Journal of Lie Theory
Volume **35** (2025) 191–205

Macdonald Identities: Revisited

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Communicated by R. Fioresi

To the memory of Professor Masatoshi Noumi

Abstract. After recalling a proof of the Macdonald identities for untwisted affine root systems, we derive the Macdonald identities for twisted affine root systems.

Mathematics Subject Classification: Primary 17B22; secondary 11F20, 17B67.

Key Words: Affine root system, Coxeter element, Dedekind η-function, Macdonald identity.

1. Introduction

In 1972, I. G. Macdonald [11] obtained several identities involving products of the Dedekind η-function in relation with affine root systems via some complicated combinatorial method. Indeed, he first classified *real* affine root systems and tried finding a generalization of the denominator identity for affine root systems, that is, an identity of the form infinite product $=$ infinite sum, where the product is taken over roots in an affine root system and the sum is taken over the Weyl group of the same affine root system. He found that it requires some '*extra factors*' in the product side. Later, V. Kac [6], R. Moody [13] and H. Garland and J. Lepowsky [5] independently showed that these 'extra factors' can be interpreted as the factors related with the *imaginary* roots. Starting from the denominator identity for each affine root system, via specialization, Macdonald obtained the so-called Macdonald identities. There are other known proofs of the Macdonald identities. For example, these identities, except for $BC_l^{(2)}$, were reproved in 1976 via some analysis on modular forms by Van Asch [16], and H. D. Fegan [2] in 1978 reproved them only for untwisted case via some analysis of the heat equation on compact simple Lie groups. The latter work relies on the result of Kostant [10] where he rewrote the Macdonald identities in such a way as to involve the heat kernel on the corresponding compact Lie group. The reader may realize the richness of the subject.

In this note, after reviewing several facts, including the proof of Macdonald identities for affine root systems of untwisted type, we present another proof of the Macdonald identity associated with a twisted affine root system, not of type $BC_l^{(2)}$, which uses unusual folding considered by K. Saito in [15] and some facts from B. Kostant's beautiful paper [9]. For $BC_l^{(2)}$ case, we present one intermediate step and state the

Research of Y. S. is supported by JSPS KAKENHI grant number JP20K03568. We are grateful to an anonymous referee for his/her careful reading and suggestions.

ISSN 0949–5932 / \$2.50

final result rapidly, as this case is too special and we believe that the reader who followed other cases may find no difficulty to fill the detail. Thus, most of this note is expository and our only modest contribution is §3.3.

Throughout this note, we don't fix any normalization of the metric, for which there is a price to pay in its presentation.

2. Affine root systems

Here, we recall some basic facts about finite and affine root systems, in particular, two types of folding procedure. We adopt the formulations in [15] for later purpose.

2.1. Generalized root system

Let F be a real vector space and $I : F \times F \to \mathbb{R}$ be a symmetric bilinear form whose signature is (l_+, l_0, l_-) for some non negative integers l_+, l_0, l_- such that not all of them are zero. As usual, for any non-isotropic vector $\alpha \in F$, we set

$$\alpha^\vee = \frac{2}{I(\alpha,\alpha)}\alpha \in F,$$

and define an isometry $s_\alpha \in \mathrm{O}(F, I)$ by $s_\alpha(\lambda) = \lambda - I(\lambda, \alpha^\vee)\alpha$.

Definition 2.1. A non-empty discrete subset R of F is called a *generalized root system* if it satisfies

(1) The lattice $Q(R)$ (called the *root lattice*), spanned by the elements of R, is full in F, i.e., $\mathbb{R} \otimes_{\mathbb{Z}} Q(R) \cong F$.
(2) For any $\alpha \in R$, one has $I(\alpha, \alpha) \neq 0$.
(3) For any $\alpha, \beta \in R$, one has $I(\alpha^\vee, \beta) \in \mathbb{Z}$.
(4) For any $\alpha \in R$, one has $s_\alpha(R) = R$.
(5) Assume that there exists two subsets R_1, R_2 of R which are orthogonal and $R_1 \cup R_2 = R$, then either R_1 or R_2 is empty.

The subgroup of $\mathrm{O}(F, I)$ generated by the reflections s_α $(\alpha \in R)$ is called the *Weyl group* of R and is denoted by $W(R)$.

A finite root system R_f is a generalized root system belonging to F_f with a positive definite metric I_f, and an *affine root system* R_{af} is a generalized root system belonging to F, containing F_f as a subspace of codimension 1, equipped with a positive semidefinite symmetric bilinear form I whose restriction to $F_f \times F_f$ is I_f. The image of R_{af} via the canonical projection $F \twoheadrightarrow F/\mathrm{rad}(I) \cong F_f$, denoted by $R_{af}/\mathrm{rad}(I)$, is called a quotient root system which can be identified with a finite root system R_f belonging to (F_f, I_f).

For a finite root system R_f of reduced type, let $(R_f)_s$ and $(R_f)_l$ be the set of short and long roots, respectively. Here and after, we fix a set of simple roots $\Pi_f = \{\alpha_i\}_{1 \leq i \leq l}$ of R_f, thus the set R_f^+ of positive roots of R_f. We also set $(R_f^+)_* = (R_f)_* \cap R_f^+$ $(* \in \{s, l\})$.

For R_f of type BC_l, the set of roots of middle length is denoted by $(R_f)_m$. We fix a set of simple roots $\Pi_f = \{\alpha_i\}_{1 \leq i \leq l}$ of R_f, hence of $(R_f)_s \cup (R_f)_m$ which is of type B_l, thus the set R_f^+ of positive roots of R_f. We assume that $\alpha_l \in (R_f)_s$.

2.2. Affine root systems

Let R_{af} be a reduced affine root system belonging to (F, I), and R_f be the finite root system of type X_l belonging to (F_f, I_f) which is isomorphic to the quotient $R_{af}/\mathrm{rad}(I)$.

Let $\delta \in \mathrm{rad}(I)$ be a generator of the lattice $\mathrm{rad}(I) \cap Q(R_{af})$. It is known (cf. [8] and/or [14]) that if

1. R_f is of type A_l $(l \geq 1)$, D_l $(l \geq 4)$ or E_6, E_7 or E_8, R_{af} is of type $X_l^{(1)}$ defined by $R_{af} = R_f + \mathbb{Z}\delta$, if
2. R_f is of type B_l, C_l, F_4 or G_2, there exists $t \in \{1, 2\}$ for R_f of type B_l, C_l or F_4 and $t \in \{1, 3\}$ for R_f of type G_2 such that R_{af} is of type $X_l^{(t)}$ defined by $R_{af} = \big((R_f)_s + \mathbb{Z}\delta\big) \cup \big((R_f)_l + t\mathbb{Z}\delta\big)$, otherwise
3. R_f is of type BC_l and R_{af} is of type $BC_l^{(2)}$ defined by

$$R_{af} = \big((R_f)_s + \mathbb{Z}\delta\big) \cup \big((R_f)_m + \mathbb{Z}\delta\big) \cup \big((R_f)_l + (1 + 2\mathbb{Z})\delta\big).$$

An affine root system of type $X_l^{(t)}$ is called *untwisted* type if $t = 1$ and *twisted* type otherwise. The number t is called the *tier number*.

