements are involutions, then it follows

$$r_B(k)\, r_C(m)\, R_B(k)\, r_C(m) = I + km\,[R_B, R_C] + km\,[R_B, R_C]\, R_C.$$

Hence $[r_B(k), R_C(m)] = I + km\,[R_B, R_C]$ by Proposition 3.1(6), and this proves (4)(i). The proofs of (4)(ii) and (iii) are direct consequences of 4(i) and Proposition 3.1.

(4)(iv) Consider the action of $r_B(k)$ and $r_{B^*}(k)$ on the subspaces $\langle e_x, e_{x+s_B} \mid x \in B \rangle$ of dimension 12 and on the subspaces $\langle e_x \mid x \in B_0 \rangle$ of dimension 15, the claim follows. The group $E = \langle U_B(k) \mid k \in K, B \in \Phi \rangle$ with the above properties is the Chevalley group of type E_6 in characteristic 2.   ∎

Definition 4.3.   We define the quadratic map $\hat{Q} : A \to A$. Let $x = \sum_{v \in \mathbb{B}} x_v e_v \in A$. Then $\hat{Q}(x) = \sum_{v \in \mathbb{B}} Q_v(x) e_v$, where $Q_v(x)$ is a quadratic form on A defined by

$$Q_v(x) = \sum_{v \in L \in \mathcal{L}} \left(\prod_{\omega \in L \setminus \{v\}} x_\omega \right).$$

In particular $Q_v(x)$ can also be written as:

$$Q_v(x) = \sum_{L = \{a, b, v\} \in \mathcal{L}} x_a x_b.$$

Lemma 4.1.   (1) *For $a, b \in A$ and $g \in E$, $a^g b^g = (ab)^{g^*}$, where g^* is the transposed inverse of g with respect to the base $\{e_x \mid x \in \mathbb{B}\}$.*

(2) *For $a \in A$, $g \in E$, $\hat{Q}(a^g) = \hat{Q}(a)^{g^*}$.*

(3) *For $k \in K$ and $q \in A$, $\hat{Q}(ka) = k^2 \hat{Q}(a)$.*

(4) *For points $x \in \mathbb{B}$, $\hat{Q}(e_x) = 0$.*

Proof.   For 1 and 2, see [10] and 3 and 4 are obvious.   ∎

5. A construction of Lie algebras embedding $D_4 < F_4 < E_6$.

Definition 5.1.   An involution $z \in W$ is a *Siegel involution* if

$$L = [V, z] := \{v^z - v \mid v \in V\}$$

is a singular line and the centralizer $C_V(z)$ of z in L is $L^\perp$.

Remark 5.2.   For a transformation X in GL(A), let X^t be the transpose of X with respect to the base $\{e_x \mid x \in \mathbb{B}\}$. For a Siegel involution $z \in W$, the map $\tau := \tau(z) : X \to ((X^t)^z)^{-1} = ((X^z)^t)^{-1}$ is an automorphism of $GL(A)$ as z acts on A by permuting the base vectors e_x i.e., $e_x{}^z = e_{x^z}$, hence $z \in GL(A)$ and induces an automorphism on $GL(A)$ that is $X \to X^z$. In particular τ normalizes $\mathbb{E}$ as $(R_B)^\tau = R_{B^*}$, where R_B is a Lie root. Further if $g \in W$, let $\tau(g)$ be the automorphism $g\tau$ of $\mathbb{E}$ and note that $\tau g = g\tau$.

Proposition 5.3.   *Let z be a Siegel involution in W. Then*

(1) *For all $v \in V$, $(v^z \mid v) = 0$.*

(2) *z fixes 12 anisotropic vectors and 12 orbits of length 2 on anisotropic vectors.*

Proof.   For $L = [V, z]$ and $v \in V$, one obtains $v^z = v + w$ for $w \in L$ and thus $Q(v) = Q(v^z) = Q(v) + (v \mid w) + Q(w) = Q(v) + (v \mid w)$. Hence $(v \mid \omega) = 0$ and $(v \mid v^z) = (v \mid v + w) = 0$. The fixed vectors of z are contained in $L^\perp$ and all anisotropic vectors in $L^\perp \backslash L$ are anisotropic vectors as Q has minimal Witt-index. Hence statement 2 holds. ∎

Remark 5.4.   Let z be a Siegel involution. Define the automorphism $\gamma = \gamma_z$ of $GL(A)$ by $\gamma(X) = \left(X^{\hat{z}}\right)^t$, where $\hat{z}$ is the linear transformation of A with $e_x^{\hat{z}} = e_{x^z}$. It is obvious that γ normalizes $\mathbb{E} \subseteq GL(A)$, as

$$R_B^\gamma = \left(R_B^{\hat{z}}\right)^t = \left(R_B^t\right)^{\hat{z}} = (R_{B^*})^z \text{ for } R_B \in \mathbb{E}.$$

Definition 5.5.   For a Siegel involution z in W, let $\mathbb{F}_z = C_{\mathbb{E}}(\gamma_z)$ be the fixed space of γ in $\mathbb{E}$.

