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Generalized Gelfand Pairs Attached to some Extensions of Heisenberg Groups

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Abstract. We show new examples of generalized Gelfand pairs of the form (K, N) by considering a family of 3-step nilpotent Lie groups $N := S \ltimes H$, where H is the $(2n+1)$ dimensional Heisenberg group, S is the subgroup of $(n \times n)$ symmetric matrices and K is a non compact, unimodular subgroup of automorphism of N. Also, we determine the automorphism group of N.

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1. Introduction

Let G be a unimodular Lie group and K a unimodular subgroup of G. Let $\mathcal{D}(G/K)$ be the space of C^∞ functions on G/K with compact support and assume that G acts on $\mathcal{D}'(G/K)$ by left translations. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G and by $\mathcal{H}^\infty$ the space of C^∞ vectors, equipped with a natural Sobolev topology. Let $\mathcal{H}^{-\infty}$ be its antidual, so $\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}$. The restriction of π to $\mathcal{H}^\infty$ gives rise to an action on $\mathcal{H}^{-\infty}$ by duality. The elements of $\mathcal{H}^{-\infty}$ are called distribution vectors.

We recall that for a wide class of Lie groups which includes nilpotent and semisimple Lie groups, any unitary representation π of G on a separable Hilbert space $\mathcal{H}$ decomposes in a unique way into a direct integral of irreducible unitary representations

$$\pi = \int_{\widehat{G}} m_\pi(\tau) d\mu(\tau),$$

where μ is a Borel measure on $\widehat{G}$ and $m_\pi : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ is the multiplicity.

The representation $(\pi, \mathcal{H})$ is called multiplicity free if the ring of continuous endomorphisms commuting with G, $End_G(\mathcal{H})$, is commutative. Equivalently $m_\pi(\tau) \leq 1$ for μ-almost all $\tau \in \widehat{G}$ (see [12]).

Definition 1.1. (G, K) is a *generalized Gelfand pair* if any unitary representation of G realized in $D'(G/K)$ decomposes as a direct integral of irreducible components without multiplicity.

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This definition was introduced by E. G. F. Thomas in [16] and it extends the classical notion of Gelfand pair when K is compact. Also from Theorem A in the same work, it is not hard to see that the Definition 1.1 is equivalent to the fact that for any irreducible representation $(\pi, \mathcal{H})$ of G realized in $\mathcal{D}'(G/K)$, the space $\mathcal{H}_K^{-\infty}$ of distribution vectors fixed by K is one dimensional. Moreover, from Theorem 1.1 in [8] it follows that a unitary representation $(\pi, \mathcal{H})$ of G admits a cyclic distribution vector fixed by K if and only if π is equivalent to an invariant Hilbert subspace of $\mathcal{D}'(G/K)$. Then the definition of a generalized Gelfand pair given in [16] is equivalent to the one introduced by G. Van Dijk (see for example [18], [17]), and which is adopted along this paper:

Definition 1.2. (G, K) is a *generalized Gelfand pair* if for any irreducible unitary representation $(\pi, \mathcal{H})$ of G, $\dim \mathcal{H}_K^{-\infty} \leq 1$.

Let N be a connected, simply connected nilpotent Lie group and K a compact subgroup of automorphisms of N. In the last decades several works focuse on Gelfand pairs of the form $(K, K \ltimes N)$ (or (K, N) in short), jointly with the corresponding spherical analysis (see [1], [2], [3], [4], [9], [13] and [19]).

One of the first results in [2] states that if (K, N) is a Gelfand pair then N is abelian or two step nilpotent.

This statement is not longer true when K is non compact: in [5], for each $m \in \mathbb{N}$, $m \geq 2$, it is exhibited a $(m+2)-$ step nilpotent Lie group N_m and a non compact subgroup H_m of $Aut(N_m)$ such that (H_m, N_m) is a generalized Gelfand pair. One has that the family $\mathfrak{n}_m := Lie(N_m)$ is one of the two families of graded filiform Lie algebras, and H_m is isomorphic to the group $\mathbb{R}^{m+1}$. The case $m = 1$, where $\mathfrak{n}_1$ corresponds to the Engel group, was studied in [10].

Moreover, in [6] is developed the spherical analysis associated to another family (N_m, K_m) of generalized Gelfand pairs, where K_m is isomorphic to the 3-dimensional Heisenberg group. For some values of m it was determined a set of generators of the algebra of left and K_m invariant differential operators, showing that the spherical distributions are not determined by the set of eingenvalues, in contrast with the compact case.

The aim of this work is to give new examples of generalized Gelfand pairs, by considering a family of 3-step nilpotent Lie groups (K, N), the Engel group being the one of minimun dimension: let H be the $2n+1$ dimensional Heisenberg group with coordinates (x, y, t), where $x, y \in \mathbb{R}^n, t \in \mathbb{R}$. It is well known that the symplectic group $Sp(n)$ acts on H by automorphisms and the group of $n \times n$ symmetric matrices S can be identified with an abelian automorphism subgroup of it. Using this identification, the action of S on H becomes

$$s \cdot (x, y, t) = (x, sx + y, t), \text{ for all } s \in S, (x, y, t) \in H.$$

The semidirect product $N := S \ltimes H$ is a 3-step nilpotent Lie group with product

$$(x, y, t, s)(x', y', t', s') = (x+x', y+y'+sx', t+t'+x \cdot (y'+sx') - x' \cdot y, s+s') \quad (1)$$

The corresponding Lie algebra is $\mathfrak{n} = \mathfrak{s} \ltimes \mathfrak{h}$ where $\mathfrak{h} = Lie(H)$ and $\mathfrak{s}$ is the space of the $n \times n$ symmetric matrix.

In this work, one of our goal is to determine the automorphism group $Aut\,(N)$ (see Theorem 3.3).

In order to explain the choice of K, we need to recall the Kirillov theory which states a one to one correspondence between the coadjoint orbits of N and $\widehat{N}$. Let us denote by $\mathfrak{n}^*$ the real dual space of $\mathfrak{n}$. For $\Lambda \in \mathfrak{n}^*$, let ρ_Λ denote the irreducible unitary representation of N associated with the coadjoint orbit $\mathcal{O}_\Lambda$. For $k \in K$, we have a new representation of N defined by $\rho_\Lambda^k(n) := \rho_\Lambda(k \cdot n)$. Let

$$K^\Lambda := \{k \in K : \rho_\Lambda^k \sim \rho_\Lambda\} = \{k \in K : k \cdot \Lambda \in \mathcal{O}_\Lambda\}$$

be the stabilizer of ρ_Λ. Thus for each $k \in K^\Lambda$ there is a unitary operator $\omega_\Lambda(k)$ such that $\rho_\Lambda^k(n) = \omega_\Lambda(k)\rho_\Lambda(n)\omega_\Lambda(k^{-1})$ for all $n \in N$. This defines a projective representation ω_Λ of K^Λ, that is,

$$\omega_\Lambda(k_1k_2) = \sigma(k_1, k_2)\omega_\Lambda(k_1)\omega_\Lambda(k_2).$$

ω_Λ is called *the intertwining representation of* ρ_Λ *or metaplectic representation* and σ *the multiplier* for the projective representation ω_Λ. Denote by $\widehat{K_\Lambda^\sigma}$ the set of (equivalent class) irreducible, unitary projective representations of K^Λ with multiplier σ.

When $K^\Lambda = K$, the Mackey's representation theory states that any irreducible unitary representation of $K \ltimes N$ is of the form

$$\rho_{\tau,\Lambda}\,(k, n) = \tau\,(k) \otimes \omega_\Lambda\,(k)\,\rho_\Lambda\,(n)\,, \tag{2}$$

where $\tau \in \widehat{K^{\overline{\sigma}}}, \overline{\sigma}$ denote the conjugate of σ, $\rho_\Lambda \in \widehat{N}$ and ω_Λ is the metaplectic representation. These representations correspond to the infinite dimensional representations ρ_Λ. When ρ_Λ is given by a character χ_Λ on N, $\rho_{\tau,\Lambda}(k, n) = \tau(k)\otimes\chi_\Lambda(n)$.

For the proof that given any irreducible unitary representation of $K \ltimes N$, the space of distribution vectors fixed by K is at most one dimensional, a crucial result is a criterion due to Mokni and Thomas, which is an analogous of a Carcano criterion for Gelfand pairs.

Theorem 1.3. [15] *Let* $(\omega; W)$, $(\gamma; V)$ *be unitary representations of* H *such that* γ *is irreducible. Let* γ^* *denote the dual representation of* γ. *Then* γ *appears in the decomposition of* ω *into irreducible components if and only if* $\gamma^* \otimes \omega$ *has a distribution vector fixed by* H *as* $(H \times H)$*-module.*

It is known that $\mathcal{H}$ is irreducible if and only if $\mathcal{H}^{-\infty}$ is irreducible (see [18], page 136). Thus, in the case $\rho_{\tau,\Lambda} = \tau \otimes \chi_\Lambda$, $\rho_{\tau,\Lambda}|_K = \tau$ and so it admits a distribution vector fixed by K if and only if τ is trivial. So, we are only interested in the representations (2) and, in these cases, according to Theorem 1.3 it is enough to show that ω_Λ is multiplicity free.

Hence, we have selected a subgroup K of $Aut\,(N)$ such that $K^\Lambda = K$ for all $\Lambda \in \mathfrak{n}^*$, see Proposition 3.6. There, we describe K as the set of 3-uple (β, B, α) with $\alpha, \beta \in \mathbb{R}^n, B \in S$ and it turns out isomorphic to the semidirec product $\mathbb{R}^n \ltimes (S \oplus \mathbb{R}^n)$.

In Section 2, we describe the exponential map $exp : \mathfrak{n} \to N$ and the coadjoint orbits of N. In Section 3, we determine $Aut(N)$, the automorphism group of N, and we characterize the subgroup K of $Aut(N)$. In Section 4, we calculate the induced representations ρ_Λ of N, the metaplectic representations ω_Λ and prove that they are irreducible proyective representations for the infinity dimensional representations ρ_Λ.

