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Geodesic Completeness of some Lorentzian Simple Lie Groups

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Abstract. We investigate geodesic completeness of left-invariant Lorentzian metrics on a simple Lie group G when there exists a left-invariant Killing vector field Z on G. Among other results, it is proved that if Z is timelike, or G is strongly causal and Z is lightlike, then the metric is complete. The situation is considerably elaborate when Z is spacelike, as our study of the special complex Lie group $SL_2(\mathbb{C})$ illustrates. We show that the existence of a lightlike vector field Z on $SL_2(\mathbb{C})$, implies geodesic completeness. When Z is spacelike and orthogonal to $\sqrt{-1}Z$, we characterize complete metrics on $SL_2(\mathbb{C})$.

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1. Introduction

Any invariant Riemannian metric on a homogeneous space G/H is known to be geodesically complete (based on a distance argument, see [16], Remark 9.37). Invariant non-Riemannian metrics, however, require additional conditions in general to be geodesically complete. For example, any invariant semi-Riemannian metric on a compact homogeneous space is complete [14, 16].

Alekseevskii and Putko [1] showed that the study of geodesic completeness of invariant metrics on homogeneous spaces could be reduced to the study of Lie groups equipped with a left-invariant semi-Riemannian metric, referred to as semi-Riemannian Lie groups.

Equations that determine geodesics in a semi-Riemannian Lie group are known as *Euler equations*. Arnold observed that the motion of a rigid body in $\mathbb{R}^3$ could be described as a motion along geodesics in the group of rotations with an invariant metric [2]. He then realized that the Euler equation could be extended to any Lie group endowed with an invariant metric.

A curve γ on a semi-Riemannian Lie group (G, q) is a geodesic if and only if $(dL_\gamma)^{-1}(\gamma')$, as a curve on the Lie algebra $\mathfrak{g}$ of G, satisfies a differential equation called the *Euler equation*. When G is semisimple (specially, simple), since the Killing

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form is non-degenerate, the Euler equation translates into a Lax equation by a linear change of variables. This is called the *dual Euler equation.* Lax equations have the advantage that their first integrals are easy to express. The dual Euler equation defines a homogeneous quadratic vector field F_{q} on the Lie algebra $\mathfrak{g}$, called the *dual Euler field.* The integral curves of F_{q} are complete if and only if the geodesics of q are complete.

In this paper we study left-invariant Lorentzian metrics on simple Lie groups. We only consider non-compact Lie groups as invariant metrics on compact Lie groups are always complete. Since there is no distance function associated to an indefinite metric, we use the notion of completeness in the sense of geodesic completeness.

The dual space $\mathfrak{g}^*$ naturally admits a Poisson structure induced on $\mathfrak{g}$ by identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the corresponding non-degenerate Killing form ([12], Ch. 7). Then the dual Euler equation on $\mathfrak{g}$ represents a Hamiltonian system with the Hamiltonian function $\frac{1}{2}\mathrm{q}^*(x,x)$, where q^* is the induced scalar product on $\mathfrak{g}^*$ and identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ by the Killing form. In general, there is no systematic method to solve a Hamiltonian system or examine completeness of its solutions. However, the more first integrals for a Hamiltonian system are available, the better the system can be described. The dual Euler equation primitively has the first integrals $\mathrm{q}^*(x,x)$ and $\mathrm{tr}(\mathrm{ad}_x^m)$, $m \in \mathbb{N}$. On $\mathfrak{sl}_2(\mathbb{R})$, because of its low dimension, these first integrals (i.e., $\mathrm{q}^*(x,x)$ and $\mathrm{tr}(\mathrm{ad}_x^2)$) is enough to guarantee the total integrability of the system. Using this fact, Bromberg and Medina [4] fully characterised the completeness of invariant Lorentzian metrics on $SL_2(\mathbb{R})$ by proving that a left-invariant Lorentzian metric q on $SL_2(\mathbb{R})$ is complete if and only if F_{q} has no non-zero fixed point, i.e. no point $x \neq 0$ where $F_{\mathrm{q}}(x) = x$. Such points are called *idempotents.*

Finding new first integrals in dimensions greater than 3, other than those mentioned above, is a major challenge. In Proposition 3.3, we recall that if there exists a left-invariant Killing vector field Z, then one can obtain an additional first integral for the dual Euler equation. We then use this new first integral along with $\mathrm{q}^*(x,x)$ and $\mathrm{tr}(\mathrm{ad}_x^m)$ to prove that ad_Z is neither diagonalizable nor nilpotent (Proposition 3.5). Moreover, we observe that if Z is lightlike, then ad_Z is compact (Lemma 3.9), allowing us to obtain a causal curve lying in a compact subspace. 'Strongly causal' manifolds do not accept such a causal curve. So, we get our first main result as follows:

Theorem 1.1. *Let (G,q) be a Lorentzian simple Lie group with a left-invariant Killing vector field Z on it. Then, q is complete in the following cases:*

(i) *when Z is timelike,*

(ii) *when G is strongly causal and Z is lightlike.*

Part (i) of the above theorem is a simple corollary of Proposition 2.1 of [18] where the authors use a more direct approach to prove their result by showing that the geodesics have bounded tangent vectors with respect to a Riemannian metric. In the case of semi-Riemannian Lie groups this means that the solutions of the dual Euler equation are bounded. Here we get Theorem 1.1 as a result of a general feature of left-invariant Killing vector fields on semisimple Lie groups, which we prove in Proposition 3.5. Theorem 1.1 holds for globally hyperbolic simple Lie groups as they are known to be strongly causal.

The case where Z is spacelike is much more challenging. To compare the situation in dimensions greater than 3 with the case of $SL_2(\mathbb{R})$, we investigate left-invariant Lorentzian metrics on $SL_2(\mathbb{C})$. To do so, we introduce a specific type of integral curves of the dual Euler field F_{q} on the Lie algebra $\mathfrak{g}$. We call an integral curve $u(t)$ of F_{q}, a *generalized conical spiral* (GCS) if it satisfies $u(s) = ru(0)$ for some $r, s \in \mathbb{R}$ with $r > 0$. It follows that the radial line generated by an idempotent is the image of a special GCS. Moreover, on $\mathfrak{sl}_2(\mathbb{R})$ such radial lines are the only examples of GCS. In particular, the result of [4] for $SL_2(\mathbb{R})$ can be re-stated as: a left-invariant Lorentzian metric q on $SL_2(\mathbb{R})$ is complete if and only if F_{q} has no GCS. We show that when there exists a left-invariant Killing vector field on $SL_2(\mathbb{R})$, then the Lorentzian metric is complete (Proposition 3.4). We then prove an analogous result for $SL_2(\mathbb{C})$ as follows.

Theorem 1.2. *Let* q *be a left-invariant Lorentzian metric on* $SL_2(\mathbb{C})$*. Suppose that there exists a left-invariant spacelike Killing vector field* Z *on* $SL_2(\mathbb{C})$ *such that* $\mathrm{q}(Z, \sqrt{-1}Z) = 0$*. Then,* q *is complete if and only if* F_{q} *has no incomplete GCS.*

If Z in Theorem 1.2 is lightlike, then $\mathrm{q}(Z, \sqrt{-1}Z) \neq 0$ and the metric q is complete. Examples in both cases of Theorem 1.2 (complete and incomplete) follow easily from Lemmas 4.10 and 4.11.

A different point of view was suggested in ([19], Theorem 1.2.8) to study and characterize completeness of invariant Lorentzian metrics on semisimple Lie groups. Let Λ^*_{q} and $\mathcal{N}$ denote, respectively, the *null cone* consisting of all null vectors in $\mathfrak{g} \cong \mathfrak{g}^*$ w.r.t q^*, and the *nilpotent cone* consisting of all nilpotent elements. In [19] it is shown that a left-invariant Lorentzian metric on $SL_2(\mathbb{R})$ is incomplete if and only if the intersection of Λ^*_{q} and $\mathcal{N}$ is transversal at some point. Using this approach, Tholozan then shows that the set U_- of incomplete left-invariant metrics and U_+ of complete metrics with bounded integral curves of the dual Euler field, are open subspaces of the space of all left-invariant semi-Riemannian metrics on $SL_2(\mathbb{R})$.

Moreover, the set of complete metrics with unbounded integral curves of the dual Euler field is a semi-algebraic variety which together with U_+ and U_- makes a partition of the space of all left-invariant semi-Riemannian metrics on $SL_2(\mathbb{R})$. It is conjectured in [19] that there exists a similar partition for the space of left-invariant semi-Riemannian metrics on any semisimple Lie group. We use Theorem 1.1 to give an example (see Example 3.11) showing that U_- is not necessarily open in general.

The paper is organised as follows: In section 2, we provide preliminaries and set the notations. Section 3 deals with the study of invariant Lorentzian metrics on simple Lie groups and the proof of Theorem 1.1. We consider invariant Lorentzian metrics on $SL_2(\mathbb{C})$, and prove Theorem 1.2, in Section 4.

2. Preliminaries

We refer the reader for details on the materials of this section to [1, 3, 4, 6, 7, 11, 12, 19, 20]. We begin by a brief review of some basic notions in Lorentzian geometry, even though most of them are defined in the general setting of semi-Riemannian manifolds. In this section we take (M, q) to be a (time-orientable) Lorentzian manifold.

According to [3], (M, q) is said to be *strongly causal* if each $p \in M$ admits a neighborhood such that no causal curve crosses it.

A causal curve $\gamma : [0, b) \to M$ (i.e., γ is non-spacelike, $\mathrm{q}(\gamma', \gamma') \leq 0$) is *future directed* if for every $t \in I$, $\gamma'(t)$ belongs to the future cone in $T_{\gamma(t)}M$. The curve γ is said to be *future imprisoned* in a compact subset $L \subset M$ if there exists some $0 < t_0 < b$, such that $\gamma([t_0, b)) \subset L$. It is *partially future imprisoned* in L if $\gamma(t_m) \in L$ for some increasing sequence $t_m \nearrow b$ in $[0, b)$.

Proposition 2.1. (Easy, see [3]) *If (M, q) is strongly causal, then no inextendible causal curve can be partially future imprisoned in any compact set.*

A smooth curve $\gamma : I \to M$ is a *geodesic* if it satisfies the equation $\nabla_{\gamma'}\gamma' = 0$, where ∇ stands for the Levi-Civita connection of q. If γ is a geodesic then $\mathrm{q}(\gamma(t), \gamma(t))$ is constant for all $t \in I$.

According to [5], a geodesic $\gamma : [0, b) \to M$, $b < +\infty$, is extendible beyond b if and only if $|\gamma'|_R = \mathrm{q}_R(\gamma'(t), \gamma'(t))$ is bounded for some, hence any, complete Riemannian metric q_R, equivalently, if there exists an increasing sequence $t_m \nearrow b$ such that $\{\gamma'(t_m)\}$ converges in TM.

A vector field $X \in \mathfrak{X}(M)$ is called *spacelike, timelike, lightlike* or *causal*, if X_p has that characteristic for every $p \in M$. A *Killing* vector field on M is a vector field X such that $\mathcal{L}_X \mathrm{q} = 0$, where $\mathcal{L}_X$ is the Lie derivation along X. Equivalently, X is Killing if its local flows are isometries. If X is Killing and $\gamma : I \to M$ is a geodesic then $\mathrm{q}(X_{\gamma(t)}, \gamma'(t))$ is constant for all $t \in I$.

Hereafter any Killing vector field will be non-zero. For a sequence $\{\gamma_n\}$ of smooth curves in a smooth manifold M, the notion of a limit curve is defined as follows: a curve γ in M is called a *limit curve* of $\{\gamma_n\}$ if there exists a subsequence $\{\gamma_m\}$ such that every neighbourhood of each point $p \in \gamma$ intersects all, but a possibly finite number, of the curves in $\{\gamma_m\}$ ([3], Ch. 3, 3.28).

In semi-Riemannian manifolds, a geodesic is uniquely determined by its initial velocity. This fact leads to the existence of a distinguished limit curve for a sequence of geodesics.