Theorem 5.6.   *$\mathbb{F}_z$ is a Lie algebra of dimension 52.*

Proof.   As γ is an automorphism of $GL(A)$ and $\mathbb{E}$, this implies $C_{\mathbb{E}}(\gamma)$ is a Lie algebra. Let s be an anisotropic vector fixed by z. Then z leaves $\{B_s, B_s^*\}$ invariant and thus $B_s^{\hat{z}} = B_s$ or $B_s^{\hat{z}} = B_s^*$. Assume that $B_s^{\hat{z}} = B_s$ and $x \in B_s$, it follows $x^z \in B_s$ and $0 = (x \mid x^z)$ which implies $B_s \subseteq C_V(z) = L^\perp$, a contradiction as B_s is a base of V. Hence $B_s^{\hat{z}} = B_s^*$ and $(R_{B_s})^\gamma = \left(R_{B_s}^{\hat{z}}\right)^t = \left(R_{B_s^*}\right)^t = R_{B_s}$. This means that z fixes 24 roots R_B with $s_B \in C_V(z)$. Hence

$$C_{\mathbb{E}}(\gamma) = C_{\mathbb{H}}(\gamma) \underset{R_B^\gamma = R_B}{\oplus} \langle R_B \rangle \underset{R_B^\gamma \neq R_B}{\oplus} \langle R_B + R_B^\gamma \rangle,$$

where $C_{\mathbb{H}}(\gamma) = \left\langle H_v \mid v \in L^\perp \right\rangle$ of dimension 4, it is the Cartan Lie subalgebra and $\dim(\mathbb{F}_z) = 4 + 24 + 24 = 52$. ∎

Theorem 5.7.   *The Lie algebra $\mathbb{F}_z$ is a Lie algebra of type F_4.*

Proof.   We define the Lie roots S_B in $\mathbb{F}_z$ by:

$$S_B = \begin{cases} R_B & \text{if } R_B \text{ is fixed by } \gamma, \\ R_B + R_B^\gamma & \text{if } R_B \text{ is not fixed by } \gamma, \end{cases}$$

and we show that $\mathbb{F}_z$ is closed under the Lie multiplication. It is obvious that $\mathbb{F}_z = C_{\mathbb{H}}(\gamma) \oplus \langle S_B \rangle$ and:

(1) If R_B and R_C are fixed by γ, then $[R_B, R_C] = 0$ or $R_D = B^{\gamma_c} = C^{\gamma_B}$ by Proposition 3.1.

(2) $[R_B, R_C + R_C^\gamma] = [R_B, R_C] + [R_B, R_C^\gamma] = [R_B, R_C] + [R_B^\gamma, R_C^\gamma]$
$$= [R_B, R_C] + [R_B, R_C]^\gamma.$$

(3) $[R_B + R_B^\gamma, R_C + R_C^\gamma] = [R_B, R_C] + [R_B, R_C]^\gamma + [R_B, R_C^\gamma] + [R_B, R_C^\gamma]^\gamma.$

(4) $[H_v, R_B] = (v \mid s_B) R_B$ if R_B is fixed.

(5) $[H_v, R_B + R_B^\gamma] = (H_v \mid s_B) R_B + \left(H_v \mid s_B^{\hat{z}}\right) R_B^\gamma = (H_v \mid s_B) R_B + (H_v^\gamma \mid s_B) R_B^\gamma$
$$= (H_v + H_v^r \mid s_B)(R_B + R_B^\gamma) = (H_v + H_v^\gamma \mid s_B) S_B.$$

This proves that $\mathbb{F}_z$ is closed under the Lie multiplication and it is a Lie algebra of type F_4. ∎

Definition 5.8. The Lie roots $\{R_B \mid R_B^\gamma = R_B\}$ are called long Lie roots.

Theorem 5.9. *The long Lie roots $\{R_B \mid R_B^\gamma = R_B\}$ generate a Lie algebra $\mathbb{E}_L$ of type D_4.*

Proof. Let L be a line and $\Phi_L = \left\{B \in \Phi \mid s_B \in L^\perp\right\}$. Then it suffices to prove that $\mathbb{E}_L$ is the subalgebra of $\mathbb{E}$, generated by R_B, $B \in \Phi_L$. So let $B, C \in \Phi_L$, we discuss the following two cases:

(i) If $B^* = C$, then it follows that $[R_B, R_C] = [R_B, R_C] \in \mathbb{H}$, the Cartan subalgebra of $\mathbb{E}$ generated by H_v, $v \in V$.