Here and after, we use the nomenclature for the reduced affine root systems due to K. Saito [15], since they reflect the structure of root systems themselves more than those of V. Kac [8].

Remark 2.2. There are several other known nomenclatures for reduced affine root systems. For the sake of reader's convenience, we clarify the relationships between them in the following table:

K. Saito ([15])	$X_l^{(1)}$	$B_l^{(2)}$	$C_l^{(2)}$	$F_4^{(2)}$	$G_2^{(3)}$	$BC_l^{(2)}$
V. Kac ([8])	$X_l^{(1)}$	$D_{l+1}^{(2)}$ $l \geq 2$	$A_{2l-1}^{(2)}$ $l \geq 3$	$E_6^{(2)}$	$D_4^{(3)}$	$A_{2l}^{(2)}$
R. Moody ([12])	$X_{l,1}$	$B_{l,2}$	$C_{l,2}$	$F_{4,2}$	$G_{2,3}$	$A_{1,2}$ $l=1$ $BC_{l,2}$ $l>1$
I. Macdonald ([11])	$X_l = X_l^\vee$ X: of type ADE X_l X: of type $BCFG$	$C_l^\vee$	$B_l^\vee$	$F_4^\vee$	$G_2^\vee$	$BC_l = BC_l^\vee$
R. Carter ([1])	$\widetilde{X}_l$	$\widetilde{C}_l^{\mathrm{t}}$	$\widetilde{B}_l^{\mathrm{t}}$	$\widetilde{F}_4^{\mathrm{t}}$	$\widetilde{G}_2^{\mathrm{t}}$	$\widetilde{C}_l'$

In the rest of this subsection, we assume that the affine root system R_{af} is of type $X_l^{(t)}$.

Set
$$\alpha_0 = \begin{cases} \delta - \theta & t = 1, \\ \delta - \theta_s & t \neq 1 \text{ and not of type } BC_l^{(2)}, \\ \delta - 2\theta_s & R_{af} \text{ of type } BC_l^{(2)}, \end{cases}$$

where $\theta \in R_f$ and $\theta_s \in (R_f)_s$ are highest roots. Then, the set

$$\Pi_{af} := \{\alpha_0\} \cup \Pi_f$$

is a set of simple roots, and the matrix $A = \big(I(\alpha_i^\vee, \alpha_j)\big)_{0 \leq i,j \leq l}$ is called the *generalized Cartan matrix*.

Let $a_i,\ a_i^\vee\ (0\leq i\leq l)$ be the positive integers such that

(1) the integers $a_0,a_1,\dots,a_l$ (resp. $a_0^\vee,a_1^\vee,\dots,a_l^\vee$), called *labels* (resp. *colabels*), are coprime to each other and

(2) they satisfy $A\begin{pmatrix}a_0\\a_1\\\vdots\\a_l\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ and $\begin{pmatrix}a_0^\vee & a_1^\vee & \cdots & a_l^\vee\end{pmatrix}A=\begin{pmatrix}0&0&\cdots&0\end{pmatrix}$.

The numbers $h(X_l^{(t)}):=\sum_{i=0}^l a_i$ and $h^\vee(X_l^{(t)}):=\sum_{i=0}^l a_i^\vee$ are called *Coxeter* and *dual Coxeter* number, respectively.

Remark 2.3. (1) For R_{af}, not of type $BC_l^{(2)}$,

(i) $a_0=a_0^\vee=1$ and the vector $\sum_{i=1}^l a_i\alpha_i$ is equal to θ if $t=1$ and to θ_s otherwise.

(ii) For $t=1$, the number $h(X_l^{(1)})$ coincides with the *Coxeter number* $h(X_l)$ of the finite root system of type X_l, i.e., the order of a Coxeter element.

(2) For R_{af} of type $BC_l^{(2)}$,

(i) $a_0=1,\ a_l=2$ and $a_0^\vee=2,\ a_l^\vee=1$.

2.3. Folding and mean-folding

Here we recall two types of folding procedures introduced by K. Saito [15]. Let R be a finite or an affine reduced root system belonging to the pair (F,I) of a real vector space with a symmetric bilinear form I, and let Γ_R be its Dynkin diagram, whose nodes are identified with simple roots. Assume that the oriented graph Γ_R admits a non-trivial finite order automorphism σ which fixes at least one node of Γ_R. Remark that the map σ is a restriction of an isometry of (F,I), which we denote by the same symbol σ. Let $\langle\sigma\rangle$ be the group generated by σ. Let $\mathrm{Tr}^{\langle\sigma\rangle}$ and $\mathrm{Tr}_{\langle\sigma\rangle}$ be the linear maps from F to its fixed point subspace $F^{\langle\sigma\rangle}$ defined by

$$\mathrm{Tr}^{\langle\sigma\rangle}(x)=\sum_{h\in\langle\sigma\rangle}h.x,\qquad \mathrm{Tr}_{\langle\sigma\rangle}(x)=\frac{1}{|\langle\sigma\rangle|}\sum_{h\in\langle\sigma\rangle}h.x.$$

The sets $\mathrm{Tr}^{\langle\sigma\rangle}(R)$ and $\mathrm{Tr}_{\langle\sigma\rangle}(R)$ are again finite or affine root systems, called the *folding* and *mean-folding* of R, respectively. Their Dynkin diagrams can be obtained by "folding" appropriately, with respect to σ, the Dynkin diagram of Γ_R.

Remark 2.4. (1) Usually, the folding of R is defined as the image of R in the $\langle\sigma\rangle$*-coinvariant* of F, and such a operation corresponds to the mean-folding $\mathrm{Tr}_{\langle\sigma\rangle}$.

(2) This automorphism σ is a restriction of an isometry of F, which also acts on the dual root system $R^\vee$ and induces an action on its Dynkin diagram $\Gamma_{R^\vee}$ denoted by $\sigma^\vee$. Two foldings are related as follows:

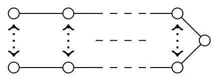
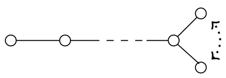
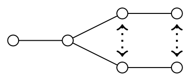
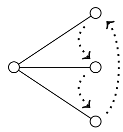
$$\mathrm{Tr}^{\langle\sigma^\vee\rangle}(R^\vee)=\Big(\mathrm{Tr}_{\langle\sigma\rangle}(R)\Big)^\vee\qquad \mathrm{Tr}_{\langle\sigma^\vee\rangle}(R^\vee)=\Big(\mathrm{Tr}^{\langle\sigma\rangle}(R)\Big)^\vee.$$

When a non-trivial diagram automorphism σ_{af} of an affine root system R_{af} can be obtained by extending trivially a diagram automorphism σ_f of the finite root system $R_f \cong R_{af}/\mathrm{rad}(I)$, namely,

$$\sigma_{af}(\alpha_i) := \begin{cases} \alpha_i & i = 0, \\ \sigma_f(\alpha_i) & 1 \le i \le l, \end{cases}$$

the (mean-)folding of (R_f, σ_f) and (R_{af}, σ_{af}) are given in the following table: set $H_f = \langle \sigma_f \rangle$ and $H_{af} = \langle \sigma_{af} \rangle$.

