Journal of Lie Theory
Volume **35** (2025) 345–358

Bounding the Norm of the Derivative of the Lie Exponential Map for Connected Lie Groups

Reza Bidar

Communicated by P. J. Olver

Abstract. Let G be a real connected Lie group with a left invariant metric d, $\mathfrak{g}$ its Lie algebra, $\exp : \mathfrak{g} \to G$ be the Lie exponential map, and ad be the adjoint representation of $\mathfrak{g}$. In this paper we use matrix algebra and Jordan normal form to derive a set of upper and lower bounds for $|d\exp_x(y)|$, $x, y \in \mathfrak{g}$ that generally are exponential type functions of the eigenvalues of ad_x. These bounds provide useful information about the exponential map and the way it relates the Euclidean metric of $\mathfrak{g}$ and the left invariant metric of G. For Lie groups for which the exponential map is a diffeomorphism, we investigate conditions under which the exponential map is a quasi-isometry. This is obviously true if G is isomorphic to $\mathbb{R}^n$. We prove that the exponential map is a quasi-isometry only when G is isomorphic to $\mathbb{R}^n$.

Mathematics Subject Classification: 22E15, 22E60.

Key Words: Lie group, exponential map, adjoint, quasi-isometry.

1. Introduction

Let G be a connected real Lie group with a left invariant Riemannian metric d, $\mathfrak{g}$ its Lie algebra as an inner product space, and $\exp : \mathfrak{g} \to G$ the Lie exponential map. For $g \in G$, let l_g denote the left multiplication by g. Given a non-zero element $x \in \mathfrak{g}$, it is well known that the differential of the exponential map at x is given by

$$d\exp_x = dl_{\exp(x)} \frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}, \tag{1}$$

Rossman [11, p.15]. Since the metric d is left invariant, it follows that for any $y \in \mathfrak{g}$

$$|d\exp_x(y)| = \left| \frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}(y) \right| .$$

Thus the problem of finding upper and lower bounds for $|d\exp_x(y)|$ would be equivalent to finding estimates for the norm of the image of

$$\frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x} .$$

We use matrix algebra and Jordan normal form to find these estimates. These bounds are generally exponential type functions of the eigenvalues of ad_x as stated in the next section.

ISSN 0949–5932 / \$2.50

Given two metric spaces (X_1, d_1) and (X_2, d_2) a continuous map $f : X_1 \to X_2$ is a (α, β)-*quasi-isometery* [4, Definition 8.14] if there exist constants $\alpha \geq 1$, and $\beta, c \geq 0$ such that for all $x, y \in X_1$

$$\frac{1}{\alpha} d_1(x,y) - \beta \leq d_2(f(x), f(y)) \leq \alpha d_1(x,y) + \beta\,, \tag{2}$$

and every point of X_2 lies in the c–neighbourhood of the image of f. In this paper we consider a somewhat stronger definition of quasi-isometry as formulated in Attie [1, Definition 1.12]. We say the exponential map is a *(smooth) quasi-isometry* or simply *quasi-isometry* if it is diffeomorphism and there is a constant $\alpha \geq 1$ such that for all $x, y \in \mathfrak{g}$:

$$\alpha^{-1}|y| \leq |d\exp_x(y)| \leq \alpha |y|\,. \tag{3}$$

By using the Riemannian exponential map one can easily see that every smooth quasi-isometry is a $(\alpha, 0)$ quasi-isometry. Knowing under which conditions the Lie exponential map satisfies (3), or is a quasi-isometry may be useful in the study of metric properties of Lie groups. Clearly if G is abelian then the exponential map satisfies (3), and if G is isomorphic to $\mathbb{R}^n$ then the exponental map is quasi-isometry. An element $x \in \mathfrak{g}$ is a *compact element* if $\mathrm{spec}(\mathrm{ad}_x) \subseteq i\mathbb{R}$. For a simply connected, solvable Lie group G, if all elements of $\mathfrak{g}$ are compact then G is a *rigid* Lie group. Under this definition $\mathbb{R}^n$ and all simply connected nilpotent Lie groups are rigid. To see some other examples of rigid Lie groups see Auslander [2]. As we will see in Proposition 2.5, for a Lie group G homeomorphic to $\mathbb{R}^n$, if there is a constant $\alpha \geq 1$ such that

$$|d\exp_x(y)| \leq \alpha|y| \quad \text{for all } x, y \in \mathfrak{g} \tag{4}$$

then G must be a rigid Lie group.

In Theorem 2.5, we prove that if the exponential map satisfies (3) then G must be abelian, and if the exponential map is quasi-isometry, then G must be isomorphic to $\mathbb{R}^n$.

2. Results

For the purpose of the method we use here, throughout the paper we work with $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$, the complexification of $\mathfrak{g}$ as detailed in section (3.4), and we consider

$$\frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\,,$$

as a linear transformation on $\mathfrak{g}^{\mathbb{C}}$ as a complex inner product space. Let $\dim(\mathfrak{g}_{\mathbb{C}}) = n$ and assume that $\hat{x} = x/|x|$ has complex eigenvalues $\lambda_1, \cdots, \lambda_n$. By the *Spectral Mapping Theorem* in Rudin [12, Theorem 10.33]) eigenvalues of $(1 - e^{-\mathrm{ad}_x})/\mathrm{ad}_x$ are $(1 - e^{-\lambda_i|x|})/\lambda_i|x|$, $1 \leq i \leq n$. We use the Jordan normal form to bound the differential of exponential map by the exponential terms related to the eigenvalues of $(1 - e^{-\mathrm{ad}_x})/\mathrm{ad}_x$ as stated in Theorem 2.1.

Theorem 2.1. *Let $x \in \mathfrak{g}$, $|x| \geq 1$. Let $\hat{x} = x/|x|$, λ_i, $1 \leq i \leq q$ be eigenvalues of $\mathrm{ad}_{\hat{x}}$, counting geometric multiplicity. Let m_i be the size of the Jordan block associated with λ_i. In addition assume that the eigenvalues have been labeled so that there is some $p \leq q$, such that for each $i > p$, $m_i > 1$, λ_i is non-zero and has zero real part.*

Let
$$\tilde\lambda_1(|x|) = \min\left\{|x|^{1-m_1}\left|\frac{1-e^{-\lambda_1|x|}}{\lambda_1|x|}\right|,\cdots,|x|^{1-m_p}\left|\frac{1-e^{-\lambda_p|x|}}{\lambda_p|x|}\right|\right\}$$
$$\tilde\lambda_2(|x|) = \min\left\{\left|\frac{1-e^{-\lambda_{p+1}|x|}}{\lambda_{p+1}|x|}\right|^{m_{p+1}},\cdots,\left|\frac{1-e^{-\lambda_q|x|}}{\lambda_q|x|}\right|^{m_q}\right\}$$
$$\tilde\lambda_3(|x|) = \max\left\{|x|^{m_1-1}\left|\frac{1-e^{-\lambda_1|x|}}{\lambda_1|x|}\right|,\cdots,|x|^{m_p-1}\left|\frac{1-e^{-\lambda_p|x|}}{\lambda_p|x|}\right|\right\}$$
$$\tilde\lambda_4(|x|) = \max\left\{|x|^{m_{p+1}-2},\cdots,|x|^{m_q-2}\right\}$$
$$\tilde\lambda_{\min}(|x|) = \min(\tilde\lambda_1(|x|),\tilde\lambda_2(|x|)),\quad \tilde\lambda_{\max}(|x|) = \max(\tilde\lambda_3(|x|),\tilde\lambda_4(|x|))$$

