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Straightening Banach-Lie-Group-Valued Almost-Cocycles

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Abstract. For a compact group $\mathbb{G}$ acting continuously on a Banach Lie group $\mathbb{U}$, we prove that maps $\mathbb{G} \to \mathbb{U}$ close to being 1-cocycles for the action can be deformed analytically into actual 1-cocycles. This recovers Hyers-Ulam stability results of Grove-Karcher-Ruh (trivial $\mathbb{G}$-action, compact Lie $\mathbb{G}$ and $\mathbb{U}$) and de la Harpe-Karoubi (trivial $\mathbb{G}$-action, $\mathbb{U}$:= invertible elements of a Banach algebra). The obvious analogues for higher cocycles also hold for abelian $\mathbb{U}$.

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Introduction

The motivation for the present note was the juxtaposition in [20, §5] of the following two results, referred to there as the *almost homomorphism* and *almost representation* theorems respectively:

- Continuous maps $\mathbb{G} \xrightarrow{\rho} \mathbb{H}$ between compact Lie groups that are *almost* morphisms in the sense that
$$\sup_{s,t\in\mathbb{G}} d(\rho(s)\rho(t),\ \rho(st)) \text{ is small}$$
for an appropriate metric d on $\mathbb{H}$ are uniformly close to actual morphisms [7, Theorem 4.3].
- Similarly, continuous almost-morphisms
$$\mathbb{G} \longrightarrow A^\times := \text{invertible elements in a Banach algebra } A$$
are close to morphisms [6, Proposition 4].

These are instances of *Hyers-Ulam stability* ([8] and its references, say), and admit a common generalization (stated formally in Corollary 2.2 below): continuous almost-morphisms from a compact group $\mathbb{G}$ to a *Banach Lie group* [12, Definition IV.I] $\mathbb{U}$ are uniformly close to morphisms. The two results can then be recovered by specializing the Banach Lie side of the picture:

- $\mathbb{U}$ compact (and hence [13, Theorem IV.3.16] finite-dimensional) yields [7, Theorem 4.3];
- while $\mathbb{U} := A^\times$ for a Banach algebra A returns [6, Proposition 4].

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The argument has a *Newton-approximation* [18, Definition 1.6] flavor, very much in the spirit of the one delivering [9, Theorem 3.1] (say). That argument suggests (and very little additional effort yields) natural generalizations: morphisms $\mathbb{G} \to \mathbb{U}$ are nothing but $\mathbb{U}$*-valued* 1*-cocycles* on $\mathbb{G}$ [15, Appendix to Chapter VII, p.123] for the trivial $\mathbb{G}$-action on $\mathbb{U}$. Dropping that triviality (Theorem2.1):

- Almost-1-cocycles $\mathbb{G} \to \mathbb{U}$ with respect to a fixed action of a compact group $\mathbb{G}$ on a Banach Lie group $\mathbb{U}$ are close to actual 1-cocycles.
- The same goes for n-cocycles with arbitrary $n \in \mathbb{Z}_{>0}$ provided $\mathbb{U}$ is abelian.

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1. Preliminaries

We follow [1, §§2.3, 3.2, 5.1 and 5.3] (and [2, Chapter III, introductory remarks]) in adopting a uniform notation for C^r (Banach) manifolds and morphisms for

$$r \in \widetilde{\mathbb{Z}}_{\geq 0} := \overline{\mathbb{Z}}_{\geq 0} \sqcup \{\omega\}, \quad \overline{\mathbb{Z}}_{\geq 0} := \mathbb{Z}_{\geq 0} \sqcup \{\infty\}, \quad \mathbb{Z}_{\geq 0} \ni n < \infty < \omega,$$

where C^r, $r \in \overline{\mathbb{Z}}_{\geq 0}$ means, as usual, r*-times continuously differentiable* and C^ω means *(real or complex) analytic.*

Write $C^r(X,Y)$ for C^r maps between $C^{\geq r}$ manifolds X and Y, with X typically finite-dimensional if $r \geq 1$; we will often abbreviate $C^0(-,-)$ to $C(-,-)$. The C^r *topology* [19, §10, Example I] on $C^r(-,-)$ is that of uniform convergence on compact sets for all derivatives up to order r. In particular, the C^∞ and C^ω topologies coincide.