Table 1: (Mean-)Folding

Type of R_f	A_{2l-1}	D_{l+1}	E_6	D_4
σ_f				
Type of $\mathrm{Tr}^{H_f}(R_f)$	B_l	C_l	F_4	G_2
Type of $\mathrm{Tr}_{H_f}(R_f)$	C_l	B_l	F_4	G_2
Type of $\mathrm{Tr}^{H_{af}}(R_{af})$	$B_l^{(2)}$	$C_l^{(2)}$	$F_4^{(2)}$	$G_2^{(3)}$
Type of $\mathrm{Tr}_{H_{af}}(R_{af})$	$C_l^{(1)}$	$B_l^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$

We will use the fact that any twisted affine root system, except for $BC_l^{(2)}$, can be obtained as a folding of an ADE-type affine root system.

Remark 2.5. Indeed, one can realize the root system R_{af} of type $BC_l^{(2)}$ as follows. Let σ_{af} be a diagram automorphism $D_{2l+2}^{(1)}$ of order 4. Then,

$$\mathrm{Tr}^{\langle \sigma_{af} \rangle}(R(D_{2l+2}^{(1)})) \cong \mathrm{Tr}_{\langle \sigma_{af} \rangle}(R(D_{2l+2}^{(1)})) \cong R(BC_l^{(2)}).$$

This automorphism σ_{af}, admitting exactly one fixed point, permutes the 4 extremal nodes and hence is not of type described above.

2.4. Affine Weyl groups

Here we extend the pair (F, I) considered in §2.2 to the real vector space $\widehat{F}$ by adding the dual of $\mathrm{rad}(I)$; $\widehat{F} := F \oplus \mathrm{rad}(I)^*$ and a symmetric bilinear form $\hat{I}$ such that $\big(\mathrm{rad}(I)^*\big)^\perp = F_f \oplus \mathrm{rad}(I)^*$ and its restriction to $\mathrm{rad}(I) \times \mathrm{rad}(I)^*$ is the dual pairing. A node α_i of the Dynkin diagram of R_{af} is called *special index* if $\delta - a_i\alpha_i \in R_{af}$.

Let $\widehat{W}(R_{af})$ be the subgroup of $\mathrm{O}(\widehat{F}, \hat{I})$ generated by reflections $\hat{s}_\alpha$ defined by

$$\hat{s}_\alpha(\lambda) = \lambda - \hat{I}(\lambda, \alpha^\vee)\alpha, \quad \alpha \in R_{af} \subset F \subset \widehat{F}, \quad \lambda \in \widehat{F}.$$

It is clear that the finite subgroup generated by $\hat{s}_\alpha$ $(\alpha \in R_f)$ is isomorphic to the finite Weyl group $W(R_f)$. For a non-isotropic vector $\gamma \in F \subset \widehat{F}$, define $t_\gamma \in \mathrm{O}(\widehat{F}, \hat{I})$ (cf. [8, (6.5.2)]) by

$$t_\gamma(\lambda) = \lambda + \hat{I}(\lambda, \delta)\gamma - \Big(\hat{I}(\lambda, \gamma) + \frac{1}{2}\hat{I}(\gamma, \gamma)\hat{I}(\lambda, \delta)\Big)\delta.$$

By direct computation, it can be verified that

1. $wt_\gamma w^{-1} = t_{w(\gamma)}$ for any $w \in W(R_f)$, and
2. $t_{\gamma_1} t_{\gamma_2} = t_{\gamma_1+\gamma_2}$ for any $\gamma_1, \gamma_2 \in F$.

First, we consider the affine root systems R_{af}, where α_0 is a speical index, i.e., R_{af} is not of type $BC_l^{(2)}$. In this case, we describe the group $\widehat{W}(R_{af})$ under the splitting $F = F_f \oplus \mathrm{rad}(I)$.

For an untwisted R_{af}, since $\hat{s}_{\alpha_0}\hat{s}_{\delta-\alpha_0} = t_{\theta^\vee}$ and $\theta \in R_f$ is a long root, it follows that $\theta^\vee \in (R_f^\vee)_s$ and its $W(R_f)$-orbit generates the coroot lattice $Q(R_f^\vee)$. Thus we have

$$\widehat{W}(R_{af}) \cong W(R_f) \ltimes Q(R_f^\vee). \tag{1}$$

Likewise, for a twisted R_{af}, not of type $BC_l^{(2)}$, as $\hat{s}_{\alpha_0}\hat{s}_{\delta-\alpha_0} = t_{\theta_s^\vee}$ and $\theta_s \in R_f$ is a short root, it follows that the $W(R_f)$-orbit of $\theta_s^\vee = (2/\hat{I}(\theta_s,\theta_s)\,\theta_s$ generate the lattice $(2/\hat{I}(\theta_s,\theta_s))\,Q(R_f)$, i.e.,

$$\widehat{W}(R_{af}) \cong W(R_f) \ltimes \Big(\frac{2}{\hat{I}(\theta_s,\theta_s)}Q(R_f)\Big). \tag{2}$$

Second, we consider the affine root systems R_{af} of type $BC_l^{(2)}$. In this case, as the unique simple short root α_l is the only special index, we shall set $\Pi_f' = \{\alpha_i\}_{0\le i\le l-1}$, $R_f' = R_{af} \cap \mathbb{Z}\Pi_f' \cong R(C_l)$ and $F_f' = \bigoplus_{i=0}^{l-1}\mathbb{R}\alpha_i$, and describe the group $\widehat{W}(R_{af})$ under the splitting $F = F_f' \oplus \mathrm{rad}(I)$.

As $\hat{s}_{\alpha_l}\hat{s}_{\delta-2\alpha_l} = t_{\theta^\vee}$, where $\theta \in R_f'$ is a highest root hence is a long root, it follows that $\theta^\vee \in (R_f')_s^\vee$ and its $W(R_f')$-orbit generates the coroot lattice $Q((R_f')^\vee)$. Thus, similarly to (1), we have

$$\widehat{W}(R_{af}) \cong W(R_f') \ltimes Q\big((R_f')^\vee\big). \tag{3}$$

Remark 2.6. It follows from the above isomorphisms that the Weyl group of R_{af}, which is a subgroup of $\mathrm{O}(F,I)$, is isomorphic to $\widehat{W}(R_{af})$.

