Automorphisms of Multiplicative Lie Algebra Extensions

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Abstract.   We discuss the inducibility problem for automorphisms of multiplicative Lie algebra extensions and show that obstruction to the inducibility of pairs of automorphisms lies in the second cohomology group of multiplicative Lie algebras. We also establish the Wells type exact sequence for multiplicative Lie algebras, which relates automorphism groups with the second cohomology group of multiplicative Lie algebras.

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1. Introduction

The concept of multiplicative Lie algebra was introduced by G. J. Ellis in [4] for giving a non-abelian version of the Magnus and Witt theorem, which essentially states that five well known universal commutator identities generate all the commutator identities of weight n. A multiplicative Lie algebra is a group (possibly non-abelian) with an extra binary operation called the multiplicative Lie product, which satisfies five identities similar to five well known universal commutator identities. The theory of multiplicative Lie algebra is interesting and distinct on its own. In recent years authors have shown interest to develop the theory for multiplicative Lie algebra. In [7, 11] theory of nilpotency and solvability for multiplicative Lie algebras have been studied. Bak et al. in [1] constructed two homology theories for multiplicative Lie algebras and compared them with the homology theories of groups and Lie rings. They also introduced the Steinberg multiplicative Lie algebra of a unital ring and proved that it is the direct product of the Steinberg group and Steinberg Lie ring. Lal and Upadhyay in [6] discussed the extension theory for multiplicative Lie algebra and derived the Schur-Hopf formula for multiplicative Lie algebra in terms of the second cohomology group. They also discussed some applications to K-theory and cyclic homology. Donadze et al. in [3] introduced the concept of non-abelian tensor product and obtained a six term exact sequence for multiplicative Lie algebras and also proved a new version of Stalling's theorem. Pandey and Upadhyay in [8] discussed the classification of multiplicative Lie products for finite groups. Pandey and Upadhyay in [9] consider a short exact sequence $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$, where H is an abelian group with

trivial multiplicative Lie product and G, K are multiplicative Lie algebra. They called $E(H, K)$ to be center extension if H is contained in the center of G, and discussed the Schreier extension theory for 2-fold extensions in two different cases and defined the second cohomology group for multiplicative Lie algebras.

C. Wells, in [12], formulated an exact sequence for the automorphism group of a given group extension, which provided a solution to the inducibility problem for a pair of automorphisms of groups. In [5, 10], authors further explored this sequence, focusing on studying the lifting and extensions of automorphisms in abelian extensions and established Wells type exact sequences for specific cases. In [2], Bardakov and Singh discussed a similar problem in the context of Lie algebras and developed Wells type sequences for Lie algebras. In this paper, our primary objective is to address the inducibility problem for multiplicative Lie algebras, as discussed in Section 2, analogous to groups [12] and Lie algebras [2]. We also see that the obstruction for a pair of automorphisms to be inducible lies in the second cohomology group of the multiplicative Lie algebra.

The rest of the paper is organized as follows. In Section 2, we provide a brief description of the second cohomology group for multiplicative Lie algebras, as outlined in [9]. Subsequently, we introduce the notion of inducible and compatible pairs for multiplicative Lie algebras, establishing a necessary and sufficient condition for a pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ to be inducible in a center extension. In Section 3, we discuss the problem under what conditions an automorphism of an ideal of multiplicative Lie algebras can be extended to the multiplicative Lie algebra itself which fixes the ideal. Following the spirit of C. Wells [12], we establish the Wells type exact sequence for multiplicative Lie algebras. As an application of these sequences, we discuss the inducibility of a pair of automorphisms. Additionally, we prove that for a split center extension, every compatible pair is inducible. We also obtain a necessary condition for the inducibility of the pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ when K is a perfect multiplicative Lie algebra. In Section 4, we prove that if the center extension $E(H, K)$ of multiplicative Lie algebras splits, then the exact sequences obtained in Section 3 also split.

2.   Exact Sequences and Inducibility of a pair of Automorphisms

In this section, we recall the definition of multiplicative Lie algebra and provide a brief overview of its extension theory.

Definition 2.1.   A *multiplicative Lie algebra* is a triple $(G, \cdot, \star)$, where G is a group with respect to the binary operation "$\cdot$" and satisfies the following conditions with respect to the binary operation "$\star$"

(1) $x \star x = e$,

(2) $x \star y \cdot z = (x \star y) \cdot {}^{y}(x \star z)$,

(3) $x \cdot y \star z = {}^{x}(y \star z) \cdot (x \star z)$,

(4) $((x \star y) \star {}^{y}z) \cdot ((y \star z) \star {}^{z}x) \cdot ((z \star x) \star {}^{x}y) = e$,

(5) ${}^{z}(x \star y) = {}^{z}x \star {}^{z}y$, for all $x, y, z \in G$.

The binary operation $\star$ is called *multiplicative Lie product* and ${}^{z}x$ denotes zxz^{-1}.

Note: A multiplicative Lie algebra is a generalization of groups and Lie algebras. Not every multiplicative Lie algebra needs to be a Lie algebra. But every Lie algebra is a multiplicative Lie algebra by defining "$\star$" to be the Lie bracket of the Lie algebra, and "$\cdot$" to be the addition of the Lie algebra.

Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension and let $t : K \longrightarrow G$ be a transversal of $E(H, K)$. Here H is an ideal of G, i.e. $x \star h \in H$ for all $x \in G$ and $h \in H$. It can be easily seen that every element $g \in G$ can be uniquely written as $g = t(x)h$ for some $h \in H$ and $x \in K$. Also, for $x, y \in K$ and $h \in H$, we have $\beta(t(xy)) = xy = \beta(t(x)t(y))$, $\beta(t(x \star y)) = x \star y = \beta(t(x) \star t(y))$ and $\beta(t(x) \star h) = x \star e = e$. Thus we get maps $f : K \times K \longrightarrow H$, $h : K \times K \longrightarrow H$ and $\Gamma_x : H \longrightarrow H$ defined by $f(x, y) = t(x)t(y)t(xy)^{-1}$, $h(x, y) = (t(x) \star t(y))(t(x \star y))^{-1}$ and $\Gamma_x(h) = t(x) \star h$. It can be easily seen that $\Gamma_x \in \mathrm{End}(H)$. Note that $\mathrm{End}(H)$ is a multiplicative Lie algebra with operations defined as $(f \cdot g)(h) = f(h) \cdot g(h)$ and $(f \star g)(h) = f(g(h)) \cdot g(f(h^{-1}))$ for all $f, g \in \mathrm{End}(H)$ and $h \in H$. Now using the definition of multiplicative Lie algebra, we can establish the following properties of the maps f, h and Γ_x:

$$f(e, x) = f(x, e) = e \tag{1}$$

$$h(x, x) = h(x, e) = h(e, x) = e \tag{2}$$

$$f(x, y)f(xy, z) = f(y, z)f(x, yz) \tag{3}$$

$$f(y^{-1}, y(x \star z))^{-1} f(y(x \star z), y^{-1}) f(x \star y, {}^y(x \star z)) h(x, y) h(x, z)$$
$$= \Gamma_x(f(y, z)) h(x, yz) \tag{4}$$

$$f(x^{-1}, x(y \star z))^{-1} f(x(y \star z), x^{-1}) f({}^x(y \star z), x \star z) h(x, z) h(y, z)$$
$$= \Gamma_z(f(x, y)^{-1}) h(xy, z) \tag{5}$$

$$\Gamma_x(f(z^{-1}, zy)^{-1} f(zy, z^{-1})) \Gamma_y(f(z^{-1}, zx) f(zx, z^{-1})^{-1}) h({}^z x, {}^z y)$$
$$= f(z^{-1}, z(x \star y))^{-1} f(z(x \star y), z^{-1}) h(x, y) \tag{6}$$

$$\Gamma_{x \star y}(f(y^{-1}, yz)^{-1} f(yz, y^{-1})) \Gamma_{y \star z}(f(z^{-1}, zx)^{-1} f(zx, z^{-1})) \Gamma_{z \star x}(f(x^{-1}, xy)^{-1}$$
$$f(xy, x^{-1})) \Gamma_z(h(x, y)^{-1}) \Gamma_x(h(y, z)^{-1}) \Gamma_y(h(z, x)^{-1}) h((x \star y), {}^y z) h((y \star z), {}^z x)$$
$$h((z \star x), {}^x y) f((x \star y) \star {}^y z, (y \star z) \star {}^z x) f(((z \star x) \star {}^x y)^{-1}, ((z \star x) \star {}^x y))) = 1 \tag{7}$$

for all x, y and $z \in K$. A triple (f, h, Γ) is called a *multiplicative center 2-cocycle* of K with coefficients in H if (f, h, Γ) satisfies the properties mentioned above. The set of all such triples is denoted by $Z^2_{ML(\Gamma)}(K, H)$, and it forms an abelian group under the operation $(f, h, \Gamma) \cdot (f', h', \Gamma) = (ff', hh', \Gamma)$. A triple (f, h, Γ) is called a *multiplicative center 2-coboundary* of K with coefficients in H if there exists an identity preserving map $\lambda : K \longrightarrow H$ such that

(1) $f(x, y) = \lambda(x)\lambda(y)\lambda(xy)^{-1}$

(2) $h(x, y) = \Gamma_x(\lambda(y))\Gamma_y(\lambda(x)^{-1})\lambda(x \star y)^{-1}$ for all $x, y \in K$.