2. Preliminary results

Let $\mathfrak{n}$ be the Lie algebra with bases

$$\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{t}_0, \mathbf{e}_{11}, \ldots, \mathbf{e}_{ij}, \ldots, \mathbf{e}_{nn}\}$$

where $1 \leq i \leq j \leq n$, and the Lie bracket is defined by

$$[\mathbf{e}_{ij}, \mathbf{e}_k] = \begin{cases} \mathbf{f}_j & \text{if } k = i \\ \mathbf{f}_i & \text{if } k = j \\ 0 & \text{if } k \neq i, j \end{cases}, \qquad [\mathbf{e}_k, \mathbf{f}_k] = 2\mathbf{t}_0$$

and zero in the other cases. It is easy to see that $\mathfrak{n} = \mathfrak{s} \ltimes \mathfrak{h}$ where $\mathfrak{h}$ is Lie algebra of the Heisenberg group $(2n+1)$-dimentional and $\mathfrak{s}$ is the space of the $n \times n$ symmetric matrix. So, $\mathcal{Z} := \mathbb{R}\mathbf{t}_0$ is the center of $\mathfrak{n}$ and

$$\mathcal{X} := \langle\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}\rangle, \quad \mathcal{Y} := \langle\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}\rangle, \quad \mathfrak{s} := \langle\{\mathbf{e}_{11}, \ldots, \mathbf{e}_{ij}, \ldots, \mathbf{e}_{nn}\}\rangle,$$

are abelian subalgebras of $\mathfrak{n}$ and

$$\mathfrak{a} = \mathcal{Y} \oplus \mathcal{Z} \oplus \mathfrak{s}$$

is the maximal abelian subalgebra of $\mathfrak{n}$.

Let N be the simply connected Lie group with Lie algebra $\mathfrak{n}$.

2.1. The exponential map

Let $\mathfrak{m}$ be the Lie algebra of all matrix $(2n+2) \times (2n+2)$ of the form:

$$\begin{pmatrix} 0 & 0 & 0 & X \\ -Y^t & 0 & X^t & T \\ S & 0 & 0 & Y \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n\times 1}$ and S is an $n \times n$ real symmetric matrix.

In the following we identify each $\mathbf{n} \in \mathfrak{n}$ with $(\mathbf{x}, \mathbf{y}, \mathbf{t}, \mathbf{s}) \in \mathbb{R}^{\frac{n^2+n}{2}+2n+1}$, its representation in the basis $\mathcal{B}$ of $\mathfrak{n}$, where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{t} = t$ and $\mathbf{s} = (s_{11}, \ldots, s_{ij}, \ldots, s_{nn})$ with $1 \leq i \leq j \leq n$. Also, by abuse of notation, we will use $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{t}$ for the corresponding column vector.

Lemma 2.1. $\varphi : \mathfrak{n} \to \mathfrak{m}, \quad (\mathbf{x}, \mathbf{y}, \mathbf{t}, \mathbf{s}) \mapsto \begin{pmatrix} 0 & 0 & 0 & \mathbf{x} \\ -\mathbf{y}^t & 0 & \mathbf{x}^t & \mathbf{t} \\ [\mathbf{s}] & 0 & 0 & \mathbf{y} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

is an isomorphism of Lie algebras.

Proof. It is easy to see that

$$\varphi[(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s}),(\mathbf{x}',\mathbf{y}',\mathbf{t}',\mathbf{s}')] = \varphi(0,0,[\mathbf{s}]\mathbf{x}'-[\mathbf{s}']\mathbf{x},2(\mathbf{x}\cdot\mathbf{y}'-\mathbf{x}'\cdot\mathbf{y}))$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -([\mathbf{s}]\mathbf{x}'-[\mathbf{s}']\mathbf{x})^t & 0 & 0 & 2(\mathbf{x}^t\mathbf{y}'-\mathbf{x}'^t\mathbf{y}) \\ 0 & 0 & 0 & [\mathbf{s}]\mathbf{x}'-[\mathbf{s}']\mathbf{x} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{x}^t[\mathbf{s}'] & 0 & 0 & -\mathbf{y}^t\mathbf{x}'+\mathbf{x}^t\mathbf{y}' \\ 0 & 0 & 0 & [\mathbf{s}]\mathbf{x}' \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{x}'^t[\mathbf{s}] & 0 & 0 & -\mathbf{y}'^t\mathbf{x}+\mathbf{x}'^t\mathbf{y} \\ 0 & 0 & 0 & [\mathbf{s}']\mathbf{x} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \mathbf{x} \\ -\mathbf{y}^t & 0 & \mathbf{x}^t & \mathbf{t} \\ [\mathbf{s}] & 0 & 0 & \mathbf{y} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & \mathbf{x}' \\ -\mathbf{y}'^t & 0 & \mathbf{x}'^t & \mathbf{t}' \\ [\mathbf{s}'] & 0 & 0 & \mathbf{y}' \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \mathbf{x}' \\ -\mathbf{y}'^t & 0 & \mathbf{x}'^t & \mathbf{t}' \\ [\mathbf{s}'] & 0 & 0 & \mathbf{y}' \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & \mathbf{x} \\ -\mathbf{y}^t & 0 & \mathbf{x}^t & \mathbf{t} \\ [\mathbf{s}] & 0 & 0 & \mathbf{y} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \varphi(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s})\varphi(\mathbf{x}',\mathbf{y}',\mathbf{t}',\mathbf{s}') - \varphi(\mathbf{x}',\mathbf{y}',\mathbf{t}',\mathbf{s}')\varphi(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s})$$

$$= [\varphi(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s}),\varphi(\mathbf{x}',\mathbf{y}',\mathbf{t}',\mathbf{s}')] \qquad \blacksquare$$

Let M be the subgroup of $(2n+2)\times(2n+2)$ matrix of the form

$$\begin{pmatrix} I & 0 & 0 & X \\ -Y^t+X^tS & 1 & X^t & T \\ S & 0 & I & Y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n\times 1}$ and S is an $n\times n$ symmetric matrix.

For each $(x,y,t,s)\in N$, by abuse of notation, we denote by x, y and t to the column vectors and by $[s]$ the $n\times n$-symmetric matrix defined in the following way:

$$\text{given}\quad s = \begin{pmatrix} s_{11} \\ s_{12} \\ \vdots \\ s_{1n} \\ s_{22} \\ \vdots \\ s_{nn} \end{pmatrix} \in \mathbb{R}^{\frac{n^2+n}{2}\times 1} \quad \text{we set} \quad [s] = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ & s_{22} & \cdots & s_{2n} \\ & & \ddots & \vdots \\ & & & s_{nn} \end{pmatrix} \tag{3}$$

Lemma 2.2. $\phi: N \to M, \quad (x,y,t,s) \mapsto \begin{pmatrix} I & 0 & 0 & x \\ -y^t+x^t[s] & 1 & x^t & t \\ [s] & 0 & I & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$

is an isomorphism of groups.

Proof.

$$\begin{aligned}
&\phi\left((x,y,t,s)(x',y',t',s')\right)\\
&=\phi(x+x',y+y'+sx',t+t'+x\cdot(y'+sx')-x'\cdot y,s+s')\\
&=\begin{pmatrix} I & 0 & 0 & x+x' \\ -(y+y'+[s]x')^t+(x+x')^t(e+e') & 1 & (x+x')^t & t+t'+x^t(y'+[s]x')-x'^t y) \\ [s]+[s'] & 0 & I & y+y'+[s]x' \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&=\begin{pmatrix} I & 0 & 0 & x+x' \\ -y^t-y'^t+x^t[s]+x^t[s']+x'^t[s'] & 1 & x^t+x'^t & t+t'+x^ty'-x'^ty+x^t[s]x' \\ [s]+[s'] & 0 & I & y+y'+[s]x' \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&=\begin{pmatrix} I & 0 & 0 & x \\ -y^t+x^t[s] & 1 & x^t & t \\ [s] & 0 & I & y \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} I & 0 & 0 & x' \\ -y'^t+x'^t[s'] & 1 & x'^t & t' \\ [s'] & 0 & I & y' \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&=\phi(x,y,t,e)\phi(x',y',t',e')
\end{aligned}$$

■

We can see that the image of the exponential map defined on $\mathfrak{m}$ is M and

$$Exp:\mathfrak{m}\to M$$

$$\begin{pmatrix} 0 & 0 & 0 & X \\ -Y^t & 0 & X^t & T \\ S & 0 & 0 & Y \\ 0 & 0 & 0 & 0 \end{pmatrix}\mapsto\begin{pmatrix} I & 0 & 0 & X \\ -Y^t+\frac{1}{2}(SX)^t & 1 & X^t & T+\frac{1}{6}X^tSX \\ S & 0 & I & Y+\frac{1}{2}SX \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So, we have the following result.

Theorem 2.3. *The exponential map $exp:\mathfrak{n}\to N$ is defined by*

$$exp(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s})=(x,y,t,s)$$

where $x=\mathbf{x},\ y=\mathbf{y}+\frac{1}{2}[\mathbf{s}]\mathbf{x},\ t=\mathbf{t}+\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\ s=\mathbf{s}$.

Proof.

$$\begin{aligned}
exp(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s})&=\left(\phi^{-1}\circ Exp\circ\varphi\right)(\mathbf{x},\mathbf{y},\mathbf{t},\mathbf{s})\\
&=\left(\phi^{-1}\circ Exp\right)\begin{pmatrix} 0 & 0 & 0 & \mathbf{x} \\ -\mathbf{y}^t & 0 & \mathbf{x}^t & \mathbf{t} \\ [\mathbf{s}] & 0 & 0 & \mathbf{y} \\ 0 & 0 & 0 & 0 \end{pmatrix}\\
&=\phi^{-1}\begin{pmatrix} I & 0 & 0 & \mathbf{x} \\ -\mathbf{y}^t+\frac{1}{2}([\mathbf{s}]\mathbf{x})^t & 1 & \mathbf{x}^t & \mathbf{t}+\frac{1}{6}\mathbf{x}^t[\mathbf{s}]\mathbf{x} \\ [\mathbf{s}] & 0 & I & \mathbf{y}+\frac{1}{2}[\mathbf{s}]\mathbf{x} \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&=(x,y,t,e),
\end{aligned}$$

where $x=\mathbf{x},\ y=\mathbf{y}+\frac{1}{2}[\mathbf{s}]\mathbf{x},\ t=\mathbf{t}+\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\ s=\mathbf{s}$. ■

Observation 2.4. $(x,y,t,s)=exp\left(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\mathbf{s}\right)$.