Let M be the lattice in the right hand side of above isomorphism $\widehat{W}(R_{af}) \cong W_f \ltimes M$. The above isomorphism is clearly given by associating $(u,\gamma) \in W_f \times M$ with $ut_\gamma \in \widehat{W}(R_{af})$. It is clear that

$$\det{}_{\widehat{F}}(ut_\gamma) = \det{}_{\widehat{F}}(u) = \det{}_{F_f}(u),$$

i.e., the determinant of an element ut_γ of $\widehat{W}(R_{af})$ depends only on its finite part.

3. The Macdonald identities

In this section, after recalling the denominator identity for finite and affine root systems and the Macdonald identity for untwisted affine root systems, we prove the Macdonald identity for twisted affine root systems, which is not of type $BC_l^{(2)}$, in a unified manner. The $BC_l^{(2)}$ case will be treated separately. We often use the symbols introduced in the previous section without reintroducing them.

3.1. Denominator identity

Here, we recall the denominator identity for finite and affine root systems, separately. First we recall the so-called *Weyl vector*. Let R_f be a reduced finite root system belonging to (F_f, I_f), $\Pi_f = \{\alpha_i\}_{1\le i\le l} \subset R_f$ the set of simple roots of R_f and R_f^+ the set of positive roots. Define the Weyl vector $\rho_f \in F_f$ by

$$\rho_f = \frac{1}{2}\sum_{\alpha\in R_f^+} \alpha.$$

It can be checked that $I_f(\rho_f, \alpha_i^\vee) = 1$ for any $1 \le i \le l$. For an affine root system R_{af} belonging to (F, I) whose quotient root system $R_{af}/\mathrm{rad}(I)$ is isomorphic to R_f (not of type BC_l), let Π_{af} be the set of simple roots recalled in §2.2. Regard R_{af} as a subset of $\widehat{F}$. For $0 \le i \le l$, let $\Lambda_i \in F_f \oplus \mathrm{rad}(I)^*$ be the vector satisfying $\hat{I}(\Lambda_i, \alpha_j^\vee) = \delta_{i,j}$. The Weyl vector $\rho \in \widehat{F}$ is defined by

$$\rho = \sum_{i=0}^{l} \Lambda_i.$$

It appears that $\rho - \rho_f \in \mathrm{rad}(I)^*$. Notice that this construction depends on the splitting $F = F_f \oplus \mathrm{rad}(I)$.

For the case when R_{af} is of type $BC_l^{(2)}$, we use another splitting $F = F_f' \oplus \mathrm{rad}(I)$ defined in §2.4. Let Π_{af} be the set of simple roots recalled in §2.2. For $0 \le i \le l$, let $\Lambda_i' \in F_f' \oplus \mathrm{rad}(I)^*$ be the vector satisfying $\hat{I}(\Lambda_i', \alpha_j^\vee) = \delta_{i,j}$. The Weyl vector $\rho \in \widehat{F}$ is defined by

$$\rho = \sum_{i=0}^{l} \Lambda_i'.$$

Recall the subset Π_f' of Π_{af} defined in §2.4. The Weyl vector ρ_f' of R_f' satisfies $\rho - \rho_f' \in \mathrm{rad}(I)^*$.

Second, we recall a certain algebra $\mathcal{E}$ over $\mathbb{C}$. For $\lambda \in \widehat{F}$, set

$$D(\lambda) = \lambda - \mathbb{Z}_{\ge 0}\Pi_f = \Big\{\, \mu \in \widehat{F} \,\Big|\, \exists\, m_i \in \mathbb{Z}_{\ge 0} \text{ s.t. } \lambda - \mu = \sum_{i=0}^{l} m_i\alpha_i. \,\Big\}.$$

The algebra $\mathcal{E}$ consists of the elements of the form $\sum_{\lambda\in\widehat{F}} c_\lambda e^\lambda$ such that there exists a finite number of vectors $\mu_1, \mu_2, \cdots, \mu_r \in \widehat{F}$ satisfying

$$\Big\{\, \lambda \in \widehat{F} \,\Big|\, c_\lambda \ne 0 \,\Big\} \subset \bigcup_{i=1}^{r} D(\mu_i).$$

The algebra structure on $\mathcal{E}$ is defined by $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$.

Now, we recall the so-called denominator identity. The denominator identity of a finite root system R_f is

$$e^{\rho_f} \prod_{\alpha\in R_f^+} \big(1 - e^{-\alpha}\big) = \sum_{u\in W(R_f)} \det{}_{F_f}(u) e^{u(\rho_f)}. \tag{4}$$

Notice that the left hand side can be rewritten as follows:

$$e^{\rho_f} \prod_{\alpha\in R_f^+} (1 - e^{-\alpha}) = \prod_{\alpha\in R_f^+} \Big(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\Big).$$

For an affine root system R_{af} of untwisted type, it is

$$\begin{aligned} &e^{\rho-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}\prod_{\alpha\in R_f^+}(1-e^{-\alpha})\prod_{n=1}^{\infty}\left((1-e^{-n\delta})^l\prod_{\alpha\in R_f}(1-e^{-\alpha-n\delta})\right)\\ &=\sum_{w\in\widehat{W}(R_{af})}\det{}_{\widehat{F}}(w)e^{w(\rho)-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}. \end{aligned}\tag{5}$$

For a twisted affine root system R_{af} not of type $BC_l^{(2)}$,

$$\begin{aligned} e^{\rho-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}&\prod_{\alpha\in R_f^+}(1-e^{-\alpha})\prod_{n=1}^{\infty}\Bigg((1-e^{-n\delta})^{|\Pi_{f,s}|}(1-e^{-tn\delta})^{|\Pi_{f,l}|}\\ &\qquad\times\prod_{\alpha\in(R_f)_s}(1-e^{-\alpha-n\delta})\prod_{\alpha\in(R_f)_l}(1-e^{-\alpha-tn\delta})\Bigg)\\ =\sum_{w\in\widehat{W}(R_{af})}&\det{}_{\widehat{F}}(w)e^{w(\rho)-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}, \end{aligned}\tag{6}$$

where we set $\Pi_{f,s}=\Pi_f\cap(R_f)_s$ and $\Pi_{f,l}=\Pi_f\cap(R_f)_l$. Here, the right hand side is a $\widehat{W}(R_{af})$ alternating sum of $e^{\rho-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}$, instead of e^ρ. This shift is made so that the vector

$$\rho':=\rho-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta$$

satisfies $\hat{I}(\rho',\rho')=0$. In this way, one may expect that the right hand side of (5) and (6) can be expressed in terms of an alternating sum involving θ constants which is the case as we will see in the next subsections.

3.2. Macdonald identity I: untwisted case

Here, we recall how one can derive the Macdonald identity for an untwisted affine root system R_{af} of type $X_l^{(1)}$.