The set of all such triples is denoted by $B^2_{ML(\Gamma)}(K, H)$ and it is a subgroup of $Z^2_{ML(\Gamma)}(K, H)$. The quotient group $H^2_{ML(\Gamma)}(K, H) = Z^2_{ML(\Gamma)}(K, H)/B^2_{ML(\Gamma)}(K, H)$ is called the second center cohomology of K with coefficient in H. Two multiplicative center 2-cocycles are said to be cohomologous if they differ by a multiplicative center 2-coboundary. Let ν be a map from K to H and $\Gamma : K \longrightarrow End(H)$ be a multiplicative Lie algebra homomorphism.

Then ν is said to be a *multiplicative center* 1-*cocycle*, if the following conditions are satisfied:

(1) $\nu(xy) = \nu(x)\nu(y)$,

(2) $\nu(x \star y) = \Gamma_x(\nu(y))\Gamma_y(\nu(x)^{-1})$ for all $x, y \in K$.

We denote the set of all such maps by $Z^1_{ML(\Gamma)}(K, H)$, which is an abelian group under the usual operation.

Note: For more details of the extension theory of multiplicative Lie algebras, readers can see [9]. In the entire paper, we consider the center extension $E(H, K)$, in which, the ideal $H \subseteq Z(G)$ and the multiplicative Lie product $\star$ on H is trivial. Thus we have

$$hg = gh, h \star k = e \text{ and } h \star g \in H \tag{8}$$

for all $h, k \in H$ and $g \in G$. We will use throughout the paper properties mentioned in equation (8) and Definition 2.1 (as well as its properties) without explicitly mentioning them, especially to show that certain maps are homomorphisms.

Now we will discuss the inducibility problem for the center extension of multiplicative Lie algebras.

Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension and let $t : K \longrightarrow G$ be a transversal of $E(H, K)$. Define $\mathrm{Aut}_H(G) = \{\phi \in \mathrm{Aut}(G) : \phi(H) = H\}$. Then $\phi \in \mathrm{Aut}_H(G)$ induces a pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$, where $\alpha(h) = \phi(h)$ for all $h \in H$ and $\eta(x) = \beta(\phi(t(x)))$ for all $x \in K$. Thus we define a map

$$\Pi : \mathrm{Aut}_H(G) \longrightarrow \mathrm{Aut}(H) \times \mathrm{Aut}(K), \text{ given by } \Pi(\phi) = (\alpha, \eta).$$

The homomorphism property of Π is easily apparent. However in the next lemma we see that η is independent of the choice of transversal t.

Lemma 2.2.   *If $\phi \in \mathrm{Aut}_H(G)$, then the map η does not depend on the choice of the transversal t.*

Proof. Let t and t' be two transversals of $E(H, K)$. Then $\beta(t(x)) = x = \beta(t'(x))$ for all $x \in K$, implying $t(x)t'(x)^{-1} \in \ker(\beta) = H$. Thus $\phi(t(x)t'(x)^{-1}) \in H = \ker(\beta)$, for $\phi \in \mathrm{Aut}_H(G)$. Therefore we have $\beta(\phi(t(x))\phi(t'(x)^{-1})) = e$, and this implies that $\beta(\phi(t(x))) = \beta(\phi(t'(x)))$. ∎

From the definition of the map Π, it follows that every $\phi \in \mathrm{Aut}_H(G)$ induces a pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$. The map Π motivates us to introduce the following definition and raises the following question.

Definition 2.3.   Let $\Pi : \mathrm{Aut}_H(G) \longrightarrow \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ be a homomorphism. A pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is called *inducible* if there exists $\phi \in \mathrm{Aut}_H(G)$ such that $\Pi(\phi) = (\alpha, \eta)$.

Question: Under what conditions a pair of multiplicative Lie algebra automorphisms in $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible?

The next result, Lemma 2.4, provides a necessary and sufficient condition for a pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ to be inducible in a center extension.

Lemma 2.4. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension of multiplicative Lie algebras. Then a pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible if and only if there exists $\lambda : K \longrightarrow H$ with $\lambda(e) = e$ such that the following conditions hold:*

(1) $f(\eta(x), \eta(y))\alpha(f(x,y))^{-1} = \lambda(xy)\lambda(x)^{-1}\lambda(y)^{-1}$

(2) $h(\eta(x), \eta(y))\alpha(h(x,y))^{-1} = \lambda(x \star y)\Gamma_{\eta(y)}(\lambda(x))\Gamma_{\eta(x)}(\lambda(y)^{-1})$

(3) $\Gamma_{\eta(x)} = \alpha\Gamma_x\alpha^{-1}$, *or* $\alpha(t(x) \star h) = t(\eta(x)) \star \alpha(h)$

for all $x, y \in K$ and $h \in H$.

Proof. Let $\phi \in \mathrm{Aut}_H(K)$ be such that $\Pi(\phi) = (\alpha, \eta)$ and $t : K \longrightarrow G$ be a transversal. Then $\beta(t(\eta(x))\phi(t(x))^{-1}) = \eta(x)\eta(x)^{-1} = e$. Since $\ker(\beta) = H$, there exists unique element $\lambda(x)$(say) in H such that $\lambda(x) = (t(\eta(x)))^{-1}\phi(t(x))$. Thus we get a map $\lambda : K \longrightarrow H$ with $\lambda(e) = e$. Now to prove (1) and (2), we will use the fact that ϕ is a multiplicative Lie algebra homomorphism and H is an abelian group with trivial multiplicative Lie product. Let $g, g' \in G$. Then $g = ht(x)$ and $g' = h't(y)$ for some $h, h' \in H$ and $x, y \in K$.

(1) Since ϕ is a group homomorphism, $\phi(gg') = \phi(g)\phi(g')$

$$\phi(ht(x)h't(y)) = \phi(ht(x))\phi(h't(y))$$
$$\phi(hh'f(x,y)t(xy)) = \phi(ht(x))\phi(h't(y))$$
$$\alpha(hh')\alpha(f(x,y))\lambda(xy)t(\eta(xy)) = \alpha(hh')\lambda(x)t(\eta(x))\lambda(y)t(\eta(y))$$
$$\alpha(f(x,y))\lambda(xy) = t(\eta(x))t(\eta(y))t(\eta(xy))^{-1}\lambda(x)\lambda(y)$$
$$\alpha(f(x,y))\lambda(xy) = f(\eta(x),\eta(y))\lambda(x)\lambda(y)$$
$$f(\eta(x),\eta(y))\alpha(f(x,y))^{-1} = \lambda(xy)\lambda(x)^{-1}\lambda(y)^{-1}.$$

(2) Similarly, we have $\phi(g \star g') = \phi(g) \star \phi(g')$, then

$$\phi(ht(x) \star kt(y)) = \phi((h \star k)(t(x) \star k)(h \star t(y))(t(x) \star t(y)))$$
$$= \phi(t(x) \star k)\phi(h \star t(y))\phi(h(x,y)t(x \star y))$$
$$= (\phi(t(x)) \star \phi(k))(\phi(h) \star \phi(t(y)))\alpha(h(x,y))\phi(t(x \star y))$$
$$= (\lambda(x)t(\eta(x)) \star \alpha(k))(\lambda(y)t(\eta(y)) \star \alpha(h))^{-1}\alpha(h(x,y))$$
$$\quad \lambda(x \star y)t(\eta(x \star y))$$
$$= (t(\eta(x)) \star \alpha(k))(t(\eta(y)) \star \alpha(h))^{-1}\alpha(h(x,y))$$
$$\quad \lambda(x \star y)t(\eta(x) \star \eta(y)).$$

and

$$\phi(ht(x)) \star \phi(kt(y)) = \alpha(h)\phi(t(x)) \star \alpha(k)\phi(t(y))$$
$$= \alpha(h)\lambda(x)t(\eta(x)) \star \alpha(k)\lambda(y)t(\eta(y))$$
$$= (t(\eta(x)) \star \alpha(k)\lambda(y)t(\eta(y)))(\alpha(h)\lambda(x) \star \alpha(k)\lambda(y)t(\eta(y)))$$
$$= (t(\eta(x)) \star \alpha(k)\lambda(y))(t(\eta(x)) \star t(\eta(y)))(\alpha(h)\lambda(x) \star t(\eta(y)))$$
$$= (t(\eta(x)) \star \lambda(y))(t(\eta(x)) \star \alpha(k))(t(\eta(x)) \star t(\eta(y)))$$
$$\quad (\lambda(x) \star t(\eta(y)))(\alpha(h) \star t(\eta(y))).$$

After comparing both, we get

$$h(\eta(x), \eta(y))\alpha(h(x,y))^{-1} = \lambda(x \star y)\Gamma_{\eta(y)}(\lambda(x))\Gamma_{\eta(x)}(\lambda(y))^{-1}.$$