Then there exist positive constants C, D, only depending on $\hat{x}$, such that for any unit $y \in \mathfrak{g}$
$$C\tilde\lambda_{\min}(|x|) \le |d\exp_x(y)| \le D\tilde\lambda_{\max}(|x|)\,. \tag{5}$$

Corollary 2.2. *Let $x \in \mathfrak{g}$ be non-zero and such that ad_x is diagonalizable. Let $\hat{x} = x/|x|$, $\lambda_1 = 0,\cdots,\lambda_p \in \mathbb{C}$ be non-repeated eigenvalues of $\mathrm{ad}_{\hat{x}}$, and*
$$\tilde\lambda_{\min}(|x|) = \min\left\{\left|\frac{1-e^{-\lambda_1|x|}}{\lambda_1|x|}\right|,\cdots,\left|\frac{1-e^{-\lambda_p|x|}}{\lambda_p|x|}\right|\right\},$$
$$\tilde\lambda_{\max}(|x|) = \max\left\{\left|\frac{1-e^{-\lambda_1|x|}}{\lambda_1|x|}\right|,\cdots,\left|\frac{1-e^{-\lambda_p|x|}}{\lambda_p|x|}\right|\right\}.$$

Then there exist positive constants C, D, only depending on $\hat{x}$, such that for any unit $y \in \mathfrak{g}$
$$C\tilde\lambda_{\min}(|x|) \le |d\exp_x(y)| \le D\tilde\lambda_{\max}(|x|)\,. \tag{6}$$

When ad_x is p-step nilpotent, the only eigenvalue is 0 and the size of the largest Jordan block is p, so we get the following corollary:

Corollary 2.3. *Let $\mathfrak{g}$ be p-step nilpotent. Given a non-zero $x \in \mathfrak{g}$,*
$$\frac{C}{|x|^{p-1}} \le |d\exp_x(y)| \le D|x|^{p-1}.$$

Let G be simply connected and solvable. In the late fifties Dixmier [5] and Saito [13] independently proved that the exponential map is injective iff it is surjective iff it is diffeomorphism, and all of these statements are equivalent to the following:
$$\mathrm{spec}(\mathrm{ad}_x) \cap i\mathbb{R} = \{0\},\quad \text{for all } x \in \mathfrak{g}\,. \tag{7}$$

This leads us to the following proposition:

Proposition 2.4. *Let G be a simply connected solvable Lie group with an injective (or surjective) exponential map. Fix a unit element $\hat{x}$ in $\mathfrak{g}$, and let $x = |x|\hat{x}$ with $|x| \ge 1$. Let m be the size of the largest Jordan block associated with $\mathrm{ad}_{\hat{x}}$. Then there exist a positive constant C such that for all units $y \in \mathfrak{g}$*
$$|d\exp_x(y)| \ge \frac{C}{|x|^m}\,. \tag{8}$$

If in addition $\mathrm{ad}_{\hat{x}}$ is diagonalizable,
$$|d\exp_x(y)| \ge \frac{C}{|x|}\,. \tag{9}$$

By using the exponential form of the eigenvalues for the differential of the exponential map we can conclude the following proposition and theorem:

Proposition 2.5. *Let G be a real connected Lie group. Assume there is a constant $\alpha \geq 1$ such that $|d\exp_x(y)| \leq \alpha|y|$ for all $x, y \in \mathfrak{g}$. Then for all $x \in \mathfrak{g}$,* $\operatorname{spec}(\operatorname{ad}_x) \subseteq i\mathbb{R}$. *If additionally G is homeomorphic to $\mathbb{R}^n$ then it is a rigid Lie group.*

Theorem 2.6. *Let G be a real connected Lie group. Assume there is a constant $\alpha \geq 1$ such that*

$$\alpha^{-1}|y| \leq |d\exp_x(y)| \leq \alpha|y|\,,$$

for all $x, y \in \mathfrak{g}$. Then G is abelian. If the exponential map is a quasi-isometry then G is ismorphic to $\mathbb{R}^n$.

3. Preliminaries

3.1. Jordan canonical form

Given a complex matrix $\boldsymbol{A} \in \mathrm{M}_n(\mathbb{C})$, let $\boldsymbol{J}$ be the Jordan normal form of $\boldsymbol{A}$. Thus there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{J}\boldsymbol{P}$, where

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_1 & & \\ & \ddots & \\ & & \boldsymbol{J}_p \end{bmatrix}, \quad \boldsymbol{J_i} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

If f is an analytic matrix function on $\mathrm{M}_n(\mathbb{C})$ for any matrix $\boldsymbol{A}$ where its norm is inside the radius of convergence of f, we have:

$$f(\boldsymbol{A}) = \boldsymbol{P}^{-1} f(\boldsymbol{J})\boldsymbol{P}, \quad f(\boldsymbol{J}) = \begin{bmatrix} f(\boldsymbol{J}_1) & & \\ & \ddots & \\ & & f(\boldsymbol{J}_p) \end{bmatrix}, \tag{10}$$

where

$$f(\boldsymbol{J_i}) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{f''(\lambda_i)}{2} & \cdots & \frac{f^{(n-1)}(\lambda_i)}{(n-1)!} \\ 0 & f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(n-2)}(\lambda_i)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda_i) & f'(\lambda_i) \\ 0 & 0 & 0 & 0 & f(\lambda_i) \end{bmatrix} \tag{11}$$

Golub [6, Theorem 11.1.1] .

3.2. Singular values

Given a linear operator T, enumerate the singular values $\{\sigma_j(\mathrm{T})\}$, $j = 1, 2, \cdots$ in a non-increasing order, and the eigenvalues $\{\lambda_j(\mathrm{T})\}$, $j = 1, 2, \cdots$ so that the moduli are non-increasing. Assume that $\mathcal{H}$ is a finite dimensional Hilber space, $\dim \mathcal{H} = n$.