First, we divide the both side of (5) by $e^\rho\prod_{\alpha\in R_f^+}(1-e^{-\alpha})$. Its left hand side simply becomes

$$e^{-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}\prod_{n=1}^{\infty}\left((1-e^{-n\delta})^l\prod_{\alpha\in R_f}(1-e^{-\alpha-n\delta})\right).$$

As for its right hand side, a direct computation shows

$$\begin{aligned} ut_\gamma(\rho)-\rho-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta&=u\big(\rho+\hat{I}(\rho,\delta)\gamma\big)-\rho-\frac{1}{2\hat{I}(\rho,\delta)}\cdot\hat{I}\big(\rho+\hat{I}(\rho,\delta)\gamma,\rho+\hat{I}(\rho,\delta)\gamma\big)\delta\\ &=u\big(\rho_f+\hat{I}(\rho,\delta)\gamma\big)-\rho_f-\frac{1}{2\hat{I}(\rho,\delta)}\cdot\hat{I}\big(\rho+\hat{I}(\rho,\delta)\gamma,\rho+\hat{I}(\rho,\delta)\gamma\big)\delta, \end{aligned}$$

for $u\in W(R_f)$ and $\gamma\in Q(R_f^\vee)$, by (1), it becomes

$$\sum_{\gamma\in Q(R_f^\vee)}\left(\frac{\sum_{u\in W(R_f)}\det_{F_f}(u)e^{u\left(\rho_f+\hat{I}(\rho,\delta)\gamma\right)-\rho_f}}{\prod_{\alpha\in R_f^+}(1-e^{-\alpha})}\right)e^{-\frac{1}{2\hat{I}(\rho,\delta)}\cdot I_f\left(\rho+\hat{I}(\rho,\delta)\gamma,\rho+\hat{I}(\rho,\delta)\gamma\right)\delta}.$$

Hence, we obtain

$$e^{-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}\prod_{n=1}^{\infty}\left(\left(1-e^{-n\delta}\right)^{l}\prod_{\alpha\in R_f}\left(1-e^{-\alpha-n\delta}\right)\right)$$
$$=\sum_{\gamma\in\hat{I}(\rho,\delta)Q(R_f^{\vee})}\left(\frac{\sum_{u\in W(R_f)}\det_{F_f}(u)e^{u(\rho_f+\gamma)}}{e^{\rho_f}\prod_{\alpha\in R_f^+}\left(1-e^{-\alpha}\right)}\right)e^{-\frac{1}{2\hat{I}(\rho,\delta)}\cdot I_f(\rho+\gamma,\rho+\gamma)\delta}. \tag{7}$$

Second, we consider the specialization $e^{\alpha_i}\mapsto 1$ $(1\le i\le l)$. For this purpose, let us calculate $\hat{I}(\rho,\delta)$ and $\hat{I}(\rho,\rho)$. By definition, we have

$$\hat{I}(\rho,\delta)=\hat{I}(\rho,\alpha_0)+\hat{I}(\rho,\theta)=\frac{\hat{I}(\theta,\theta)}{2}\left(1+\sum_{i=1}^{l}a_i^{\vee}\right)=\frac{\hat{I}(\theta,\theta)}{2}h^{\vee}(X_l^{(1)}), \tag{8}$$

from which it follows from Freudenthal de Vries' strange formula (cf. §47.10 of [4], [3] or Exercise 6.4 of [14]) that

$$\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}=\frac{2}{I_f(\theta,\theta)}\cdot\frac{I_f(\rho_f,\rho_f)}{2h^{\vee}(X_l^{(1)})}=\frac{\dim\mathfrak{g}(R_f)}{24},$$

where $\mathfrak{g}(R_f)$ is the simple Lie algebra of type X_l. Thus, the left hand side of (7) becomes

$$e^{-\frac{\dim\mathfrak{g}(R_f)}{24}\delta}\prod_{n=1}^{\infty}\left(1-e^{-n\delta}\right)^{l+|R_f|}=\left(e^{-\frac{1}{24}\delta}\prod_{n=1}^{\infty}\left(1-e^{-n\delta}\right)\right)^{\dim\mathfrak{g}(R_f)}=\eta\left(e^{-\delta}\right)^{\dim\mathfrak{g}(R_f)},$$

where $\eta(q)$ is the Dedekind eta function: $\eta(q)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ (usually, viewed as a function on τ in the upper half plane, where $q=e^{2\pi i\tau}$). Recalled that the *weight lattice* of R_f is defined by

$$P(R_f)=\left\{\lambda\in F_f\,\middle|\,I_f(\lambda,\alpha_i^{\vee})\in\mathbb{Z}\ \ 1\le\forall i\le l\right\}.$$

For the right hand side, since $Q(R_f^{\vee})$ is a sublattice of $\frac{2}{I_f(\theta,\theta)}Q(R_f)\subset F_f$, it follows from (8) that $\hat{I}(\rho,\delta)Q(R_f^{\vee})\subset h^{\vee}(X_l^{(1)})Q(R_f)$. Hence, it is sufficient to compute the specialization $e^{\alpha_i}\mapsto 1$ $(1\le i\le l)$ of

$$\frac{\sum_{u\in W(R_f)}\det_{F_f}(u)e^{u(\rho_f+\lambda)}}{e^{\rho_f}\prod_{\alpha\in R_f^+}\left(1-e^{-\alpha}\right)}\in\bigoplus_{\mu\in P(R_f)}\mathbb{R}e^{\mu},$$

for any $\lambda\in P(R_f)$. Regard e^{μ} $(\mu\in P(R_f))$ as a function on F_f defined by $e^{\mu}(x)=e^{I_f(\mu,x)}$, and evaluate the above fraction at $t\rho_f$ $(t\in\mathbb{R})$. As

$$\begin{aligned}&\left(\sum_{u\in W(R_f)}\det{}_{F_f}(u)e^{u(\lambda+\rho_f)}\right)(t\rho_f)\\&=\sum_{u\in W(R_f)}\det{}_{F_f}(u)e^{I_f\left(u(\lambda+\rho_f),t\rho_f\right)}=\sum_{u\in W(R_f)}\det{}_{F_f}(u^{-1})e^{I_f\left(u^{-1}(\rho_f),t(\lambda+\rho_f)\right)}\\&=\left(\sum_{u\in W(R_f)}\det{}_{F_f}(u)e^{u(\rho_f)}\right)\left(t(\lambda+\rho_f)\right),\end{aligned}$$

it follows from (4) that

$$\left(\frac{\sum_{u\in W(R_f)}\det_{F_f}(u)e^{u(\rho_f+\lambda)}}{e^{\rho_f}\prod_{\alpha\in R_f^+}(1-e^{-\alpha})}\right)(t\rho_f)=\prod_{\alpha\in R_f^+}\left(\frac{e^{\frac{t}{2}I_f(\lambda+\rho_f,\alpha)}-e^{-\frac{t}{2}I_f(\lambda+\rho_f,\alpha)}}{e^{\frac{t}{2}I_f(\rho_f,\alpha)}-e^{-\frac{t}{2}I_f(\rho_f,\alpha)}}\right)$$
$$\xrightarrow{t\to 0}\prod_{\alpha\in R_f^+}\frac{I_f(\lambda+\rho_f,\alpha)}{I(\rho_f,\alpha)}.$$