(3) Let $x \in K$ and $h \in H$. Then

$$\Gamma_{\eta(x)}(h) = t(\eta(x)) \star h = \lambda(x)\phi(t(x)) \star h = {}^{\lambda(x)}(\phi(t(x)) \star h)(\lambda(x) \star h)$$

$$= \phi(t(x)) \star \phi(\phi^{-1}(h)) = \phi(t(x) \star \phi^{-1}(h)) = \alpha(t(x) \star \alpha^{-1}(h))$$

$$= \alpha(\Gamma_x(\alpha^{-1}(h))) = \alpha\Gamma_x\alpha^{-1}(h).$$

Thus $\Gamma_{\eta(x)} = \alpha\Gamma_x\alpha^{-1}$, or $\Gamma_{\eta(x)}\alpha = \alpha\Gamma_x$, and this implies $\alpha(t(x)\star h) = t(\eta(x))\star\alpha(h)$. Conversely if $g \in G$, then $g = ht(x)$ for some $h \in H$ and $x \in K$. Thus we define a map $\phi : G \longrightarrow G$ by $\phi(g) = \alpha(h)\lambda(x)t(\eta(x))$. Clearly, $\phi(h) = \alpha(h)$ for all $h \in H$ and $\eta(x) = \beta(t(\eta(x))) = \beta(\lambda(x)^{-1}\phi(t(x))) = \beta(\phi(t(x)))$ for all $x \in K$. Now we will prove that ϕ is an automorphism of G. Let $g_1, g_2 \in G$. Then $g_1 = ht(x)$ and $g_2 = kt(y)$ for some $x, y \in K$ and $h, k \in H$. Consider

$$\phi(ht(x)kt(y)) = \phi(t(xy)f(x,y)hk) = t(\eta(xy))\lambda(xy)\alpha(f(x,y)hk)$$

$$= t(\eta(x)\eta(y))\lambda(x)\lambda(y)\alpha(f(x,y))^{-1}f(\eta(x),\eta(y))\alpha(f(x,y))\alpha(hk)$$

$$= t(\eta(x))t(\eta(y))f(\eta(x),\eta(y))^{-1}f(\eta(x),\eta(y))\lambda(x)\lambda(y)\alpha(h)\alpha(k)$$

$$= t(\eta(x))t(\eta(y))\lambda(x)\lambda(y)\alpha(h)\alpha(k)$$

$$= \phi(ht(x))\phi(kt(y)).$$

and

$$\phi(t(x)h) \star \phi(t(y)k) = t(\eta(x))\lambda(x)\alpha(h) \star t(\eta(y))\lambda(y)\alpha(k)$$

$$= {}^{t(\eta(x))}(\lambda(x)\alpha(h) \star t(\eta(y))\lambda(y)\alpha(k))(t(\eta(x)) \star t(\eta(y))\lambda(y)\alpha(k))$$

$$= {}^{\lambda(x)}(\alpha(h) \star t(\eta(y))\lambda(y)\alpha(k))(\lambda(x) \star t(\eta(y))\lambda(y)\alpha(k))$$

$$(t(\eta(x)) \star t(\eta(y)))^{t(\eta(y))}(t(\eta(x)) \star \lambda(y)\alpha(k))$$

$$= (\alpha(h) \star t(\eta(y)))^{t(\eta(y))}(\alpha(h) \star \lambda(y)\alpha(k))(\lambda(x) \star t(\eta(y)))$$

$$(t(\eta(x)) \star t(\eta(y)))(t(\eta(x)) \star \lambda(y))^{\lambda(y)}(t(\eta(x)) \star \alpha(k))$$

$$= (\alpha(h) \star t(\eta(y)))(\alpha(h) \star \lambda(y))(\alpha(h) \star \alpha(k))(\lambda(x) \star t(\eta(y)))$$

$$(t(\eta(x)) \star t(\eta(y)))(t(\eta(x)) \star \lambda(y))(t(\eta(x)) \star \alpha(k))$$

$$= t(\eta(x \star y))h(\eta(x),\eta(y))(\lambda(x) \star t(\eta(y)))(t(\eta(x)) \star \lambda(y))$$

$$(t(\eta(x)) \star \alpha(k))(\alpha(h) \star t(\eta(y))).$$

$$\phi(t(x)h \star t(y)k) = \phi((t(x) \star t(y))(t(x) \star k)(h \star t(y))(h \star k))$$

$$= \phi(t(x \star y)h(x,y))\alpha(t(x) \star k)\alpha(h \star t(y))$$

$$= t(\eta(x \star y))\lambda(x \star y)\alpha(h(x,y))(t(\eta(x)) \star \alpha(k))(\alpha(h) \star t(\eta(y)))$$

$$= t(\eta(x \star y))h(\eta(x),\eta(y))(\lambda(x) \star t(\eta(y)))(t(\eta(x)) \star \lambda(y))$$

$$(t(\eta(x)) \star \alpha(k))(\alpha(h) \star t(\eta(y))).$$

Thus ϕ is a multiplicative Lie algebra homomorphism. Let $g \in \ker(\phi)$. Then we have $\phi(g) = \phi(t(x)h) = t(\eta(x))\lambda(x)\alpha(h) = e$; this implies $t(\eta(x)) \in H$, that is $\beta(t(\eta(x))) = \eta(x) = e$; this implies $x = e$ for $\eta \in \mathrm{Aut}(K)$. Thus $\alpha(h) = e$, implying implies $h = e$. Hence $g = e$ and ϕ is injective. Now we will show that ϕ is surjective, if $g \in G$, then $g = t(y)h'$ for some $y \in K$ and $h' \in H$. Since $\eta \in \mathrm{Aut}(K)$, there exists $y' \in K$ such that $\phi(y') = y$, also $\lambda(y')^{-1}h' \in H$, implying that there exists $h \in H$ such that $\alpha(h) = \lambda(y')^{-1}h'$. Now we compute

$$\phi(t(y')h) = t(\phi(y'))\lambda(y')\alpha(h) = t(y)\lambda(y')\lambda(y')^{-1}h' = t(y)h' = g.$$

Thus ϕ is surjective. Therefore $\phi \in \mathrm{Aut}_H(G)$. ∎

Concluding from the previous Lemma 2.4, we can assert that every inducible pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ satisfies $\Gamma_{\eta(x)} = \alpha \Gamma_x \alpha^{-1}$, indicating that $\Gamma_{\eta(x)}$ is a homomorphism from H to H. Consequently, we introduce the following definition.

Definition 2.5. A pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is called *compatible* if it satisfies $\alpha \Gamma_x \alpha^{-1} = \Gamma_{\eta(x)}$ for all $x \in K$. In other words, the following diagram commutes

$$
\begin{array}{ccc}
H & \xrightarrow{\ \eta\ } & G \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \Gamma} \\
\mathrm{End}(H) & \xrightarrow{\ \theta\ } & \mathrm{End}(H)
\end{array}
$$

where $\theta(u) = \alpha u \alpha^{-1}$ for all $u \in \mathrm{End}(H)$. We denote the set of all compatible pairs by C^L.

Remark 2.6. The set of all compatible pairs

$$C^L = \{(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K) : \alpha \Gamma_x \alpha^{-1} = \Gamma_{\eta(x)} \text{ for all } x \in K\}$$

is a subgroup of $\mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

Remark 2.7. Since $\eta \in \mathrm{Aut}(K)$, then after replacing x with $\eta^{-1}(x)$ in (1) and (2) of Lemma 2.4, we have

$$f(x,y)\alpha(f(\eta^{-1}(x), \eta^{-1}(y)))^{-1} = \lambda(\eta^{-1}(xy))\lambda(\eta^{-1}(x))^{-1}\lambda(\eta^{-1}(y))^{-1}$$

$$f(x,y)(\alpha(f(\eta^{-1}(x), \eta^{-1}(y))))^{-1} = \lambda'(xy)\lambda'(x)^{-1}\lambda'(y)^{-1}, \quad \text{and}$$

$$h(x,y)\alpha(h(\eta^{-1}(x), \eta^{-1}(y)))^{-1} = \lambda(\eta^{-1}(x) \star \eta^{-1}(y))\Gamma_y(\lambda(\eta^{-1}(x)))\Gamma_x(\lambda(\eta^{-1}(y)^{-1}))$$

$$= \lambda'(x \star y)\Gamma_y(\lambda'(x))\Gamma_x(\lambda'(y)^{-1}),$$

where $\lambda' = \lambda \eta^{-1}$. It can be seen that $(\alpha(f(\eta^{-1}, \eta^{-1})), \alpha(h(\eta^{-1}, \eta^{-1}))) \in Z^2_{ML(\Gamma)}(K, H)$. Thus multiplicative center 2-cocycles (f, h) and $(\alpha(f(\eta^{-1}, \eta^{-1})), \alpha(h(\eta^{-1}, \eta^{-1})))$ are cohomologous to each other.

From the Remark 2.7 and Lemma 2.4 we have the following corollary.