Singular values of a linear operator defined on a finite dimensional Hilbert space are related to the norm of the operator on some subspaces of $\mathcal{H}$. This is known as minimax principle for singular values (See Bhatia [3, Chapter III]):

Proposition 3.1 (The minimax principle for singular values). *Given any operator on a finite dimensional Hilbert space $\mathcal{H}$, $\dim\mathcal{H} = n$, we have for $1 \leq j \leq n$:*

$$\sigma_j(\mathrm{T}) = \max_{\mathcal{M}:\dim\mathcal{M}=j}\ \min_{x\in\mathcal{M},|x|=1} |\mathrm{T}(x)| = \min_{\mathcal{N}:\dim\mathcal{N}=n-j+1}\ \max_{x\in\mathcal{N},|x|=1} |\mathrm{T}(x)|. \tag{12}$$

In particular, the minimax principle implies that:

$$\sigma_1(\mathrm{T}) = \max_{|x|=1} |\mathrm{T}(x)| = \|\mathrm{T}\|, \quad \sigma_n(\mathrm{T}) = \min_{|x|=1} |\mathrm{T}(x)|. \tag{13}$$

3.3. Lie groups homeomorphic to $\mathbb{R}^n$

Let $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ be the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Lie groups homeomorphic to $\mathbb{R}^n$ are either solvable or a semidirect product of a solvable group and copies of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ as stated in the following Theorem (See Onishchik [7, Theorem 3.2.]):

Theorem 3.2. *Let G be connected Lie group. Then the following conditions are euivalent:*

(i) *the Lie group G is diffeomorphic to $\mathbb{R}^n$, $n = \dim G$;*

(ii) *the Lie group G is contractible;*

(iii) *the maximal subgroup of G is trivial;*

(iv) *the Levi decomposition for the group G is of the form $G = S \ltimes R$, where the radical R is simply connected, and the Levi subgroup S is isomorphic to $\underbrace{\mathcal{A}\times\cdots\times\mathcal{A}}_{m}$, $\mathcal{A} = \widetilde{\mathrm{SL}}_2(\mathbb{R})$ for some $m \geq 0$.*

3.4. Complexification

Let G be a real Lie group, $\mathfrak{g}$ its Lie algebra as the tangent space at identity with the inner product $\langle\cdot,\cdot\rangle$ induced by the left invariant Riemannian metric of G. The tensor prodcut $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}\otimes_{\mathbb{C}}\mathbb{C}$ is called the complexification of $\mathfrak{g}$. For further discussion see Knapp [9, Chapter 1]. When working with inner product spaces it may be easier to work with an alternative representation for $\mathfrak{g}^{\mathbb{C}}$, also called the complex hull of $\mathfrak{g}$. For further discussions see Naimark [10, Chapter X]. Elements of $\mathfrak{g}^{\mathbb{C}}$ can be considered as $u + iv$, $u, v \in \mathfrak{g}$. The operations in $\mathfrak{g}^{\mathbb{C}}$ are then defined by:

1. $(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$,
2. $(\alpha + i\beta)(u + iv) = \alpha u - \beta v + i(\alpha v + \beta u)$, for any $\alpha + i\beta \in \mathbb{C}$,
3. $[u_1 + iv_1, u_2 + iv_2] = [u_1, u_2] - [v_1, v_2] + i([u_1, v_2] + [v_1, u_2])$.

Certain important properties of an algebra are preserved under complexification: $\mathfrak{g}^{\mathbb{C}}$ is nilpotent, solvable or semi-simple if and only if $\mathfrak{g}$ has this property. Any linear transformation, or Lie algebra homomorphism $\mathrm{T} : \mathfrak{g} \to \mathfrak{g}$ can be naturally extended to $\mathrm{T} : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ through

$$\mathrm{T}(u + iv) = \mathrm{T}(u) + i\mathrm{T}(v).$$

The real inner product $\langle\cdot,\cdot\rangle$ can be naturally extended to a complex inner product; that is a map $\langle\cdot,\cdot\rangle : \mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$ which is positive definite, linear in the first argument, and has conjugate symmetry: for all $x, y \in \mathfrak{g}^{\mathbb{C}}$, $\langle x, y\rangle = \overline{\langle y, x\rangle}$. One can easily verify that

$$|u + iv|^2 = |u|^2 + |v|^2 . \tag{14}$$

It can be easily seen that an orthonormal Basis $\mathcal{F}$ in $\mathfrak{g}$ as a real vector space, is also an orthonormal basis of $\mathfrak{g}^{\mathbb{C}}$ as a complex vector space.

4. Proof of the results

We use the following notations throughout this section:

Notations. Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra.

1. Norm on $\mathfrak{g}^{\mathbb{C}}$ is defined by (14). $\mathrm{End}(\mathfrak{g}^{\mathbb{C}})$ is the space of endomorphisms of $\mathfrak{g}^{\mathbb{C}}$ with the operator norm.
2. For any $a \in \mathbb{C}$, $x \in \mathfrak{g}^{\mathbb{C}}$, $\mathrm{T} \in \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$, $|a|$, $|x|$, and $\|\mathrm{T}\|$ represent the absolute value of a, the norm of the element x, and the operator norm of the linear map T, respectively. $\|\mathrm{T}\|_{\mathfrak{g}}$ represents the norm of $\mathrm{T}|_{\mathfrak{g}}$ and $\mathrm{spec}(\mathrm{T})$ is the spectrum of T.
3. For a given basis $\mathcal{F}$ in $\mathfrak{g}^{\mathbb{C}}$ $[\mathrm{T}]_{\mathcal{F}}$ is the matrix representation of T in basis $\mathcal{F}$.
4. $\|\boldsymbol{A}\|$ and $\|\boldsymbol{A}\|_F$ are the 2-norm and Frobenius norm of $\boldsymbol{A} \in \mathrm{M}_n(\mathbb{C})$, respectively (See Bhatia [3, Chapter I]).
5. For $z \in \mathbb{C}$, $\boldsymbol{A} \in \mathrm{M}_n(\mathbb{C})$, $\mathrm{T} \in \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$, let
$$\mathrm{E}(z) := \frac{1-e^{-z}}{z}, \quad \mathrm{E}(\boldsymbol{A}) := \frac{1-e^{-\boldsymbol{A}}}{\boldsymbol{A}}, \quad \mathrm{E}(\mathrm{T}) := \frac{1-e^{-\mathrm{T}}}{\mathrm{T}}.$$
In particular, $\mathrm{E}(0) = 1$ and $\mathrm{E}(\mathbf{0})$ is the identity matrix or operator. Further, $\sigma_{\min}$ represents the smallest singular value of $\boldsymbol{A}$ or T.
6. For two real valued functions $f(t)$ and $g(t)$ defined on a real interval I, we say "$f(t)$ is bounded from above by $g(t)$", or "$g(t)$ is bounded from below by $f(t)$" if there is $M > 0$ such that $|f(t)| \leq M|g(t)|$ for all t in I.