2.2. Orbits

For $\mathbf{h} \in \mathfrak{h}$, let $(x_1,\dots,x_n,y_1,\dots,y_n,t,0) \in \mathbb{R}^{2n+1+\frac{n^2+n}{2}}$ be its representation in the basis $\mathcal{B}$ and let $ad_{\mathbf{h}}$ be the operator $ad(\mathbf{h})$. So, for $1 \le i < j \le n$ we have that

$$ad^k_{\mathbf{h}}(\mathbf{e}_{ij}) = \begin{cases} -x_i\mathbf{f}_j - x_j\mathbf{f}_i & \text{if } k=1,\\ -4x_ix_j\mathbf{t}_0 & \text{if } k=2,\\ 0 & \text{if } k\ge 3;\end{cases}$$

for $1 \le i \le n$ we have

$$ad^k_{\mathbf{h}}(\mathbf{e}_{ii}) = \begin{cases} -x_i\mathbf{f}_i & \text{if } k=1,\\ -2x_ix_i\mathbf{t}_0 & \text{if } k=2,\\ 0 & \text{if } k\ge 3;\end{cases} \qquad ad^k_{\mathbf{h}}(\mathbf{f}_i) = \begin{cases} 2x_i\mathbf{t}_0 & \text{if } k=1,\\ 0 & \text{if } k\ge 2;\end{cases}$$

$$ad^k_{\mathbf{h}}(\mathbf{e}_i) = \begin{cases} -2y_i\mathbf{t}_0 & \text{if } k=1,\\ 0 & \text{if } k\ge 2;\end{cases}$$

and $ad^k_{\mathbf{h}}(\mathbf{t}_0)=0$ for all $k\ge 1$. Similarly, for $\mathbf{s}\in\mathfrak{s}$ let

$$(0,0,0,s_{11},\dots,s_{ij},\dots,s_{nn}) \in \mathbb{R}^{2n+1+\frac{n^2+n}{2}}$$

be its representation in the basis $\mathcal{B}$, let us denote by $ad_{\mathbf{s}}$ the operator $ad(\mathbf{s})$. Then,

$$ad^k_{\mathbf{s}}(\mathbf{e}_i) = \begin{cases} \sum_{j=1}^n s_{ij}\mathbf{f}_j & \text{if } k=1,\\ 0 & \text{if } k\ge 2;\end{cases}$$

and $ad^k_{\mathbf{s}}(\mathbf{f}_i) = ad^k_{\mathbf{s}}(\mathbf{t}_0) = ad^k_{\mathbf{s}}(\mathbf{e}_{ij}) = 0$ for all $k\ge 1$. Therefore,

$$Exp(ad_{\mathbf{h}})(\mathbf{e}_{ij}) = \begin{cases} \mathbf{e}_{ij} - x_i\mathbf{f}_j - x_j\mathbf{f}_i - 2x_ix_j\mathbf{t}_0, & \text{if } i<j\\ \mathbf{e}_{ii} - x_i\mathbf{f}_i - x_ix_i\mathbf{t}_0, & \text{if } i=j\end{cases},$$

$$Exp(ad_{\mathbf{h}})(\mathbf{e}_i) = \mathbf{e}_i - 2y_i\mathbf{t}_0, \quad Exp(ad_{\mathbf{h}})(\mathbf{f}_i) = \mathbf{f}_i + 2x_i\mathbf{t}_0,$$

$$Exp(ad_{\mathbf{h}})(\mathbf{t}_0) = \mathbf{t}_0, \quad Exp(ad_{\mathbf{s}})(\mathbf{e}_{ij}) = \mathbf{e}_{ij}, \quad Exp(ad_{\mathbf{s}})(\mathbf{e}_i) = \mathbf{e}_i + \sum_{j=1}^n e_{ij}\mathbf{f}_j,$$

$$Exp(ad_{\mathbf{s}})(\mathbf{f}_i) = \mathbf{f}_i \quad\text{and}\quad Exp(ad_{\mathbf{s}})(\mathbf{t}_0) = \mathbf{t}_0$$

For $\mathbf{n}\in\mathfrak{n}$ we set $n = exp(\mathbf{n}) = (x,y,t,s) \in N$, thus

$$Ad(exp(\mathbf{n})) = Ad(x,y,t,s) = Ad(x,y,t,0)Ad(0,0,0,s) = Exp(ad_{\mathbf{h}}) \circ Exp(ad_{\mathbf{s}}),$$

where $exp(\mathbf{h}) = (x,y,t,0)$ and $exp(\mathbf{s}) = (0,0,0,s)$. So,

$$\begin{aligned} Ad(n)(\mathbf{e}_{ij}) &= \mathbf{e}_{ij} - x_i\mathbf{f}_j - x_j\mathbf{f}_i - 2x_ix_j\mathbf{t}_0\\ Ad(n)(\mathbf{e}_{ii}) &= \mathbf{e}_{ii} - x_i\mathbf{f}_i - x_ix_i\mathbf{t}_0\\ Ad(n)(\mathbf{e}_i) &= exp(ad_{\mathbf{h}})\Big(\mathbf{e}_i + \sum_{j=1}^n s_{ij}\mathbf{f}_j\Big)\\ &= exp(ad_{\mathbf{h}})(\mathbf{e}_i) + \sum_{j=1}^n s_{ij}\, exp(ad_{\mathbf{h}})\,(\mathbf{f}_j)\\ &= \mathbf{e}_i + \sum_{j=1}^n s_{ij}\mathbf{f}_j + 2\Big(-y_i + \sum_{j=1}^n s_{ij}x_j\Big)\mathbf{t}_0\\ Ad(n)(\mathbf{f}_i) &= \mathbf{f}_i + 2x_i\mathbf{t}_0, \quad Ad(n)(\mathbf{t}_0) = \mathbf{t}_0. \end{aligned}$$

Let $i : \mathfrak{h} \to \mathfrak{n}$ be the inclusion map, with $i^* : \mathfrak{n}^* \to \mathfrak{h}^*$ its dual. $\mathfrak{n}^*$ has coordinates $(\mu, \nu, \lambda, \sigma)$, where $\mu,\ \nu \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\sigma \in \mathbb{R}^{\frac{n^2+n}{2}}$ also, $\mathfrak{n}$ has coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{t}, \mathbf{s})$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{t} \in \mathbb{R}$ and $\mathbf{s} \in \mathbb{R}^{\frac{n^2+n}{2}}$. The pairing between $\mathfrak{n}$ and $\mathfrak{n}^*$ is given by

$$(\mu, \nu, \lambda, \sigma)(\mathbf{x}, \mathbf{y}, \mathbf{t}, \mathbf{s}) = \mu \cdot \mathbf{x} + \nu \cdot \mathbf{y} + \lambda \mathbf{t} + \sigma \cdot \mathbf{s}$$

where $\mu \cdot \mathbf{x} = \sum_{i=1}^{n} \mu_i x_i$, $\nu \cdot \mathbf{y} = \sum_{i=1}^{n} \nu_i y_i$, $\lambda \mathbf{t} = \lambda t$ and $\sigma \cdot \mathbf{s} = \sum_{i \le j} \sigma_{ij} s_{ij}$.

So, if $\Lambda = (\mu, \nu, \lambda, \sigma) \in \mathfrak{n}^*$ then

$$\begin{aligned}
\Lambda(Ad(n))(\mathbf{e}_{ij}) &= \sigma_{ij} - x_i \nu_j - x_j \nu_i - 2 x_i x_j \lambda \\
\Lambda(Ad(n))(\mathbf{e}_{ii}) &= \sigma_{ii} - x_i \nu_i - x_i x_i \lambda \\
\Lambda(Ad(n))(\mathbf{e}_i) &= \mu_i + \sum_{j=1}^{n} s_{ij} \nu_j + 2\Big(- y_i + \sum_{j=1}^{n} s_{ij} x_j \Big) \lambda \\
\Lambda(Ad(n))(\mathbf{f}_i) &= \nu_i + 2 x_i \lambda \\
\Lambda(Ad(n))(\mathbf{t}_0) &= \lambda
\end{aligned}$$

Proposition 2.5. *Let Λ be in $\mathfrak{n}^*$,*

1. *If $\Lambda = (0, 0, \lambda, \sigma)$ with $\lambda \neq 0$, the coadjoint orbit is*
   $$\mathcal{O}_{\lambda,\sigma} = \{(\mu, \nu, \lambda, \sigma_{\nu,\lambda}) : \nu, \mu \in \mathbb{R}^n\}$$
   where $[\sigma_{\nu,\lambda}] = [\sigma] - \frac{1}{2\lambda} \nu^t \nu + \frac{1}{4\lambda} I^\nu$ and $I^\nu_{ij} = \nu_i^2 \delta_{ij}$.
2. *If $\Lambda = (0, \nu, 0, \sigma)$ with $\nu \neq 0$, we have that*
   $$\mathcal{O}_{\nu,\sigma} = \{(\mu, \nu, 0, \sigma_{\nu,x}) : x, \mu \in \mathbb{R}^n\}$$
   where $[\sigma_{\nu,x}] = [\sigma] - x^t \nu - \nu^t x$.
3. *If $\Lambda = (\mu, 0, 0, \sigma)$, we have $\mathcal{O}_{\mu,\sigma} = \{(\mu, 0, 0, \sigma)\}$.*