Corollary 2.8. *Let* $E(H, K) \equiv e \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ \beta\ } K \longrightarrow e$ *be a center extension. A compatible pair* (α, η) *is inducible if and only if multiplicative center 2-cocycles* (f, h) *and* $(\alpha \circ f \circ (\eta^{-1}, \eta^{-1}), \alpha \circ h \circ (\eta^{-1}, \eta^{-1}))$ *are cohomologous.*

An extension $E(H, K) \equiv e \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ \beta\ } K \longrightarrow e$ of multiplicative Lie algebra H by K is called *central extension* if $H \subseteq Z(G) \cap LZ(G)$, where $LZ(G)$ is the *multiplicative center* of G defined as $LZ(G) = \{x \in G : x \star y = e \text{ for all } y \in G\}$. Thus, if $E(H, K)$ is a central extension, then the map $\Gamma : K \longrightarrow \mathrm{End}(H)$ is given by $\Gamma_x(h) = t(x) \star h = e$ for all $x \in K$ and $h \in H$. For a central extension, Lemma 2.4 takes the following form.

Lemma 2.9. *Let* $E(H, K) \equiv e \longrightarrow H \xrightarrow{\ i\ } G \xrightarrow{\ \beta\ } K \longrightarrow e$ *be a central extension of multiplicative Lie algebras. Then a pair* $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ *is inducible if and only if there exists* $\lambda : K \longrightarrow H$ *with* $\lambda(e) = e$ *such that the following conditions hold:*

(1) $f(\eta(x), \eta(y))\alpha(f(x,y))^{-1} = \lambda(xy)\lambda(x)^{-1}\lambda(y)^{-1}$

(2) $h(\eta(x), \eta(y))\alpha(h(x,y))^{-1} = \lambda(x \star y)$ *for all* $x, y \in K$.

3. Wells type exact sequence and inducible problem

In this section, we discuss the lifting of automorphisms in the context of center extensions of multiplicative Lie algebras. Also we establish the Wells type exact sequence and see that obstruction to inducibility of a pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ lies in $H^2_{ML(\Gamma)}(K, H)$.

Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension and let $t : K \longrightarrow G$ be a transversal. Then $E(H, K)$ induces a multiplicative center 2-cocycle $(f, h, \Gamma) \in Z^2_{ML(\Gamma)}(K, H)$. Define $C^L_1 = \{\alpha \in \mathrm{Aut}(H) : (\alpha, I) \in C^L\}$ and $C^L_2 = \{\eta \in \mathrm{Aut}(K) : (I, \eta) \in C^L\}$. This implies

$$\alpha \in C^L_1 \iff \alpha \Gamma_x = \Gamma_x \alpha, \text{ or } \alpha(t(x) \star h) = t(x) \star \alpha(h) \text{ for all } x \in K, \ h \in H$$

and $\quad \eta \in C^L_2 \iff \Gamma_x = \Gamma_{\eta(x)}, \text{ or } t(x) \star h = t(\eta(x)) \star h \text{ for all } x \in K, \ h \in H.$

For any $\alpha \in C^L_1$ and $\eta \in C^L_2$, we can define maps $r_\alpha, r'_\alpha, r_\eta, r'_\eta : K \times K \longrightarrow H$ by

$$r_\alpha(x, y) = f(x, y)\alpha(f(x, y))^{-1}, \ r'_\alpha(x, y) = h(x, y)\alpha(h(x, y))^{-1}$$

and $\quad r_\eta(x, y) = f(\eta(x), \eta(y))f(x, y)^{-1}, \ r'_\eta(x, y) = h(\eta(x), \eta(y))h(x, y)^{-1}.$

Lemma 3.1. *Let (f, g, Γ) be a multiplicative center 2-cocycle corresponding to the extension $E(H, K)$. Then $(r_\alpha, r'_\alpha, \Gamma)$ and $(r_\eta, r'_\eta, \Gamma)$ are multiplicative center 2-cocycles, i.e. $(r_\alpha, r'_\alpha, \Gamma)$ and $(r_\eta, r'_\eta, \Gamma) \in Z^2_{ML(\Gamma)}(K, H)$.*

Proof. To prove that $(r_\alpha, r'_\alpha, \Gamma) \in Z^2_{ML(\Gamma)}(K, H)$. We need to prove that $(r_\alpha, r'_\alpha, \Gamma)$ satisfies the equations from (1) to (7) of 2-cocycle. Clearly (1) and (2) hold. For (3), consider

$$\begin{aligned}
r_\alpha(x, y)r_\alpha(xy, z) &= f(x, y)\alpha(f(x, y))^{-1}f(xy, z)\alpha(f(xy, z))^{-1} \\
&= f(x, y)f(xy, z)\alpha(f(x, y)(xy, z))^{-1} \\
&= f(y, z)f(x, yz)\alpha(f(y, z)f(x, yz))^{-1} \\
&= f(y, z)\alpha(f(y, z))^{-1}f(x, yz)\alpha(f(x, yz))^{-1} \\
&= r_\alpha(y, z)r_\alpha(x, yz).
\end{aligned}$$

For (4) consider

$$r_\alpha(y^{-1}, y(x \star z))^{-1}r_\alpha(y(x \star z), y^{-1})r_\alpha(x \star y, {}^y(x \star z))r'_\alpha(x, y)r'_\alpha(x, z)$$
$$= f(y^{-1}, y(x \star z))^{-1}\alpha(f(y^{-1}, y(x \star z)))f(y(x \star z), y^{-1})\alpha(f(y(x \star z), y^{-1}))^{-1}$$
$$\quad f(x \star y, {}^y(x \star z))\alpha(f(x \star y, {}^y(x \star z)))^{-1}h(x, y)\alpha(h(x, y))^{-1}h(x, z)\alpha(h(x, z))^{-1}$$
$$= f(y^{-1}, y(x \star z))^{-1}f(y(x \star z), y^{-1})f(x \star y, {}^y(x \star z))h(x, y)h(x, z)\alpha(f(y^{-1}, y(x \star z)))$$
$$\quad \alpha(f(y(x \star z), y^{-1}))^{-1}\alpha(f(x \star y, {}^y(x \star z)))^{-1}\alpha(h(x, y))^{-1}\alpha(h(x, z))^{-1}$$
$$= \Gamma_x(f(y, z))h(x, yz)\alpha(\Gamma_x(f(y, z))h(x, yz))^{-1}$$
$$= \Gamma_x(f(y, z)\alpha(f(y, z))^{-1})h(x, yz)\alpha(h(x, yz))^{-1} \text{ (For } \alpha\Gamma_x = \Gamma_x\alpha)$$
$$= \Gamma_x(r_\alpha(y, z))r'_\alpha(x, yz).$$

In the same way, we can prove (5). Now for (6) consider

$$\Gamma_x(r_\alpha(z^{-1}, zy)^{-1}r_\alpha(zy, z^{-1}))\Gamma_y(r_\alpha(z^{-1}, zx)r_\alpha(zx, z^{-1})^{-1})r'_\alpha({}^zx, {}^zy)$$
$$= \Gamma_x(f(z^{-1}, zy)^{-1}\alpha(f(z^{-1}, zy))f(zy, z^{-1})\alpha(f(zy, z^{-1}))^{-1})$$
$$\quad \Gamma_y(f(z^{-1}, zx)\alpha(f(z^{-1}, zx))^{-1}f(zx, z^{-1})^{-1}\alpha(f(zx, z^{-1})))h({}^zx, {}^zy)\alpha(h({}^zx, {}^zy))^{-1}$$

$$\begin{aligned}
&= \Gamma_x(f(z^{-1}, zy)^{-1}f(zy, z^{-1}))\Gamma_y(f(z^{-1}, zx)f(zx, z^{-1})^{-1})h(^zx, {}^z y) \\
&\quad \alpha(\Gamma_x(f(z^{-1}, zy)f(zy, z^{-1})^{-1})\Gamma_y(f(z^{-1}, zx)^{-1})f(zx, z^{-1})h(^zx, {}^z y)^{-1}) \\
&= f(z^{-1}, z(x \star y))^{-1}f(z(x \star y), z^{-1})h(x, y)\alpha(f(z^{-1}, z(x \star y)) \\
&\quad f(z(x \star y), z^{-1})^{-1}h(x, y)^{-1}) \\
&= r_\alpha(z^{-1}, z(x \star y))^{-1}r_\alpha(z(x \star y), z^{-1})r'_\alpha(x, y).
\end{aligned}$$

By the similar computation equation (7) holds. Thus $(r_\alpha, r'_\alpha, \Gamma) \in Z^2_{ML(\Gamma)}(K, H)$. Similarly, we can prove that $(r_\eta, r'_\eta, \Gamma) \in Z^2_{ML(\Gamma)}(K, H)$. ∎