Proposition 2.2. ([17]) *Let (M, q) be a Lorentzian manifold.*
For $m = 1, 2, \ldots$, let $\gamma_m : [0, b_m) \to M$, $b_m \leq +\infty$, be a sequence of geodesics such that γ_m is inextendible beyond b_m. If $\gamma_m'(0)$ converges to x in TM, then the geodesic in M with initial velocity x is a limit curve of $\{\gamma_m\}$.

The remainder of this section reviews key structures and results related to semisimple Lie algebras and the dual Euler equation, which will be used in the following sections.

Let G be a (real) Lie group and $\mathfrak{g}$ its Lie algebra, which is the vector space $\mathfrak{X}_L(G)$ of left-invariant vector fields on G, equipped with the Lie bracket of vector fields. Given vector fields $X, Y, Z, \ldots \in \mathfrak{g}$, we use the lowercase letters $x, y, z, \ldots$ to denote the corresponding elements $X_e, Y_e, Z_e, \ldots \in T_eG$. Since left-invariant vector fields are uniquely determined by their values at the identity, i.e. $T_eG \cong \mathfrak{g}$, throughout the paper we identify any $X, Y, Z, \ldots \in \mathfrak{g}$ with $x, y, z, \ldots$ and use them interchangeably whenever needed.

Cartan's criterion states that the algebra $\mathfrak{g}$ is semisimple if and only if its Killing form $\mathrm{K} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by $\mathrm{K}(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$ is non-degenerate. The Killing form is a symmetric bilinear form invariant under all automorophisms of $\mathfrak{g}$. Another equivalent condition on $\mathfrak{g}$ to be semisimple is that $\mathfrak{g}$ has no non-zero abelian ideal.

If $\mathfrak{g}$ has no proper ideal at all, then it is by definition a simple Lie algebra. The following theorem, which is a special case of Corollary 2.3 in [20], states that, in general, a real simple Lie algebra does not have 'large' subalgebras.

Theorem 2.3. ([20]) *Let $\mathfrak{g}$ be a real simple Lie algebra. Then $\mathfrak{g}$ has a codimension one subalgebra if and only if it is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.*

For the rest of the section $\mathfrak{g}$ is a semisimple Lie algebra unless otherwise specified. The Killing form allows us to identify $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by using the correspondence $x \mapsto \mathrm{K}(x,\cdot)$. Here $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad}_x(z) := [x,z]$, is the *adjoint representation* of $\mathfrak{g}$ on itself. For each $g \in G$, let $C_g : G \to G$ be the function sending each $h \in G$ to ghg^{-1}, C_g is an inner automorphism, and since $C_g = L_g \circ R_{g^{-1}}$ is a diffeomorphism, it is an automorphism of the Lie group G. The differential of C_g at the identity is denoted by Ad_g, it is also an automorphism of Lie algebra $\mathfrak{g}$ of G, *adjoint representation* of G is defined by $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$. One can see that ad is the differential of Ad at identity element $e \in G$, acting on the tangent space T_eG, which is, by definition, $\mathfrak{g}$. Since the Killing form K is invariant under all automorphism of $\mathfrak{g}$, it satisfies $\mathrm{K}(y,[x,z]) = \mathrm{K}([y,x],z)$ for all $x,y,z \in \mathfrak{g}$ ([16], p. 301–302).

An element $x \in \mathfrak{g}$ is called *diagonalizable*, respectively *nilpotent*, if the adjoint operator ad_x is diagonalizable, respectively nilpotent. Moreover, an element $x \in \mathfrak{g}$ is called *compact* if the one-parameter subgroup $\mathrm{Ad}(\exp(tx))$ lies in a commutative compact subgroup of $\mathrm{Ad}(G)$. Equivalently, x is compact if the eigenvalues of ad_x are all purely imaginary.

We recall some classical simple results from Linear Algebra that we will use hereafter.

Remark 2.4. Let E be a linear endomorphism of a real or complex finite-dimensional vector space;

(i) if E is nilpotent or if it is diagonalizable, then so is ad_E;

(ii) if $\mathrm{tr}(E^m) = 0$ for every positive integer m, then E is nilpotent;

(iii) if there is a linear endomorphisem L such that $\mathrm{ad}_E L = E$, then E is nilpotent. ■

Also it will be useful to recall have the following classical facts on Lie algebras $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{sl}_2(\mathbb{C})$.

Remark 2.5. (i) Up to conjugation, respectively up to the adjoint action of $SL_2(\mathbb{C})$, elements of $SL_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ can be given as:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \text{ and respectively: } p_a = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ or } s_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

with $a \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$. So elements of $\mathfrak{sl}_2(\mathbb{C})$ are either diagonalizable (those of type s_λ) or nilpotent (those of type p_a). Then, by part (i) of Remark 2.4, ad_{p_a} (resp. ad_{s_λ}) is nilpotent (resp. diagonalizable). Therefore, elements of $\mathfrak{sl}_2(\mathbb{R})$, a subset of $\mathfrak{sl}_2(\mathbb{C})$, are either semi-simple (i.e. diagonalizable on the algebraic closure) or nilpotent. One should note that s_λ consists of several classes under the adjoint action. We also see that any $z \in \mathfrak{sl}_2(\mathbb{C})$ is compact if and only if $z = s_\lambda$ with $\lambda \in \sqrt{-1}\mathbb{R}$.

(ii) A useful basis of $\mathfrak{sl}_2(\mathbb{F})$ for $\mathbb{F}=\mathbb{C}$ and $\mathbb{F}=\mathbb{R}$ is (x,y,ξ) with:

$$x = p_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = p_1^\top = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \xi = s_{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

They satisfy in particular: $[\xi,x]=x$, $[\xi,y]=-y$ and $[x,y]=2\xi$. Furthermore, we have $c_{\mathfrak{sl}_2(\mathbb{F})}(\xi)=\operatorname{Ker}(\operatorname{ad}_\xi)=\operatorname{span}_{\mathbb{F}}\{\xi\}$, $[\xi,\mathfrak{sl}_2(\mathbb{C})]=\operatorname{Im}(\operatorname{ad}_\xi)=\operatorname{span}_{\mathbb{F}}\{x,y\}$, and $[\operatorname{Im}(\operatorname{ad}_\xi),\operatorname{Im}(\operatorname{ad}_\xi)]\subseteq\operatorname{Ker}(\operatorname{ad}_\xi)$. For $x,y\in\mathfrak{sl}_2(\mathbb{F})$, $[x,y]=0$ if and only if x and y are $\mathbb{F}$-collinear

(iii) Consider $\mathfrak{sl}_2(\mathbb{C})$ as a real Lie algebra and define its Killing form K by $\mathrm{K}(x,y)=2\operatorname{Re}(\operatorname{tr}(x.y))$, where $\operatorname{Re}(\lambda)$ is the real part of a complex number λ. Then $\operatorname{sign}(\mathrm{K})=(3,3)$ and $\operatorname{Mat}(\mathrm{K})=\operatorname{diag}(1,-1)$ on $\operatorname{Ker}(\operatorname{ad}_\xi)=\operatorname{span}_{\mathbb{R}}\{\xi,\sqrt{-1}\xi\}$, and the direct sum $\operatorname{Im}(\operatorname{ad}_\xi)\oplus\operatorname{Ker}(\operatorname{ad}_\xi)$ is K-orthogonal. ■

The subset of nilpotents in $\mathfrak{g}$, denoted by $\mathcal{N}$, is invariant under the adjoint action of G on $\mathfrak{g}$ defined by $g\cdot x=\operatorname{Ad}_g(x)$ for every $x\in\mathfrak{g}$. Therefore $\mathcal{N}$ is the union of all the nilpotent orbits of the adjoint action. We refer to $\mathcal{N}$ as the *nilpotent cone*, for it clearly contains the radial line $\mathbb{R}x$ for any $x\in\mathcal{N}$. One can see that the tangent space to any orbit $O(x)\subset\mathcal{N}$ is given by

$$T_x(O(x)) = T_x\operatorname{Ad}_G(x) = \{\operatorname{ad}_y(x) : y\in\mathfrak{g}\} = [x,\mathfrak{g}], \quad \forall x\in\mathcal{N}. \tag{1}$$

Let q be a left-invariant semi-Riemannian metric on G. Using the correspondence between left-invariant tensor fields on G and tensors on its tangent space at the identity element, one may think of q as a non-degenerate symmetric bilinear form on $\mathfrak{g}$. Then, associated to q, there is a unique K-symmetric isomorphism A_{q} on $\mathfrak{g}$ such that $\mathrm{q}(x,y)=\mathrm{K}(x,A_{\mathrm{q}}y)$ for all $x,y\in\mathfrak{g}$. We denote by q^* the induced bilinear form on $\mathfrak{g}^*$. Identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form as above, the isomorphism associated to q^* is A_{q}^{-1}, that is, $\mathrm{q}^*(x,y)=\mathrm{K}(x,A_{\mathrm{q}}^{-1}y)$ for all $x,y\in\mathfrak{g}$.

We denote by Λ_{q}^* the null cone determined by q^* in $\mathfrak{g}$:

$$\Lambda_{\mathrm{q}}^* = \{x\in\mathfrak{g} : \mathrm{q}^*(x,x)=0\} = \{x\in\mathfrak{g} : \mathrm{K}(x,A_{\mathrm{q}}^{-1}x)=0\}.$$

The null cone is a hypersurface of $\mathfrak{g}$ and its tangent space at $0\neq x\in\Lambda_{\mathrm{q}}^*$ is given by

$$T_x\Lambda_{\mathrm{q}}^* = \{y\in\mathfrak{g} : \mathrm{q}^*(x,y)=0\}. \tag{2}$$

By (1) and (2) and using the K-symmetry of A_{q}^{-1} one can see that the two cones $\mathcal{N}$ and Λ_{q}^* are transversal at $x\in\Lambda_{\mathrm{q}}^*\cap\mathcal{N}$ if and only if $\mathrm{K}([x,y],A_{\mathrm{q}}^{-1}x)\neq 0$ for some $y\in\mathfrak{g}$, and by the non-degeneracy of K, and as ad_x is K-skew symmetric, it is equivalent to $[x,A_{\mathrm{q}}^{-1}x]\neq 0$.

The position of the cones $\mathcal{N}$ and Λ_{q}^* with respect to each other is related to the metric completeness:

Proposition 2.6. ([19], Lemma 1.2.6) *Let* q *be a left-invariant semi-Riemannian metric on a semisimple Lie group* G*. If* $\Lambda_{\mathrm{q}}^*\cap\mathcal{N}=\{0\}$*, then* q *is complete.*

Given a curve $\gamma(t)$ in G, one can define a curve $u(t)$ in $\mathfrak{g}$ by $u(t):=(dL_{\gamma(t)})^{-1}(\gamma'(t))$ where, L_g is the left multiplication by g in G. We may sometimes refer to $u(t)$ as the *reflection* of $\gamma(t)$ in $\mathfrak{g}$, or, call $\gamma(t)$ the reflection geodesic of $u(t)$. For any vector field

Y along γ the covariant derivative $\nabla_{\gamma'(t)}Y(t)$ of Y with respect to the Levi-Civita connection ∇ of q on G, and the differential of the curve $v(t) = (dL_{\gamma(t)})^{-1}(Y(t))$ in $\mathfrak{g}$ are related by the following formula:

$$(dL_{\gamma(t)})^{-1}\left(\nabla_{\gamma'(t)}Y(t)\right) = v'(t) + \nabla_{u(t)}v(t). \tag{3}$$

where, for $x, y \in \mathfrak{g}$, $\nabla_x y$ is defined by considering x and y as left-invariant vector fields on G and then evaluating $\nabla_x y$ at $e \in G$. So, in particular, the curve $\gamma(t)$ is a geodesic of G (w.r.t q) if and only if $u'(t) = -\nabla_{u(t)}u(t)$. Using the Koszul formula

$$\nabla_x y = \frac{1}{2}\left\{[x,y] - (\mathrm{ad}_x)^* y - (\mathrm{ad}_y)^* x\right\}, \quad x, y \in \mathfrak{g}, \tag{4}$$

with $(\mathrm{ad}_x)^* = -A_{\mathrm{q}}^{-1} \circ \mathrm{ad}_x \circ A_{\mathrm{q}}$ being the transpose of ad_x w.r.t q, $\nabla_{u(t)}u(t) = A_{\mathrm{q}}^{-1}\left[A_{\mathrm{q}}u(t), u(t)\right]$, and by a linear change of variable $w(t) = A_{\mathrm{q}}u(t)$ (obtained by pushing the vector field on the dual $\mathfrak{g}^*$ by $x \mapsto \mathrm{q}(x,.)$, and pulling it back on $\mathfrak{g}$ by $x \mapsto \mathrm{K}(x,.)$, which is possible as K is non-degenerate), the equation $u'(t) = -\nabla_{u(t)}u(t)$ turns into

$$w'(t) = \left[w(t), A_{\mathrm{q}}^{-1}w(t)\right]. \tag{5}$$

Thus, $\gamma(t)$ is a geodesic in G if and only if $w(t)$ and $A_{\mathrm{q}}^{-1}w(t)$ satisfy (5). Equation (5) is referred to as the *dual Euler equation* (but sometimes also itself as the Euler (-Arnold) equation in the literature) and, accordingly, the vector field on $\mathfrak{g}$ defined by $F_{\mathrm{q}} : x \mapsto [x, A_{\mathrm{q}}^{-1}x]$, which is a homogeneous quadratic vector field, is called the *dual Euler field.* Hence the completeness of the metric q can be reformulated as follows.