3. Automorphisms of $\mathfrak{s} \ltimes \mathfrak{h}$

We begin by recalling that the descending central series of a Lie algebra $\mathfrak{g}$ is defined by

$$\mathfrak{g}^0 := \mathfrak{g}, \mathfrak{g}^1 := [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{g}^0], \ldots, \mathfrak{g}^n := [\mathfrak{g}, \mathfrak{g}^{n-1}].$$

Besides if $T \in Aut(\mathfrak{g})$ then for $n \in \mathbb{N}$,

$$T(\mathfrak{g}^n) = \mathfrak{g}^n \tag{4}$$

Theorem 3.1. *Let $\theta \in Aut(\mathfrak{n})$ then*

(i) $\theta(\mathcal{Z}) = \mathcal{Z}$,

(ii) $\theta(\mathcal{Y} \oplus \mathcal{Z}) = \mathcal{Y} \oplus \mathcal{Z}$,

(iii) $\theta(\mathfrak{a}) = \mathfrak{a}$, *Therefore, the matrix of* θ *in the bases* $\mathcal{B}$ *is*

$$\begin{pmatrix} A & 0 & 0 & 0 \\ B & F & 0 & G \\ \alpha & \beta & a & \gamma \\ C & 0 & 0 & H \end{pmatrix}$$

with $A, F \in Gl(n,\mathbb{R}),\ a \neq 0,\ H \in Gl\left(\frac{n^2+n}{2},\mathbb{R}\right),\ \alpha,\beta \in M(1\times n,\mathbb{R}),$

$$\gamma \in M\left(1\times \frac{n^2+n}{2},\mathbb{R}\right) \quad \textit{and} \quad C \in M\left(\frac{n^2+n}{2}\times n,\mathbb{R}\right).$$

Proof. The descending central series of $\mathfrak{n}$ is

$$\mathfrak{n}^2 = [\mathfrak{n},\mathfrak{n}] = \mathcal{Y} \oplus \mathcal{Z}, \quad \mathfrak{n}^3 = [\mathfrak{n},\mathfrak{n}^2] = \mathcal{Z}.$$

Then, by (4) we have proved the items (i) and (ii).
To prove the item (iii), it is enough to prove that $\theta(\mathbf{e}_{ij}) \in \mathfrak{a}$ for all $i \leq j$. As

$$\theta(\mathbf{e}_{ij}) = a_1^{ij}\,\mathbf{e}_1 + \cdots + a_n^{ij}\,\mathbf{e}_n + \mathbf{u}, \tag{5}$$

with $\mathbf{u} \in \mathfrak{a}$. We will prove that $a_k^{ij} = 0$ for all $k = 1,\ldots,n$.
By item (ii) there are $\mathbf{v}_1,\ldots,\mathbf{v}_n \in \mathcal{Y} \bigoplus \mathcal{Z}$ such that

$$\theta(\mathbf{v}_k) = \mathbf{f}_k. \tag{6}$$

By (5) and (6), we have that

$$0 = \theta([\mathbf{e}_{ij},\mathbf{v}_k]) = [\theta(\mathbf{e}_{ij}),\mathbf{f}_k] = a_k^{ij}\,2\mathbf{t}_0,$$

hence $a_k^{ij} = 0$ for all $k = 1,\ldots,n$ and $i \leq j$. ■

We now describe $Aut_0(\mathfrak{n}) := \{\theta \in Aut(\mathfrak{n}) : \theta(\mathbf{t}_0) = \mathbf{t}_0\}$.
Given $\theta \in Aut_0(\mathfrak{n})$, for each $h \in \mathfrak{h}$ we have that

$$\theta(h) = \pi_\theta(h) + s_\theta(h)$$

where $\pi_\theta(h)$ is the projection of $\theta(h)$ on $\mathfrak{h}$ and $s_\theta(h)$ is the projection of $\theta(h)$ on $\mathfrak{s}$. Note that

$$\theta|_{\mathcal{Y}\oplus\mathcal{Z}} = \pi_\theta\,|_{\mathcal{Y}\oplus\mathcal{Z}}\,, \tag{7}$$

since $s_\theta\,|_{\mathcal{Y}\oplus\mathcal{Z}} \equiv 0$. Moreover, π_θ is injective, that is, if $x \in \mathcal{X}$ then

$$\pi_\theta(x) = 0 \Rightarrow Ax + Bx + \alpha x = 0 \Rightarrow Ax = 0 \Rightarrow x = 0.$$

We denote by $s\cdot\mathfrak{h}$ the action of an element $s \in \mathfrak{s}$ on $\mathfrak{h}$. Furthermore, note that by definition of the bracket we have for all $s \in \mathfrak{s}$ and $h \in \mathfrak{h}$

$$s\cdot h = [s,h] \in \mathcal{Y}. \tag{8}$$

Proposition 3.2. *If* $\theta \in Aut_0(\mathfrak{n})$ *then* $\pi_\theta \in Aut_0(\mathfrak{h})$ *and*

$$s_\theta(h_1)\cdot\pi_\theta(h_2) = s_\theta(h_2)\cdot\pi_\theta(h_1) \tag{9}$$

Proof. Given $h_1, h_2 \in \mathfrak{h}_n$, as $\mathfrak{h}_n$ is 2-step nilpotent we have that

$$[h_1, h_2],\ [\pi_\theta(h_1), \pi_\theta(h_2)] \in \mathcal{Z}.$$

By (7) we get that

$$\begin{aligned}\pi_\theta([h_1, h_2]) &= \theta[h_1, h_2] = [\theta(h_1), \theta(h_2)]\\ &= [\pi_\theta(h_1) + s_\theta(h_1), \pi_\theta(h_2) + s_\theta(h_2)]\\ &= [\pi_\theta(h_1), \pi_\theta(h_2)] + s_\theta(h_1) \cdot \pi_\theta(h_2) - s_\theta(h_2) \cdot \pi_\theta(h_1). \qquad (10)\end{aligned}$$

By (8), $s_\theta(h_1) \cdot \pi_\theta(h_2) - s_\theta(h_2) \cdot \pi_\theta(h_1) \in \mathcal{Y}$ although $[\pi_\theta(h_1), \pi_\theta(h_2)] \in \mathcal{Z}$ and $\pi_\theta([h_1, h_2]) = \theta[h_1, h_2] = [h_1, h_2] \in \mathcal{Z}$. Therefore, from (10) it follows that

$$\pi_\theta([h_1, h_2]) = [\pi_\theta(h_1), \pi_\theta(h_2)], \qquad (11)$$
$$s_\theta(h_1) \cdot \pi_\theta(h_2) = s_\theta(h_2) \cdot \pi_\theta(h_1), \qquad (12)$$

for all $h_1, h_2 \in \mathfrak{h}$. The proposition follows from (11) and (12). ■

Next, as $\pi_\theta \in Aut_0(\mathfrak{h})$ we have that the matrix of θ in the bases $\mathcal{B}$ is of the form

$$\begin{pmatrix} A & 0 & 0 & 0\\ B & (A^{-1})^t & 0 & G\\ \alpha & \beta & 1 & \gamma\\ C & 0 & 0 & H \end{pmatrix} \qquad (13)$$

where BA^{-1} is a symmetric matrix. We are looking for conditions on C, G, H and γ in order that the above matrix defines an automorphism on $\mathfrak{n}$. We begin by noting that (9) is equivalent to the fact that $C^l A^t$ is a symmetric matrix for all $l = 1, \dots, n$ where

$$C^l = \begin{pmatrix} c_{1l,1} & c_{1l,2} & \cdots & c_{1l,n}\\ c_{2l,1} & c_{2l,2} & \cdots & c_{2l,n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{l-1l,1} & c_{l-1l,2} & \cdots & c_{l-1l,n}\\ c_{ll,1} & c_{ll,2} & \cdots & c_{ll,n}\\ c_{ll+1,1} & c_{ll+1,2} & \cdots & c_{ll+1,n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{ln,1} & c_{ln,2} & \cdots & c_{ln,n} \end{pmatrix}$$

In fact, for all $n = h \oplus s$ and $n' = h' \oplus s'$ we have that

$$\begin{aligned}\theta[h \oplus s, h' \oplus s'] &= \theta[h, h'] + \theta[h, s'] + \theta[s, h']\\ &= \pi_\theta[h, h'] + \pi_\theta(s \cdot h') - \pi_\theta(s' \cdot h)\end{aligned}$$

where we have used (ii) of Theorem 3.1.

On the other hand,

$$\begin{aligned}&[\theta(h \oplus s), \theta(h' \oplus s')] = [\theta(h), \theta(h')] + [\theta(h), \theta(s')] + [\theta(s), \theta(h')] + [\theta(s), \theta(s')]\\ &= [\pi_\theta(h), \pi_\theta(h')] + [\pi_\theta(h), \theta(s')] + [\theta(s), \pi_\theta(h')]\end{aligned}$$

where we have used (9) and the fact that $s \cdot \theta(s') = 0$ for all $s, s' \in \mathfrak{s}$ since $\theta(s) \in \mathfrak{a}$.

So, as $\pi_\theta \in Aut_0(\mathfrak{h})$ by Proposition 3.2, $\theta \in Aut_0(\mathfrak{n})$ if and only if

$$\theta(s \cdot h) = [\theta(s), \pi_\theta(h)] \tag{14}$$

By the definition of the bracket, we can rewrite equation (14) in coordinates as

$$(A^{-1})^t(s \cdot x) + \langle \beta, s \cdot x \rangle \, \mathbf{t}_0 = [Gs, Ax] + [Hs, Ax] \tag{15}$$

for all $s \in \mathfrak{s}$ and $x \in \mathcal{X}$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$.
Moreover, as $(A^{-1})^t(s \cdot x), [Hs, Ax] \in \mathcal{Y}$ and $\langle \beta, (s \cdot x) \rangle \, \mathbf{t}_0, [Gs, Ax] \in \mathcal{Z}$ we have that the equality (15) is equivalent to

$$(A^{-1})^t(s \cdot A^{-1}x) = [Hs, x] \tag{16}$$

$$\left\langle \beta, (s \cdot A^{-1}x) \right\rangle \mathbf{t}_0 = [Gs, x] \tag{17}$$

for all $x \in \mathcal{X}$ and $s \in \mathfrak{s}$, therefore H and G are uniquely determined by A and β since the bracket Lie $[\cdot, \cdot]$ defines a non degenerate bilinear form.