Setting
$$d_{X_l}(\lambda)=\prod_{\alpha\in R_f^+}\frac{I_f(\lambda+\rho_f,\alpha)}{I(\rho_f,\alpha)}$$

for $\lambda\in P(R_f)$, we see that the right hand side of (7) becomes

$$\sum_{\gamma\in\hat{I}(\rho,\delta)Q(R_f^\vee)}d_{X_l}(\gamma)e^{-\frac{I_f(\rho_f+\gamma,\rho_f+\gamma)}{2h^\vee(X_l^{(1)})}\cdot\frac{2}{I_f(\theta,\theta)}\delta}.$$

Thus, we obtain the so-called Macdonald identity:

Theorem 3.1 (cf. Theorem 8.7 in [11])**.** *Let R_f be a finite root system of type X_l and R_{af} be an affine root system of type $X_l^{(1)}$.*

$$\eta\left(e^{-\delta}\right)^{\dim\mathfrak{g}(R_f)}=\sum_{\gamma\in\hat{I}(\rho,\delta)Q(R_f^\vee)}d_{X_l}(\gamma)e^{-\frac{I_f(\rho_f+\gamma,\rho_f+\gamma)}{2h^\vee(X_l^{(1)})}\cdot\frac{2}{I_f(\theta,\theta)}\delta}.\qquad(9)$$

Notice that $\dim\mathfrak{g}(R_f)=(h(X_l)+1)\left|\Pi_f\right|$ by Corollary 6.8 and Theorem 8.4 of [9].

3.3. Macdonald identity II: twisted case

Here, we basically follow the same idea as in the last subsection to obtain the Macdonald identity for a twisted affine root system $R_{af}=R(X_l^{(t)})$ not of type $BC_l^{(2)}$. First, we divide the both side of (6) by $e^\rho\prod_{\alpha\in R_f^+}(1-e^{-\alpha})$. Its left hand side simply becomes

$$e^{-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}\prod_{n=1}^\infty\left(\left(1-e^{-n\delta}\right)^{|\Pi_{f,s}|}\left(1-e^{-tn\delta}\right)^{|\Pi_{f,l}|}\prod_{\alpha\in(R_f)_s}\left(1-e^{-\alpha-n\delta}\right)\prod_{\alpha\in(R_f)_l}\left(1-e^{-\alpha-tn\delta}\right)\right).$$

As for its right hand side, by a similar computation in the last subsection together with (2), it becomes

$$\sum_{\gamma\in\frac{2}{\hat{I}(\theta_s,\theta_s)}Q(R_f)}\left(\frac{\sum_{u\in W(R_f)}\det_{F_f}(u)e^{u\left(\rho_f+\hat{I}(\rho,\delta)\gamma\right)-\rho_f}}{\prod_{\alpha\in R_f^+}(1-e^{-\alpha})}\right)e^{-\frac{1}{2\hat{I}(\rho,\delta)}\cdot I_f\left(\rho+\hat{I}(\rho,\delta)\gamma,\rho+\hat{I}(\rho,\delta)\gamma\right)\delta}.$$

Hence, we obtain

$$e^{-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}\prod_{n=1}^{\infty}\left(\left(1-e^{-n\delta}\right)^{|\Pi_{f,s}|}\left(1-e^{-tn\delta}\right)^{|\Pi_{f,l}|}\prod_{\alpha\in(R_f)_s}\left(1-e^{-\alpha-n\delta}\right)\prod_{\alpha\in(R_f)_l}\left(1-e^{-\alpha-tn\delta}\right)\right)$$

$$=\sum_{\gamma\in 2\frac{\hat{I}(\rho,\delta)}{\hat{I}(\theta_s,\theta_s)}Q(R_f)}\left(\frac{\sum_{u\in W(R_f)}\det_{F_f}(u)e^{u(\rho_f+\gamma)}}{e^{\rho_f}\prod_{\alpha\in R_f^+}\left(1-e^{-\alpha}\right)}\right)e^{-\frac{1}{2\hat{I}(\rho,\delta)}\cdot I_f(\rho+\gamma,\rho+\gamma)\delta}. \tag{10}$$

Second, we consider the specialization $e^{\alpha_i}\mapsto 1$ $(1\le i\le l)$. For this purpose, let us calculate $\hat{I}(\rho,\rho)$, and $\hat{I}(\rho,\delta)$. A similar computation to (8) yields

$$\hat{I}(\rho,\delta)=\frac{\hat{I}_f(\theta_s,\theta_s)}{2}h^\vee(X_l^{(t)}), \tag{11}$$

In particular, we have $\displaystyle \frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}=\frac{2}{I_f(\theta_s,\theta_s)}\cdot\frac{I_f(\rho_f,\rho_f)}{2h^\vee(X_l^{(t)})}.$

Let us compute $\displaystyle \frac{I_f(\rho_f,\rho_f)}{2h^\vee(X_l^{(t)})}.$

For this purpose, we realize R_f and R_{af} as the image of a folding, explained in §2.3. Let $\widetilde{F}_f$ be a real vector space, containing F_f as a subspace, equipped with positive definite metric $\tilde{I}_f$ whose restriction to $F_f\times F_f$ gives I_f. Let $\widetilde{R}_f$ be the finite root system belonging to $(\widetilde{F}_f,\tilde{I}_f)$ with a set of simple roots $\widetilde{\Pi}_f$ whose Dynkin diagram admits a non-trivial automorphism $\tilde{\sigma}_f$ such that one has $\widetilde{F}_f^H=F_f$, $\mathrm{Tr}^H(\widetilde{R}_f)=R_f$ and $\mathrm{Tr}^H(\widetilde{\Pi}_f)=\Pi_f$, where we set $H=\langle\tilde{\sigma}_f\rangle$. In particular, it implies $\mathrm{Tr}^H(\widetilde{R}_f^+)=R_f^+$. Hence, in particular, we have the following:

1. $(R_f)_s=\left\{\mathrm{Tr}^H(\alpha)\,\middle|\,\alpha\in\left(\widetilde{R}_f\right)^H\right\}$ and $(R_f)_l=\left\{\mathrm{Tr}^H(\alpha)\,\middle|\,\alpha\in\widetilde{R}_f\setminus\left(\widetilde{R}_f\right)^H\right\}$.
2. Set $\tilde{\rho}_f=\frac{1}{2}\sum_{\alpha\in\widetilde{R}_f^+}\alpha$. Then,
$$\tilde{\rho}_f=\frac{1}{2}\Bigg(\sum_{\alpha\in(\widetilde{R}_f^+)^H}\alpha+\sum_{\alpha\in\widetilde{R}_f^+\setminus(\widetilde{R}_f^+)^H}\alpha\Bigg)=\frac{1}{2}\Bigg(\sum_{\alpha\in(R_f^+)_s}\alpha+\sum_{\alpha\in(R_f^+)_l}\alpha\Bigg)=\rho_f.$$
3. Let $\tilde{\theta}\in\widetilde{R}_f^+$ be the highest root. One has $\tilde{\theta}=\mathrm{Tr}^H(\tilde{\theta})=\theta_s$.