Consider a compact group $\mathbb{G}$ and a Banach Lie group $\mathbb{U}$, the former acting continuously on the latter by automorphisms via

$$\mathbb{G} \times \mathbb{U} \ni (s,x) \longmapsto {}^{s}x \in \mathbb{U}.$$

We employ the language of group cohomology fairly liberally: cocycles, coboundaries, etc., as can be recalled briefly from [14, Chapter 9], [15, Chapter VII and its Appendix], [3, Chapter III], [4, Chapter IV] and many other sources. In particular, given the above setup and $\rho \in C(\mathbb{G}^n,\ \mathbb{U})$, for abelian $\mathbb{U}$ write (e.g. [15, §VII.3, equation (**)])

$$\begin{aligned} \delta\rho(s_0,\ s_1,\ \cdots,\ s_n) := {}^{s_0}\rho(s_1,\ \cdots,\ s_n) &- \rho(s_0 s_1,\ s_2,\ \cdots,\ s_n) \\ &+ \rho(s_0, s_1 s_2,\ \cdots,\ s_n) \\ &- \cdots \\ &+ (-1)^{n+1}\rho(s_0, s_1,\ \cdots,\ s_{n-1}) \end{aligned} \tag{1}$$

for the *coboundary* of ρ. This can be made sense of for *non*-abelian $\mathbb{U}$ as well if $n \leq 1$ (per the discussion on non-abelian cohomology in [15, Appendix to Chapter VII]): (1) is then interpreted as

$$\delta x(s) := {}^{s}x \cdot x^{-1} \text{ for } n = 0$$

(with $x \in \mathbb{U}$ regarded as a function $\mathbb{G}^0 \to \mathbb{U}$) and

$$\delta\rho(s_0, s_1) = \rho(s_0) \cdot {}^{s_0}\rho(s_1) \cdot \rho(s_0 s_1)^{-1} \text{ for } n = 1.$$

Write
$$C_Z(\mathbb{G}^n,\ \mathbb{U}) := \{\text{continuous } \mathbb{G} \xrightarrow{\rho} \mathbb{U} \mid \delta\rho \equiv 1\}$$

and similarly with C^r in place of $C := C^0$ when $\mathbb{G}$ is Lie.

2. Analytic deformations of almost-morphisms

For subsets $S \subseteq \mathbb{U}$ and $B \subseteq GL(\mathfrak{u})$ of a Banach Lie group and the general Lie group of its Lie algebra $\mathfrak{u} := \mathit{Lie}(\mathbb{U})$ respectively, we write

$$S_{\mathrm{Ad}\subset B} := \{s \in S \mid \mathrm{Ad}(s) \in B\}, \quad \mathbb{U} \xrightarrow{\text{Ad:=the \textit{adjoint representation} [2, §III.3.12]}} GL(\mathfrak{u})$$

The same notation applies to spaces of maps $X \to \mathbb{U}$: an $\mathrm{Ad} \subset B$ subscript indicates maps taking values in B after composing with Ad.

Theorem 2.1. *Let $\mathbb{G}$ be a compact group, $n \in \mathbb{Z}_{>0}$, and $\mathbb{U}$ a Banach Lie group, abelian if $n \geq 2$. Let also $B \subset GL(\mathfrak{u})$ be a bounded open subset.*

1. *For every neighborhood $V \ni 1 \in C(\mathbb{G}^n,\ \mathbb{U})$ there is a neighborhood $W \ni 1 \in C(\mathbb{G}^{n+1}, \mathbb{U})$ and an analytic, uniformly continuous map*
$$\left(\delta^{-1}W\right)_{\mathrm{Ad}\subset B} \ni \rho \longmapsto \rho' \in C_Z(\mathbb{G}^n,\ \mathbb{U}), \quad \rho' \cdot \rho^{-1} \in V \text{ throughout.}$$
2. *Moreover, if $\mathbb{G}$ is Lie (and hence also an analytic manifold) the analogous statements hold upon substituting C^r $(1 \leq r \leq \omega)$ for C^0 throughout.*