(ii) If $B^* \neq C$, then $[R_B, R_C] = 0$ or R_D for $D \in \Phi$, by Proposition 3.1 and $s_B + s_C \in L^\perp$, this implies $D \in \Phi_L$. Hence

$$\mathbb{E}_L = \mathbb{H}_L \underset{B \in \Phi}{\oplus} \langle R_B \rangle = C_{\mathbb{H}}(\gamma) \underset{B^\gamma = B}{\oplus} \langle R_B \rangle,$$

where $\mathbb{H}_L = \mathbb{E}_L \cap \mathbb{H} = \langle [R_B, R_B^*], B \in \Phi_L \rangle$ of dimension 4.

As $|\Phi_L| = 2\left|L^\perp \backslash L\right| = 24$, then it follows that dimension $\mathbb{E}_L$ is 28, and from the relations in $\mathbb{E}$ and F_4, we see that $\mathbb{E}_L$ is of type D_4. This completes the proof of the theorem. ∎

6. The Chevalley group of type D_4

Definition 6.1. The group $E_L = \langle U_B \mid B \in \Phi_L \rangle$ is defined to be the Chevalley group of type D_4.

Theorem 6.2. *The group E_L is isomorphic to $\Omega_8^+(K)$.*

Proof. Let $L = [V, z] \in \mathcal{L}$ where z is a Siegel involution. If $x \in \mathbb{B}$, Let $V_x = \{y \in \mathbb{B} \mid y \notin L \text{ and } (x \mid y) = 0\}$, then using the geometric properties of the generalized quadrangle $(\mathbb{B}, \mathcal{L})$, one can easily see $|V_x| = 8$, and for points $x, y \in V_x$ with $(y \mid w) = 1$, one has $y + w$ is an anisotropic vector in $L^\perp$. Conversely any anisotropic vector $s \in L^\perp$ is such a sum. Hence , let $s \in L^\perp \backslash L$, $s = w + y$ for $w, y \in V_x$ and B is a root base with $s_B = s = w + y$, $y \in B$, $w \in B^*$. So if $r = r_B(k)$, $k \in K$, is an element in E_L, then r leaves the space $\langle e_P \mid P \in V_x \rangle$ invariant and induces a Siegel involution on this space, as $e_y^r = e_y + k e_w$, $e_w^r = e_w$ as

$s \in B^*$, $e_{y+x}^r = e_{y+x}$, $e_{w+x}^r = r_{w+z} + ke_x$ and $e_u^r = e_u$ for all elements $u \in V_x$. Hence the four lines through x may be seen as $\{x, x_i, y_i\}$, $i = 1, 2, 3, 4$. Let $A_x := A(V_x)$ be the space spanned by V_x, then if $a \in A_x = \sum_{i=1}^{4} k_i e_{x_i} + \sum_{i=1}^{4} m_i e_{y_i}$, where $k_i, m_i \in K$, then

$$\hat{Q}(a) = \sum_{i=1}^{4} k_i m_i e_x = Q_x(a)e_x.$$

Hence $(A_x, \hat{Q}|_{A_x})$ is an orthogonal space of "+" type, as $A_x = \langle e_{x_1}, e_{x_1+x} \rangle \oplus \langle e_{x_2}, e_{x_2+x} \rangle \oplus \langle e_{x_3}, e_{x_3+x} \rangle \oplus \langle e_{x_4}, e_{x_4+x} \rangle$. This means that $\hat{Q}$ induces a non-degenerate quadratic form on A_x and the restriction $r|_{A_x}$ preserves $\hat{Q}$ as

$$\hat{Q}(a^r) = \hat{Q}(a)^{r^*} = Q_x(a)e_x^{r^*} = Q_x(a)e_x.$$

Moreover $r|_{A_x}$ preserves the restriction $Q|_{A_x}$ and hence induces a Siegel involution on A_x and all Siegel involutions with respect to the base $\{e_y \mid y \in V_x\}$ can thus be obtained. Hence E_L induces a group isomorphic to $\Omega_8^+(K)$ on $\langle e_y \mid y \in V_x \rangle$ and acts faithfully. Hence $E_L \cong \Omega_8^+(K)$. This completes the proof. ∎

Corollary 6.3. *Let W_L be the stabilizer in W of L. Then W_L permutes the roots $r_B, B \in \Phi_L$. This implies that W_L normalizes the group E_L and $\langle E_L, W_L \rangle = E_L W_L$ is the triality group of shape $\Omega_3^+(K)$. S_3 where W_L induces S_3 on L. (Compare with [10, Theorem 3.3].)*

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