Using Lemma 3.1, we define the following maps $\chi_1 : C_1^L \longrightarrow H^2_{ML(\Gamma)}(K, H)$ and $\chi_2 : C_2^L \longrightarrow H^2_{ML(\Gamma)}(K, H)$ by

$$\chi_1(\alpha) = (r_\alpha, r'_\alpha, \Gamma)B^2_{ML(\Gamma)}(K, H)$$
$$\chi_2(\eta) = (r_\eta, r'_\eta, \Gamma)B^2_{ML(\Gamma)}(K, H).$$

Lemma 3.2. *If $(r_\alpha, r'_\alpha, \Gamma)$ and $(r_\eta, r'_\eta, \Gamma)$ are multiplicative center 2-cocycles, then the maps χ_1 and χ_2 are well defined.*

Proof. To prove that χ_1 and χ_2 are well defined, we need to show that χ_1 and χ_2 are independent of the transversal t. Let t and t' be two transversals of the extension $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ with $t(e) = t'(e) = e$. Then there exist maps $f, f', h, h' : K \times K \longrightarrow H$ defined by $f(x, y) = t(x)t(y)t(xy)^{-1}$, $f'(x, y) = t'(x)t'(y)t'(xy)^{-1}$, $h(x, y) = (t(x) \star t(y))t(x \star y)^{-1}$ and by $h'(x, y) = (t'(x) \star t'(y))t'(x \star y)^{-1}$. Since $\beta(t(x)) = \beta(t'(x)) = x$, $t(x)(t'(x))^{-1} \in \ker(\beta) = H$. Thus there exist a unique $\lambda(x) \in H$, this implies we can define a map $\lambda : K \longrightarrow H$ with $\lambda(e) = e$ such that $t(x) = \lambda(x)t'(x)$ for all $x \in K$. Hence

$$f(x, y) = \lambda(x)\lambda(y)\lambda(xy)^{-1}f'(x, y),$$

and $\qquad h(x, y) = \Gamma_x(\lambda(y))\Gamma_y(\lambda(x))^{-1}\lambda(x \star y)^{-1}h'(x, y).$

Let $\alpha \in C_1^L$ and $\eta \in C_2^L$. Suppose the maps $r_\alpha, r'_\alpha, r_\eta, r'_\eta : K \times K \longrightarrow H$ associated with t are the same as defined before. For t', define $s_\alpha, s'_\alpha, s_\eta, s'_\eta : K \times K \longrightarrow H$ as $s_\alpha(x, y) = f'(x, y)\alpha(f'(x, y))^{-1}$, $s'_\alpha(x, y) = h'(x, y)\alpha(h'(x, y))^{-1}$, $s_\eta(x, y) = f'(\eta(x), \eta(y))f'(x, y)^{-1}$ and $s'_\eta(x, y) = h'(\eta(x), \eta(y))h'(x, y)^{-1}$. Then

$$\begin{aligned}
r_\alpha(x, y)s_\alpha(x, y)^{-1} &= f(x, y)\alpha(f(x, y))^{-1}(f'(x, y)\alpha(f'(x, y))^{-1})^{-1} \\
&= h(x, y)h'(x, y)^{-1}\alpha(h(x, y)h'(x, y)^{-1})^{-1} \\
&= (\lambda(x)\lambda(y)\lambda(xy)^{-1})\alpha(\lambda(x)\lambda(y)\lambda(xy)^{-1})^{-1}
\end{aligned}$$

and

$$\begin{aligned}
r'_\alpha(x, y)s'_\alpha(x, y)^{-1} &= h(x, y)\alpha(h(x, y))^{-1}(h'(x, y)\alpha(h'(x, y))^{-1})^{-1} \\
&= h(x, y)h'(x, y)^{-1}\alpha(h(x, y)h'(x, y)^{-1})^{-1} \\
&= \Gamma_x(\lambda(y))\Gamma_y(\lambda(x))^{-1}\lambda(x \star y)^{-1}\alpha(\Gamma_x(\lambda(y))\Gamma_y(\lambda(x))^{-1}\lambda(x \star y)^{-1})^{-1} \\
&= \Gamma_x(\lambda(y)\alpha(\lambda(y))^{-1})\Gamma_y(\lambda(x)\alpha(\lambda(x))^{-1})^{-1}\lambda(x \star y)^{-1}(\alpha(\lambda(x \star y))^{-1})^{-1}.
\end{aligned}$$

Now define a map $\lambda' : K \longrightarrow H$ by $\lambda'(x) = \lambda(x)\alpha(\lambda(x))^{-1}$; clearly $\lambda'(e) = e$.

Then we have
$$r_\alpha(x,y)s_\alpha(x,y)^{-1} = \lambda'(x)\lambda'(y)\lambda'(xy)^{-1},$$
and
$$r'_\alpha(x,y)s'_\alpha(x,y)^{-1} = \Gamma_x(\lambda'(y))\Gamma_y(\lambda'(x))^{-1}\lambda'(x\star y)^{-1}.$$

Thus $(r_\alpha s_\alpha^{-1}, r'_\alpha s'^{-1}_\alpha, \Gamma) \in B^2_{ML(\Gamma)}(K,H)$. Hence the map χ_1 is well defined. Similarly we can show that χ_2 is well defined. ∎

Note: The maps χ_1 and χ_2 need not be homomorphisms, as group operation for C_1^L and C_2^L is composition not pointwise addition, whereas $\ker(\chi_1)$ and $\ker(\chi_2)$ have the usual meaning.

Notations: The following notations are used in the remaining part of the paper.

$\mathrm{Aut}_H(G) = \{\phi \in \mathrm{Aut}(G) : \phi(H) = H\}$

$\mathrm{Aut}^H(G) = \{\phi \in \mathrm{Aut}(G) : \phi(h) = h \text{ for all } h \in H\}$

$\mathrm{Aut}^{H,K}(G) = \{\phi \in \mathrm{Aut}(G) : \phi(h) = h \text{ and } \phi(x)x^{-1} \in H \text{ for all } x \in G, \ h \in H\}$

$\mathrm{Aut}_H^K(G) = \{\phi \in \mathrm{Aut}(G) : \phi(H) = H \text{ and } \phi(x)x^{-1} \in H \text{ for all } x \in G\}.$

With the help of the above discussion we are able to address the following problem in the next theorem for multiplicative Lie algebras.

Problem: Let $E(H,K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension. Then

(1) under what conditions a pair $(\alpha, I) \in C^L$ is inducible? In other words, under what conditions $\alpha \in C_1^L$ can be lifted to an automorphism of G, which fixes H?

(2) under what conditions a pair $(\eta, I) \in C^L$ is inducible? In other words, under what conditions $\eta \in C_2^L$ can be lifted to an automorphism ϕ of G such that $\beta(\phi(t(x))) = x$ for all $x \in K$?

Theorem 3.3. *Let $E(H,K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension. Then there exist the following exact sequences*
$$e \longrightarrow \mathrm{Aut}^{H,K}(G) \xrightarrow{i} \mathrm{Aut}_H^K(G) \xrightarrow{\Pi_1} C_1^L \xrightarrow{\chi_1} H^2_{ML(\Gamma)}(K,H)$$
$$e \longrightarrow \mathrm{Aut}^{H,K}(G) \xrightarrow{i} \mathrm{Aut}^H(G) \xrightarrow{\Pi_2} C_2^L \xrightarrow{\chi_2} H^2_{ML(\Gamma)}(K,H),$$
where the maps $\Pi_1(\phi) = \alpha$ and $\Pi_2(\phi) = \eta$.

Proof. Clearly, both the sequences are exact at the first two terms. Now we prove the exactness of the first sequence at the third term. Let $\phi \in \mathrm{Aut}_H^K(G)$. Then $\Pi_1(\phi) \in C_1^L$ and $(\chi_1\Pi_1)(\phi) = \chi_1(\Pi_1(\phi)) = \chi_1(\alpha) = (r_\alpha, r'_\alpha, \Gamma)B^2_{ML(\Gamma)}(K,H)$, where $\alpha = \phi|_H$. Since $\phi \in \mathrm{Aut}_H^K(G)$, we have $\eta(x) = \beta(\phi(t(x))) = x$. Now

$$\begin{aligned}
r_\alpha(x,y) &= f(x,y)\alpha(f(x,y))^{-1}\\
&= f(\eta(x),\eta(y))\alpha(f(x,y))^{-1} \ (\because \ \eta(x)=x)\\
&= \lambda(xy)\lambda(x)^{-1}\lambda(y)^{-1} \ (\text{by Lemma 2.4})
\end{aligned}$$

$$\begin{aligned}
r'_\alpha(x,y) &= h(x,y)\alpha(h(x,y))^{-1}\\
&= h(\eta(x),\eta(y))\alpha(h(x,y))^{-1} \ (\because \ \eta(x)=x)\\
&= \lambda(x\star y)\Gamma_y(\lambda(x))\Gamma_x(\lambda(y))^{-1} \ (\text{by Lemma 2.4}).
\end{aligned}$$

Thus $(r_\alpha, r'_\alpha, \Gamma) \in B^2_{ML(\Gamma)}(K, H, \Gamma)$, this implies $\mathrm{Im}(\Pi_1) \subseteq \ker(\chi_1)$. Let $\alpha \in C_1^L$ such that $\chi_1(\alpha) \in B^2_{ML(\Gamma)}(K, H)$. Then Remark 2.7 implies that cocycles (f, h) and $(\alpha \circ f, \alpha \circ h)$ are cohomologous. Thus Corollary 2.8 implies that (α, I) is inducible. Hence there exists $\phi \in \mathrm{Aut}_H(G)$ such that $\Pi(\phi) = (\alpha, I)$. Clearly $\phi \in \mathrm{Aut}_H^K(G)$ with $\Pi_1(\phi) = \alpha$. Therefore $\ker(\chi_1) \subseteq \mathrm{Im}(\phi_1)$.