Theorem 3.3. *A matrix in* $Gl\left(2n + 1 + \frac{n(n+1)}{2}, \mathbb{R}\right)$ *of the form*

$$\begin{pmatrix} A & 0 & 0 & 0 \\ B & F & 0 & G_{A,\beta} \\ \alpha & \beta & a & \gamma \\ C & 0 & 0 & H_{A,a} \end{pmatrix}$$

defines an automorphism $\theta \in Aut(\mathfrak{n})$ *if and only if*

(i) $C^l A^t$ *is a symmetric matrix for all* $l = 1, \ldots, n$,

(ii) $F = a(A^{-1})^t$ *and* BA^t *is a symmetric matrix,*

(iii) $-2G_{A,\beta}s = (A^{-1})^t[s]\beta^t$ *for all* $s \in \mathbb{R}^{\frac{n^2+n}{2} \times 1}$,

(iv) $[H_{A,a}s] = a(A^{-1})^t[s]A^{-1}$ *for all* $s \in \mathbb{R}^{\frac{n^2+n}{2} \times 1}$.

Proof. Item (i) is proved in Section 5, item (ii) is a consequence of Proposition 3.2 and items (iii) and (iv) follow from (16) and (17). ■

Observation 3.4. *As* N *is a simply connected Lie group, it follows that*

$$Aut(N) = \{exp \circ k \circ \exp^{-1} : k \in Aut(\mathfrak{n})\}$$

where exp is the exponential map.

3.1. A non abelian subgroup K of $Aut(\mathfrak{n})$

By Mackey representation theory, $\widehat{K \ltimes N}$ is easier to describe in the case that

$$k \cdot \Lambda \in \mathcal{O}_\Lambda$$

for all $k \in K$, $\Lambda \in \mathfrak{n}^*$. So, the aim of this section is to determine these possible automorphisms subgroups.

Note that for $k \in Aut(\mathfrak{n})$ and $\Lambda = (\mu, \nu, \lambda, \sigma) \in \mathfrak{n}^*$, $k \cdot \Lambda \in \mathcal{O}_\Lambda$ if and only if $k^t(\mu, \nu, \lambda, \sigma) \in \mathcal{O}_\Lambda$. Thus, if $\Lambda = (0, 0, \lambda, \sigma)$ with $\lambda \neq 0$ we have that

$$k^t(0, 0, \lambda, \sigma) \in \mathcal{O}_\Lambda \Leftrightarrow \lambda[\gamma] + [H\sigma] = [\sigma] - \frac{\lambda}{2}\beta^t\beta + \frac{\lambda}{4}I^\beta$$

If $\Lambda = (0, \nu, 0, \sigma)$ with $\nu \neq 0$ we have that

$$k^t(0, \nu, 0, \sigma) \in \mathcal{O}_\Lambda \Leftrightarrow \begin{cases} A^{-1}\nu = \nu \\ [G^t\nu] + [H\sigma] = [\sigma] - x^t\nu - \nu^t x \end{cases}$$

Finally, if $\Lambda = (\mu, 0, 0, \sigma)$ we have that

$$k^t(\mu, 0, 0, \sigma) \in \mathcal{O}_\Lambda \Leftrightarrow \begin{cases} A^t\mu + C^t\sigma = \mu \\ H^t\sigma = \sigma \end{cases}$$

So, for all $\Lambda \in \mathfrak{n}^*$

$$k \cdot \Lambda \in \mathcal{O}_\Lambda \Leftrightarrow \begin{cases} A = I, F = I, H = I, C = 0 \\ [\gamma] = -\frac{1}{2}\beta^t\beta + \frac{1}{4}I^\beta \\ [\nu G] = -x^t\nu - \nu^t x \end{cases}$$

Lemma 3.5. *For $\beta \in \mathbb{R}^n$ we set $[\gamma_\beta] = -\frac{1}{2}\beta^t\beta + \frac{1}{4}I^\beta$. Then,*

$$\gamma_{\beta+\beta'} = \gamma_\beta + \gamma_{\beta'} + \beta G_{\beta'}.$$

The proof is in Section 5.

Proposition 3.6. *Let K be the subgroup of $Aut_0(\mathfrak{n})$ defined by*

$$K = \left\{ \begin{pmatrix} I & 0 & 0 & 0 \\ B & I & 0 & G_\beta \\ \alpha & \beta & 1 & \gamma_\beta \\ 0 & 0 & 0 & I \end{pmatrix} \in Aut(\mathfrak{n}) : [\gamma_\beta] = -\frac{1}{2}\beta^t\beta + \frac{1}{4}I^\beta \right\} \tag{18}$$

then $K^\Lambda = K$, for all $\Lambda \in \mathfrak{n}^*$. Moreover, K is isomorphic to $\mathbb{R}^n \ltimes (\mathcal{S} \oplus \mathbb{R}^n)$ where $\mathcal{S}$ is the space of $n \times n$-symmetric matrices.

Proof. Note that

$$\begin{pmatrix} I & 0 & 0 & 0 \\ B & I & 0 & G_\beta \\ \alpha & \beta & 1 & \gamma_\beta \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ B' & I & 0 & G_{\beta'} \\ \alpha' & \beta' & 1 & \gamma_{\beta'} \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ B + B' & I & 0 & G_{\beta+\beta'} \\ \alpha' + \alpha + \beta B' & \beta + \beta' & 1 & \gamma_{\beta+\beta'} \\ 0 & 0 & 0 & I \end{pmatrix}$$

since $G_\beta + G_{\beta'} = G_{\beta+\beta'}$ by item (iii) of Theorem 3.3 and $\gamma_\beta + \gamma_{\beta'} + \beta G_{\beta'} = \gamma_{\beta+\beta'}$ by Lemma 3.5. Then, identifying each matrix $\begin{pmatrix} I & 0 & 0 & 0 \\ B & I & 0 & G_\beta \\ \alpha & \beta & 1 & \gamma_\beta \\ 0 & 0 & 0 & I \end{pmatrix}$ with the term (β, B, α) we have that

$$(\beta, B, \alpha)(\beta', B', \alpha') = (\beta + \beta', B + B', \alpha + \alpha' + \beta B') \tag{19}$$

which proves the proposition. ■

4. The metaplectic representation of K

4.1. The induced representation of N

Here we follow the description of the Kirillov one to one correspondence between the set of coadjoint orbits and $\widehat{N}$ [11].

For Λ in $\mathfrak{n}_m^*$, let B_Λ be the skew-symmetric form defined by

$$B_\Lambda(u;v) := \Lambda([u,v]),\ u,v \in \mathfrak{n}_m.$$

Let $\mathfrak{M}_\Lambda$ be a maximal isotropic subspace of B_Λ, and set $M_\Lambda = exp(\mathfrak{M}_\Lambda)$. Defining on M_Λ the character $\chi_\Lambda(exp(u)) = e^{i\Lambda u}$, the irreducible representation of N corresponding to $\mathcal{O}_\Lambda$ is the induced representation $\rho_\Lambda = Ind_{M_\Lambda}^N(\chi_\Lambda)$.

We recall that the induced representation is the pair $(\rho_\Lambda, H_\Lambda)$ where H_Λ is the completion of

$$\{f \in C_c(N) : f(gh) = \chi_\Lambda(h^{-1})f(g) \text{ for all } h \in M_\Lambda,\ g \in N\}$$

with respect to the scalar product

$$\langle f_1, f_2\rangle = \int_{N/M_\Lambda} f_1(u)\overline{f_2(u)}du,$$

and the action is the left regular representation, that is

$$[\rho_\Lambda(g)f]\,(g') = f(g^{-1}g'), \quad g, g' \in N.$$

In our case, for $\Lambda \neq (\mu, 0, 0, \sigma)$, we have

$$\mathfrak{M}_\Lambda = \mathfrak{a},$$

since $\mathfrak{a}$ is an abelian ideal of codimention one. Now, by Theorem 2.3,

$$M_\Lambda = exp(\mathfrak{M}_\Lambda) = \{(0,y,t,s) \in N\},$$

the character χ_Λ on M_Λ is defined by

$$\chi_\Lambda(0,y,t,s) = \chi_\Lambda\left(exp(\mathbf{0},\mathbf{y},\mathbf{t},\mathbf{s})\right) = e^{i\Lambda(\mathbf{0},\mathbf{y},\mathbf{t},\mathbf{s})}, \tag{20}$$

and the representation space is isomorphic to $L^2(\mathbb{R}^n)$.
Furthermore, note that

$$(x,y,t,s)^{-1} = (-x, -y + [s]x, -t, -s)$$

For $f \in L^2(\mathbb{R}^n)$ we have that

$$\begin{aligned}
&\rho_\Lambda(x,y,t,s)f(u) = f\left((x,y,t,s)^{-1}(u,0,0,0)\right)\\
&= f\left((-x, -y + [s]x, -t, -s)(u,0,0,0)\right)\\
&= f\left(u - x, -y - [s](u-x), -t + u\cdot y - u\cdot[s]x + x\cdot[s]u, -s\right)\\
&= f\left((u-x,0,0,0)(0, -y - [s](u-x), -t + (2u-x)\cdot(y - [s]x) + u\cdot[s]u, -s)\right)\\
&= \chi_\Lambda(0, -y - [s](u-x), -t + (2u-x)\cdot(y - ex) + u\cdot[s]u, -s)^{-1}f(u-x)\\
&= \chi_\Lambda(0, y + [s](u-x), t - (2u-x)\cdot(y - [s]x) - u\cdot[s]u, s)f(u-x).
\end{aligned} \tag{21}$$