Consider the real vector space $\widetilde{F}=\widetilde{F}_f\oplus\mathrm{rad}(I)$ equipped with a positive semidefinite metric $\tilde{I}$ such that $\tilde{I}|_{\tilde{F}_f\times\tilde{F}_f}=\tilde{I}_f$ and $\tilde{I}|_{F\times F}=I$, where $F=F_f\oplus\mathrm{rad}(I)$ is clearly a subspace of $\widetilde{F}$. The automorphism $\tilde{\sigma}_f$ extends to the automorphism $\tilde{\sigma}_{af}$ of the affine root system $\widetilde{R}_{af}:=\widetilde{R}_f+\mathbb{Z}\delta$, fixing $\delta\in\mathrm{rad}(I)$. By definition, one has $\mathrm{Tr}^H(\widetilde{R}_{af})=R_{af}$. Let Y_N be the type of the root system $\widetilde{R}_f$. Then, the type of the affine root system $\widetilde{R}_{af}$ is $Y_N^{(1)}$ and

Lemma 3.2. $h^\vee(X_l^{(t)})=h^\vee(Y_N^{(1)})$.

Proof. Let $\widetilde{\Pi}_{af}=\{\beta_j\}_{0\le j\le N}$ be the set of simple roots of $\widetilde{R}_{af}$ containing $\widetilde{\Pi}_f$ and satisfying $\mathrm{Tr}^H(\widetilde{\Pi}_{af})=\Pi_{af}$. Let $\{(a_j)_{Y_N}\}_{0\le j\le N}$ and $\{(a_j^\vee)_{Y_N}\}_{0\le j\le N}$ be the labels and colabels, respectively, of $\widetilde{\Pi}_{af}$.

By definition, $h^\vee(Y_N^{(1)}) = \sum_{j=0}^N (a_i^\vee)_{Y_N}$. Since $\widetilde{R}_f$ hence $\widetilde{R}_{af}$ is of simply-laced type, it follows that $(a_j^\vee)_{Y_N} = (a_j)_{Y_N}$ for any $0 \le j \le N$, whereas for R_f, one has

$$\theta_s^\vee = \frac{2}{I_f(\theta_s,\theta_s)}\theta_s = \frac{2}{I_f(\theta_s,\theta_s)}\Big(\sum_{i=1}^l a_i\alpha_i\Big) = \sum_{\alpha_i\in\Pi_{f,s}} a_i\alpha_i^\vee + \sum_{\alpha_i\in\Pi_{f,l}} ta_i\alpha_i^\vee$$

which implies

$$a_i^\vee = \begin{cases} a_i & \alpha_i \in \Pi_{f,s}, \\ ta_i & \alpha_i \in \Pi_{f,l}. \end{cases}$$

As $\tilde\theta = \mathrm{Tr}^H(\widetilde{\theta}) = \theta_s$, it follows that $(a_j)_{Y_N} = a_i$ if $\mathrm{Tr}^H(\beta_j) = \alpha_i$.

Since, $t = |\langle\tilde\sigma_f\rangle| = \big|H.\beta\big|$ if $\beta \in \widetilde{R}_f \setminus \widetilde{R}_f^H$, we obtain

$$\begin{aligned} h^\vee(X_l^{(t)}) &= 1 + \sum_{i=1}^l a_i^\vee = 1 + \sum_{\substack{1\le i\le l\\ \alpha_i\in\Pi_{f,s}}} a_i + \sum_{\substack{1\le i\le l\\ \alpha_i\in\Pi_{f,l}}} ta_i \\ &= 1 + \sum_{\substack{1\le j\le N\\ \beta_j\in\widetilde{\Pi}_f^H}} (a_j)_{Y_N} + \sum_{\substack{1\le j\le N\\ \beta_j\in\widetilde{\Pi}_f\setminus\widetilde{\Pi}_f^H}} (a_j)_{Y_N} = h^\vee(Y_N^{(1)}). \end{aligned}$$ ∎

By this lemma and Freudenthal de Vries' strange formula for $\widetilde{R}_f$, we obtain

$$\frac{I_f(\rho_f,\rho_f)}{2h^\vee(X_l^{(t)})} = \frac{\tilde I_f(\tilde\rho_f,\tilde\rho_f)}{2h^\vee(Y_N^{(1)})} = \frac{\dim\mathfrak{g}(Y_N)}{24}\cdot\frac{\tilde I(\tilde\theta,\tilde\theta)}{2},$$

which implies

$$\frac{\hat I(\rho,\rho)}{2\hat I(\rho,\delta)} = \frac{2}{I_f(\theta_s,\theta_s)}\cdot\frac{\dim\mathfrak{g}(Y_N)}{24}\cdot\frac{\tilde I_f(\tilde\theta,\tilde\theta)}{2} = \frac{\dim\mathfrak{g}(Y_N)}{24}.$$

Now, we compute $|(R_f)_s|$ and $|(R_f)_l|$. Let $c \in W(R_f)$ be a Coxeter element, and $\langle c\rangle$ the group generated by c. Consider the action of $\langle c\rangle$ on R_f. By Theorem 8.1 in [9], each $\langle c\rangle$-orbit contains a positive root, say α, which is sent to a negative root, i.e., $c(\alpha) \in -R_f^+$. Such a positive root can be expressed as follows.

Suppose that we have $c = s_{\alpha_l}s_{\alpha_{l-1}}\cdots s_{\alpha_1}$. Then, there exists $1 \le j \le l$ such that $\alpha = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_{j-1}}(\alpha_j)$. Moreover, by Corollay 8.2, ibid., each $\langle c\rangle$-orbit contains exactly $h(X_l)$ elements. Thus, we have

Lemma 3.3. $|(R_f)_s| = h(X_l)|\Pi_{f,s}|$ *and* $|(R_f)_l| = h(X_l)|\Pi_{f,l}|$.

We now express $\dim\mathfrak{g}(Y_N)$ in terms of $|\Pi_{f,s}|$ and $|\Pi_{f,l}|$. Since $t = |\langle\tilde\sigma_f\rangle| = \big|H.\beta\big|$ if $\beta \in \widetilde{R}_f \setminus \widetilde{R}_f^H$, one has

$$\begin{aligned} |\tilde R_f| &= \big|(\widetilde{R}_f)^{\langle\tilde\sigma_f\rangle}\big| + \big|\widetilde{R}_f \setminus \widetilde{R}_f^{\langle\tilde\sigma_f\rangle}\big| = \big|(R_f)_s\big| + t\big|(R_f)_l\big|, \\ N = \big|\widetilde{\Pi}_f\big| &= \big|\widetilde{\Pi}_f^{\langle\tilde\sigma_f\rangle}\big| + \big|\widetilde{\Pi}_f \setminus \widetilde{\Pi}_f^{\langle\tilde\sigma_f\rangle}\big| = \big|\Pi_{f,s}\big| + t\big|\Pi_{f,l}\big|. \end{aligned}$$