Theorem 2.7. (Easy, see [1]) *Let (G, q) be a semi-Riemannian semisimple Lie group. Then, the following are equivalent:*

(i) *the metric* q *is (geodesically) complete,*

(ii) *the solutions of the (dual) Euler equation* (5) *are complete,*

(iii) *the (dual) Euler field F_{q} is complete.*

In particular, if solutions of (5)*, or equivalently, integral curves of F_{q} are bounded, then* q *is complete.*

By Theorem 2.7 and the equivalent conditions mentioned in Section 2 for geodesic completeness, one gets:

Proposition 2.8. *Let (G, q) be a semi-Riemannian semisimple Lie group. Suppose that $u : [0, b) \to \mathfrak{g}$, $b < +\infty$, is a solution of the dual Euler equation. Then, the following are equivalent:*

(i) *u is extendible beyond b,*

(ii) *$\|u(t)\|$ is bounded for some, hence any, norm $\|.\|$ obtained from a positive definite scalar product on $\mathfrak{g}$,*

(iii) *there exists a sequence $t_m \nearrow b$ in $[0, b)$ such that $\{u(t_m)\}$ converges in $\mathfrak{g}$.*

Speaking in terms of dynamical systems, if $v' = f(v)$ is a dynamical system, where $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map, and $v(t)$ is a trajectory of the system, then a point

$y \in \mathbb{R}^n$ is called an *ω-limit point* of $v(t)$, if there exists a diverging increasing sequence $\{t_m\}$ such that $v(t_m) \to y$. In particular, if $v(t)$ has no ω-limit point, then it is unbounded. The *ω-limit set* of a system is the set of all its ω-limit points. So the solution $u(t)$ of the dual Euler equation in Proposition 2.8 is extendible if and only if it has an ω-limit point.

In the proof of Proposition 4.8, we project the dual Euler equation on a hypersurface to obtain a linear system. The ω-limit set of a linear system is computable, [11]. An special case is the following.

Proposition 2.9. ([11]) *Let $X' = AX$ be a linear dynamical system on $\mathbb{R}^n$ with $A \in \mathbb{R}^{n\times n}$. If an eigenvalue of A has positive real part, then the ω-limit set of the system is empty.*

An *idempotent* of the dual Euler field F_{q} is by definition an element $x \in \mathfrak{g}$ such that $F_{\mathrm{q}}(x) = x$. If γ is a geodesic in G starting at $e \in G$ such that $\gamma'(0)$ is an idempotent, then γ runs on a one-parameter subgroup of G, namely, γ is a reparametrization of $\exp(tx)$. It turns out that γ is always incomplete. Indeed, γ is explicitly given as follows: the integral curve $u(t)$ of the dual Euler field F_{q} with initial condition $u(0) = x$ is $u(t) = \alpha(t)x$, where $\alpha(t) = 1/(1-t)$. The corresponding geodesic of $u(t)$ in G is $\gamma(t) = \exp(\beta(t)A_{\mathrm{q}}^{-1}x)$ where β satisfies $\beta'(t) = \alpha(t)$ and $\beta(0) = 1$. Clearly, γ is incomplete because u is incomplete.

Remark 2.10. Any semisimple Lie algebra $\mathfrak{g}$ endowed with the Lie Poisson bracket obtained from the Lie bracket is a (linear) Poisson manifold and the dual Euler equation (5) is a Hamiltonian system of differential equations with Hamiltonian function $\mathrm{q}^*(x,x) = \mathrm{K}(x, A_{\mathrm{q}}^{-1}x)$. A function $f : \mathfrak{g} \to \mathbb{R}$ is called a *first integral* or, as physicists wish to call, a *constant of the motion* for the dual Euler equation if it is constant on each solution of the equation. The functions $\mathrm{tr}(\mathrm{ad}_x^m)$, for $m = 1, 2, \ldots$, and $\mathrm{q}^*(x,x)$ are first integrals of (5).

If the dual Euler equation has an unbounded solution $u : [0,b) \to \mathfrak{g}$, then there is a non-zero element θ in $\mathfrak{g}$ that is a zero for all homogenuous first integrals of any degree $k \geq 1$. Indeed, taking an arbitrary norm $\|.\|$ on $\mathfrak{g}$, we can choose a sequence $\{t_m\}$ in $[0,b)$ such that $\|u(t_m)\|$ tends to $+\infty$ and that $u(t_m)/\|u(t_m)\|$ tends to some θ. Let f be a homogeneous first integral of degree $k \geq 1$ and C its constant value on $\{u(t_m), t \in \mathbb{R}\}$, by continuity of f,

$$f(\theta) = \lim_{m\to\infty} \frac{1}{\|u(t_m)\|^k} f(u(t_m)) = \lim_{m\to\infty} \frac{1}{\|u(t_m)\|^k} C = 0.$$

In particular, this holds with $f = \mathrm{q}^*(.,.)$ and $f = \mathrm{tr}(\mathrm{ad}_x^m)$ for all $m > 0$. ∎

3. Lorentzian simple Lie groups

It is well known that the isometry group of a semi-Riemannian manifold is a Lie group whose Lie algebra is anti-isomorphic to the Lie algebra of complete Killing vector fields on the manifold ([16], Proposition 9.33). On a Riemannian simple Lie group (G, q) any Killing vector field Z can be written as $Z = Z_L + Z_R$ with Z_L a left- and Z_R a right-invariant vector field on G, [10, 15]. Any right-invariant vector field on a semi-Riemannian Lie group is Killing as its flow is given by left translations of one-parameter subgroups ([9], p. 257). Therefore, in this case, the set of left-invariant Killing vector fields determines how rich is the supply of all Killing vector fields. In the non-Riemannian case, an analogous result [8] states that if (G, q) is a compact Lorentzian simple Lie group then $Iso^o(G, \mathrm{q}) \subset G \times G$, implies that a Killing vector field on G admits a similar decomposition as a sum of a pair of left- and right- invariant vector fields. Even though a similar decomposition for Killing vector fields does not hold in general on an arbitrary semi-Riemannian Lie group, knowing the effects of the existence of a left-invariant Killing vector field is interesting. For instance, it is classical that each left-invariant Killing vector field on a simple Lorentzian (in fact, (semi-)Riemannian) Lie group provides an additional first integral for the corresponding dual Euler equation.

In this section we study the geodesic completeness of simple Lorentzian Lie groups admitting a left-invariant Killing vector field, such that

$$\dim(Iso(G, \mathrm{q}) \cap InnAut(G)) \geq 1.$$

We obtain some characterizations of left-invariant Killing vector fields and prove Theorem 1.1 via Proposition 3.6 and Corollary 3.10.

Unless otherwise stated, in this section G denotes a semisimple Lie group equipped with a left-invariant Lorentzian metric q.

We begin with stating an equivalent algebraic condition on a left-invariant vector field on G to be Killing.

Lemma 3.1. *A vector field $z \in \mathfrak{g}$ is Killing if and only if $(\mathrm{ad}_z)^* = -\mathrm{ad}_z$, which amounts to: $A_\mathrm{q} \circ \mathrm{ad}_z = \mathrm{ad}_z \circ A_\mathrm{q}$. In particular, this implies $[z, A_\mathrm{q} z] = A_\mathrm{q}[z, z] = 0$.*

Proof. Let z be a Killing vector field. Then for every pair of left-invariant vector fields x, y on G one has

$$0 = (\mathcal{L}_z \,\mathrm{q})(x, y) = z\,\mathrm{q}(x, y) - \mathrm{q}([z, x], y) - \mathrm{q}([z, y], x), \tag{6}$$

where $\mathcal{L}_z$ is the Lie tensor derivation in the z-direction. The function $\mathrm{q}(x, y)$ is constant, so we have $0 = z\,\mathrm{q}(x, y)$ which yields $\mathrm{q}(\mathrm{ad}_z\, x, y) = \mathrm{q}(-\mathrm{ad}_z\, y, x)$. It then follows that

$$(\mathrm{ad}_z)^* = -\mathrm{ad}_z\,. \tag{7}$$

On the other hand, one has $(\mathrm{ad}_z)^* = -A_\mathrm{q}^{-1} \circ \mathrm{ad}_z \circ A_\mathrm{q}$. Comparing the last two equalities, one can see that $\mathrm{ad}_z \circ A_\mathrm{q} = A_\mathrm{q} \circ \mathrm{ad}_z$ is equivalent to (6) on $\mathfrak{X}_L(G)$. One can extend this equivalence to $\mathfrak{X}(G)$ using a global frame on G obtained from any basis for $\mathfrak{X}_L(G)$. ∎

Remark 3.2. The property $A_\mathrm{q} \circ \mathrm{ad}_z = \mathrm{ad}_z \circ A_\mathrm{q}$ means that $\mathrm{Im}(\mathrm{ad}_z)$ and $\mathrm{Ker}(\mathrm{ad}_z)$ are A_q-stable, and $\mathrm{Im}(\mathrm{ad}_z) \perp_\mathrm{q} \mathrm{Ker}(\mathrm{ad}_z)$. Since for every $x \in \mathrm{Ker}(\mathrm{ad}_z)$ and $y \in \mathfrak{g}$, we have $\mathrm{q}(x, [z, y]) = \mathrm{K}(A_\mathrm{q} x, [z, y]) = \mathrm{K}([A_\mathrm{q} x, z], y) = \mathrm{K}(A_\mathrm{q}[x, z], y) = 0$. ∎

On a semi-Riemannian manifold (M, q) with a Killing vector field Z, $\mathrm{q}(Z, \gamma'(t))$ is constant along a given geodesic $\gamma(t)$ ([16], Lemma 9.26). In the case of a semi-Riemannian Lie group with a left-invariant Killing vector field $Z \in \mathfrak{X}_L(G)$, this first integral, translated in the setting of the dual Euler equation, becomes the function $\mathrm{K}(\cdot, z)$. Proposition 3.3 proves directly that the latter is a first integral.