As detailed in subsection 3.4, the inner product of $\mathfrak{g}$ induced by the metric d, may be extended to an inner product in $\mathfrak{g}^{\mathbb{C}}$. Moreover $(1 - e^{-\mathrm{ad}_x})/\mathrm{ad}_x$ can be regarded as a linear transformation on $\mathfrak{g}^{\mathbb{C}}$. Since the problem of estimating the norm the exponential map reduces to estimating the norm of a linear map over the Lie algebra, we can do estimates in $\mathfrak{g}^{\mathbb{C}}$ and then derive the bounds for the real case.

In order to prove Theorem 2.1 we first introduce and prove the necessary lemmas:

Lemma 4.1. *Let* $f(z) = \dfrac{1-e^{-z}}{z}$, $z \in \mathbb{C}$. *Given* $k > 0$, *and a complex number* z_0 :

(1) *If* $z_0 \neq 0$ *and* $Re(z_0) = 0$, *there is constant* $A_k > 0$ *such that for all real numbers* $t \geq 1$, $\left|f^{(k)}(tz_0)\right| < \dfrac{A_k}{t}$.

(2) *If* $z_0 = 0$, *or* $Re(z_0) \neq 0$, *there is a constant* $B_k > 0$ *such that for all real numbers* $t \geq 0$, $\left|f^{(k)}(tz_0)\right| < B_k|f(tz_0)|$.

Proof. Noting $f(0)=1$, the case $z_0=0$ is trivial. Assume $k>0$ and a fixed $z_0\neq 0$ are given. Using the power series of the analytic function $f(z)$, the function $\left|f^{(k)}(tz_0)\right|$ is continuous and takes a maximum over $0\le t\le 1$. As well, when $\mathrm{Re}(z_0)\neq 0$, $\left|f^{(k)}(tz_0)/f(tz_0)\right|$ is continuous and takes a maximum over $0\le t\le 1$. So without loss of generality we may assume $t\ge 1$. For simplifyng the notation let $z=tz_0$. A quick induction shows that for any $k>0$

$$f^{(k)}(z)=\frac{(-1)^k k!}{z^{k+1}}+\frac{z^k+p(z)}{z^{k+1}}e^{-z}=\frac{1}{t}\left(\frac{(-1)^k k!}{t^k z_0^{k+1}}+\frac{t^k z_0^k+p(tz_0)}{t^k z_0^{k+1}}e^{-tz_0}\right),$$

where $p(z)$ is a polynomial of degree $k-1$. When $\mathrm{Re}(z_0)=0$, the function in the parenthesis has bounded norm on $[1,\infty)$ proving statement 1. When $\mathrm{Re}(z_0)\neq 0$ we have:

$$\begin{aligned}\left|\frac{f^k(z)}{f(z)}\right| &= \left|\left(\frac{(-1)^k k!}{z^k}+\frac{z^k+p(z)}{z^k}e^{-z}\right)\frac{1}{1-e^{-z}}\right|\\ &\le \left|\frac{(-1)^k k!}{z^k}\right|\left|\frac{e^z}{e^z-1}\right|+\left|\frac{z^k+p(z)}{z^k}\right|\left|\frac{1}{e^z-1}\right|.\end{aligned}$$

As $t\to\infty$, we have $\dfrac{(-1)^k k!}{z^k}\to 0$, $\dfrac{z^k+p(z)}{z^k}\to 1$, and

$$\mathrm{Re}(z_0)>0: \frac{e^z}{e^z-1}\to 1,\quad \frac{1}{e^z-1}\to 0 \qquad \mathrm{Re}(z_0)<0: \frac{e^z}{e^z-1}\to 0,\quad \frac{1}{e^z-1}\to -1.$$

Thus the continuous function $t\to\left|\dfrac{f^k(tz_0)}{f(tz_0)}\right|$ is bounded on $[0,\infty)$ proving the second statement. ■

Lemma 4.2. *Let*

$$\boldsymbol{A}=\begin{bmatrix} a_0 & ta_1 & t^2a_2 & \dots & t^n a_n\\ 0 & a_0 & ta_1 & \cdots & t^{n-1}a_{n-1}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & a_0 & ta_1\\ 0 & 0 & \cdots & 0 & a_0\end{bmatrix}$$

where $t\ge 0$ is a real number and $a_0=a_0(t),\cdots,a_n=a_n(t)$ are complex valued functions of t. When $a_0\neq 0$, the inverse of $\boldsymbol{A}$ has the form

$$\boldsymbol{A}^{-1}=\begin{bmatrix} \frac{1}{a_0} & tb_1 & t^2b_2 & \dots & t^n b_n\\ 0 & \frac{1}{a_0} & tb_1 & \cdots & t^{n-1}b_{n-1}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{1}{a_0} & tb_1\\ 0 & 0 & \cdots & 0 & \frac{1}{a_0}\end{bmatrix}$$

where $b_i=b_i(t)$ are complex valued functions of t. Let $I=\{t|t\ge 1, a_0(t)\neq 0\}$.

1. *Assume that for $i=1,\cdots,n$ there exist $k_i>0$ such that $|a_i(t)|\le k_i|a_0(t)|$ for all t. Then there exists $l_i>0$ such that $b_i(t)\le l_i|a_0(t)|^{-1}$ for all $t\in I$. In addition, there is a constant $l>0$ such that $\|\boldsymbol{A}^{-1}\|_F\le lt^n|a_0(t)|^{-1}$ for all $t\in I$.*

2. *Assume that for $i = 0, \cdots, n$ there exist $k_i > 0$ such that $|a_i(t)| \le k_i/t$ for all $t > 0$. Then there exists $l_i > 0$ such that $b_i(t) \le l_i t^{-i}|a_0(t)|^{-(i+1)}$ for all $t \in I$. In addition, there is a constant $l > 0$ such that $\|\boldsymbol{A}^{-1}\|_F \le l|a_0(t)|^{-(n+1)}$ for all $t \in I$.*

Proof. Let $b_0 = 1/a_0$ and $l_0 = 1$, and to simplify the notation let $\tilde{a}_i = t^i a_i$, $\tilde{b}_i = t^i b_i$. Let

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

be the $(n+1) \times (n+1)$ Jordan block with the eigenvalues zero. Then

$$\boldsymbol{A} = \sum_{i=0}^{n} \tilde{a}_i \boldsymbol{J}^i,$$

where $\boldsymbol{J}^0 = \boldsymbol{I}$ is the identity matrix. If $\boldsymbol{AB}$ is another matrix that form as

$$\boldsymbol{AB} = \sum_{i=1}^{n} \tilde{b}_i \boldsymbol{J}^i, \quad \tilde{b_0} \neq 0,$$

re-indexing by powers of $\boldsymbol{J}$, and using the fact that $\boldsymbol{J}^n = \boldsymbol{0}$, we get

$$\boldsymbol{AB} = \sum_{i=1}^{n} \tilde{a}_i \boldsymbol{J}^i \sum_{j=0}^{n} \tilde{b}_j \boldsymbol{J}^j = \sum_{m=0}^{n} \left(\sum_{i=0}^{m} \tilde{a}_i \tilde{b}_{m-i} \right) \boldsymbol{J}^m.$$