Now we prove the exactness of the second sequence at the third term. Let $\phi \in \mathrm{Aut}^H(G)$. Then $\Pi_2(\phi) = \eta \in C_2^L$ and $(\chi_2 \Pi_2)(\phi) = \chi_2(\eta) = (r_\eta, r'_\eta, \Gamma) B^2_{ML(\Gamma)}(K, H)$, where $\eta(x) = \beta(\phi(t(x)))$ for all $x \in K$. Since $\eta = \Pi_2(\phi) \in C_2^L$, $\Gamma_x = \Gamma_{\eta(x)}$ for all $x \in K$, this implies $\alpha = \phi|_H = I_H$. Now

$$r_\eta(x, y) = f(\eta(x), \eta(y)) f(x, y)^{-1} = f(\eta(x), \eta(y)) \alpha(f(x, y))^{-1}$$
$$= \lambda(xy) \lambda(x)^{-1} \lambda(y)^{-1} \text{ (by Lemma 2.4)},$$

$$r'_\eta(x, y) = h(\eta(x), \eta(y)) h(x, y)^{-1} = h(\eta(x), \eta(y)) \alpha(h(x, y))^{-1}$$
$$= \lambda(x \star y)(\lambda \star t(\eta(y)))^{-1}(t(\eta(x)) \star \lambda(y))^{-1} \text{ (by Lemma 2.4)}$$
$$= \lambda(x \star y) \Gamma_y(\lambda(x)) \Gamma_x(\lambda(y))^{-1}.$$

This implies $(r_\eta, r'_\eta, \Gamma) \in B^2_{ML(\Gamma)}(K, H)$. Thus $\mathrm{Im}(\Pi_2) \subseteq \ker(\chi_2)$. Let $\eta \in \ker(\chi_2)$ be such that $\chi_2(\eta) \in B^2_{ML(\Gamma)}(K, H)$. Remark 2.7, implies that cocycles (f, h) and $(f \circ (\eta^{-1}, \eta^{-1}), h \circ (\eta^{-1}, \eta^{-1}))$ are cohomologous. Thus Corollary 2.8 implies that (I, η) is inducible. Hence there exist $\phi \in \mathrm{Aut}_H(G)$ such that $\Pi(\phi) = (I, \eta)$. Clearly $\eta \in \mathrm{Aut}_H^K(G)$ with $\Pi_2(\phi) = \eta$. Therefore $\ker(\chi_2) \subseteq \mathrm{Im}(\phi_2)$. ∎

Remark 3.4. From the last Theorem 3.3, we can conclude that $\alpha \in C_1^L$ and $\eta \in C_2^L$ can be lifted to an automorphism in $\mathrm{Aut}_H(G)$, respectively if and only if $\alpha \in \ker(\chi_1)$ and $\eta \in \ker(\chi_2)$.

Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension and let $(\alpha, \eta) \in C^L$. Define maps $s_{\alpha, \eta}, s'_{\alpha, \eta} : K \times K \longrightarrow H$ by

$$s_{\alpha, \eta}(x, y) = r_\alpha(x, y) r_\eta(x, y) = f(\eta(x), \eta(y)) \alpha(f(x, y))^{-1}$$

and $\qquad s'_{\alpha, \eta}(x, y) = r'_\alpha(x, y) r'_\eta(x, y) = h(\eta(x), \eta(y)) \alpha(h(x, y))^{-1}.$

Using Lemma 3.1, $(s_{\alpha, \eta}, s'_{\alpha, \eta}, \Gamma) \in Z^2_{ML(\Gamma)}(K, H)$. We know that different 2-cocycles associated with the same extension differ by an 2-coboundary. Thus we define a map $\chi : C^L \longrightarrow H^2_{ML(\Gamma)}(K, H)$ by

$$\chi(\alpha, \eta) = (s_{\alpha, \eta}, s'_{\alpha, \eta}, \Gamma) B^2_{ML(\Gamma)}(K, H).$$

From Lemma 3.2 we observe that χ is independent of the choice of transversal t. Thus χ is well defined.

Remark 3.5. If $\chi(\alpha, \eta)$ is trivial for the pair $(\alpha, \eta) \in C^L$, where $\eta \in \mathrm{Aut}(K)$, then we can think x as $\eta^{-1}(x)$. Then Corollary 2.8 implies that, (f, h) and $(\alpha \circ f \circ (\eta^{-1}, \eta^{-1}), \alpha \circ h \circ (\eta^{-1}, \eta^{-1}))$ are cohomologous cocycles. Thus, the compatible pair $(\alpha, \eta) \in C^L$ is inducible if and only if $\chi(\alpha, \eta)$ is trivial. Therefore, we can say that $\chi(\alpha, \eta)$ is an obstruction for the inducibility of the compatible pair (α, η).

Lemma 3.6. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension. Then $Z^1_{ML(\Gamma)}(K, H) \cong \mathrm{Aut}^{H,K}(G)$ as groups.*

Proof.   Define $\delta : Z^1_{ML(\Gamma)}(K, H) \longrightarrow \text{Aut}(G)$ by $\delta(\nu) = \phi_\nu$, where $\phi_\nu : G \longrightarrow G$ is defined by $\phi_\nu(ht(x)) = h\nu(x)t(x)$, $h \in H$ and $x \in K$. To show that $\phi_\nu \in \text{Aut}(G)$ let $g = ht(x)$ and $g' = h't(y) \in G$. Then

$$\phi_\nu(ht(x)h't(y)) = \phi_\nu(hh't(x)t(y)) = hh'f(x,y)\nu(xy)t(xy)$$
$$= hh'f(x,y)\nu(x)\nu(y)f(x,y)^{-1}t(x)t(y) = h\nu(x)t(x)h'\nu(y)t(y)$$
$$= \phi_\nu(ht(x))\phi_\nu(h't(y)).$$

and

$$\phi_\nu(ht(x) \star h't(y)) = \phi_\nu((h \star t(y))(t(x) \star h')h(x,y)(t(x \star y)))$$
$$= (t(y) \star h)^{-1}(t(x) \star h')h(x,y)\nu(x \star y)t(x \star y)$$
$$= (t(y) \star h)^{-1}(t(x) \star h')h(x,y)(t(x) \star \nu(y))(t(y) \star \nu(x))^{-1}h(x,y)^{-1}(t(x) \star t(y))$$
$$= (t(y) \star h)^{-1}(t(x) \star h')(t(x) \star \nu(y))(t(y) \star \nu(x))^{-1}(t(x) \star t(y))$$
$$= (h \star h')(h \star t(y))(t(x) \star h')(t(x) \star \nu(y))(\nu(x) \star t(y))(t(x) \star t(y))$$
$$= (h\nu(x)t(x) \star h'\nu(y)t(y)) = \phi_\nu(ht(x)) \star \phi_\nu(h't(y)).$$

Thus ϕ_ν is a multiplicative Lie algebra homomorphism. Let $g = ht(x) \in G$ and $\phi_\nu(ht(x)) = h\nu(x)t(x) = e$. Then $t(x) \in H$; this implies $x = \beta(t(x)) = e$ and $h = e$. Thus $g = e$, i.e. ϕ_ν is injective. If $ht(x) \in G$, then $\phi_\nu(h\lambda(x)^{-1}t(x)) = ht(x)$. Thus ϕ_ν is surjective. Hence $\phi_\nu \in \text{Aut}(G)$. Since $\phi_\nu(h) = h$ for all $h \in H$ and $\phi_\nu(ht(x))(ht(x))^{-1} = h\nu(x)t(x)t(x)^{-1}h^{-1} = \nu(x) \in H$. Therefore $\phi_\nu \in \text{Aut}^{H,K}(G)$. Now we prove that δ is a group isomorphism. Let ν_1 and $\nu_2 \in Z^1_{ML(\Gamma)}(K, H)$.

$$\delta(\nu_1 + \nu_2)(g) = \phi_{\nu_1+\nu_2}(ht(x)) = h(\nu_1 + \nu_2)(x)t(x)$$
$$= h\nu_1(x)t(x) + h\nu_2(x)t(x) = (\delta(\nu_1) + \delta(\nu_2))(g).$$