Proposition 3.3. *Let $z \in \mathfrak{g}$ be a Killing vector field. Then,* $\mathrm{K}(\cdot, z)$ *is a first integral of the dual Euler equation of* q*. Equivalently:* $\mathrm{K}([y, A_{\mathrm{q}}^{-1}y], z) = 0$ *for all* $y \in \mathfrak{g}$*.*

Proof. We need to show that $\mathrm{K}(u(t), z)$ is constant for any solution $u(t)$ of the dual Euler equation of q. Let $y \in \mathfrak{g}$, then, we have

$$\mathrm{K}([y, A_{\mathrm{q}}^{-1}y], z) = \mathrm{K}(y, [A_{\mathrm{q}}^{-1}y, z]),$$

since $A_{\mathrm{q}}^{-1} \circ \mathrm{ad}_z = \mathrm{ad}_z \circ A_{\mathrm{q}}^{-1}$ (Lemma 3.1) and A_{q}^{-1} is K-symmetric,

$$\mathrm{K}(y, [A_{\mathrm{q}}^{-1}y, z]) = \mathrm{K}(y, A_{\mathrm{q}}^{-1}[y, z]) = \mathrm{K}(A_{\mathrm{q}}^{-1}y, [y, z]).$$

Hence $$\mathrm{K}([y, A_{\mathrm{q}}^{-1}y], z) = \mathrm{K}(A_{\mathrm{q}}^{-1}y, [y, z]) = \mathrm{K}([A_{\mathrm{q}}^{-1}y, y], z),$$

and thus $$0 = \mathrm{K}([y, A_{\mathrm{q}}^{-1}y], z), \quad \text{for all } y \in \mathfrak{g}. \tag{8}$$

In particular, $$\frac{d}{dt}\mathrm{K}(u(t), z) = \mathrm{K}(u'(t), z) = \mathrm{K}([u(t), A_{\mathrm{q}}^{-1}u(t)], z) = 0,$$

which implies that $\mathrm{K}(u(t), z)$ is constant. ∎

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ has the lowest dimension among all (non-compact) simple Lie algebras and, up to isomorphism, it is the only three-dimensional simple Lie algebra of non-compact type. In the following proposition and subsequent examples, we see that, in the presence of a left-invariant Killing vector field, the behavior of a left-invariant Lorentzian metric in dimension three is different from that in higher dimensions. For this reason, we consider the case of dimension three separately and for the rest of the section we assume that $\dim(G) > 3$.

We show that in dimension three, the existence of a Killing vector field implies metric completeness.

Proposition 3.4. *Let* (G, q) *be a non-compact* 3*-dimensional Lorentzian simple Lie group. If there exists a left-invariant Killing vector field on* G*, then* q *is geodesically complete.*

Proof. Since the geodesic equation is presented in terms of the dual Euler equation on the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, we may, for simplicity, take $G = SL_2(\mathbb{R})$ as the corresponding Lie group of $\mathfrak{sl}_2(\mathbb{R})$.

Let $z \in \mathfrak{sl}_2(\mathbb{R})$ be a Killing vector field. If q is not complete, then according to Theorem 2 in [4] there exists a nilpotent $0 \neq x \in \mathfrak{sl}_2(\mathbb{R})$ such that $[x, A_{\mathrm{q}}^{-1}x] = x$. Take the basis (x, y, ξ) as in Remark 2.5, that is,

$$[\xi, x] = x, \quad [\xi, y] = -y, \quad [x, y] = 2\xi.$$

The above bracket relations can be used to check that $(x,\ y,\ \xi)$ is a pseudo-orthogonal basis for $\mathfrak{sl}_2(\mathbb{R})$ with respect to the Killing form K. By Proposition 3.3 one gets $\mathrm{K}(z, x) = \mathrm{K}(z, [x, A_{\mathrm{q}}^{-1}x]) = 0$, yielding $z = ax + b\xi$ for some $a, b \in \mathbb{R}$, and $[z, x] = bx$. We then obtain

$$\begin{aligned} bx = [z, x] &= [z, [x, A_{\mathrm{q}}^{-1}x]] = [[z, x], A_{\mathrm{q}}^{-1}x] + [x, [z, A_{\mathrm{q}}^{-1}x]] \\ &= b[x, A_{\mathrm{q}}^{-1}x] + [x, A_{\mathrm{q}}^{-1}[z, x]] = 2bx. \end{aligned}$$

which gives $b = 0$ and $z = ax$, implying that $0 = [z, A_{\mathrm{q}}^{-1}z] = a^2[x, A_{\mathrm{q}}^{-1}x] = a^2x$. Hence, $a = 0$ and, consequently, $z = 0$; a contradiction. ∎

Suppose that $(x,\ y,\ \xi)$ is a basis for $\mathfrak{sl}_2(\mathbb{R})$ as in the proof of Proposition 3.4. Let q be the left-invariant Lorentzian metric on $SL_2(\mathbb{R})$ whose associated isomorphism is given by

$$A_{\mathrm{q}}^{-1} = \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

with $0 \neq a, b$, in the basis $(x,\ y,\ \xi)$. Then one can see that ad_x commutes with A_{q} and, thus, by Lemma 3.1, the nilpotent element x defines a left-invariant Killing vector field on $SL_2(\mathbb{R})$. Similarly, if the metric q is associated to

$$A_{\mathrm{q}}^{-1} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix},$$

then the left-invariant vector field obtained from the diagonalizable element ξ is Killing. As the following proposition shows, such examples, namely, Killing vector fields generated by nilpotent or diagonalizable elements, exist just on $SL_2(\mathbb{R})$.

We denote by $c_{\mathfrak{g}}(z)$ the centralizer of z in $\mathfrak{g}$; $c_{\mathfrak{g}}(z) = \mathrm{Ker}(\mathrm{ad}_z) = \{x \in \mathfrak{g} : [z, x] = 0\}$ and $[z, \mathfrak{g}] = \mathrm{ad}_z(\mathfrak{g}) = \{[z, x] : x \in \mathfrak{g}\} = \mathrm{Im}(\mathrm{ad}_z)$.

Proposition 3.5. *Let G be a simple Lorentzian Lie group with* $\dim(G) > 3$. *If $z \in \mathfrak{g}$ is a non-zero Killing vector field, then,* ad_z *is neither diagonalizable nor nilpotent.*

Proof. Suppose that ad_z is diagonalizable. Then, one has the decomposition $\mathfrak{g} = c_{\mathfrak{g}}(z) \oplus [z, \mathfrak{g}]$. The operator ad_z is skew-adjoint with respect to K and thus the decomposition is K-orthogonal. Hence, K is non-degenerate on $c_{\mathfrak{g}}(z)$ and $[z, \mathfrak{g}]$. Moreover, by Remark 3.2, the decomposition is also orthogonal with respect to the Lorentzian metric q, which also shows that q is non-degenerate on $c_{\mathfrak{g}}(z)$ and $[z, \mathfrak{g}]$.

Let λ and μ be two non-zero eigenvalues of ad_z with eigenspaces W_λ and W_μ, respectively. Then for every $x \in W_\lambda$ and $y \in W_\mu$, we have, using Lemma 3.1

$$\lambda\, \mathrm{q}(x, y) = \mathrm{q}(\mathrm{ad}_z\, x, y) = \mathrm{q}(x, \mathrm{ad}_z^*\, y) = -\,\mathrm{q}(x, \mathrm{ad}_z\, y) = -\mu\, \mathrm{q}(x, y). \tag{9}$$

If $\lambda = \mu$, it follows that $\mathrm{q}(x, y) = 0$ and $\mathrm{q}(x, x) = 0$ for any $x, y \in W_\lambda$. This shows that W_λ is one-dimensional and also that q is Lorentzian on $[z, \mathfrak{g}]$.

On the other hand, if $\lambda \neq \mu$, then as above one gets $0 = \mathrm{q}(x, x) = \mathrm{q}(y, y)$ for any $x \in W_\lambda$ and $y \in W_\mu$, which since q is Lorentzian yields $\mathrm{q}(x, y) \neq 0$ and hence $\mu = -\lambda$. So, either ad_z has just one non-zero eigenvalue and, then, $c_{\mathfrak{g}}(z)$ is a codimension one subalgebra of $\mathfrak{g}$, or it has two non-zero eigenvalues $\pm\lambda$. In the latter case one can see that $\mathbb{R}x \oplus c_{\mathfrak{g}}(z)$, with $\mathrm{ad}_z\, x = \lambda x$, is a subalgebra of $\mathfrak{g}$ of codimension one. It then follows from Theorem 2.3 that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. This contradicts our assumption that $\dim(G) > 3$.

Now suppose that ad_z is nilpotent. Then, there exists some $m \geq 3$ so that $\mathrm{ad}_z^{m-1} \neq 0$ and $\mathrm{ad}_z^m = 0$. Let $k = \frac{m}{2}$ if m is even and $k = [\frac{m}{2}] + 1$ when m is odd. Then for every $x, y \in \mathfrak{g}$:

$$\mathrm{q}(\mathrm{ad}_z^k\, x, \mathrm{ad}_z^k\, y) = \mathrm{q}(x, (\mathrm{ad}_z^k)^* \circ \mathrm{ad}_z^k\, y) = \mathrm{q}(x, (-1)^k\, \mathrm{ad}_z^{2k}\, y) = 0.$$

So, $\mathrm{Im}(\mathrm{ad}_z^k)$ is a isotropic subspace of $\mathfrak{g}$ and, since q is Lorentzian, it must be a one-dimensional subspace. Note also that by Lemma 3.1, $\mathrm{q}(z, \mathrm{ad}_z^k x) = 0$ for all $x \in \mathfrak{g}$.

On the other hand, since $[z, A_\mathrm{q} z] = 0$ (Lemma 3.1), $[\mathrm{ad}_z, \mathrm{ad}_{A_\mathrm{q} z}] = \mathrm{ad}_{[z, A_\mathrm{q} z]} = 0$, we see that ad_z and $\mathrm{ad}_{A_\mathrm{q} z}$ commute and, therefore, $(\mathrm{ad}_z \circ \mathrm{ad}_{A_\mathrm{q} z})^m = \mathrm{ad}_z^m \circ \mathrm{ad}_{A_\mathrm{q} z}^m = 0$. Then $\mathrm{ad}_z \circ \mathrm{ad}_{A_\mathrm{q} z}$ is nilpotent. One then gets

$$\mathrm{q}(z, z) = \mathrm{K}(z, A_\mathrm{q} z) = \mathrm{tr}(\mathrm{ad}_z \circ \mathrm{ad}_{A_\mathrm{q} z}) = 0.$$

Thus, z is a lightlike element which is also orthogonal to $\mathrm{Im}(\mathrm{ad}_z^k)$ and, hence, $\mathrm{Im}(\mathrm{ad}_z^k) = \mathbb{R}z$. This, in particular, yields $\mathrm{ad}_z^{k+1} = 0$ which, according to the definitions of k and m, concludes that $m = k + 1$ and, consequently, we get $k = 2$ and $m = 3$. As a result, we get that the kernel of (the operator) $\mathrm{ad}_z^2 : \mathfrak{g} \to \mathfrak{g}$ is of codimension one.

Now we show that $\mathrm{Ker}(\mathrm{ad}_z^2)$ is a subalgebra of $\mathfrak{g}$. Again by Lemma 3.1, for any $y \in \mathrm{Ker}(\mathrm{ad}_z^2)$, we have $\mathrm{q}(\mathrm{ad}_z y, \mathrm{ad}_z y) = -\mathrm{q}(\mathrm{ad}_z^2 y, y) = 0$, and $\mathrm{q}(z, \mathrm{ad}_z y) = -\mathrm{q}(\mathrm{ad}_z^* z, y) = 0$. This, together with the fact that z is lightlike, gives $\mathrm{ad}_z y = \lambda z$ for some real number λ. Let $\mathrm{ad}_z y_i = \lambda_i y_i$ for $i = 1, 2$. Then by the Jacobi identity, we have

$$\mathrm{ad}_z[y_1, y_2] = [\mathrm{ad}_z y_1, y_2] + [y_1, \mathrm{ad}_z y_2] = 2\lambda_1\lambda_2 z.$$

So, $[y_1, y_2] \in \mathrm{Ker}(\mathrm{ad}_z^2)$ for every $y_1, y_2 \in \mathrm{Ker}(\mathrm{ad}_z^2)$, hence $\mathrm{Ker}(\mathrm{ad}_z^2)$ is a subalgebra of codimension one in $\mathfrak{g}$.

Now, having a codimension one subalgebra, Theorem 2.3 implies that $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, which is a contradiction. ∎

The following proposition gives Part (i) of Theorem 1.1.

Proposition 3.6. *Let (G, q) be a Lorentzian simple Lie group with $\dim(G) > 3$. If there exists a timelike Killing vector field $z \in \mathfrak{g}$, then solutions of the corresponding Euler equation are bounded. In particular, q is geodesically complete.*

Proof. Any orbit O of the Euler field remains in a quadric $Q = \{x : \mathrm{q}(x, x) = cst\}$; moreover, if z is a Killing field, by the remark before Proposition 3.3, it remains in a hyperplane $H = \{x : \mathrm{q}(z, x) = cst\}$. Now if z is timelike, H is spacelike, so $Q \cap H$ is a sphere. Hence O is bounded, thus complete and the metric is complete as well by Theorem 2.7. ∎

We now turn to the case where z is lightlike.

