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Semigroups in Semi-Simple Lie Groups: Flag Type and Estimation of Cocycles

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Abstract. The flag type of a semigroup S of a noncompact semisimple Lie group is an algebraic tool related to the geometry of the invariant control set determined by S on the flag manifolds of G. In the present paper, we show that it is possible to recover the flag type by studying the existence of lower bounds for cocycles on the maximal flag manifold.

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1. Introduction

Let G be a noncompact semisimple Lie group G with a finite center and $S \subset G$ a subsemigroup with nonempty interior. The purpose of this paper is to characterize the *flag type* of S by means of lower bounds of cocycles over the flag manifolds of G.

The concept of the flag type of a semigroup has proved to be valuable for the study of semigroups in semisimple Lie groups (see [12, 15, 20, 18]) and its applications to dynamical systems and harmonic analysis (see [10, 11, 13, 19, 17]). It appears in the study of the action of a semigroup S on the flag manifolds of G. Precisely, we have the following result, proved in [20, Theorem 4.3], ensuring its existence and uniqueness (see also [16, 14, 15]).

Theorem / Definition 1.1 *Let $S \subset G$ be a proper semigroup with* $\mathrm{int} S \neq \emptyset$*. Then there exists a unique flag manifold $\mathbb{F}_{\Theta(S)}$ satisfying the following two conditions:*

1. *$\mathbb{F}_{\Theta(S)}$ is maximal among the flag manifolds $\mathbb{F}_\Theta$ such that the unique S-invariant control set $C_\Theta \subset \mathbb{F}_\Theta$ is contractible in the sense that there exists $g \in \mathrm{int} S$ such that $g^n C_\Theta$ shrinks to a point as $n \to +\infty$.*
2. *$\mathbb{F}_{\Theta(S)}$ is minimal among the flag manifolds $\mathbb{F}_\Theta$ such that $C = \pi^{-1}(C_\Theta)$ is the invariant control set in the maximal flag manifold $\mathbb{F}$.*

The flag manifold $\mathbb{F}_{\Theta(S)}$ and the subset of roots $\Theta(S)$ are called, indistinguishably, the flag type *of the semigroup S.*

All the conditions for the flag type of S use in an essential way that S has a nonempty interior.

In this paper we prove another equivalent condition that relates the flag type to lower bounds of cocycles. The statement of this condition does not require in advance that

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the semigroup have a nonempty interior and hence makes sense for other classes of semigroups.

Let $G = KAN$ be an Iwasawa decomposition. By this decomposition, the minimal parabolic subgroup is $P = MAN$ where M is the centralizer of A in K and $\mathbb{F} = G/P = K/M$. The K-invariant cocycles over the flag manifolds are defined after this decomposition as follows: Define the map $\rho : G \times K \to A$ by

$$gu = k\rho\left(g, u\right) n \in KAN$$

and put $\mathsf{a}\left(g, u\right) = \log \rho\left(g, u\right) \in \mathfrak{a}$. These maps are right M-invariant in the second variable; hence, they factor to maps (with the same notation) $\rho : G \times \mathbb{F} \to A$ and $\mathsf{a} : G \times \mathbb{F} \to \mathfrak{a}$. For $\lambda \in \mathfrak{a}^{*}$ we write $\mathsf{a}_{\lambda}\left(g, x\right) = \lambda \mathsf{a}\left(g, x\right)$ and $\rho_{\lambda}\left(g, x\right) = \mathrm{e}^{\mathsf{a}_{\lambda}(g,x)}$.

Next we state the main result of this paper.

Theorem 3.1 *Let $S \subset G$ be a semigroup with $\mathrm{int}S \neq \emptyset$ and write $\mathbb{F}_{\Theta(S)}$, $\Theta\left(S\right) \subset \Sigma$, for its flag type. Denote by C the invariant control set of S in the maximal flag manifold $\mathbb{F}$. If $x_0 \in C_0$, then*

$$\inf_{g \in S} \rho_{\alpha}\left(g, x_0\right) > 0$$

for any $\alpha \in \Sigma \setminus \Theta\left(S\right)$. Conversely, $\inf_{g \in S} \rho_{\alpha}\left(g, x_0\right) = 0$ if $\alpha \in \Theta\left(S\right)$.

Let us mention that the existence of a positive lower bound for cocycles on flag manifolds has already been used in [17] in the study of the moment Lyapunov exponents of the i.i.d. random product on G defined by a probability measure. This leads to several important results concerning the spectral radii of compact operators on Banach spaces. Another importance of the previous result is that for the statement of the conditions in this theorem, it is not required that the semigroup have a nonempty interior. The condition can be stated for any semigroup that has a unique invariant control set in $\mathbb{F}$. Thus, apart from its intrinsic interest for the structure of the semigroups with nonempty interiors, the above theorem can open the way to the understanding of semigroups in a more general context. We have in mind the completely irreducible and contracting semigroups that arise in the study of random products in semi-simple Lie groups (see Guivarc'h-Raugi [5], Abels-Margulis-Soifer [1], and Gol'dsheid-Margulis [4]). In case G is an algebraic group, irreducibility plus contractibility is equivalent to saying that the semigroup is Zariski dense.