Thus δ is a group homomorphism. Let $\delta(\nu) = I_\nu$. Then $h\nu(x)t(x) = ht(x)$; this implies $\nu(x) = e$ for all $x \in K$. Thus δ is injective. Let $\phi \in \text{Aut}^{H,K}(G)$. Then $\beta(\phi(t(x))t(x)^{-1}) = e$, $\phi(t(x))t(x)^{-1} \in \ker(\beta) = H$, this implies that there exists unique element say $\nu(x) \in H$ such that $\phi(t(x)) = \nu(x)t(x)$. Thus we have a map $\nu : K \longrightarrow H$ with $\nu(e) = e$ such that $\phi(t(x)) = \nu(x)t(x)$. Since ϕ is a multiplicative Lie algebra homomorphism. Thus

$$\phi(t(x)t(y)) = \phi(t(x))\phi(t(y))$$
$$\phi(f(x,y)t(xy)) = \nu(x)t(x)\nu(y)t(y)$$
$$f(x,y)\nu(xy)t(xy) = \nu(x)\nu(y)f(x,y)t(xy)$$
$$\nu(xy) = \nu(x)\nu(y),$$

and
$$\phi(t(x) \star t(y)) = \phi(t(x)) \star \phi(t(y))$$
$$\phi(h(x,y)t(x \star y)) = \nu(x)t(x) \star \nu(y)t(y)$$
$$h(x,y)\nu(x \star y)t(x \star y) = (\nu(x) \star \nu(y))(\nu(x) \star t(y))(t(x) \star \nu(y))(t(x) \star t(y))$$
$$\nu(x \star y) = (t(x) \star \nu(y))(t(y) \star \nu(x))^{-1}$$
$$\nu(x \star y) = \Gamma_x(\nu(y))\Gamma_y(\nu(x)^{-1}).$$

Thus $\nu \in Z^1_{ML(\Gamma)}(K, H)$. Since $\phi(ht(x)) = \phi(h)\phi(t(x)) = h\nu(x)t(x) = \delta(\nu)$ for $h \in H$ and $x \in K$. Thus δ is surjective. Hence $Z^1_{ML(\Gamma)}(K, H) \cong \text{Aut}^{H,K}(G)$.   ∎

Theorem 3.7. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension. Then the following sequence is exact*

$$e \longrightarrow Z^1_{ML(\Gamma)}(K, H) \xrightarrow{i} \mathrm{Aut}_H(G) \xrightarrow{\Pi} C^L \xrightarrow{\chi} H^2_{ML(\Gamma)}(K, H),$$

where $\chi(\alpha, \eta) = (s_{\alpha,\eta}, s'_{\alpha,\eta}, \Gamma) B^2_{ML(\Gamma)}(K, H)$ and $\Pi(\phi) = (\alpha, \eta)$.

Proof. Clearly, the sequence is exact at the first two terms. For the third term, let $\phi \in \mathrm{Aut}_H(G)$ be such that $\Pi(\phi) = (\alpha, \eta)$. By Lemma 2.4, we have

$$f(\eta(x), \eta(y)) \alpha(f(x, y))^{-1} = \lambda(xy) \lambda(x)^{-1} \lambda(y)^{-1},$$

and $\qquad h(\eta(x), \eta(y)) \alpha(h(x, y))^{-1} = \lambda(x \star y) \Gamma_{\eta(y)}(\lambda(x)) \Gamma_{\eta(x)}(\lambda(y)^{-1}),$

where $\lambda : K \longrightarrow H$ with $\lambda(e) = e$ this implies $s_{\alpha,\eta}(x, y) = \lambda(xy) \lambda(x)^{-1} \lambda(y)^{-1}$ and $s'_{\alpha,\eta}(x, y) = \lambda(x \star y) \Gamma_{\eta(y)}(\lambda(x)) \Gamma_{\eta(x)}(\lambda(y)^{-1})$. Thus $(s_{\alpha,\eta}, s'_{\alpha,\eta}, \Gamma) \in B^2_{ML(\Gamma)}(K, H)$. Hence $(\alpha, \eta) \in \ker(\chi)$. Let $(\alpha, \eta) \in \ker(\chi)$ such that

$$\chi(\alpha, \eta) = (s_{\alpha,\eta}, s'_{\alpha,\eta}, \Gamma) B^2_{ML(\Gamma)}(K, H) = B^2_{ML(\Gamma)}(K, H);$$

this implies $(s_{\alpha,\eta}, s'_{\alpha,\eta}, \Gamma) \in B^2_{ML(\Gamma)}(K, H)$, then there exist $\lambda : K \longrightarrow H$ with $\lambda(e) = e$ such that

$$s_{\alpha,\eta}(x, y) = \lambda(xy)^{-1} \lambda(x) \lambda(y) \text{ and } s'_{\alpha,\eta}(x, y) = \lambda(x \star y) \Gamma_{\eta(y)}(\lambda(x)) \Gamma_{\eta(x)}(\lambda(y)^{-1}).$$

Then by Lemma 2.4, the pair (α, η) is inducible. Thus there exists $\phi \in \mathrm{Aut}_H(G)$ such that $\Pi(\phi) = (\alpha, \eta)$. Hence $(\alpha, \eta) \in \mathrm{Im}(\Pi)$. ∎

Corollary 3.8. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a split center extension. Then every compatible pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible.*

Proof. Since $E(H, K)$ is a split extension, the maps $f : K \times K \longrightarrow H$ and $g : K \times K \longrightarrow H$ are both trivial. Hence the map χ is also trivial. From Theorem 3.7, we get that Π is surjective. ∎

Theorem 3.9. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a central extension. Then the following sequence is exact*

$$e \longrightarrow Z^1_{ML(\Gamma)}(K, H) \xrightarrow{i} \mathrm{Aut}_H(G) \xrightarrow{\Pi} \mathrm{Aut}(H) \times \mathrm{Aut}(K) \xrightarrow{\chi} H^2_{ML(\Gamma)}(K, H),$$

where $\chi(\alpha, \eta) = (s_{\alpha,\eta}, s'_{\alpha,\eta}, \Gamma) B^2_{ML(\Gamma)}(K, H)$ and $\Pi(\phi) = (\alpha, \eta)$.

Proof. Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a central extension. Then $\Gamma_x(h) = t(x) \star h = e$, this implies that if $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$, we have $\alpha \Gamma_x \alpha^{-1} = \Gamma_{\eta(x)}$. Thus $(\alpha, \eta) \in C^L$. Hence $\mathrm{Aut}(H) \times \mathrm{Aut}(K) = C^L$. The exactness of the sequence follows from Theorem 3.7 and Lemma 2.9. ∎

Corollary 3.10. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a split central extension. Then every pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible.*

Definition 3.11. [6] Let $E(R, K) \equiv e \longrightarrow R \xrightarrow{i} F \xrightarrow{\nu} K \longrightarrow e$ be a free presentation of a multiplicative Lie algebra K. Then the *Schur multiplier* of K is defined as $\dfrac{([F, F](F \star F)) \cap R}{[R, F](R \star F)}$ and denoted by $\tilde{M}(K)$.

Recall from [[6], Corollary 6.3], if K is a finite multiplicative Lie algebra, then $\tilde{M}(K) \cong H^2_{ML}(K, \mathbb{C}^\star)$. Also, a multiplicative Lie algebra K is said to be *perfect* if $K = [K, K](K \star K)$. Now we mention two Propositions from [6] as we will use them in our next corollary.

Proposition 3.12. [6] *Every perfect multiplicative Lie algebra K admits a unique universal central extension. More precisely, if a free presentation of K is given by the extension $E(R, K)$, then a universal central extension by K is given by*

$$e \longrightarrow \frac{([F,F](F \star F)) \cap R}{[R,F](R \star F)} \xrightarrow{\bar{i}} \frac{([F,F](F \star F))}{[R,F](R \star F)} \xrightarrow{\bar{\nu}} K \longrightarrow e.$$

Proposition 3.13. [6] *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a central extension. Then $E(H, K)$ is a universal central extension by K if and only if G is perfect and every central extension by G splits.*

Corollary 3.14. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a central extension, where K is a finite multiplicative Lie algebra. If K is perfect and $\tilde{M}(K) = e$. Then every pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible.*

Proof. Let $E(R, K) \equiv e \longrightarrow R \xrightarrow{i} F \xrightarrow{\nu} K \longrightarrow e$ be a free presentation of K. Then by Proposition 3.12, we have the following universal central extension

$$e \longrightarrow \frac{([F,F](F \star F)) \cap R}{[R,F](R \star F)} \xrightarrow{\bar{i}} \frac{([F,F](F \star F))}{[R,F](R \star F)} \xrightarrow{\bar{\nu}} K \longrightarrow e.$$

Also, we have $\tilde{M}(K) = e$, this implies

$$e \longrightarrow e \longrightarrow \frac{([F,F](F \star F))}{[R,F](R \star F)} \xrightarrow{\bar{\nu}} K \longrightarrow e$$

is a universal central extension and $e \longrightarrow e \longrightarrow K \xrightarrow{i} K \longrightarrow e$ is also a universal central extension. Thus by Proposition 3.13, every central extension by K splits. Hence $H^2_{ML(\Gamma)}(K, H) = e$. Therefore by Theorem 3.9, every pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible. $\blacksquare$