Lemma 3.7. *Let (G, q) be a Lorentzian simple Lie group with $\dim(G) > 3$. If there exists a lightlike Killing vector field $z \in \mathfrak{g}$, such that $A_\mathrm{q} z$ is lightlike, then, the solutions of the corresponding dual Euler equation are bounded. In particular, q is geodesically complete.*

Proof. Suppose that $z \in \mathfrak{g}$ is a Killing vector field and z and $A_\mathrm{q} z$ are both lightlike and the dual Euler equation has an unbounded solution. By Remark 2.10, there is a non-zero $\theta \in \mathfrak{g}$ such that $\mathrm{q}^*(\theta, \theta) = \mathrm{tr}(\mathrm{ad}_\theta^m) = 0$ for all m, in particular ad_θ is nilpotent. Also by Proposition 3.3, $\mathrm{K}(z, .)$ is a first integral, then $\mathrm{K}(z, \theta) = 0$. So we have $\mathrm{q}(A_\mathrm{q}^{-1}\theta, A_\mathrm{q}^{-1}\theta) = \mathrm{q}^*(\theta, \theta) = 0$ and $\mathrm{q}(A_\mathrm{q}^{-1}\theta, z) = \mathrm{K}(\theta, z) = 0$, thus $A_\mathrm{q}^{-1}\theta$ is lightlike in $z^\perp$. Now q is Lorentzian and z is lightlike, therefore $A_\mathrm{q}^{-1}\theta = \lambda z$ for some $\lambda \in \mathbb{R}^*$, i.e. $\theta = \lambda A_\mathrm{q} z$.

Since $\mathrm{q}(A_{\mathrm{q}}z, z) = \mathrm{K}(A_{\mathrm{q}}z, A_{\mathrm{q}}z) = \lambda^{-2}\,\mathrm{K}(\theta,\theta) = \lambda^{-2}\,\mathrm{tr}(\mathrm{ad}_\theta^2) = 0$, z and $A_{\mathrm{q}}z$ are orthogonal to each other and lightlike vectors. As q is Lorentzian, they are collinear. Therefore z and θ are collinear, so that z is nilpotent, contradicting Proposition 3.5. Thus, all solutions of the dual Euler equation are bounded and by Theorem 2.7, q is complete. ■

An immediate consequence of the above proposition is the following corollary.

Corollary 3.8. *Let (G,q) be a Lorentzian simple Lie group with $\dim(G) > 3$. If there exists a lightlike Killing vector field $z \in \mathfrak{g}$ such that z is an eigenvector of A_{q}, then* q *is geodesically complete.*

As we saw in Proposition 3.5, diagonalizable and nilpotent vectors in T_eG can not be extended to a left-invaraint Killing vector field on G. In the next result we see that only compact elements can generate a left-invariant lightlike Killing vector field.

Lemma 3.9. *Let (G,q) be a Lorentzian simple Lie group with $\dim(G) > 3$. If $z \in \mathfrak{g}$ is a non-spacelike Killing vector field, then z is a compact element.*

Proof. Recall that the centralizer $c_{\mathfrak{g}}(z)$ and $\mathrm{Im}(\mathrm{ad}_z) = [z,\mathfrak{g}]$ are perpendicular with respect to the Killing form, i.e. $c_{\mathfrak{g}}(z)^{\perp_{\mathrm{K}}} = [z,\mathfrak{g}]$. We also know from Proposition 3.5 that z is not nilpotent. Hence by part (iii) in Remark 2.4, $z \notin [z,\mathfrak{g}]$. Then there is an $x_0 \in c_{\mathfrak{g}}(z)$ such that $0 \neq \mathrm{K}(x_0, z) = \mathrm{q}({A_{\mathrm{q}}}^{-1}x_0, z)$. Since z is non-spacelike, $\mathrm{span}\{{A_{\mathrm{q}}}^{-1}x_0, z\}$ is Lorentzian. By Remark 3.2, $[z,\mathfrak{g}] \subseteq (\mathrm{span}\{{A_{\mathrm{q}}}^{-1}x_0, z\})^{\perp_{\mathrm{q}}}$, thus the subspace $[z,\mathfrak{g}]$ is spacelike.

The metric q is positive definite on $[z,\mathfrak{g}]$ and the operator ad_z, being skew-symmetric with respect to q, has only purely imaginary eigenvalues on $[z,\mathfrak{g}]$, so it is compact. ■

The following corollary proves the second part of Theorem 1.1.

Corollary 3.10. *Let (G,q) be a Lorentzian simple Lie group with $\dim(G) > 3$. If G is strongly causal and there exists a left-invariant lightlike Killing vector field z on G, then* q *is geodesically complete.*

Proof. If q is incomplete then it follows from Lemma 3.9 that z is a compact element, that is, the one-parameter subgroup $\mathrm{Ad}(\exp(tz))$ is included in a compact subgroup of $\mathrm{Ad}(G)$. Since G is simple, it is a finite covering group of $\mathrm{Ad}(G)$, ([7], Ch. 1, 1.3). Therefore the causal curve $\exp(tz)$ is also included in a compact subset of G, which is a contradiction by Proposition 2.1, since G is strongly causal. ■

The sole existence of a left-invariant lightlike Killing vector field does not imply the completeness of the metric. In the following example we define an incomplete metric which has a left-invariant lightlike Killing vector field. This shows that the space of incomplete left-invariant metrics as a subspace of all left-invariant metrics on a simple Lie group is not open in general.

Example 3.11. Let $\mathfrak{g} = \mathfrak{o}(4,1)$ be the Lie algebra of the isometry group of the 4-dimensional hyperbolic space and $z \in \mathfrak{g}$ a unit compact element. Then $c_{\mathfrak{g}}(z) \cong \mathfrak{o}(2,1) \oplus \mathbb{R}z \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}z$ and $[z,\mathfrak{g}]$ has a decomposition $[z,\mathfrak{g}] = W_1 \oplus W_2$ where W_1 and W_2 are ad_z-invariant subspaces and the Killing form is positive definite on W_1 and negative definite on W_2. Take the basis (x,y,ξ) for $\mathfrak{sl}_2(\mathbb{R})$ as in Remark 2.5. Then the Killing form on $c_{\mathfrak{g}}(z)$ is given by

$$K|_{\mathbb{R}z\oplus\mathbb{R}y\oplus\mathbb{R}x\oplus\mathbb{R}\xi} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Define q_0 to be the left-invariant Lorentzian metric on $G = O(4,1)$ with the associated operator A_0 given by

$$A_0|_{W_1} = \mathrm{Id}, \quad A_0|_{W_2} = -\mathrm{Id}, \quad A_0|_{\mathbb{R}z\oplus\mathbb{R}y\oplus\mathbb{R}x\oplus\mathbb{R}\xi} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The system of the dual Euler equations on the invariant subspace $\mathrm{span}\{x,y,\xi,z\}$, in the coordinates (x_0,y_0,ξ_0,z_0) corresponding to the basis (x,y,ξ,z), reduces to

$$\begin{aligned} x_0'(t) &= \xi_0(t)(y_0(t) - x_0(t)), \\ y_0'(t) &= \xi_0(t)(y_0(t) + z_0(t)), \\ \xi_0'(t) &= -2x_0(t)z_0(t) - 2y_0^2(t), \\ z_0'(t) &= 0, \end{aligned}$$

with an incomplete solution as follows

$$x_0(t) = 1 - \frac{2}{(1-t)^2}, \quad y_0(t) = -z_0(t) = 1, \quad \xi_0(t) = \frac{2}{t-1}.$$

Note that z is a left-invariant lightlike Killing vector field with respect to q_0.

The space $\mathcal{M}$ of all left-invariant semi-Riemannian metrics for which z is a Killing vector field can be described as the space of matrices A_{q} satisfying $[\mathrm{ad}_z, A_{\mathrm{q}}] = 0$ and thus $\mathcal{M}$ is a submanifold of the space of all left-invariant semi-Riemannian metrics on G. We show that the space U_- of all incomplete left-invariant metrics on G, is not an open subset. We do this by showing that $U_- \cap \mathcal{M}$ is not open in $\mathcal{M}$.

Define $A_\epsilon := A_0 + \epsilon dz \otimes z$, for each $\epsilon > 0$. A_ϵ represents a left-invariant Lorentzian metric $\mathrm{q}_\epsilon = \mathrm{q}_0 - \epsilon dz^2$ on G for which z is a left-invariant timelike Killing vector field and by Theorem 1.1, $\mathrm{q}_\epsilon \in U_+ \cap \mathcal{M}$ (U_+ the space of all complete left-invariant metrics on G with bounded integral curves of the Euler field). On the other hand, from its definition $\mathrm{q}_0 \in U_- \cap \mathcal{M}$ and $\mathrm{q}_{\frac{1}{n}} \to \mathrm{q}_0$ which implies that $U_- \cap \mathcal{M} \subset \mathcal{M}$ is not open. ■

We conclude this section with the following remark.

Remark 3.12. Suppose that there exists a left-invariant lightlike Killing vector field z on a Lorentzian simple Lie group (G,q). Then by Lemma 3.1, $F_{\mathrm{q}}(A_{\mathrm{q}}(z)) = [z, A_{\mathrm{q}}z] = 0$. Hence, $A_{\mathrm{q}}z$ is an equilibrium point of the Hamiltonian dynamical system associated with the metric q on $\mathfrak{g}$. Moreover, it can be seen from the proof of Lemma 3.7 that every unbounded trajectory of the dual Euler equation is 'asymptotically' tangent to the line spanned by $A_{\mathrm{q}}z$. ■

4. Left-invariant Lorentzian metrics on $SL_2(\mathbb{C})$

In this section, we study completeness of left-invariant Lorentzian metrics on $SL_2(\mathbb{C})$ and prove Theorem 1.2 via Proposition 4.8.

The real Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the real vector space

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\},$$

equipped with the Lie bracket $[x, y] = xy - yx$. We begin by proving the following proposition which shows that Corollary 3.10 holds for $SL_2(\mathbb{C})$ without the assumption of strong causality.

Proposition 4.1. *Let* q *be a left-invariant Lorentzian metric on* $SL_2(\mathbb{C})$. *If there exists a left-invariant causal Killing vector field on* $SL_2(\mathbb{C})$, *then solutions of the dual Euler equation are bounded and, in particular,* q *is geodesically complete.*

Proof. Let $z \in \mathfrak{sl}_2(\mathbb{C})$ be a Killing vector field such that $\mathrm{q}(z, z) \leq 0$. By Proposition 3.6, we just need to consider the case $\mathrm{q}(z, z) = 0$. In this case, as in the proof of Lemma 3.7, it follows that if there exists an unbounded solution for the dual Euler equation, then $A_{\mathrm{q}} z$ is nilpotent. On the other hand, since, by Lemma 3.1, $A_{\mathrm{q}} \circ \mathrm{ad}_z = \mathrm{ad}_z \circ A_{\mathrm{q}}$, we get $[z, A_{\mathrm{q}} z] = 0$, which means that $A_{\mathrm{q}} z = \lambda z$ for some $\lambda \in \mathbb{C}$, since in $\mathfrak{sl}_2(\mathbb{C})$ only $\mathbb{C}$-linearly dependent elements commute. So, z is also nilpotent; contrary to Proposition 3.5. Hence, any solution of the dual Euler equation is bounded and q is geodesically complete by Theorem 2.7. ∎

Remark 4.2. When investigating the completeness of a left-invariant metric q, one can study $C_g^*(\mathrm{q})$, the pullback of q by an inner automorphism C_g of G, where $C_g(h) = ghg^{-1}$. If $x(t)$ is a solution of the dual Euler field of q, then $\mathrm{Ad}_{g^{-1}} x(t)$ is a solution of the dual Euler field of $C_g^*(\mathrm{q})$. ∎

In the following lemma we obtain representations for A_{q}^{-1} and ad_z which are more convenient to use. In particular, Lemma 3.9 holds for any Killing vector field on $SL_2(\mathbb{C})$.