So, the condition that $\boldsymbol{AB}$ is the inverse of $\boldsymbol{A}$, i.e., that $\boldsymbol{AB} = \boldsymbol{I}$, is that the inner sum is 1 for $m = 0$ and 0 for $m > 0$. In terms of the coefficients,

$$\begin{aligned}
1 &= \tilde{a}_0 \tilde{b}_0 \\
0 &= \tilde{a}_0 \tilde{b}_1 + \tilde{a}_1 \tilde{b}_0 \\
0 &= \tilde{a}_0 \tilde{b}_2 + \tilde{a}_1 \tilde{b}_1 + \tilde{a}_2 \tilde{b}_0 \\
&\vdots \\
0 &= \tilde{a}_0 \tilde{b}_n + \tilde{a}_1 \tilde{b}_{n-1} + \cdots + \tilde{a}_{n-1} \tilde{b}_1 + \tilde{a}_n \tilde{b}_0
\end{aligned}$$

Proof of (1). $\tilde{b_0} = 1/\tilde{a}_0 = 1/a_0$, $\tilde{b_1} = -\tilde{a}_1 \tilde{b}_0/\tilde{a}_0 = t(-a_1/a_0^2)$, so $b_1 = -a_1/a_0^2$, and $|b_1| = |a_1/a_0^2| \le k_1 1/|a_0|$, defining $l_1 = k_1$. From the last equation, and using induction we get

$$\tilde{b_n} = -\frac{1}{a_0} \sum_{i=1}^{n} \tilde{a}_i \tilde{b}_{n-i} = -\frac{1}{a_0} \sum_{i=1}^{n} t^i a_i t^{n-i} b_{n-i} = \left(-\frac{1}{a_0} \sum_{i=1}^{n} a_i b_{n-i} \right) t^n. \tag{15}$$

Setting b_n the term in the parenthesis, and using induction we get:

$$|b_n| \le \frac{1}{|a_0|} \sum_{i=1}^{n} |a_i||b_{n-i}| \le \frac{1}{|a_0|} \sum_{i=1}^{n} k_i |a_0| l_{n-i} \frac{1}{|a_0|} = \left(\sum_{i=1}^{n} k_i l_{n-i} \right) \frac{1}{|a_0|},$$

defining $l_n = \sum_{i=1}^{n} k_i l_{n-i}$.

Since all entries of $\boldsymbol{A}^{-1}$ are bounded by $t^n|a_0(t)|^{-1}$, so $\|\boldsymbol{A}^{-1}\|_F$ is also bounded by $t^n|a_0(t)|^{-1}$.

Proof of (2). $|b_1| = |a_1/a_0^2| \le k_1/(t|a_0|^2)$, defining $l_1 = k_1$. From equation 15 and using induction it follows that:

$$\begin{aligned}|\tilde{b_n}| &\le \frac{1}{|a_0|}\sum_{i=1}^{n}|\tilde{a}_i||\tilde{b}_{n-i}| \le \frac{1}{|a_0|}\sum_{i=1}^{n}k_i t^{i-1}\frac{l_{n-i}}{|a_0|^{n-i+1}}\\ &= \frac{1}{|a_0|^{n+1}}\sum_{i=1}^{n}k_i l_{n-i}t^{i-1}|a_0|^{i-1} \le \frac{1}{|a_0|^{n+1}}\sum_{i=1}^{n}k_i k_0^{i-1}l_{n-i},\end{aligned}$$

defining $l_n = \sum_{i=1}^{n} k_i k_0^{i-1} l_{n-i}$. Finally,

$$\begin{aligned}t^i b_i &\le l_i|a_0(t)|^{-(i+1)} = l_i|a_0(t)|^{-(n+1)}|a_0(t)|^{n-i}\\ &\le l_i|a_0(t)|^{-(n+1)}(k_0/t)^{n-i} \le l_i k_0^n|a_0(t)|^{-(n+1)},\end{aligned}$$

thus all entries of $\boldsymbol{A}^{-1}$ are bounded by $|a_0(t)|^{-(n+1)}$, so is $\|\boldsymbol{A}^{-1}\|_F$. ■

Lemma 4.3. *Let*

$$\boldsymbol{J_i} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

be a Jordan block of size $n > 1$, and $\boldsymbol{J_i} = [\lambda_i]$ be the Jordan block of size $n = 1$.

1. *Let $n > 1$, $\lambda_i \neq 0$ and $Re(\lambda_i) = 0$. Then there are constants $C_i, D_i > 0$ such that $\|\mathrm{E}(t\boldsymbol{J_i})\| \le C_i t^{n-2}$, and $\sigma_{\min}(\mathrm{E}(t\boldsymbol{J_i})) \ge D_i\,|\mathrm{E}(t\lambda_i)|^n$ for all $t \ge 1$.*
2. *Let $n = 1$, or $\lambda_i = 0$, or $Re(\lambda_i) \neq 0$. Then there are constants $C_i, D_i > 0$ such that $\|\mathrm{E}(t\boldsymbol{J_i})\| \le C_i t^{n-1}\,|\mathrm{E}(t\lambda_i)|$ and $\sigma_{\min}(\mathrm{E}(t\boldsymbol{J_i})) \ge \frac{D_i\,|\mathrm{E}(t\lambda_i)|}{t^{n-1}}$ for all $t \ge 1$.*

Proof. If $n = 1$, $\|\mathrm{E}(t\boldsymbol{J_i})\| = |\mathrm{E}(t\lambda_i)|$ so the case is trivial. For $n > 1$ we have:

$$\mathrm{E}(t\boldsymbol{J_i}) = \begin{bmatrix} \mathrm{E}(t\lambda_i) & t\mathrm{E}'(t\lambda_i) & t^2\frac{\mathrm{E}''(t\lambda_i)}{2} & \cdots & t^{n-1}\frac{\mathrm{E}^{(n-1)}(t\lambda_i)}{(n-1)!} \\ 0 & \mathrm{E}(t\lambda_i) & t\mathrm{E}'(t\lambda_i) & \cdots & t^{n-2}\frac{\mathrm{E}^{(n-2)}(t\lambda_i)}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \mathrm{E}(t\lambda_i) & t\mathrm{E}'(t\lambda_i) \\ 0 & 0 & 0 & 0 & \mathrm{E}(t\lambda_i) \end{bmatrix}$$

If $\mathrm{Re}(\lambda_i) = 0$ we obtain by Lemma 4.1

$$\left|t^k\,\frac{\mathrm{E}^{(k)}(t\lambda_i)}{k!}\right| \le \frac{A_k}{t}\,\frac{t^k}{k!} \le A_k t^{n-2}, \quad k = 1, \cdots, n-1.$$