So, we get

- For $\Lambda = (0,0,\lambda,\sigma)$ with $\lambda \neq 0$,
$$\rho_\Lambda(x,y,t,s)f(u) = e^{i\lambda[t-(2u-x)\cdot(y-[s]x)-u\cdot[s]u]}e^{i\sigma\cdot s}f(u-x).$$
- For $\Lambda = (0,\nu,0,\sigma)$ with $\nu \neq 0$,
$$\rho_\Lambda(x,y,t,s)f(u) = e^{i\nu\cdot(y+[s](u-x))}e^{i\sigma\cdot s}f(u-x).$$
- For $\Lambda = (\mu,0,0,\sigma)$, note that $\mathfrak{M}_\Lambda = \mathfrak{n}$ then
$$\rho_\Lambda = \chi_\Lambda. \tag{22}$$

4.2. Computation of ω_Λ

By Proposition 3.6, we have that

$$K^\Lambda = K \quad \forall \Lambda \in \mathfrak{n}^*.$$

In this situation, Mackey's representation theory states that any irreducible unitary representation of the semidirect product $K \ltimes N$ is of the form

$$\rho_{\sigma,\Lambda} := \tau \otimes \omega_\Lambda \rho_\Lambda,$$

where ω_Λ is a projective representation with multiplier σ that satisfies

$$\rho^k_\Lambda(n) = \omega_\Lambda(k)\rho_\Lambda(n)\omega_\Lambda(k^{-1}), \quad \forall n \in N,\ k \in K,\ \tau \in \widehat{K^\sigma} \text{ and } \rho_\Lambda \in \widehat{N}.$$

In order to obtain explicitly ω_Λ, by (19), we write

$$(\beta,B,\alpha) = (\beta,0,0)(0,B,0)(0,0,\alpha-\beta B)$$

Moreover, for any $v \in \mathbb{R}^n$ and B a $n\times n$-symmetric matrix, we denote by L_v, U_v and R_B to the operators defined on $L^2(\mathbb{R}^n)$ by

$$L_v f(u) = f(u-v), \quad U_v f(u) = e^{iv\cdot u}f(u), \quad R_B f(u) = e^{-iBu\cdot u}f(u)$$

Lemma 4.1. $\gamma_\beta \cdot s + \frac{1}{4}\beta\cdot[s]\beta = 0$.

The proof is in Section 5.

- Case $\Lambda = (0,0,\lambda,\sigma)$ with $\lambda \neq 0$:

$$\begin{aligned}
\left[\rho^\beta_\Lambda(x,y,t,s)f\right](u) &= \left[\rho_\Lambda\left((\beta,0,0)^{-1}(x,y,t,s)\right)f\right](u)\\
&= \left[\rho_\Lambda\left((-\beta,0,0)(x,y,t,s)\right)f\right](u)\\
&= \left[\rho_\Lambda exp\left((-\beta,0,0)\left(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\mathbf{s}\right)\right)f\right](u)\\
&= \left[\rho_\Lambda exp\left(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x}-G_\beta\mathbf{s},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x}-\beta\cdot\mathbf{y}+\frac{1}{2}\beta\cdot[\mathbf{s}]\mathbf{x}+\gamma_\beta\cdot\mathbf{s},\mathbf{s}\right)f\right](u)\\
&= \left[\rho_\Lambda(x,y-G_\beta s,t-\beta\cdot y+\frac{1}{2}\beta\cdot[s]x+\gamma_\beta\cdot s,s)f\right](u)\\
&= e^{i\lambda\left[t-\beta\cdot y+\frac{1}{2}\beta\cdot[s]x+\gamma_\beta\cdot s-(2u-x)\cdot\left(y-G_\beta s-[s]x\right)-u\cdot[s]u\right]}e^{i\sigma\cdot s}f(u-x)
\end{aligned}$$

$$= e^{i\lambda\left[-\beta\cdot y+\frac{1}{2}\beta\cdot[s]x+\gamma_\beta\cdot s+(2u-x)\cdot G_\beta s\right]}e^{i\lambda[t-(2u-x)\cdot(y-[s]x)-u\cdot[s]u]}e^{i\sigma\cdot s}f(u-x)$$

$$= e^{i\lambda\left[-\frac{1}{2}\beta\cdot[s]x-x\cdot G_\beta s+\gamma_\beta\cdot s+\frac{1}{4}\beta\cdot[s]\beta+2u\cdot(G_\beta s)+\beta\cdot[s]u\right]}\left[\rho_\Lambda(x,y,t,s)\left(L_{\frac{1}{2}\beta}f\right)\right]\left(u+\frac{1}{2}\beta\right)$$

$$= e^{i\lambda\left[-\frac{1}{2}[s]\beta\cdot x-G_\beta s\cdot x+(2G_\beta s)\cdot u+[s]\beta\cdot u+\gamma_\beta\cdot s+\frac{1}{4}\beta\cdot[s]\beta\right]}\left[\rho_\Lambda(x,y,t,s)\left(L_{\frac{1}{2}\beta}f\right)\right]\left(u+\frac{1}{2}\beta\right)$$

$$= \left[L_{-\frac{1}{2}\beta}\rho_\Lambda(x,y,t,s)L_{\frac{1}{2}\beta}f\right](u)$$

where in the last equality we have used item (iii) of Theorem 3.3 and Lemma 4.1. Moreover,

$$\begin{aligned}
\left[\rho^\alpha_\Lambda(x,y,t,s)f\right](u) &= \left[\rho_\Lambda(0,0,\alpha)^{-1}(x,y,t,s)f\right](u)\\
&= [\rho_\Lambda(0,0,-\alpha)(x,y,t,s)f](u)\\
&= \left[\rho_\Lambda exp\big((0,0,-\alpha)\big(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\mathbf{e}\big)\big)f\right](u)\\
&= \left[\rho_\Lambda exp\big(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x}-\alpha\cdot\mathbf{x},\mathbf{s}\big)f\right](u)\\
&= \left[\rho_\Lambda(x,y,t-\alpha\cdot x,s)f\right](u)\\
&= e^{i\lambda[t-\alpha\cdot x-(2u-x)\cdot(y-[s]x)]}e^{i\sigma\cdot s}f(u-x)\\
&= e^{-i\lambda\alpha\cdot u}e^{i\lambda(t-(2u-x)\cdot(y-[s]x))}e^{i\sigma\cdot s}e^{i\lambda\alpha\cdot(u-x)}f(u-x)\\
&= \left[U_{-\lambda\alpha}\rho_\Lambda(x,y,t,s)U_{\lambda\alpha}f\right](u),
\end{aligned}$$

and

$$\begin{aligned}
\left[\rho^B_\Lambda(x,y,t,s)f\right](u) &= \left[\rho_\Lambda(0,B,0)^{-1}(x,y,t,s)f\right](u)\\
&= \left[\rho_\Lambda(0,-B,0)(x,y,t,s)f\right](u)\\
&= \left[\rho_\Lambda exp\big((0,-B,0)\big(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\mathbf{s}\big)\big)f\right](u)\\
&= \left[\rho_\Lambda exp\big(\mathbf{x},\mathbf{y}-\frac{1}{2}[\mathbf{s}]\mathbf{x}-B\mathbf{x},\mathbf{t}-\frac{1}{6}\mathbf{x}\cdot[\mathbf{s}]\mathbf{x},\mathbf{s}\big)f\right](u)\\
&= \left[\rho_\Lambda(x,y-Bx,t,s)f\right](u)\\
&= e^{i\lambda(t-(2u-x)\cdot(y-Bx-[s]x)-u\cdot[s]u)}e^{i\sigma\cdot s}f(u-x)\\
&= e^{i\lambda(2u-x)\cdot Bx}e^{i\lambda(t-(2u-x)\cdot(y-[s]x)-u\cdot[s]u)}e^{i\sigma\cdot s}f(u-x)\\
&= e^{i\lambda(2u-x)\cdot Bx+B(u-x)\cdot(u-x)}\left[\rho_\Lambda(x,y,t,s)R_{\lambda B}f\right](u)\\
&= \left[R_{-\lambda B}\rho_\Lambda(x,y,t,s)R_{\lambda B}f\right](u).
\end{aligned}$$

Note that $\omega_\Lambda(\beta,0,0)=L_{\frac{1}{2}\beta}$, $\omega_\Lambda(0,0,\alpha)=U_{\lambda\alpha}$ and $\omega_\Lambda(0,B,0)=R_{\lambda B}$. Thus, we set ω_Λ defined by

$$\omega_\Lambda(\beta,B,\alpha)f(u)=\left[L_{\frac{1}{2}\beta}R_{\lambda B}U_{\lambda(\alpha-\beta B)}f\right](u)\,. \tag{23}$$