Corollary 3.15. *Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a central extension, where K is a cyclic group of order m and H is an abelian group of order n such that m and n are coprime. Then every pair $(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K)$ is inducible.*

Proof. We know that the Schur multiplier $M(K)$ of a cyclic group is trivial. Thus by corollary [3.5, [6]] we have $H^2_{ML(\Gamma)}(K, H) \cong H^2(K, H) \times Hom(\wedge^2 K, H)$, where $H^2(K, H)$ is the second cohomology group of K with coefficient in H and $\wedge^2 K$ is non-abelian exterior square group of K. Since m and n are coprime, $H^2(K, H) = e$. Since $M(K) = e$, then $\wedge^2 K = [K, K] = e$. Therefore $H^2_{ML(\Gamma)}(K, H) = e$. By Theorem 3.7, the map Π is surjective. $\blacksquare$

4. Split exact sequences

Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a center extension.
If $C_1^{L^*} = \{\alpha \in C_1^L : \chi_1(\alpha) = e\}$ and $C_2^{L^*} = \{\alpha \in C_2^L : \chi_2(\eta) = e\}$, then, by Theorem 3.3, we have the following exact sequences:

$$e \longrightarrow \mathrm{Aut}^{H,K}(G) \xrightarrow{i} \mathrm{Aut}_H^K(G) \xrightarrow{\Pi_1} C_1^{L^*} \longrightarrow e \tag{9}$$

$$e \longrightarrow \mathrm{Aut}^{H,K}(G) \xrightarrow{i} \mathrm{Aut}^H(G) \xrightarrow{\Pi_2} C_2^{L^*} \longrightarrow e \tag{10}$$

Let $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ be a central extension and let $C^{L^*} = \{(\alpha, \eta) \in \mathrm{Aut}(H) \times \mathrm{Aut}(K) : \chi(\alpha, \eta) = e\}$. Then by Theorem 3.7, we have the following exact sequence:

$$e \longrightarrow Z_{ML(\Gamma)}^1(K, H) \xrightarrow{i} \mathrm{Aut}_H(G) \xrightarrow{\Pi} C^{L^*} \longrightarrow e \tag{11}$$

Theorem 4.1. *If exact sequence $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ splits, then the sequences (9) and (10) also split. Also, if $E(H, K)$ is a central extension and splits, then the sequence (11) also splits.*

Proof.   Since $E(H, K) \equiv e \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow e$ splits, we have a multiplicative Lie algebra homomorphism $t : K \longrightarrow G$ such that $\beta \circ t = I_K$. To show that sequence (9) splits. Define a map $\delta_1 : C_1^{L^*} \longrightarrow \mathrm{Aut}_H^K(G)$ as $\delta_1(\alpha) = \phi_1$, where $\phi_1 : G \longrightarrow G$ defined as $\phi_1(t(x)h) = t(x)\alpha(h)$, $h \in H$ and $x \in K$. Let $g_1 = t(x_1)h_1$ and $g_2 = t(x_2)h_2 \in G$ for some $x_1, x_2 \in K$ and $h_1, h_2 \in H$. Then

$$\phi_1(g_1 g_2) = \phi_1(t(x_1 x_2)h_1 h_2) = t(x_1 x_2)\alpha(h_1 h_2) = t(x_1)\alpha(h_1)t(x_2)\alpha(h_2) = \phi_1(g_1)\phi_1(g_2)$$

and $\begin{aligned}[t] \phi_1(g_1 \star g_2) &= \phi_1(t(x_1)h_1 \star t(x_2)h_2) \\ &= \phi_1((t(x_1) \star t(x_2))(h_1 \star t(x_2))(t(x_1) \star h_2)) \ (\because \ H \subseteq Z(G)) \\ &= t(x_1 \star x_2)\alpha(h_1 \star t(x_2))\alpha(t(x_1) \star h_2) \ (\because \ t \text{ - homomorphism}) \\ &= t(x_1) \star t(x_2)(\alpha(h_1) \star t(x_2))(t(x_1) \star \alpha(h_2)) \ (\because \ \alpha \in C_1^L) \\ &= t(x_1)\alpha(h_1) \star t(x_2)\alpha(h_2) = \phi_1(g_1) \star \phi_1(g_2). \end{aligned}$

Thus χ_1 is a multiplicative Lie algebra homomorphism. Let $h \in H$. Then we have $\phi_1(h) = \alpha(h)$, implying $\phi_1(H) = H$. Consider

$$\phi_1(g)g^{-1} = \phi_1(t(x)h)(t(x)h)^{-1} = t(x)\alpha(h)(ht(x))^{-1} = \alpha(h)h^{-1} \in H.$$

Hence $\phi_1 \in \mathrm{Aut}_H^K(G)$ and $\Pi_1 \circ \delta_1 = I_{C_1^{L^*}}$. To show that the sequence (10) splits, define a map $\delta_2 : C_2^{L^*} \longrightarrow \mathrm{Aut}^H(G)$ by $\delta_2(\eta) = \phi_2$, where $\phi_2 : G \longrightarrow G$ given by $\phi(g) = \phi_2(t(x)h) = t(\eta(x))h$. Then we can easily see that ϕ_2 is a group homomorphism. Consider

$$\begin{aligned} \phi_2(g_1 \star g_2) &= \phi_2(t(x_1)h_1 \star t(x_2)h_2) \\ &= \phi_2((t(x_1) \star t(x_2))(t(x_1) \star h_2)(h_1 \star t(x_2))) \\ &= t(\eta(x_1) \star \eta(x_2))(t(x_1) \star h_2)(h_1 \star t(x_2)) \\ &= t(\eta(x_1)) \star t(\eta(x_2))(t(\eta(x_1)) \star h_2)(h_1 \star t(\eta(x_2))) \\ &= t(\eta(x_1))h_1 \star t(\eta(x_2))h_2 = \phi_2(g_1) \star \phi_2(g_2). \end{aligned}$$

Thus ϕ_2 is a multiplicative Lie algebra homomorphism, also $\phi_2(h) = h$. Hence $\phi_2 \in \mathrm{Aut}^H(G)$ and $\Pi_2 \circ \phi_2 = I_{C_2^{L^*}}$. Finally for the sequence (11), define a map $\delta : C^{L^*} \longrightarrow \mathrm{Aut}_H(G)$ by $\delta(\alpha, \eta) = \phi$, where $\phi : G \longrightarrow G$ is given by $\phi(g) = \phi(t(x)h) = t(\eta(x))\alpha(h)$. Since $H \subseteq Z(G) \cap LZ(G)$, i.e. $h \star g = e$ for all $g \in G$ and $h \in H$. Then, ϕ is a group homomorphism and

$$\phi(g_1 \star g_2) = \phi(t(x_1 \star x_2)) = t(\eta(x_1 \star x_2)) = \phi(g_1) \star \phi(g_2).$$

Thus ϕ is a multiplicative Lie algebra homomorphism, also $\phi(h) = \alpha(h)$. Hence $\phi \in \mathrm{Aut}_H(G)$ and $\Pi \circ \delta = I_{C^{L^*}}$. ∎

References

[1]   A. Bak, G. Donadze, N. Inassaridze, M. Ladra: *Homology of multiplicative Lie rings*, J. Pure Appl. Algebra 208 (2007) 761–777.

[2]   V. G. Bardakov, M. Singh: *Extensions and automorphisms of Lie algebras*, J. Algebra Appl. 16/5 (2017), art.no. 1750162, 15p.

[3]   G. Donadze, N. Inassaridze, M. Ladra, A. M. Vieites: *Exact sequences in homology of multiplicative Lie rings and a new version of Stallings*, J. Pure Appl. Algebra 222 (2018) 1786–1802.

[4]   G. J. Ellis: *On five well known commutator identities*, J. Aust. Math. Soc., Ser. A 54 (1993) 1–19.

[5]   P. Jin: *Automorphisms of groups*, J. Algebra 312 (2007) 562–569.

[6]   R. Lal, S. K. Upadhyay: *Multiplicative Lie algebra and Schur multiplier*, J. Pure Appl. Algebra 223 (2019) 3695–3721.

[7]   M. S. Pandey, R. Lal, S. K. Upadhyay: *Lie commutator, solvability and nilpotency in multiplicative Lie algebras*, J. Algebra Appl. 20 (2021), art.no. 2150138, 11p.

[8]   M. S. Pandey, S. K. Upadhyay: *Classification of multiplicative Lie algebra structures on a finite group*, Colloq. Math. 168 (2022) 25–34.

[9]   M. S. Pandey, S. K. Upadhyay: *Theory of extensions of multiplicative Lie algebras*, J. Lie Theory 31 (2022) 637–658.

[10] I. B. S. Passi, M. Singh, M. K. Yadav: *Automorphisms of abelian group extensions*, J. Algebra 324 (2010) 820–830.

[11] F. Point, P. Wantiez: *Nilpotency criteria for multiplicative Lie algebra*, J. Pure Appl. Algebra 111 (1996) 229–243.

[12] C. Wells: *Automorphisms of group extensions*, Trans. Amer. Math. Soc. 155 (1971) 189–194.

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