Thus there is a constant C_i such that

$$\|\mathrm{E}(t\boldsymbol{J_i})\| \leq \|\mathrm{E}(t\boldsymbol{J_i})\|_F \leq C_i t^{n-2}.$$

When $\mathrm{E}(t\lambda_i) = 0$, $\sigma_{\min}(\mathrm{E}(t\boldsymbol{J_i}) = 0$ and the case is trivial. Assume $\mathrm{E}(t\lambda_i) \neq 0$. By Lemma 4.1, $\frac{\mathrm{E}^{(k)}(t\lambda_i)}{(k-1)!}$, $k = 0, 1, \cdots, n-1$ are bounded by $1/t$, thus statement 2 of Lemma 4.2 implies that

$$\sigma_{\min}(\mathrm{E}(t\boldsymbol{J_i})) = \frac{1}{\|\mathrm{E}(t\boldsymbol{J_i})^{-1}\|} \geq \frac{1}{\|\mathrm{E}(t\boldsymbol{J_i})^{-1}\|_F} \geq \frac{1}{l\,|\mathrm{E}(t\lambda_i)|^{-n}},$$

proving the first statement.

If $\lambda_i = 0$ or $\mathrm{Re}(\lambda_i) \neq 0$, By Lemma 4.1,

$$\left| t^k \frac{\mathrm{E}^{(k)}(t\lambda_i)}{k!} \right| \leq B_k\,|\mathrm{E}(t\lambda_i)|\,\frac{t^k}{k!} \leq B_k t^{n-1}\,|\mathrm{E}(t\lambda_i)|, \quad k = 1, \cdots, n-1.$$

Thus there is a constant C_i such that

$$\|\mathrm{E}(t\boldsymbol{J_i})\| \leq \|\mathrm{E}(t\boldsymbol{J_i})\|_F \leq C_i t^{n-1}\,|\mathrm{E}(t\lambda_i)|.$$

By Lemma 4.1, $\frac{\mathrm{E}^{(k)}(t\lambda_i)}{(k-1)!}$, $k = 0, 1, \cdots, n-1$ are bounded by $|\mathrm{E}(t\lambda_i)|$, thus statement 1 of Lemma 4.2 implies that

$$\sigma_{\min}(\mathrm{E}(t\boldsymbol{J_i})) = \frac{1}{\|\mathrm{E}(t\boldsymbol{J_i})^{-1}\|} \geq \frac{1}{\|\mathrm{E}(t\boldsymbol{J_i})^{-1}\|_F} \geq \frac{1}{l t^{n-1}\,|\mathrm{E}(t\lambda_i)|^{-1}},$$

proving the second statement. ∎

Lemma 4.4. *Let*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_1} & & \\ & \ddots & \\ & & \boldsymbol{A_q} \end{bmatrix}$$

be a diagonal block matrix. Let $\sigma_{i1} \geq \sigma_{i2} \cdots \geq \sigma_{in_i}$, $i = 1, \cdots, q$ be the singular values of $\boldsymbol{A}_i$ in non-ascending order. Singular values of $\boldsymbol{A}$ is the collection

$$\{\sigma_{ij} \mid 1 \leq i \leq q,\, 1 \leq j \leq n_i\}.$$

In particular $\|\boldsymbol{A}\| = \max\{\|\boldsymbol{A_1}\|, \cdots, \|\boldsymbol{A_q}\|\}$.

Proof. Singular values of $\boldsymbol{A}$ are the square roots of the eigenvalues of

$$\boldsymbol{A}^*\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_1^*}\boldsymbol{A_1} & & \\ & \ddots & \\ & & \boldsymbol{A_q^*}\boldsymbol{A_q} \end{bmatrix}$$

which are the collection of eigenvalues of $\boldsymbol{A}_i^*\boldsymbol{A}_i$, $1 \leq i \leq q$. ∎

Lemma 4.5. *Let*

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J_1} & & \\ & \ddots & \\ & & \boldsymbol{J_q} \end{bmatrix}$$

be a Jordan normal form, λ_i the eigenvalue of $\boldsymbol{J_i}$ so that for $i \geq p$, $\boldsymbol{J_i}$ has size > 1, λ_i is non-zero and has zero real part. Let

$$\begin{aligned} \tilde{\lambda}_1(t) &= \min\left\{t^{1-m_1}|\mathrm{E}(t\lambda_1)|, \cdots, t^{1-m_{p-1}}|\mathrm{E}(t\lambda_{p-1})|\right\} \\ \tilde{\lambda}_2(t) &= \min\left\{|\mathrm{E}(t\lambda_p)|^{m_p}, \cdots, |\mathrm{E}(t\lambda_q)|^{m_q}\right\} \\ \tilde{\lambda}_3(t) &= \max\left\{t^{m_1-1}\left|\mathrm{E}(t\lambda_1)\right|, \cdots, t^{m_{p-1}-1}\left|\mathrm{E}(t\lambda_{p-1})\right|\right\} \\ \tilde{\lambda}_4(t) &= \max\left\{t^{m_p-2}, \cdots, t^{m_q-2}\right\} \\ \tilde{\lambda}_{\min}(t) &= \min(\tilde{\lambda}_1(t), \tilde{\lambda}_2(t)) \\ \tilde{\lambda}_{\max}(t) &= \max(\tilde{\lambda}_3(t), \tilde{\lambda}_4(t)) \end{aligned}$$

Then there exist positive constants C', D', such that

$$\|\mathrm{E}(t\boldsymbol{J})\| \leq C'\tilde{\lambda}_{\max}(t) \quad \text{and} \quad \sigma_{\min}(\mathrm{E}(t\boldsymbol{J})) \geq D'\tilde{\lambda}_{\min}(t)$$

Proof. We have:

$$\mathrm{E}(t\boldsymbol{J}) = \begin{bmatrix} \mathrm{E}(t\boldsymbol{J_1}) & & \\ & \ddots & \\ & & \mathrm{E}(t\boldsymbol{J_q}) \end{bmatrix}.$$

From Lemma 4.4 it follows that

$$\|\mathrm{E}(t\boldsymbol{J})\| = \max\{\|\mathrm{E}(t\boldsymbol{J_1})\|, \cdots, \|\mathrm{E}(t\boldsymbol{J_q})\|\}$$

and thus by lemma 4.3 it is bounded by $\tilde{\lambda}_{\max}(t)$ for all $t \geq 1$. Similarly, from Lemma 4.4 it follows that

$$\sigma_{\min}(\mathrm{E}(t\boldsymbol{J})) = \min\{\sigma_{\min}(\mathrm{E}(t\boldsymbol{J_1})), \cdots, \sigma_{\min}(\mathrm{E}(t\boldsymbol{J_q}))\}$$

so by Lemma 4.3 it is bounded from below by $\tilde{\lambda}_{\min}(t)$ for all $t \geq 1$. ∎