In order to prove that ω_Λ is a projective representation we note that

$$\begin{aligned}
&[\omega_\Lambda(\beta_1,B_1,\alpha_1)\omega_\Lambda(\beta_2,B_2,\alpha_2)f](u)\\
&= [L_{\frac{1}{2}\beta_1}R_{\lambda B_1}U_{\lambda(\alpha_1-\beta_1B_1)}\omega_\Lambda(\beta_2,B_2,\alpha_2)f](u)\\
&= e^{i\lambda(\alpha_1-\beta_1B_1)\cdot(u-\frac{1}{2}\beta_1)}[L_{\frac{1}{2}\beta_1}R_{\lambda B_1}\omega_\Lambda(\beta_2,B_2,\alpha_2)f](u)
\end{aligned}$$

$$
\begin{aligned}
&= e^{i\lambda(\alpha_1-\beta_1 B_1)\cdot(u-\frac{1}{2}\beta_1)}\left[L_{\frac{1}{2}\beta_1}R_{\lambda B_1}\left(u\mapsto e^{i\lambda(\alpha_2-\beta_2 B_2)\cdot(u-\frac{1}{2}\beta_2)}L_{\frac{1}{2}\beta_2}R_{\lambda B_2}f(u)\right)\right](u)\\
&= e^{i\lambda(\alpha_1-\beta_1 B_1)\cdot(u-\frac{1}{2}\beta_1)}\left[L_{\frac{1}{2}\beta_1}\left(u\mapsto e^{i\lambda(\alpha_2-\beta_2 B_2+)\cdot(u-\frac{1}{2}\beta_2)}R_{\lambda B_1}L_{\frac{1}{2}\beta_2}R_{\lambda B_2}f(u)\right)\right](u)\\
&= e^{i\lambda(\alpha_1-\beta_1 B_1)\cdot(u-\frac{1}{2}\beta_1)}\\
&\qquad \left[L_{\frac{1}{2}\beta_1}\left(u\mapsto e^{i\lambda(\alpha_2-\beta_2 B_2)\cdot(u-\frac{1}{2}\beta_2)}e^{-i\lambda\beta_2\cdot B_1 u}e^{i\frac{1}{4}\lambda\beta_2\cdot B_1\beta_2}L_{\frac{1}{2}\beta_2}R_{\lambda B_1}R_{\lambda B_2}f(u)\right)\right](u)\\
&= e^{i\lambda(\alpha_1-\beta_1 B_1)(u-\frac{1}{2}\beta_1)}e^{i\lambda(\alpha_2-\beta_2 B_2)(u-\frac{1}{2}\beta_1-\frac{1}{2}\beta_2)}e^{-i\lambda\beta_2\cdot B_1(u-\frac{1}{2}\beta_1)}e^{i\frac{1}{4}\lambda\beta_2\cdot B_1\beta_2}\\
&\qquad [L_{\frac{1}{2}\beta_1}L_{\frac{1}{2}\beta_2}R_{\lambda B_1}R_{\lambda B_2}f](u)\\
&= e^{i\lambda\left[(\alpha_1-\beta_1 B_1)\cdot\frac{1}{2}\beta_2-\frac{1}{4}\beta_2\cdot B_1\beta_2\right]}e^{i\lambda[\alpha_1+\alpha_2+\beta_1 B_2-(\beta_1+\beta_2)(B_1+B_2)]\cdot(u-\frac{1}{2}\beta_1-\frac{1}{2}\beta_2)}\\
&\qquad [L_{\frac{1}{2}(\beta_1+\beta_2)}R_{\lambda(B_1+B_2)}f](u)\\
&= e^{i\lambda\left[(\alpha_1-\beta_1 B_1)\cdot\frac{1}{2}\beta_2-\frac{1}{4}\beta_2\cdot B_1\beta_2\right]}\left[\omega_\Lambda(\beta_1+\beta_2,B_1+B_2,\alpha_1+\alpha_2+\beta_1 B_2)f\right](u)\\
&= \sigma_\lambda\left((\beta_1,B_1,\alpha_1),(\beta_2,B_2,\alpha_2)\right)\left[\omega_\Lambda\left((\beta_1,B_1,\alpha_1)(\beta_2,B_2,\alpha_2)\right)f\right](u),
\end{aligned}
$$

where σ_λ is defined by

$$\sigma_\lambda\left((\beta_1,B_1,\alpha_1),(\beta_2,B_2,\alpha_2)\right)=e^{i\lambda\left[\frac{1}{2}(\alpha_1-\beta_1 B_1)\cdot\beta_2-\frac{1}{4}\beta_2\cdot B_1\beta_2\right]},$$

and it is easy to check that

$$
\begin{aligned}
&\sigma_\lambda\left((\beta_1,B_1,\alpha_1),(\beta_2,B_2,\alpha_2)(\beta_3,B_3,\alpha_3)\right)\sigma_\lambda\left((\beta_2,B_2,\alpha_2),(\beta_3,B_3,\alpha_3)\right)\\
&\quad=\sigma_\lambda\left((\beta_1,B_1,\alpha_1)(\beta_2,B_2,\alpha_2),(\beta_3,B_3,\alpha_3)\right)\sigma_\lambda\left((\beta_1,B_1,\alpha_1),(\beta_2,B_2,\alpha_2)\right).
\end{aligned}
$$

- Case $\Lambda=(0,\nu,0,\sigma)$ with $\nu\neq 0$:

$$
\begin{aligned}
\left[\rho^\beta_\Lambda(x,y,t,s)f\right](u) &= \left[\rho_\Lambda(x,y-G_\beta s,t-\beta\cdot y+\frac{1}{2}\beta\cdot[s]x+\gamma_\beta\cdot e,e)f\right](u)\\
&= e^{i(-\nu\cdot[s](u-x)+\sigma\cdot e)}e^{i\nu\cdot\left(y-G_\beta s\right)}f(u-x)\\
&= e^{i(-\nu\cdot[s](u-x)+\sigma\cdot s)}e^{-i\nu\cdot G_\beta s}e^{i\nu\cdot y}f(u-x)\\
&= e^{i(-\nu\cdot[s](u-x)+\sigma\cdot s)}e^{i\nu\cdot[s]\frac{1}{2}\beta}e^{i\nu\cdot y}f(u-x)\quad\text{by (iii) of Theorem 3.3}\\
&= e^{i(-\nu\cdot[s]\left(u-\frac{1}{2}\beta-x\right)+\sigma\cdot s)}e^{i\nu\cdot y}f\left(u-\frac{1}{2}\beta+\frac{1}{2}\beta-x\right)\\
&= \left[L_{\frac{1}{2}\beta}\rho_\Lambda(x,y,t,s)L_{-\frac{1}{2}\beta}f\right](u)
\end{aligned}
$$

$$
\begin{aligned}
\left[\rho^\alpha_\Lambda(x,y,t,s)f\right](u) &= \left[\rho_\Lambda(x,y,t-\alpha x,s)f\right](u)\\
&= e^{i(-\nu\cdot[s](u-x)+\sigma\cdot s)}e^{i\nu\cdot y}f(u-x)\\
&= \left[\rho_\Lambda(x,y,t,s)\,(f)\right](u)
\end{aligned}
$$

$$
\begin{aligned}
\left[\rho^B_\Lambda(x,y,t,s)f\right](u) &= \left[\rho_\Lambda(x,y-Bx,t,s)f\right](u)\\
&= e^{i(-\nu\cdot[s](u-x)+\sigma\cdot s)}e^{i\nu\cdot(y-Bx)}f(u-x)\\
&= e^{i(-\nu\cdot[s](u-x)+\sigma\cdot s)}e^{-i\nu\cdot Bx}e^{i\nu\cdot y}f(u-x)\\
&= e^{-i\nu\cdot Bu}e^{i(-\nu\cdot[s](u-x)+\sigma\cdot s)}e^{i\nu\cdot y}e^{i\nu\cdot B(u-x)}f(u-x)\\
&= U_{-B\nu}\left[\rho_\Lambda(x,y,t,s)U_{B\nu}f\right](u)
\end{aligned}
$$

So, we define ω_Λ by

$$\omega_\Lambda(\beta, B, \alpha)f(u) = \left[L_{-\frac{1}{2}\beta}U_{B\nu}f\right](u). \tag{24}$$

It is easy to check that

$$\begin{aligned}
&[\omega_\Lambda(\beta_1, B_1, \alpha_1)\omega_\Lambda(\beta_2, B_2, \alpha_2)f](u)\\
&= e^{i\nu\cdot B_1\left(u+\frac{1}{2}\beta_1\right)}e^{i\nu\cdot B_2\left(u+\frac{1}{2}\beta_2\right)}f\left(u+\frac{1}{2}\beta_2+\frac{1}{2}\beta_1\right)\\
&= e^{-i\nu\cdot\frac{1}{2}(B_2\beta_1+B_1\beta_2)}e^{i\nu\cdot(B_1+B_2)\left(u+\frac{1}{2}\beta_1+\frac{1}{2}\beta_2\right)}f\left(u+\frac{1}{2}\beta_2+\frac{1}{2}\beta_1\right)\\
&= e^{-i\frac{1}{2}\nu\cdot(B_2\beta_1+B_1\beta_2)}\omega_\Lambda((\beta_1, B_1, \alpha_1)(\beta_2, B_2, \alpha_2))f(u)\\
&= \sigma_\nu((\beta_1, B_1, \alpha_1), (\beta_2, B_2, \alpha_2))\omega_\Lambda((\beta_1, B_1, \alpha_1)(\beta_2, B_2, \alpha_2))f(u),
\end{aligned}$$

where σ_ν is a multiplier and it is defined by

$$\sigma_\nu((\beta_1, B_1, \alpha_1), (\beta_2, B_2, \alpha_2)) = e^{-i\frac{1}{2}\nu\cdot(B_2\beta_1+B_1\beta_2)}.$$

- Case $\Lambda = (\mu, 0, 0, \sigma)$:

Since $\mathcal{O}_\Lambda = \{(\mu, 0, 0, \sigma)\}$ and K fixes the orbits, by (22) $\rho_\Lambda^k = \rho_\Lambda$ for all $k \in K$ then, ω_Λ is the identity.

Theorem 4.2. *(K, N) is a generalized Gelfand pair.*

Proof. By Theorem 1.3, it is enough to see that ω_Λ is multiplicity free. In fact, note that if $V \subset L^2(\mathbb{R}^n)$ is a non trivial closed subspace, invariant by L_β and U_α for all $\beta, \alpha \in \mathbb{R}^n$ then, $V = L^2(\mathbb{R}^n)$ (see [7], page 160).

Thus, for $\Lambda = (0, 0, \lambda, \sigma)$ or $\Lambda = (0, \nu, 0, \sigma)$ we have that ω_Λ is an irreducible representation. Finally, for $\Lambda = (\mu, 0, 0, \sigma)$ the result is trivial. ∎

5. Proof of technical results

Proof of Theorem 3.3 item (i).