We first record the consequence mentioned in the introduction. Specializing to $n = 1$ and a *trivial* $\mathbb{G}$-action on $\mathbb{U}$, 1-cocycles are nothing but topological-group morphisms $\mathbb{G} \to \mathbb{U}$; thus:

Corollary 2.2. *Let $\mathbb{G}$ be a compact group, $\mathbb{U}$ a Banach Lie group, and $B \subset GL(\mathfrak{u})$ be a bounded open subset.*

1. *For every neighborhood $V \ni 1 \in C(\mathbb{G},\ \mathbb{U})$ there is a neighborhood $W \ni 1 \in C(\mathbb{G}^2, \mathbb{U})$ and an analytic, uniformly continuous map*
$$\left(\delta^{-1}W\right)_{\mathrm{Ad}\subset B} \ni \rho \longmapsto \rho' \in \mathrm{TopGp}(\mathbb{G}, \mathbb{U}), \quad \rho' \cdot \rho^{-1} \in V \text{ throughout.}$$
2. *If $\mathbb{G}$ is Lie the analogous statements hold upon substituting C^r $(1 \leq r \leq \omega)$ for C^0.*

Closeness estimates in the Lie algebra $\mathfrak{u} := \mathit{Lie}(\mathbb{U})$ are with respect to a complete norm $\|\cdot\|$ thereon with

$$\|[x, y]\| \leq C\|x\| \cdot \|y\|, \quad \forall x, y \in \mathfrak{u} \quad \text{for some } C > 0;$$

such a norm always exists [2, §III.3.7, Corollary to Proposition 24], and we can of course always scale to $C = 1$. $\mathbb{U}$ is itself completely metrizable [2, §III.1.1, Proposition 1], and the *exponential map* [2, §III.4.3, Theorem 4]

$$\mathfrak{u} \xrightarrow{\exp = e^{(\cdot)}} \mathbb{U}$$

implements an analytic isomorphism between origin neighborhoods of $\mathfrak{u}$ and $\mathbb{U}$. Closeness between "small" elements thus transports over back and forth between the two spaces.

Proof of Theorem 2.1. We focus on the case $n = 1$; the abelianness of $\mathbb{U}$ makes the argument, if anything, even simpler otherwise. Moreover, the analytic nature of the construction $\rho \mapsto \rho'$ will make it clear that that construction preserves r-fold continuous differentiability or indeed analyticity, so that part (2) need not be addressed separately.

The assumption is that

$$\rho(s)\cdot\ {}^{s}\rho(t) = e^{\beta(s,t)}\rho(st), \quad s,t\in\mathbb{G} \tag{2}$$

with $\left(\mathbb{G}^2 \ni (s,t) \longmapsto \beta(s,t)\in\mathfrak{u}\right) = O(\varepsilon)$ for small $\varepsilon > 0$

in standard *big-oh notation* [5, §3.2]:

$$\|\beta(-,-)\| \le K\varepsilon \ \text{ uniformly for some } K>0, \tag{3}$$

valid universally, for any $\varepsilon > 0$, provided the latter is sufficiently small.

The goal is to produce

$$(\mathbb{G}\ni s\longmapsto \alpha(s)\in\mathfrak{u}) = O(\varepsilon)$$

again, attached analytically to ρ, so that

$$\begin{aligned} e^{\alpha(s)}\cdot e^{s\triangleright\alpha(t)}\cdot\rho(s)\cdot\ {}^{s}\rho(t) &= e^{\alpha(s)}\rho(s)\cdot\ {}^{s}\left(e^{\alpha(t)}\rho(t)\right)\\ &= e^{\alpha(st)}\rho(st)\\ &= e^{\alpha(st)}\cdot e^{-\beta(s,t)}\cdot\rho(s)\ {}^{s}\rho(t), \quad s,t\in\mathbb{G}\end{aligned} \tag{4}$$

meaning that $$e^{\alpha(s)}\cdot e^{s\triangleright\alpha(t)} = e^{\alpha(st)}\cdot e^{-\beta(s,t)}, \quad s,t\in\mathbb{G},$$