Lemma 4.6. *Let V be a finite dimensional complex inner product space, and $\mathrm{P}, \mathrm{Q}, \mathrm{T} \in \mathrm{End}_{\mathbb{C}}(V)$, P and Q invertible. Then*

$$\sigma_{\min}(\mathrm{PTQ}) \geq \sigma_{\min}(\mathrm{P})\sigma_{\min}(\mathrm{T})\sigma_{\min}(\mathrm{Q}). \tag{16}$$

Proof. Without loss of generality we can assume T is invertible (otherwise the right side of inequality is zero). We have:

$$\min_{|y|=1}|\mathrm{TQ}(y)| = \min_{|y|=1}\left|\mathrm{T}\left(\frac{\mathrm{Q}(y)}{|\mathrm{Q}(y)|}\right)|\mathrm{Q}(y)|\right| \geq \min_{|y|=1}|\mathrm{T}(y)| \cdot \min_{|y|=1}|\mathrm{Q}(y)|,$$

repeating the argument one more time it follows that

$$\min_{|y|=1}|\mathrm{PTQ}(y)| \geq \min_{|y|=1}|\mathrm{P}(y)| \cdot \min_{|y|=1}|\mathrm{T}(y)| \cdot \min_{|y|=1}|\mathrm{Q}(y)|,$$

which gives the desired inequality. ∎

We may proceed to prove Theorem 2.1:

Proof of Theorem 2.1. Let $\mathcal{F}$ be an arbitray orthonormal basis for $\mathfrak{g}^{\mathbb{C}}$. Let $\boldsymbol{J}$ be the Jordan normal form for $\mathrm{ad}_{\hat{x}}$; so there is an invertible complex matrix $\boldsymbol{P}$ such that $[\mathrm{ad}_{\hat{x}}]_{\mathcal{F}} = \boldsymbol{P}^{-1}\boldsymbol{J}\boldsymbol{P}$, $[\mathrm{ad}_x]_{\mathcal{F}} = \boldsymbol{P}^{-1}|x|\boldsymbol{J}\boldsymbol{P}$ and thus

$$\left[\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right]_{\mathcal{F}} = \boldsymbol{P}^{-1}\mathrm{E}\left(|x|\boldsymbol{J}\right)\boldsymbol{P}.$$

So using Lemma 4.5 we have:

$$\begin{aligned}|d\exp_x(y)| &\leq \left\|\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right\| = \|\boldsymbol{P}^{-1}\mathrm{E}\left(|x|\boldsymbol{J}\right)\boldsymbol{P}\| \\ &\leq \|\boldsymbol{P}^{-1}\|\,\|\mathrm{E}\left(|x|\boldsymbol{J}\right)\|\,\|\boldsymbol{P}\| \leq \|\boldsymbol{P}^{-1}\|\|\boldsymbol{P}\|D'\tilde{\lambda}_{\max}(|x|)\,,\end{aligned}$$

defining $D = \|\boldsymbol{P}^{-1}\|\|\boldsymbol{P}\|C'$. In addition, using Lemma 4.6 it follows that:

$$\begin{aligned}|d\exp_x(y)| &\geq \min_{|y|=1,\, y\in\mathfrak{g}^{\mathbb{C}}}\left|\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}(y)\right| = \sigma_{\min}\left(\frac{1-e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}\right) \\ &= \sigma_{\min}\left(\boldsymbol{P}^{-1}\mathrm{E}\left(|x|\boldsymbol{J}\right)\boldsymbol{P}\right) \geq \sigma_{\min}\left(\boldsymbol{P}^{-1}\right)\sigma_{\min}(\mathrm{E}\left(|x|\boldsymbol{J}\right))\sigma_{\min}(\boldsymbol{P}) \\ &\geq \sigma_{\min}\left(\boldsymbol{P}^{-1}\right)\sigma_{\min}(\boldsymbol{P})C'\tilde{\lambda}_{\min}(|x|)\,,\end{aligned}$$

defining $C = \sigma_{\min}\left(\boldsymbol{P}^{-1}\right)\sigma_{\min}(\boldsymbol{P})C'$. ■

Proof of Proposition 2.4. We follow the notations of Theorem 2.1. In addition, assume that the eigenvalues of $\mathrm{ad}_{\hat{x}}$ are labeled so that for some $r \geq 0$, $\lambda_1 = \cdots = \lambda_r = 0$. By (7) all non-zero eigenvalues of $\mathrm{ad}_{\hat{x}}$ have non-zero real part. Notice that for non-zero λ_j, $\left|1-e^{-\lambda_j|x|}\right|$ is strictly positive, and as $|x|\to\infty$,

$$\mathrm{Re}(\lambda_j) < 0 : \left|1-e^{-\lambda_j|x|}\right| \to \infty \quad \mathrm{Re}(\lambda_j) > 0 : \left|1-e^{-\lambda_j|x|}\right| \to 1\,.$$

Thus C_0 defined by

$$C_0 = \min_{j=1,\cdots,q}\left\{\inf\left|1-e^{-\lambda_j|x|}\right|, |x|\geq 1\right\}$$

is positive which leads to the inequalities

$$\begin{aligned}\tilde{\lambda}_1(|x|) &\geq \min\left\{\frac{1}{|x|^{m_1-1}},\cdots,\frac{1}{|x|^{m_r-1}},\frac{C_0}{|\lambda_{r+1}||x|^{m_{r+1}}},\cdots,\frac{C_0}{|\lambda_p||x|^{m_p}}\right\}, \\ \tilde{\lambda}_2(|x|) &\geq \min\left\{\frac{C_0^{m_{p+1}}}{|\lambda_{p+1}x|^{m_{p+1}}},\cdots,\frac{C_0^{m_q}}{|\lambda_q x|^{m_q}}\right\}.\end{aligned}$$

Consequently, for some $C_1 > 0$, $\tilde{\lambda}_{min}(|x|) \geq C_1/|x|^m$, proving (8). When $\mathrm{ad}_{\hat{x}}$ is diagonalizable, $m = 1$ so (9) follows. ■

Proof of Proposition 2.5. Assume that there is some non-zero $x\in\mathfrak{g}$ such that $\mathrm{spec}(\mathrm{ad}_x) \subsetneq i\mathbb{R}$. Let $\hat{x} = x/|x|$. Then $\mathrm{ad}_{\hat{x}}$ has an eigenvalue $\lambda_1 = \alpha + i\beta$ where $\alpha \neq 0$. Switching x to $-x$ if necessary we may assume $\alpha < 0$. Then $\mathrm{E}(\lambda_1|x|)$ is an eigenvalue of $\mathrm{E}(\mathrm{ad}_x)$. Let $y = y_1 + iy_2 \in \mathfrak{g}^{\mathbb{C}}$ be an eigenvector associated with λ_1, so that $\mathrm{ad}_{\hat{x}}(y) = \lambda_1 y$. Using the power series for $\mathrm{E}(\mathrm{ad}_x)$ it follows that

$$\mathrm{E}(\mathrm{ad}_x)(y) = \mathrm{E}(\lambda_1 |x|) y\,.$$

Thus $|\mathrm{E}(\mathrm{ad}_x)(y)| \to \infty$, as $|x| \to \infty$. Since

$$|\mathrm{E}(\mathrm{ad}_x)(y)| = |\mathrm{E}(\mathrm{ad}_x)(y_1) + i\mathrm{E}(\mathrm{ad}_x)(y_2)| \le |\mathrm{E}(\mathrm{ad}_x)(y_1)| + |\mathrm{E}(\mathrm{ad}_x)(y_2)|\,,$$

it follows that as $|x| \to \infty$:

$$|\mathrm{E}(\mathrm{ad}_x)(y_1)| = |d\exp_x(y_1)| \to \infty \quad \text{or} \quad |\mathrm{E}(\mathrm{ad}_x)(y_2)| = |d\exp_x(y_2)| \to \infty\,,$$

contradicting (4).