Lemma 4.3. *Let* z *be a left-invariant Killing vector field on* $(SL_2(\mathbb{C}), \mathrm{q})$ *and* $c_{\mathfrak{g}}(z)$ *be the centralizer of* z *in* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. *Then one has,*

(i) $\mathfrak{sl}_2(\mathbb{C})$ *is decomposed orthogonally as* $c_{\mathfrak{g}}(z) \oplus [z, \mathfrak{sl}_2(\mathbb{C})]$ *with respect to* K *and* q. *In particular,* q *is positive definite on* $[z, \mathfrak{sl}_2(\mathbb{C})]$ *and* z *is a compact element of* $\mathfrak{sl}_2(\mathbb{C})$, *i.e.* $z = c\sqrt{-1}\xi = c\sqrt{-1}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ *with* $c \in \mathbb{R}^*$.

(ii) *In a basis* $(e, [\sqrt{-1}\xi, e], \mu e, \mu[\sqrt{-1}\xi, e])$ *for* $[z, \mathfrak{sl}_2(\mathbb{C})]$, *where* μ *is an appropriate unit complex number,* K *reads* $\mathrm{diag}(1, 1, -1, -1)$ *and* ad_z *and* A_{q}^{-1} *read:*

$$\mathrm{ad}_z \big|_{[z,\mathfrak{g}]} = \left(\begin{array}{cc|cc} 0 & -c & & \\ c & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & -c \\ \multicolumn{2}{c|}{0} & c & 0 \end{array}\right), \quad A_{\mathrm{q}}^{-1}\big|_{[z,\mathfrak{g}]} = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & b & \\ 0 & & & b \end{pmatrix}, \quad ab < 0. \tag{10}$$

Moreover, in the basis $(\xi, \sqrt{-1}\xi)$ *of* $c_{\mathfrak{g}}(z)$, A_{q}^{-1} *reads:*

$$A_{\mathrm{q}}^{-1}\big|_{c_{\mathfrak{g}}(z)} = \begin{pmatrix} d_1 & d_2 \\ -d_2 & d_3 \end{pmatrix}, \tag{11}$$

where $d_1 < 0$ *(resp.* $d_1 = 0$, $d_1 > 0$*) when* z *is spacelike (resp. lightlike, timelike.)*

Proof. From Proposition 3.5 we know that ad_z is not nilpotent as an $\mathbb{R}$-linear transformation, therefore, is not nilpotent as a $\mathbb{C}$-linear transformation. So, by part (i) of Remark 2.5 and by Remark 4.2, up to conjugation by some element g of $SL_2(\mathbb{C})$, one may assume that z is represented by $z = s_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, where $\lambda \in \mathbb{C}^*$. Then one has

$$\mathfrak{sl}_2(\mathbb{C}) = c_{\mathfrak{g}}(z) \oplus [z, \mathfrak{sl}_2(\mathbb{C})] = c_{\mathfrak{g}}(\xi) \oplus [\xi, \mathfrak{sl}_2(\mathbb{C})].$$

Hence, by part (iii) of Remark 2.5, and Remark 3.2, the decomposition is K- and q-orthogonal.

It can be seen that $\mathrm{ad}_z^2 = \lambda^2 \mathrm{ad}_\xi^2 = \lambda^2 \,\mathrm{Id}$ on $[z, \mathfrak{sl}_2(\mathbb{C})]$. Also ad_z^2 is symmetric with respct to q. Thus, for every $x, y \in [z, \mathfrak{sl}_2(\mathbb{C})]$, $\mathrm{q}(\lambda^2 x, y) = \mathrm{q}(x, \lambda^2 y)$. Now suppose that $\mathrm{Im}(\lambda^2) \neq 0$, i.e. $\lambda^2 \notin \mathbb{R}$, then by Remark 2.5(iii), we have

$$\mathrm{K}(x, \sqrt{-1}A_{\mathrm{q}}y) = \mathrm{K}(\sqrt{-1}x, A_{\mathrm{q}}y) = \mathrm{q}(\sqrt{-1}x, y) = \mathrm{q}(x, \sqrt{-1}y) = \mathrm{K}(x, A_{\mathrm{q}}\sqrt{-1}y),$$

so $\sqrt{-1} \circ A_{\mathrm{q}} = A_{\mathrm{q}} \circ \sqrt{-1}$ on $[z, \mathfrak{sl}_2(\mathbb{C})]$. Besides, $c_{\mathfrak{g}}(z)$ is A_{q}-stable (Remark 3.2) and on it, $\mathrm{ad}_{\sqrt{-1}z} = 0$, hence on $\mathfrak{sl}_2(\mathbb{C})$, we have

$$\mathrm{ad}_{\sqrt{-1}z} \circ A_{\mathrm{q}} = \sqrt{-1} \circ \mathrm{ad}_z \circ A_{\mathrm{q}} = \sqrt{-1} \circ A_{\mathrm{q}} \circ \mathrm{ad}_z = A_{\mathrm{q}} \circ \sqrt{-1} \circ \mathrm{ad}_z = A_{\mathrm{q}} \circ \mathrm{ad}_{\sqrt{-1}z}.$$

By Lemma 3.1, this is equivalent to $\sqrt{-1}Z$ being a Killing vector field, and so is $\xi \in \mathrm{span}_{\mathbb{R}}\{z, \sqrt{-1}z\}$. But ξ is also diagonal, which is a contradiction by Proposition 3.5. Therefore, $\lambda^2 \in \mathbb{R}$, and again by Proposition 3.5, λ can not be a real number, so $\lambda \in \sqrt{-1}\mathbb{R}$ and z is compact.

Now we show that q is positive definite on $[z, \mathfrak{sl}_2(\mathbb{C})]$. Let x be a timelike vector on $[z, \mathfrak{sl}_2(\mathbb{C})]$. Since $(\mathrm{ad}_z)^* = -\mathrm{ad}_z$ and $\lambda^2 \in -\mathbb{R}^+$, we have

$$\mathrm{q}(\mathrm{ad}_z x, \mathrm{ad}_z x) = -\mathrm{q}(\mathrm{ad}_z^2 x, x) = -4\lambda^2 \mathrm{q}(x, x) < 0 \quad \text{and} \quad \mathrm{q}(\mathrm{ad}_z x, x) = 0.$$

So one can see that q is negative definite on $\mathrm{span}_{\mathbb{R}}\{x, \mathrm{ad}_z x\}$, which is a two dimensional subspace of $[z, \mathfrak{sl}_2(\mathbb{C})]$. This can't happen because index of q is at most one on $[z, \mathfrak{sl}_2(\mathbb{C})]$. Hence for every $x \in [z, \mathfrak{sl}_2(\mathbb{C})]$, $\mathrm{q}(x, x) > 0$. This proves part (i).

Since A_{q}^{-1} is q-symmetric and q is positive definite on $[z, \mathfrak{sl}_2(\mathbb{C})]$, there is an orthogonal basis (e_1, e_2, e_3, e_4) of $[z, \mathfrak{sl}_2(\mathbb{C})]$ such that $A_{\mathrm{q}}^{-1} = \mathrm{diag}(a_1, a_2, a_3, a_4)$.

Let W_i be the eigenspace associated with the eigenvalue a_i. For any $x \in W_i \setminus \{0\}$, we have

$$a_i \mathrm{K}(x, x) = \mathrm{K}(x, A_{\mathrm{q}}^{-1}x) = \mathrm{q}(A_{\mathrm{q}}^{-1}x, A_{\mathrm{q}}^{-1}x) > 0.$$

On the other hand, since $A_{\mathrm{q}} \circ \mathrm{ad}_z = \mathrm{ad}_z \circ A_{\mathrm{q}}$, we get $A_{\mathrm{q}}^{-1}[z, x] = [z, A_{\mathrm{q}}^{-1}x] = a_i[z, x]$, which implies that $[z, x] \in W_i$. The vectors x and $[z, x]$ are linearly independent, for otherwise, x is nilpotent (Remark 2.4, (iii)) and one has $\mathrm{K}(x,x)=0$. In particular, $\dim(W_i)$ is at least 2. Therefore $A_{\mathrm{q}}^{-1}=\mathrm{diag}(a,a,b,b)$ or $A_{\mathrm{q}}^{-1}=\mathrm{diag}(a,a,a,a)$. Since $\mathrm{sign}(\mathrm{q})=(0,4)$ (q is positive definite) and $\mathrm{sign}(\mathrm{K})=(2,2)$ on $[\xi,\mathfrak{sl}_2(\mathbb{C})]$, $A_{\mathrm{q}}^{-1}=\mathrm{diag}(a,a,b,b)$ with $ab<0$, in an orthogonal basis (e_1,e_2,e_3,e_4) with $e_2=[\sqrt{-1}\xi,e_1]$ and $e_4=[\sqrt{-1}\xi,e_3]$. Thus

$$[e_1+e_3,A_{\mathrm{q}}^{-1}(e_1+e_3)]=(b-a)[e_1,e_3],\quad [e_1+e_4,A_{\mathrm{q}}^{-1}(e_1+e_4)]=(b-a)[e_1,e_4]. \quad (12)$$

From part (ii) of Remark 2.5, we know that for any $x,y\in[\xi,\mathfrak{sl}_2(\mathbb{C})]$, $[x,y]\in c_{\mathfrak{g}}(\xi)=\mathrm{span}_{\mathbb{R}}\{\xi,\sqrt{-1}\xi\}$, and from (8) and (12), we get $0=\mathrm{K}(z,[e_1,e_3])=\mathrm{K}(z,[e_1,e_4])$. This together with the non-degeneracy of K on $c_{\mathfrak{g}}(\xi)$ implies that $[e_1,e_3]$ and $[e_1,e_4]$ are $\mathbb{R}$-linearly dependent. Thus, $[e_1,c_3e_3+c_4e_4]=0$ for some $(c_3,c_4)\in\mathbb{R}^2$, and it follows (Remark 2.5, (ii)) that $\mu e_1\in\mathrm{span}_{\mathbb{R}}\{e_3,e_4\}$ for some unit element $\mu=\mu_1+\sqrt{-1}\mu_2\in\mathbb{C}\setminus\mathbb{R}$, in particular $\mathrm{K}(e_1,\mu e_1)=0$.

So, we can use $(e_1,\ [\sqrt{-1}\xi,e_1],\ \mu e_1,\ \mu[\sqrt{-1}\xi,e_1])$ as a K-orthonormal basis for $[z,\mathfrak{sl}_2(\mathbb{C})]$. Thus, in this basis ad_z and A_{q}^{-1} admit the representations in (10).