where the ε-almost-action '$\triangleright$' is defined by

$$e^{s\triangleright x} := \rho(s)\cdot\ {}^{s}(e^{x})\cdot\rho(s)^{-1} = \rho(s)\cdot e^{\ {}^{s}x}\cdot\rho(s)^{-1}, \quad s\in\mathbb{G},\ x\in\mathfrak{u}:$$

an instance of *twisting* an action by a cocycle [17, §I.5.3] (in this case an approximate one). As hinted above, we will construct α by successive approximation as

$$\alpha = \log\left(\cdots e^{\alpha_2}\cdot e^{\alpha_1}\right), \quad \alpha_n = O(\varepsilon^n), \tag{5}$$

where

- there is a single constant $K > 0$ as in (3), pertinent to all instances of $O(\varepsilon)$, $O(\varepsilon^2)$, etc., valid throughout the proof;
- and the convergence of (5) follows from this via the *Baker-Campbell-Hausdorff (or BCH) formula* ([13, Definition IV.1.3], [16, Part I, §§IV.7 and IV.8 and part II, §V.4]).

We construct the requisite $\alpha_1 = O(\varepsilon)$ first, and then proceed recursively. Repeated application of (2) yields

$$\begin{aligned} e^{s\triangleright\beta(t,u)}e^{\beta(s,tu)}\rho(stu) &= \rho(s)\cdot e^{\ {}^{s}\beta(t,u)}\cdot\ {}^{s}\rho(tu)\\ &= \rho(s)\cdot\ {}^{s}\left(\rho(t)\cdot\ {}^{t}\rho(u)\right)\\ &= \rho(s)\cdot\ {}^{s}\rho(t)\cdot\ {}^{st}\rho(u)\\ &= e^{\beta(s,t)}\cdot\rho(st)\cdot\ {}^{st}\rho(u)\\ &= e^{\beta(s,t)}\cdot e^{\beta(st,u)}\cdot\rho(stu),\end{aligned}$$

meaning that $$e^{s\triangleright\beta(t,u)}\cdot e^{\beta(s,tu)} = e^{\beta(s,t)}\cdot e^{\beta(st,u)}$$
and hence β is an *(ε^2)-almost-cocycle* with respect to '$\triangleright$':

$$\delta_{\triangleright}\beta(s,t,u) := s\triangleright\beta(t,u) - \beta(st,u) + \beta(s,tu) - \beta(s,t) = O(\varepsilon^2).$$

The usual (e.g. [11, I, proof of Theorem 2.3]) Haar-averaging procedure then also ensures that it is a coboundary of an $O(\varepsilon)$ cochain α_1 to order ε^2: setting

$$\alpha_1(t) := -\int_{\mathbb{G}} s\triangleright\beta(s^{-1},t)\ \mathrm{d}\mu_{\mathbb{G}}(s),\quad \mu_{\mathbb{G}} := \text{Haar probability measure on } \mathbb{G}, \tag{6}$$

we have both $\alpha_1 = O(\varepsilon)$ and

$$\delta_{\triangleright}\alpha_1 + \beta = O(\varepsilon^2),\quad \delta_{\triangleright}\alpha_1(s,t) := s\triangleright\alpha_1(t) - \alpha_1(st) + \alpha_1(s). \tag{7}$$

We remind the reader how that computation functions in the broader context of n-cocycles for $\mathbb{G}$-actions $(s,v)\mapsto {}^{s}v$ on Banach spaces E; this will both sketch how the present argument functions in the higher-n case and justify (7).
Assume, to that end, that θ is an n-cocycle ($n\ge 1$) for such an action, so (1) holds (with θ in place of ρ). The claim is that

$$\delta\eta = \theta \quad \text{for}\quad \eta(s_1,\cdots,s_{n-1}) := \int_{\mathbb{G}} {}^{s}\theta(s^{-1},s_1,\cdots,s_{n-1})\ \mathrm{d}\mu_{\mathbb{G}}(s).$$