Let K be the maximal compact subgroup of G. By Cartan-Iwasawa-Malcev theorem detailed in [14], a connected real Lie group is homeomorphic to $\mathbb{R}^m \times K$. This implies that if G is homeomorphic to $\mathbb{R}^n$, then K must have the homotopy type of a point, so it must be contractible. But the only contractible connected compact Lie group is the trivial group [8]. Hence K is trivial. By Theorem 3.2, $G = S \ltimes R$, where the radical R is simply connected, and the Levi subgroup S is isomorphic to

$$\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{m}, \quad \mathcal{A} = \widetilde{\mathrm{SL}}_2(\mathbb{R}), \qquad m \ge 0\,.$$

Assume $m > 0$. Then $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is a subgroup of G, thus its Lie algebra which is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$, is a subalgebra of $\mathfrak{g}$. But $\mathfrak{sl}_2(\mathbb{R})$ contains elements for which $\mathrm{ad}_{\boldsymbol{x}}$ has eigenvalues with non-zero real parts. For example, a quick calculation shows for

$$\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\mathrm{ad}_{\boldsymbol{x}}$ has the eigenvalue $\lambda_1 = 2$, contradicting the fact all elements of $\mathfrak{g}$ must be compact. Thus $m = 0$ and G is solvable. ∎

Proof of Theorem 2.6. By Proposition 2.5, for all $x \in \mathfrak{g}$, $\mathrm{spec}(\mathrm{ad}_x) \subseteq i\mathbb{R}$. By (3) for all $x \in \mathfrak{g}$, $d\exp_x$ must be invertible. We claim there is no element in $\mathfrak{g}$ with a non-zero purely imaginary eigenvalue. To the contrary assume that $x_1 \in \mathfrak{g}$ has a non-zero eigenvalue $\lambda_1 = i\theta$. Then $x_2 = (2\pi/\theta)x_1$ has the eigenvalue $\lambda_2 = 2\pi i$ implying $\mathrm{E}(\mathrm{ad}_{x_2})$ and thus $d\exp_{x_2}$ are not invertible which is a contradiction. Therefore for all $x \in \mathfrak{g}$, $\mathrm{spec}(\mathrm{ad}_x) = \{0\}$ and $\mathfrak{g}$ must be nilpotent.

We claim that $\mathfrak{g}$ is one-step nilpotent so is abelian. Assume to the contrary that $\mathfrak{g}$ is p-step nilpotent, $p > 1$, and consider fixed unit elements $\hat{x}, y \in \mathfrak{g}$, such that $\mathrm{ad}_{\hat{x}}^{p-1}(y) \ne 0$. Let $x = |x|\hat{x}$. Then we have

$$\begin{aligned} |d\exp_x(y)| &= \left| \frac{1 - e^{-\mathrm{ad}_x}}{\mathrm{ad}_x}(y) \right| = \left| \sum_{k=0}^{p-1} \frac{(-1)^k}{(k+1)!} |x|^k \mathrm{ad}_{\hat{x}}^k(y) \right| \\ &= \left| \sum_{k=0}^{p-1} \frac{(-1)^k |x|^{k-p+1}}{(k+1)!} \mathrm{ad}_{\hat{x}}^k(y) \right| |x|^{p-1}\,. \end{aligned}$$

Thus $|d\exp_x(y)| \to \infty$, as $|x| \to \infty$, contradicting the fact that the exponential map is a quasi-isometry. Since G is connected, it also must be abelian.

Finally, when the exponential map is a quasi-isometry G is diffeomorphic to $\mathbb{R}^n$, so by Theorem 3.2 the maximal compact subgroup of G is trivial. It is well known that a connected abelian Lie group is isomorphic to $\mathbb{R}^n \times (S^1)^k$ for some $n, k \geq 0$. Therefore G must be isomorphic to $\mathbb{R}^n$. ■

References

[1] O. Attie: *Quasi-isometry classification of some manifolds of bounded geometry*, Math. Zeitschrift 216 (1994) 501–527.

[2] L. Auslander: *An exposition of the structure of solvmanifolds. I: Algebraic theory*, Bull. Amer. Math. Soc. 79 (1973) 227–261.

[3] R.-R. Bhatia: *Matrix Analysis*, Springer, New York (1997).

[4] M.-R. Bridson, A. Haefliger: *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften 319, Springer, Berlin (1999).

[5] J. Dixmier: *L'application exponentielle dans les groupes de Lie résolubles*, Bull. Math. Soc. France 85 (1957) 113–121.

[6] G.-H. Golub, C.-F. Van Loan: *Matrix Computation*, John Hopkins University Press, New York (1996).

[7] V.-V. Gorbatsevitch, A.-L. Onishchik, E.-B. Vinberg: *Lie Groups and Lie Algebras. III: Structure of Lie Groups and Lie Algebras*, Springer, New York (1994).

[8] B. Hoffmann: *A compact contractible topological group is trivial*, Archiv Math. 32 (1979) 585–587.

[9] A.-W. Knapp: *Lie Groups Beyond an Introduction*, Birkhäuser, Boston (1996).

[10] M.-A. Naimark, A.-I. Štern: *Theory of Group Representations*, Springer, New York (1982).

[11] W. Rossmann: *Lie Groups: An Introduction Through Linear Groups*, Oxford University Press, Oxford (2002).

[12] W. Rudin: *Functional Analysis*, McGraw-Hill, New York (1991).

[13] M. Saito: *Sur certains groupes de Lie résolubles*, Sci. Papers Coll. Gen. Educ., Univ. Tokyo 7 (1957) 1–11.

[14] M. Stroppel: *Locally Compact Groups*, European Mathematical Society, Zürich (2006).

Reza Bidar, Faculty of Mathematics, University of Dayton, Dayton, Ohio, U.S.A.;
rbidar1@udayton.edu, mrez.bidar@gmail.com.

Received October 7, 2024
and in final form February 11, 2025