In the basis $(\xi,\sqrt{-1}\xi)$ of $c_{\mathfrak{g}}(z)$, in which $\mathrm{Mat}(\mathrm{K}\,|_{c_{\mathfrak{g}}(z)})=\mathrm{diag}(1,-1)$ (see Remark 2.5 (iii)), one has:

$$A_{\mathrm{q}}^{-1}\big|_{c_{\mathfrak{g}}(z)}=\begin{pmatrix} d_1 & d_2\\ -d_2 & d_3\end{pmatrix},\quad A_{\mathrm{q}}\big|_{c_{\mathfrak{g}}(z)}=\frac{1}{\delta}\begin{pmatrix} d_3 & -d_2\\ d_2 & d_1\end{pmatrix},\quad \mathrm{q}\,\big|_{c_{\mathfrak{g}}(z)}=\frac{1}{\delta}\begin{pmatrix} d_3 & -d_2\\ -d_2 & -d_1\end{pmatrix}$$

where δ denotes $\det(A_{\mathrm{q}}^{-1}|_{c_{\mathfrak{g}}(z)})$. As q, hence also q^*, is Lorentzian on $c_{\mathfrak{g}}(z)$, the determinant of $\mathrm{Mat}(\mathrm{q}^*)=\mathrm{Mat}(\mathrm{K}).\mathrm{Mat}(A_{\mathrm{q}}^{-1})$ on $c_{\mathfrak{g}}(z)$ is negative, namely: $\delta\det(\mathrm{diag}(1,-1))<0$. So $\delta>0$. Now we read on Mat(q) that $\mathrm{q}(z,z)=-\frac{d_1}{\delta}$; its sign is opposite to that of d_1, which gives the result. ∎

Remark 4.4. Take the decomposition

$$\mathrm{span}_{\mathbb{R}}\{\xi,\sqrt{-1}\xi\}\oplus\mathrm{span}_{\mathbb{R}}\{e,[\sqrt{-1}\xi,e]\}\oplus\mathrm{span}_{\mathbb{R}}\{\mu e,[\sqrt{-1}\xi,\mu e]\}$$

of $\mathfrak{sl}_2(c)$ as in Lemma 4.3, we observed that its components are (i) invariant under A_{q} and (ii) orthogonal to each other with respect to K and q. As a matter of fact, for any pair of symmetric (or alternate) bilinear form $\{A,B\}$ with A non-degenerate on a vector space V, there is a unique and maximal decomposition on V satisfying properties (i), (ii) and more (see [13]). ∎

Corollary 4.5. *Let* q *be a left-invariant Lorentzian metric on* $SL_2(\mathbb{C})$. *Then the dimension of the Lie algebra of the left-invariant Killing vector fields on* $(SL_2(\mathbb{C}),\mathrm{q})$ *is at most one.*

Proof. Denote the Lie algebra of the left-invariant Killing vector fields by $\mathfrak{h}$. The signature of the Killing form K is $(3,3)$ on $\mathfrak{sl}_2(\mathbb{C})$ and, by Lemma 4.3, K is negative definite on $\mathfrak{h}$, so $\dim(\mathfrak{h})\le 3$. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ does not have any compact subalgebra with dimension 2. Now suppose $\dim(\mathfrak{h})=3$. Three dimensional compact Lie algebras are isomorphic to $\mathfrak{o}(3)$, [4]. So without loss of generality we can assume

$$\mathfrak{h}=\mathrm{span}_{\mathbb{R}}\left\{e_1=\begin{pmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\end{pmatrix},\ e_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ e_3=\begin{pmatrix}0 & \sqrt{-1}\\ \sqrt{-1} & 0\end{pmatrix}\right\}.$$

One can see that $\mathfrak{h}^{\perp_\kappa} = \sqrt{-1}\mathfrak{h}$, $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{h} \oplus \sqrt{-1}\mathfrak{h}$ and $[x, y] \neq 0$ for every non collinear $x, y \in \mathfrak{h}$. By Lemma 3.1 we obtain $[x, A_\mathrm{q}x] = A_\mathrm{q}[x, x] = 0$ and $[x, A_\mathrm{q}\sqrt{-1}x] = A_\mathrm{q}[x, \sqrt{-1}x] = 0$ for every $x \in \mathfrak{h}$. Hence

$$A_\mathrm{q}(x), A_\mathrm{q}(\sqrt{-1}x) \in c_\mathfrak{g}(x) = \mathrm{span}_\mathbb{R}\{x, \sqrt{-1}x\},$$

i.e. $A_\mathrm{q}x = \lambda x$ and $A_\mathrm{q}\sqrt{-1}x = \gamma\sqrt{-1}x$ for some $(\lambda, \gamma) \in \mathbb{C}^2$. Thus the decomposition $\mathfrak{sl}_2(\mathbb{C}) = \oplus_{i=1}^3 c_\mathfrak{g}(e_i)$ is q-orthogonal, and also by Lemma 4.3, q is Lorentzian on each $c_\mathfrak{g}(e_i)$. Therefore $\mathrm{sign}(\mathrm{q}) = (3, 3)$ on $\mathfrak{sl}_2(\mathbb{C})$, and q is not Lorentzian. ■

As mentioned in Section 2, an idempotent of the dual Euler field F_q is an element $x \in \mathfrak{g} \cong T_eG$ which is the initial velocity of a non-constant geodesic γ starting at $e \in G$ and running on a one parameter subgroup of G. In this case, the graph of the reflected curve $u(t)$ of $\gamma(t)$ is just the radial half-line $\mathbb{R}^+x$.
To establish our next result, we introduce a family of curves which includes the above mentioned reflected curves as special cases.

Definition 4.6. Let F be a homogeneous quadratic vector field on a vector space $\mathbb{V}$, and $u(t)$ be an integral curve of F. Then, $u(t)$ is called a generalized conical spiral (GCS) of F, if there exist $r, s \in \mathbb{R}$ with $r > 0$, so that $u(s) = ru(0)$. ■

Note that since F is homogeneous quadratic, one gets $u(ks) = r^k u(0)$ for any positive integer k. Here are some features of GCS curves.

Proposition 4.7. *Let F be a homogeneous quadratic vector field on a vector space $\mathbb{V}$. Let $x \in \mathbb{V}$ and $u : I \subset \mathbb{R} \to \mathbb{V}$ be a GCS of F with $u(s) = ru(0)$ for some $r, s \in \mathbb{R}$. Then:*

(i) *if $F(x) = x$, then $\mathbb{R}x$ is a GCS for any $s > 0$, with $r = \dfrac{1}{1-s}$,*
(ii) *if $r = 1$, the curve $u(t)$ is an ordinary s-periodic curve and, therefore, complete,*
(iii) *if $r \neq 1$, the curve $u(t)$ is incomplete.*

Moreover, when $\mathbb{V} = \mathfrak{g}$ is a Lie algebra with Lie group G and $F = F_\mathrm{q}$ is the dual Euler field of a left-invariant Lorentzian metric q *on G, we have*

(iv) *if $r \neq 1$, the curve $u(t)$ is nilpotent and $\mathrm{q}^*(u(t), u(t)) = 0$ for any $t \in I$.*

Proof. Parts (i) and (ii) are trivial. For part (iii), let $r > 1$ and $[0, b)$ be the maximal domain of the curve $u(t)$. For each positive integer k, define $\tilde{u} := u\big|_{[0,s]}$ and let $s_k = \sum_{j=0}^{k-1}(s/r^j)$. Then $u(t) = r^k\tilde{u}(r^k(t - s_k))$, where $t \in [s_k, s_k + (s/r^k)]$. This yields $b = sr/(r-1)$. So, $u(t)$ is incomplete.

Similarly, if $r < 1$ then one can see that $u(t)$ cannot be extended beyond a finite time in the negative direction.

Finally, to prove (iv) let $u(t)$ be a GCS curve of the dual Euler field $F_\mathrm{q}(x) = [x, A_\mathrm{q}x]$. We have $\mathrm{q}^*(u(s), u(s)) = r^2\,\mathrm{q}^*(u(0), u(0))$. On the other hand, since $\mathrm{q}^*(x, x)$ is a first integral, one gets $\mathrm{q}^*(u(s), u(s)) = \mathrm{q}^*(u(0), u(0))$. Putting together, we conclude that $(r^2 - 1)\,\mathrm{q}^*(u(0), u(0)) = 0$. Since $r \neq 1$, one has $\mathrm{q}^*(u(0), u(0)) = 0$. Since $\mathrm{q}^*(u(t), u(t))$ is constant, we get $\mathrm{q}^*(u(t), u(t)) = 0$ for any $t \in I$.

In a similar way, one can use the first integrals $\mathrm{tr}(\mathrm{ad}^m_{u(t)})$ to show that $u(t)$ is nilpotent. ■

By Lemma 4.3, if $\mathrm{q}(z, \sqrt{-1}z) = 0$, then $d_2 = 0$ and $A_{\mathrm{q}}^{-1} = \mathrm{diag}(d_1, d_3, a, a, b, b)$ in the basis $(\xi, \sqrt{-1}\xi, e, [\sqrt{-1}\xi, e], \mu e, \mu[\sqrt{-1}\xi, e])$. Now we restate Theorem 1.2 in the setting of Lemma 4.3.

Proposition 4.8. *Let* q *be a left-invariant Lorentzian metric on* $SL_2(\mathbb{C})$ *with* $\sqrt{-1}\xi$ *as a compact spacelike Killing vector field, and* $A_{\mathrm{q}}^{-1}\sqrt{-1}\xi = d_3\sqrt{-1}\xi$. *Then* q *is geodesically complete if and only if its dual Euler field has no incomplete GCS.*

We prove Proposition 4.8 via Lemmas 4.9-4.11. First in Lemma 4.9 we consider solutions of the dual Euler equation in the zero level set $\mathrm{K}(x, \sqrt{-1}\xi) = 0$, which is decomposed as $\mathbb{R}\xi \oplus [\xi, \mathfrak{sl}_2(\mathbb{C})]$. Such a solution can be written as $u(t) = f(t)\xi + v(t)$ where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth function and $v : I \to [\xi, \mathfrak{sl}_2(\mathbb{C})]$ is the projection of $u(t)$ on $[\xi, \mathfrak{sl}_2(\mathbb{C})]$ by the projection map $\pi : \mathbb{R}\xi \oplus [\xi, \mathfrak{sl}_2(\mathbb{C})] \to [\xi, \mathfrak{sl}_2(\mathbb{C})]$.

Lemma 4.9. *The dual Euler equation of* q *on the hyperplane* $\mathrm{K}(\sqrt{-1}\xi, x) = 0$ (*that is stable under the dual Euler flow by Proposition* 3.3) *can be written as*

$$[u, A_{\mathrm{q}}^{-1}u] = \varphi(v)\xi + fV(v), \tag{13}$$

where $\varphi : [\xi, \mathfrak{sl}_2(\mathbb{C})] \to \mathbb{R}$, $v = \pi(u)$, $f = \mathrm{K}(u, \xi)/\,\mathrm{K}(\xi, \xi)$ *and* V *is a linear operator on* $[\xi, \mathfrak{sl}_2(\mathbb{C})]$. *Furthermore, if the system* $y' = V(y)$ *has unbounded solutions, then the solutions of the dual Euler equation on the hyperplane* $\mathrm{K}(x, \sqrt{-1}\xi) = 0$ *are also unbounded.*

Proof. Given a solution $u(t) = f(t)\xi + v(t)$ of the dual Euler equation, we have

$$[u, A_{\mathrm{q}}^{-1}u] = f^2[\xi, A_{\mathrm{q}}^{-1}\xi] + f[v, A_{\mathrm{q}}^{-1}\xi] + f[\xi, A_{\mathrm{q}}^{-1}v] + [v, A_{\mathrm{q}}^{-1}v]. \tag{14}$$

In equation (14) one has $[\xi, A_{\mathrm{q}}^{-1}\xi] = 0$ and $[v, A_{\mathrm{q}}^{-1}\xi], [\xi, A_{\mathrm{q}}^{-1}v] \in [\xi, \mathfrak{sl}_2(\mathbb{C})] = c_{\mathfrak{g}}(\xi)^{\perp}$ with respect to K and $[v, A_{\mathrm{q}}^{-1}v] \in c_{\mathfrak{g}}(\xi) = \mathrm{span}_{\mathbb{R}}\{\xi, \sqrt{-1}\xi\}$ (that comes from Remark 2.5(ii) and A_{q}-stability of $[\xi, \mathfrak{sl}_2(\mathbb{C})]$ by Lemma 4.3). Since $\sqrt{-1}\xi$ is a Killing vector field, by Proposition 3.3 we have $0 = \mathrm{K}(\sqrt{-1}\xi,\ [u, A_{\mathrm{q}}^{-1}u]) = \mathrm{K}(\sqrt{-1}\xi, [v, A_{\mathrm{q}}^{-1}v])$ which yields $[v, A_{\mathrm{q}}^{-1}v] = \varphi(v)\xi$ for some $\varphi : [\xi, \mathfrak{sl}_2(\mathbb{C})] \to \mathbb{R}$. Thus, (14) could be rewritten as

$$[u, A_{\mathrm{q}}^{-1}u] = \varphi(v)\xi + f\left([v, A_{\mathrm{q}}^{-1}\xi] + [\xi, A_{\mathrm{q}}^{-1}v]\right). \tag{15}$$

We denote by V the linear vector field on $[\xi, \mathfrak{sl}_2(\mathbb{C})]$ obtained from the second term of (15), that is,

$$V : [\xi, \mathfrak{sl}_2(\mathbb{C})] \to [\xi, \mathfrak{sl}_2(\mathbb{C})], \quad y \mapsto [\xi, (A_{\mathrm{q}}^{-1} - d_1\,\mathrm{Id})y] + [\sqrt{-1}\xi, d_2 y], \tag{16}$$

in which $A_{\mathrm{q}}^{-1}\xi = (d_1 - \sqrt{-1}d_2)\xi$. In the above formula, $\mathrm{Id} : [\xi, \mathfrak{sl}_2(\mathbb{C})] \to [\xi, \mathfrak{sl}_2(\mathbb{C})]$ is the identity map.