By (9) we have that for all i, j

$$\begin{gathered}
s_\theta(\mathbf{e}_i)\cdot\pi_\theta(\mathbf{e}_j) = s_\theta(\mathbf{e}_j)\cdot\pi_\theta(\mathbf{e}_i)\\
\left(\sum_{l=1}^{n}\sum_{m=l}^{n}c_{lm,i}\mathbf{e}_{lm}\right)\cdot\left(\sum_{k=1}^{n}a_{k,j}\mathbf{e}_k\right) = \left(\sum_{l=1}^{n}\sum_{m=l}^{n}c_{lm,j}\mathbf{e}_{lm}\right)\cdot\left(\sum_{k=1}^{n}a_{k,i}\mathbf{e}_k\right)\\
\sum_{l=1}^{n}\sum_{m=l}^{n}\sum_{k=1}^{n}c_{lm,i}a_{k,j}\mathbf{e}_{lm}\cdot\mathbf{e}_k = \sum_{l=1}^{n}\sum_{m=l}^{n}\sum_{k=1}^{n}c_{lm,j}a_{k,i}\mathbf{e}_{lm}\cdot\mathbf{e}_k\\
\sum_{l=1}^{n}\sum_{m=l}^{n}c_{lm,i}a_{l,j}\mathbf{f}_m + \sum_{l=1}^{n}\sum_{m=l+1}^{n}c_{lm,i}a_{m,j}\mathbf{f}_l = \sum_{l=1}^{n}\sum_{m=l}^{n}c_{lm,j}a_{l,i}\mathbf{f}_m + \sum_{l=1}^{n}\sum_{m=l+1}^{n}c_{lm,j}a_{m,i}\mathbf{f}_l\\
\sum_{m=1}^{n}\sum_{l=1}^{m}c_{lm,i}a_{l,j}\mathbf{f}_m + \sum_{l=1}^{n}\sum_{m=l+1}^{n}c_{lm,i}a_{m,j}\mathbf{f}_l = \sum_{m=1}^{n}\sum_{l=1}^{m}c_{lm,j}a_{l,i}\mathbf{f}_m + \sum_{l=1}^{n}\sum_{m=l+1}^{n}c_{lm,j}a_{m,i}\mathbf{f}_l
\end{gathered}$$

$$\sum_{l=1}^{n}\sum_{m=1}^{l} c_{ml,i}a_{m,j}\mathbf{f}_l + \sum_{l=1}^{n}\sum_{m=l+1}^{n} c_{lm,i}a_{m,j}\mathbf{f}_l = \sum_{l=1}^{n}\sum_{m=1}^{l} c_{ml,j}a_{m,i}\mathbf{f}_l + \sum_{l=1}^{n}\sum_{m=l+1}^{n} c_{lm,j}a_{m,i}\mathbf{f}_l$$

$$\sum_{l=1}^{n}\left(\sum_{m=1}^{l} c_{ml,i}a_{m,j} + \sum_{m=l+1}^{n} c_{lm,i}a_{m,j}\right)\mathbf{f}_l = \sum_{l=1}^{n}\left(\sum_{m=1}^{l} c_{ml,j}a_{m,i} + \sum_{m=l+1}^{n} c_{lm,j}a_{m,i}\right)\mathbf{f}_l$$

Then,

$$\sum_{m=1}^{l} c_{ml,i}a_{m,j} + \sum_{m=l+1}^{n} c_{lm,i}a_{m,j} = \sum_{m=1}^{l} c_{ml,j}a_{m,i} + \sum_{m=l+1}^{n} c_{lm,j}a_{m,i}, \qquad \text{for all } l = 1,\dots,n$$

So, for all $l = 1,\dots,n$ we have that

$$C^l[\cdot,i]\cdot A[\cdot,j] = C^l[\cdot,j]\cdot A[\cdot,i] \qquad \forall i,j$$

That is, $C^l A^t$ is a symmetric matrix. ■

Proof of Lemma 3.5. If $1 \le i < j \le n$ then

$$\begin{aligned}
(\gamma_{\beta+\beta'})_{ij} &= -\frac{1}{2}(\beta_i+\beta'_i)(\beta_j+\beta'_j)\\
&= -\frac{1}{2}(\beta_i\beta_j+\beta'_i\beta_j+\beta_i\beta'_j+\beta'_i\beta'_j)\\
&= (\gamma_\beta)_{ij} + (\gamma_{\beta'})_{ij} - \frac{1}{2}(\beta_i\beta'_j+\beta_j\beta'_i)\\
&= (\gamma_\beta)_{ij} + (\gamma_{\beta'})_{ij} + \beta_i(G_{\beta'})_{i,ij} + \beta_j(G_{\beta'})_{j,ij}\\
&= (\gamma_\beta)_{ij} + (\gamma_{\beta'})_{ij} + \sum_{k=1}^{n}\beta_k(G_{\beta'})_{k,ij}\\
&= (\gamma_\beta)_{ij} + (\gamma_{\beta'})_{ij} + (\beta G_{\beta'})_{ij}
\end{aligned}$$

and

$$\begin{aligned}
(\gamma_{\beta+\beta'})_{ii} &= -\frac{1}{4}(\beta_i+\beta'_i)(\beta_i+\beta'_i)\\
&= -\frac{1}{4}(\beta_i\beta_i+\beta'_i\beta_i+\beta_i\beta'_i+\beta'_i\beta'_i)\\
&= (\gamma_\beta)_{ii} + (\gamma_{\beta'})_{ii} - \frac{1}{2}(\beta_i\beta'_i)\\
&= (\gamma_\beta)_{ii} + (\gamma_{\beta'})_{ii} + \beta_i(G_{\beta'})_{i,ii}\\
&= (\gamma_\beta)_{ii} + (\gamma_{\beta'})_{ii} + \sum_{k=1}^{n}\beta_k(G_{\beta'})_{k,ii}\\
&= (\gamma_\beta)_{ii} + (\gamma_{\beta'})_{ii} + (\beta G_{\beta'})_{ii}
\end{aligned}$$ ■

Proof of Lemma 4.1.

$$\begin{aligned}
\gamma_\beta\cdot s + \frac{1}{4}\beta\cdot s\beta &= \sum_{i\le j}\gamma_{ij}s_{ij} + \frac{1}{4}\sum_{i=1}^{n}\beta_i([s]\beta)_i\\
&= \sum_{i=1}^{n}\gamma_{ii}s_{ii} + \sum_{i<j}\gamma_{ij}s_{ij} + \frac{1}{4}\sum_{i=1}^{n}\beta_i([s]\beta)_i
\end{aligned}$$

$$= -\frac{1}{4}\sum_{i=1}^{n}\beta_i\beta_i s_{ii} - \frac{1}{2}\sum_{i<j}\beta_i\beta_j s_{ij} + \frac{1}{4}\sum_{i=1}^{n}\beta_i\sum_{j=1}^{n}s_{ij}\beta_j$$
$$= 0$$ ∎

References

[1] F. Astengo, B. Di Blasio, F. Ricci: *Gelfand transforms of polyradial functions on the Heisenberg group*, J. Funct. Anal. 251 (2007) 772–791.

[2] C. Benson, J. Jenkins, G. Ratcliff: *On Gelfand pairs associated with solvable Lie groups*, Trans. Amer. Math. Soc. 321 (1990) 85–116.

[3] C. Benson, J. Jenkins, G. Ratcliff: *Bounded K-spherical function on Heisenberg group*, J. Funct. Anal. 105 (1992) 409–443.

[4] C. Benson, J. Jenkins, G. Ratcliff: *The orbit method and Gelfand pairs associated with nilpotent Lie groups*, J. Geom. Anal. 9 (1999) 569–582.

[5] S. Campos, J. García, L. Saal: *Generalized Gelfand pairs associated to m-step nilpotent Lie groups*, J. Geom. Anal. 33 (2023, art.no. 54.

[6] S. Campos, J. García, L. Saal: *Spherical analysis attached to some m-step nilpotent Lie group*, J. Fourier Analysis Appl. 30 (2024), art.no. 20.

[7] A. Deitmar: *A First Course in Harmonic Analysis*, Universitext, Springer, Berlin (2005).

[8] J. Faraut: *Distributions sphériques sur les espaces hyperboliques*, J. Math. Pures Appl. 58 (1979) 369–444.

[9] V. Fischer, F. Ricci, O. Yakimova: *Nilpotent Gelfand pairs and spherical transforms of Schwartz functions. III: Isomorphisms between Schwartz spaces under Vinberg condition*, arXiv 1210.7962 (2012).

[10] A. Gallo, L. Saal: *A generalized Gelfand pair attached to a 3-step nilpotent Lie group*, J. Fourier Anal. Appl. 26/4 (2020), art.no. 62, 11p.

[11] A. A. Kirillov: *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys 17 (1962) 53–104.

[12] T. Kobayashi: *Multiplicity free representations and visible actions on complex manifolds*, Publ. RIMS Kyoto Univ. 41 (2005) 497–549.

[13] J. Lauret: *Gelfand pairs attached to representations of compact Lie groups*, Transform. Groups 5 (2000) 307–324.

[14] G. W. Mackey: *Unitary Group Representations in Physics, Probability, and Number Theory*, Mathematics Lecture Note Series Vol. 55, Benjamin/Cummings, Reading (1978).

[15] K. Mokni, E. G. F. Thomas: *Paires de Guelfand généralisées associées au groupe d'Heisenberg*, J. Lie Theory 8 (1998) 325–334.

[16] E. G. F. Thomas: *The theorem of Bochner Schwartz Godement for generalized Gelfand pairs*, in: *Functional Analysis: Surveys and Recent Results III*, Proc. Conf., Paderborn 1983, North-Holland Mathematical Studies Vol. 90, Elsevier, Amsterdam (1984) 291–304.

[17] G. Van Dijk: *Group representations on spaces of distributions*, Russian J. Math. Phys. 2 (1994) 57–68.

[18] G. Van Dijk: *Introduction to Harmonic Analysis and Generalized Gelfand Pairs*, Studies in Mathematics Vol. 36, De Gruyter, Berlin (2009).

[19] E. B. Vinberg: *Commutative homogeneous spaces and co-isotropic symplectic actions*, Russian Math. 56 (2001) 1–60.

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