Indeed, compressing $\int_{\mathbb{G}}\bullet(s)\ \mathrm{d}\mu_{\mathbb{G}}(s)$ down to $\int_s$, we have

$$\begin{aligned}
\delta\eta(s_0,\cdots,s_{n-1}) = {}& {}^{s_0}\eta(s_1,\cdots,s_{n-1}) && - \eta(s_0s_1,s_2,\cdots,s_{n-1})\\
&&& + \eta(s_0,s_1s_2,\cdots,s_{n-1})\\
&&& - \cdots \qquad\qquad (8)\\
&&& + (-1)^{n+1}\eta(s_0,s_1,\cdots,s_{n-2})\\
= {}& \int_s {}^{s_0s}\theta(s^{-1},s_1,\cdots,s_{n-1}) && - \int_s {}^{s}\theta(s^{-1},s_0s_1,s_2,\cdots,s_{n-1})\\
&&& + \int_s {}^{s}\theta(s^{-1},s_0,s_1s_2,\cdots,s_{n-1})\\
&&& - \cdots \qquad\qquad (9)\\
&&& + (-1)^{n+1}\int_s {}^{s}\theta(s^{-1},s_0,s_1,\cdots,s_{n-2}).
\end{aligned}$$

The cocycle property for θ turns this into

$$\int_s {}^{s_0s}\theta(s^{-1},s_1,\cdots,s_{n-1}) + \int_s \theta(s_0,s_1,\cdots,s_{n-1}) - \int_s {}^{s}\theta(s^{-1}s_0,s_1,\cdots,s_{n-1}).$$

The two outer terms cancel by Haar left invariance, while the normalization $\mu(\mathbb{G})=1$ identifies the middle term with $\theta(s_0,\cdots,s_{n-1})$. If instead of an action ${}^{s}(\bullet)$ and a cocycle we work with an ε-action $\triangleright(\bullet)$ and an ε^2-cocycle $\theta = O(\varepsilon)$ for $\delta_{\triangleright}$ (as in (6)), the equality (8) = (9) becomes

$$(9) - (8) = O(\varepsilon^2).$$

This, applied to $\theta := \beta$ (with $n = 1$) and $\eta := -\alpha_1$, is what yields (7). In other words, with α_1 in place of α, (4) holds to order ε^2.
We can now proceed recursively, making the substitutions

$$\rho \rightsquigarrow \rho_1 := e^{\alpha_1}\rho, \quad \beta \rightsquigarrow \beta_1 := \beta + \delta_\triangleright \alpha_1, \quad \varepsilon \rightsquigarrow \varepsilon^2. \tag{10}$$

To check the desired estimates, note that $\beta_1 = O(\varepsilon^2)$ by (7) and

$$\begin{aligned}
\rho_1(s) \cdot {}^s\rho_1(t) &= e^{\alpha_1(s)}\rho(s) \cdot e^{{}^s\alpha_1(t)} \cdot {}^s\rho(t) \\
&= e^{\alpha_1(s)} \cdot e^{s\triangleright\alpha_1(t)} \cdot \rho(s) \cdot {}^s\rho(t) \\
&= e^{\alpha_1(s)} \cdot e^{s\triangleright\alpha_1(t)} \cdot e^{\beta(s,t)} \cdot \rho(st) \quad \text{by (2)} \\
&= e^{\alpha_1(s)} \cdot e^{s\triangleright\alpha_1(t)} \cdot e^{\beta(s,t)} \cdot e^{-\alpha_1(st)}\rho_1(st) \quad \text{by (10)} \\
&\simeq_{\varepsilon^2} e^{\alpha_1(s)-\alpha_1(st)+s\triangleright\alpha_1(t)+\beta(s,t)}\rho_1(st) \quad \text{by BCH} \\
&= e^{\beta_1(s,t)}\rho_1(st) \quad \text{by (10) again.}
\end{aligned}$$

The uniformity of the constants featuring implicitly in the $O(\varepsilon^n)$ for varying n follows directly from the boundedness of $\{s\triangleright\}_{s\in\mathbb{G}}$ ensured by restricting attention to $(\delta^{-1}W)_{\mathrm{Ad}\subset B}$, and the $\alpha_n = O(\varepsilon^n)$ are all constructed analytically (via (6)) in terms of the initial data. ■

Remark 2.3. We saw in the course of the proof of Theorem 2.1 that almost-cocycles are close to coboundaries, so in particular close to actual cocycles. This is presumably the type of result alluded to in passing in [10, Remark immediately preceding §3.4] (which in turn also refers back to [6]).

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