Let $\tilde{v}(t)$ be the solution of $\tilde{v}'(t) = V(\tilde{v}(t))$ with initial condition $\tilde{v}(0) = y_0$. Suppose that $u(t) = f(t)\xi + v(t)$ is the solution of the dual Euler equation with $u(0) = f(0)\xi + y_0$, then by (13), $v'(t) = f(t)V(v(t))$. And, each orbit $u(\mathbb{R})$ of the dual Euler flow is contained in one of the two open half-spaces separated by the hyperplane $f^{-1}(0)$, and its projection $v(\mathbb{R}) = \pi(u(\mathbb{R}))$ is contained in an orbit $\tilde{v}(\mathbb{R})$ of the vector field V with a possibly different parametrization: $\tilde{v}(\int f(t)dt) = v(t)$. In particular, if a solution of $\tilde{v}'(t) = V(\tilde{v}(t))$ are unbounded, then the corresponding solution of the dual Euler equation is unbounded as well. ■

Let $\mu = \mu_1 + \sqrt{-1}\mu_2$ with $(\mu_1, \mu_2) \in \mathbb{R}^2$ be the constants in Part (ii) of Lemma 4.3 and V be the operator in (16) with $d_2 = 0$. To simplify the calculations we consider $\mu_2 V$ instead of V in the following. The operator $\mu_2 V$ has the representation

$$\begin{pmatrix} 0 & -\mu_1(a-d_1) & 0 & -(b-d_1) \\ \mu_1(a-d_1) & 0 & (b-d_1) & 0 \\ 0 & (a-d_1) & 0 & \mu_1(b-d_1) \\ -(a-d_1) & 0 & -\mu_1(b-d_1) & 0 \end{pmatrix}.$$

in the K-orthonormal basis $(e, [\sqrt{-1}\xi, e], \mu e, \mu[\sqrt{-1}\xi, e])$, and its characteristic polynomial is given by

$$P_V(x) = (x^2 + (\mu_1^2 - 1)d)^2 + \mu_1^2(a-b)^2 x^2,$$

where $d = (d_1 - b)(d_1 - a)$. In what follows we consider different possibilities for d and in each case we examine the intersection of the nilpotent cone

$$\mathcal{N} = \{x \in \mathfrak{sl}_2(\mathbb{C}) : \mathrm{tr}(x^2) = 0\}$$

(Remark 2.5, part (i)) and the null cone Λ_{q}^* in the hyperplane $\mathrm{K}(x, \sqrt{-1}\xi) = 0$.

Lemma 4.10. *If $d \leq 0$, then the solutions of the dual Euler equation are complete.*

Proof. Let $A_{\mathrm{q}}^{-1} = \mathrm{diag}(d_1, d_3, a, a, b, b)$ and $(f_1, f_2, x_1, x_2, y_1, y_2)$ denote the coordinates in the basis $(\xi, \sqrt{-1}\xi, e, [\sqrt{-1}\xi, e], \mu e, \mu[\sqrt{-1}\xi, e])$. Since $\mathrm{K}(x, \sqrt{-1}\xi) = 0$, $f_2 = 0$. Using $\mathrm{K}(x,x) = 2\,\mathrm{Re}(\mathrm{tr}(x^2)) = 0$ and $\mathrm{K}(x, \sqrt{-1}x) = -2\,\mathrm{Im}(\mathrm{tr}(x^2)) = 0$ for $x \in \mathcal{N}$, and $\mathrm{q}^*(x,x) = \mathrm{K}(x, A_{\mathrm{q}}^{-1}x) = 0$ for $x \in \Lambda_{\mathrm{q}}^*$, one obtains the following system of equations:

$$\begin{aligned} x_1^2 + x_2^2 - y_1^2 - y_2^2 + f_1^2 &= 0 \\ x_1 y_1 + x_2 y_2 + \mu_1(x_1^2 + x_2^2 + y_1^2 + y_2^2) &= 0 \\ a x_1^2 + a x_2^2 - b y_1^2 - b y_2^2 + d_1 f_1^2 &= 0 \end{aligned} \tag{17}$$

from which we get $(a - d_1)(x_1^2 + x_2^2) + (d_1 - b)(y_1^2 + y_2^2) = 0$, having only the trivial solution for $d < 0$. Hence, by Remark 2.10, when $d < 0$, the solutions of dual Euler equation are bounded.

Now suppose that $d = 0$. The constants a and b in (10) have different signs. There is no loss of generality in assuming $a > 0$ and $b < 0$. One then obtains $d_1 = b$. So $(a-b)(x_1^2 + x_2^2) = 0$, hence $x_1 = x_2 = 0$, which by the last equation in (17) implies $\mu_1(y_1^2 + y_2^2) = 0$. If $\mu_1 \neq 0$, the system of equations (17) has just the trivial solution. So let $\mu_1 = 0$ and $x = f_1\xi + f_2\sqrt{-1}\xi + y$ with $y = (x_1, x_2, y_1, y_2)$ representing an element of $[\xi, \mathfrak{sl}_2(\mathbb{C})]$ in the basis $(e, [\sqrt{-1}\xi, e], \mu e, \mu[\sqrt{-1}\xi, e])$, then

$$[x, A_{\mathrm{q}}^{-1}x] = f_1\left([y, A_{\mathrm{q}}^{-1}\xi] + [\xi, A_{\mathrm{q}}^{-1}y]\right) + f_2\left([y, A_{\mathrm{q}}^{-1}\sqrt{-1}\xi] + [\sqrt{-1}\xi, A_{\mathrm{q}}^{-1}y]\right) + [y, A_{\mathrm{q}}^{-1}y],$$

where $[y, A_{\mathrm{q}}^{-1}y] = \dfrac{1}{\mu_2}(a-b)(x_2y_1 - x_1y_2)\xi$ and

$$[\xi, A_{\mathrm{q}}^{-1}y] + [y, A_{\mathrm{q}}^{-1}\xi] = (\frac{1}{\mu_2}(a-b)x_2)\mu e - (\frac{1}{\mu_2}(a-b)x_1)\mu[\sqrt{-1}\xi, e],$$

$$\begin{aligned}{[\sqrt{-1}\xi, A_{\mathrm{q}}^{-1}y] + [y, A_{\mathrm{q}}^{-1}\sqrt{-1}\xi]} &= (\frac{-1}{\mu_2}(a-d_3)x_2)e + (\frac{1}{\mu_2}(a-d_3)x_1)[\sqrt{-1}\xi, e] \\ &+ (\frac{-1}{\mu_2}(b-d_3)y_2)\mu e + (\frac{1}{\mu_2}(b-d_3)y_1)\mu[\sqrt{-1}\xi, e].\end{aligned}$$

Therefore, the dual Euler equation is given by

$$\begin{aligned} f_1' &= \frac{1}{\mu_2}(a-b)(x_2y_1 - y_2x_1), & f_2' &= 0,\\ x_1' &= \frac{-1}{\mu_2}(a-d_3)f_2x_2, & y_1' &= \frac{1}{\mu_2}(a-b)f_1x_2 - \frac{1}{\mu_2}(b-d_3)f_2y_2,\\ x_2' &= \frac{1}{\mu_2}(a-d_3)f_2x_1, & y_2' &= \frac{-1}{\mu_2}(a-b)f_1x_1 + \frac{1}{\mu_2}(b-d_3)f_2y_1. \end{aligned}$$

The above equations give $x_1' = \frac{-1}{\mu_2}(a-d_3)f_2x_2$ and $x_2' = \frac{1}{\mu_2}(a-d_3)f_2x_1$, which yields $x_1'x_1 + x_2'x_2 = 0$. Thus, $x_1^2 + x_2^2 = r^2$ for some constant r. Also from the first integral $\mathrm{K}(x, \sqrt{-1}x)$, we have $-f_1f_2 + x_1y_1 + x_2y_2 = c_0$ for some constant c_0. From the above system we get

$$\begin{aligned} \frac{\mu_2}{(a-b)}f_1'' &= x_2'y_1 - y_2'x_1 + x_2y_1' - y_2x_1'\\ &= (a-b)f_1(x_1^2 + x_2^2) + (a-b)f_2(x_1y_1 + x_2y_2)\\ &= (a-b)(r^2 + f_2^2)f_1 + (a-b)f_2c_0, \end{aligned}$$

Hence, for any solution $u(t) = f_1(t)\xi + f_2(0)\sqrt{-1}\xi + v(t)$ of the dual Euler equation, $f_1(t)$ is complete. Since f_1 is the coefficient in the timelike direction and $\mathrm{q}^*(u(t), u(t))$ is constant, the curve $v(t) = (x_1(t), x_2(t), y_1(t), y_2(t))$ also turns out to be complete. ■

Next, we consider the case $d > 0$.

Lemma 4.11. *If $d > 0$, then q is complete if and only if F_{q} has no incomplete GCS.*

Proof. First we show that when $\mu_1^2 > d(a+b-2d_1)^{-2}$, the system of equations (17) has only the trivial solution. From (17) we get

$$x_1^2 + x_2^2 = \frac{b-d_1}{a-b}\, f_1^2, \quad y_1^2 + y_2^2 = \frac{a-d_1}{a-b}\, f_1^2, \quad x_1y_1 + x_2y_2 = \frac{2d_1 - a - b}{a-b}\, f_1^2\mu_1. \quad (18)$$

Then by the Schwarz inequality we have $\mu_1^2 \le d(a+b-2d_1)^{-2}$. On the other hand, from the discriminant of $P_V(x)$ one can see that if $\mu_1^2 < 4d(a+b-2d_1)^{-2}$, the eigenvalues of V are complex with non-zero real parts. Since $P_V(x) = P_V(-x)$, two of them have a positive real part. So by Proposition 2.9 the ω-limit of the system $y' = V(y)$ is empty and its solutions are unbounded, hence by Lemma 4.9 the solutions of F_{q} are unbounded in the hyperplane $\mathrm{K}(\sqrt{-1}\xi, x) = 0$. Now let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, by (18) we have

$$\|\mathbf{x}\|^2 = \frac{b-d_1}{a-b}\, f^2, \quad \|\mathbf{y}\|^2 = \frac{a-d_1}{a-b}\, f^2, \quad \langle \mathbf{x}, \mathbf{y}\rangle = \frac{2d_1 - a - b}{a-b}\, f^2\mu_1.$$

So we can determine $\mathbf{y}$ by $(\mathbf{x}, f)$, $\mathbf{y} = \frac{\langle \mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|}\mathbf{x} \pm \frac{\|\mathbf{x}\times\mathbf{y}\|}{\|\mathbf{x}\|}\mathbf{x}^\perp$, where $\mathbf{x}^\perp = (-x_2, x_1)$.

Thus we have two copies of $S^1 \times \mathbb{R}$ which intersect each other only at the origin point, so $\mathcal{N} \cap \Lambda_{\mathrm{q}}^* \cap \{x : \mathrm{K}(x, \sqrt{-1}\xi) = 0\} \setminus \{0\}$ is diffeomorphic to four disjoint copy of $S^1 \times \mathbb{R}$. Therefore every unbounded solution $u(t)$ with

$$0 \ne u(0) \in (\mathcal{N} \cap \Lambda_{\mathrm{q}}^*) \cap \{x : \mathrm{K}(x, \sqrt{-1}\xi) = 0\}$$

must lie in one of these components. So, either the projection of $u(t)$ on S^1 covers the whole S^1, or we have an idempotent. In either case $u(t)$ is an incomplete GCS. In fact the idempotent occurs when $\mu_1 = 0$, since just in this case the eigenvalues of $V(x)$ are real and F_{q} can have a radial solution. ∎

Lemmas 4.10 and 4.11 together with Theorem 2.7 complete the proof of Proposition 4.8.

Remark 4.12. In the proof of Lemma 4.11, when $\mu_1 \neq 0$, one obtains an incomplete solution of the dual Euler equation which is not generated by an idempotent. In particular, unlike on $SL_2(\mathbb{R})$, there are incomplete left-invariant Lorentzian metrics on $SL_2(\mathbb{C})$ with no idempotent. ∎

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