Hence, Lemma 3.3 implies

$$\dim\mathfrak{g}(Y_N) = N + |\tilde R_f| = (h(X_l)+1)\big(\big|\Pi_{f,s}\big| + t\big|\Pi_{f,l}\big|\big).$$

Hence, the left hand side of (10) becomes

$$e^{-\frac{1}{24}(h(X_l)+1)\left(\left|\Pi_{f,s}\right|+t\left|\Pi_{f,l}\right|\right)\delta}\prod_{n=1}^{\infty}\left(\left(1-e^{-n\delta}\right)^{(h(X_l)+1)\left|\Pi_{f,s}\right|}\left(1-e^{-tn\delta}\right)^{(h(X_l)+1)\left|\Pi_{f,l}\right|}\right)$$
$$=\left(\eta\left(e^{-\delta}\right)^{\left|\Pi_{f,s}\right|}\eta\left(e^{-t\delta}\right)^{\left|\Pi_{f,l}\right|}\right)^{h(X_l)+1}.$$

As for the right hand side of (10), (11) implies

$$2\,\frac{\hat{I}(\rho,\delta)}{\hat{I}(\theta_s,\theta_s)}=\frac{2}{I_f(\theta_s,\theta_s)}\cdot\frac{I_f(\theta_s,\theta_s)}{2}h^{\vee}(X_l^{(t)})=h^{\vee}(X_l^{(t)})$$

from which a similar argument to the last subsection implies that the right hand side of (10) becomes

$$\sum_{\gamma\in h^{\vee}(X_l^{(t)})Q(R_f)}d_{X_l}(\gamma)e^{-\frac{I_f(\rho_f+\gamma,\rho_f+\gamma)}{2h^{\vee}(X_l^{(t)})}\cdot\frac{2}{I_f(\theta_s,\theta_s)}\delta}.$$

Thus, we obtain the so-called Macdonald identity:

Theorem 3.4 (cf. Theorem 8.11 in [11]). *Let R_{af} be a twisted affine root system of type $X_l^{(t)}$ with $t\neq 1$ and R_f be its finite quotient system. Assume that R_{af} is not of type $BC_l^{(2)}$.*

$$\left(\eta\left(e^{-\delta}\right)^{\left|\Pi_{f,s}\right|}\eta\left(e^{-t\delta}\right)^{\left|\Pi_{f,l}\right|}\right)^{h(X_l)+1}=\sum_{\gamma\in h^{\vee}(X_l^{(t)})Q(R_f)}d_{X_l}(\gamma)e^{-\frac{I_f(\rho_f+\gamma,\rho_f+\gamma)}{2h^{\vee}(X_l^{(t)})}\cdot\frac{2}{I_f(\theta_s,\theta_s)}\delta}. \quad (12)$$

3.4. Type $BC_l^{(2)}$

Here we treat the case when R_{af} is of type $BC_l^{(2)}$ rapidly. As in the last subsections, we assume that R_{af} belongs to $(F,I)\subset(\widehat{F},\hat{I})$.

Let $(R_{af})_s,(R_{af})_m$ and $(R_{af})_l$ be the subset of short, middle length and long roots of R_{af}, respectively. Set $(R'_f)_*=R'_f\cap(R_{af})_*$ $(*\in\{m,l\})$. By definition,

$$(R_{af})_s=\left\{\frac{1}{2}\big(\alpha+(2n-1)\delta\big)\,\middle|\,\alpha\in(R'_f)_l,\ n\in\mathbb{Z}\right\},$$
$$(R_{af})_m=(R'_f)_m+\mathbb{Z}\delta,\qquad (R_{af})_l=(R'_f)_l+2\mathbb{Z}\delta.$$

The denominator identity of type $BC_l^{(2)}$ gives

$$e^{\rho-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}\prod_{\alpha\in(R'_f)^+}(1-e^{-\alpha})\prod_{n=1}^{\infty}\Big(\big(1-e^{-n\delta}\big)^l$$
$$\times\prod_{\alpha\in(R'_f)_m}\big(1-e^{\alpha-n\delta}\big)\prod_{\alpha\in(R'_f)_l}\big(1-e^{\alpha-2n\delta}\big)\big(1-e^{\frac{1}{2}(\alpha-(2n-1)\delta)}\big)\Big)$$
$$=\sum_{w\in\widehat{W}(R_{af})}\det{}_{\widehat{F}}(w)e^{w(\rho)-\frac{\hat{I}(\rho,\rho)}{2\hat{I}(\rho,\delta)}\delta}.$$

Dividing the both sides by $e^\rho \prod_{\alpha\in(R'_f)^+}(1-e^{-\alpha})$ and specializing $e^{-\alpha_i}$ $(0 \le i \le l-1)$ to 1, we obtain the so-called the Macdonald identity:

Theorem 3.5 (cf. Type BC_l (6)(b) of Appendix 1 in [11]). *Let R_{af} be an affine root system of type $BC_l^{(2)}$ and R'_f be a finite root subsystem of type C_l.*

$$\left(\eta\left(e^{-\frac{1}{2}\delta}\right)^2\eta\left(e^{-\delta}\right)^{2l-3}\eta\left(e^{-2\delta}\right)^2\right)^l = \sum_{\gamma\in\hat{I}(\rho,\delta)Q((R'_f)^\vee)} d_{C_l}(\gamma)e^{-\frac{1}{2\hat{I}(\rho,\delta)}\hat{I}(\rho+\gamma,\rho+\gamma)\delta}. \tag{13}$$

Details are left to the reader.

3.5. Postscript

The article [11] contains quite intriguing computations.

For example, I. Macdonald has classified affine root systems even of non-reduced type and he obtained 4 types: $R(BCC_l)$, $R(C^\vee BC_l)$, $R(BB_l^\vee)$ and $R(C^\vee C_l)$, but its denominator identities were not given. In 1978, V. Kac [7] considered a certain class of infinite dimensional Lie superalgebras which contains 4 types of affine Lie superalgebras: $B^{(1)}(0,l)$, $A^{(4)}(0,2l)$, $A^{(2)}(0,2l-1)$ and $C^{(2)}(l+1)$. Indeed, up to $\mathbb{Z}/2\mathbb{Z}$-structure, we have the next correspondence:

Non-reduced type	BCC_l	$C^\vee BC_l$	$BB_l^\vee$	$C^\vee C_l$
Affine super	$B^{(1)}(0,l)$	$A^{(4)}(0,2l)$	$A^{(2)}(0,2l-1)$	$C^{(2)}(l+1)$

where those in the same column are the same root system ! Moreover, their Macdonald identities coincides with those for types $BC_l^{(2)}$, $B_l^{(2)}$, $B_l^{(1)}$ and $B_l^{(2)}$, respectively. Notice that these 4 affine root systems of reduced type are root subsystems of the corresponding non-reduced consisting of the indivisible roots (cf. [11]). There still seems to have many hidden structures to be explored in [11].

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Received September 11, 2024
and in final form January 5, 2025