The paper is structured as follows: In Section 2 we obtain lower bounds for the cocycles where the associated functional belongs to the cone generated by fundamental weights outside the flag type. In particular, a very useful result for rank-one groups is obtained. Such a result allows us to restrict the proof of our main result to the fibers of some specific fibration between flag manifolds. We finish the section analyzing when such a lower bound is uniform. By a concrete example, we show that uniformity does not hold in general; however, one can find a coset of the semigroup where uniformity holds.

In Section 3, we outline and demonstrate our primary findings regarding the flag type with the lower bounds of cocycles on the flag manifolds. We start by showing that the cocycles associated with roots in the flag type of the semigroup admit no lower bound. In consequence, we look at roots outside the flag type. By looking at a fibration of the maximal flag onto the flag associated with the values of the cocycle, for a fixed point in the core of the invariant control set, coincide with a cocycle defined on the invariant control set of a rank-one group. This fact, together with the results in Section 3, allows us to obtain the desired lower bound. In order to make the paper self-contained and more fluent, an appendix is available, where we introduce the basic results and notations related to the semisimple theory. We also use such a section to recall alternative definitions of the flag type of semigroups and the cocycles induced by the Iwasawa decomposition on the flag manifolds.

2. Lower bounds of cocycles

In this section we present a series of lemmas that will help us to prove our main results. Although some of these lemmas were already proved in [17], in order to keep the paper self-contained, we reproduce their proof here again.

Let us consider $S \subset G$ as a semigroup with $\mathrm{int}S \neq \emptyset$ and denote by C the unique invariant control set in the maximal flag manifold. In a partial flag manifold $\mathbb{F}_\Theta$ the invariant control set is denoted by C_Θ.

For x in a control set, we write $S_x = \{g \in S : gx = x\}$ and $S_x^\circ = \{g \in \mathrm{int}S : gx = x\}$. The core of a control set is the subset of those x such that $S_x^\circ \neq \emptyset$. The core of the control set C_Θ is denoted by $(C_\Theta)_0$.

The proof of the lower bound in Theorem 3.1 is based on the following lemma that reduces the estimate to the isotropy group.

Lemma 2.1. *Let S be a semigroup with $\mathrm{int}S \neq \emptyset$ and denote by C its invariant control set in the maximal flag manifold $\mathbb{F} = G/P$. Take $x_0 \in C_0$ and $\lambda \in \mathfrak{a}^*$ and suppose that there exists $d > 0$ such that $\rho_\lambda(g, x_0) > d$ for all $g \in S_{x_0}^\circ$.*
Then there exists $c > 0$ such that $\rho_\lambda(g, x_0) > c$ for all $g \in S$.

Proof. Suppose by contradiction the existence of a sequence $g_k \in S$ with $\rho_\lambda(g_k, x_0) \to 0$. Since C is a compact and invariant subset, it can be assumed that $g_k x_0 \to y$ with $y \in C$. If $g \in S$, then

$$\rho_\lambda(gg_k, x_0) = \rho_\lambda(g, g_k \cdot x_0)\rho_\lambda(g_k, x_0) \to 0,$$

because the function $z \mapsto \rho_\lambda(g, z)$ is bounded. Moreover, the fact that $x_0 \in C_0$ and $C \subset \mathrm{cl}(Sy)$ implies that

$$\emptyset \neq (\mathrm{int}S)^{-1}x_0 \cap Sy \quad \Longrightarrow \quad gy = x_0, \text{ for some } g \in \mathrm{int}S.$$

Hence, we can substitute g_k by gg_k and assume that $g_k \in \mathrm{int}S$ and $g_k x \to x_0$. Since $x_0 \in C_0$, the semigroup $S_{x_0}^\circ$ is not empty. Take $g_0 \in S_{x_0}^\circ$ and a compact neighborhood W of g_0 in $\mathrm{int}S$. We have that $U = W^{-1}x_0$ is a neighborhood of x_0 in $\mathbb{F}$ because $g_0 x_0 = x_0$. By construction, for every $z \in U$ there exists $h \in W$ such that $x_0 = hz$. Write

$$r = \sup\{\rho_\lambda(h, z) : h \in W, z \in \mathbb{F}\},$$

which is finite by compactness.

Now, let k be large enough so that $g_k x_0 \in U$ and $\rho_\lambda(g_k, x_0) < d/2r$.

Then there exists $h \in W$ such that $hg_k x_0 = x_0$, and we have

$$\rho_\lambda (hg_k, x_0) = \rho_\lambda(h, g_k x_0)\rho_\lambda(g_k, x_0) \leq r\rho_\lambda (g_k, x_0) < \frac{rd}{2r} = \frac{d}{2},$$

contradicting the assumption. ■

To get a first application of Lemma 2.1, we introduce the following notation: Write the simple system of roots as $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ and let $\Phi = \{\mu_1, \ldots, \mu_l\} \subset \mathfrak{a}^*$ be the set of corresponding fundamental weights (see Appendix A). Take $\Theta \subset \Sigma$ with $\Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_j}\}$ and let $\Phi_\Theta = \{\mu_{i_1}, \ldots, \mu_{i_j}\}$ be the set of fundamental weights with the same indices as those in Θ. Equivalently

$$\Phi \setminus \Phi_\Theta = \{\mu \in \Phi : \forall \alpha \in \Theta,\ \langle \alpha, \mu \rangle = 0\}.$$

Denote by $(\mathfrak{a}^*_\Theta)^+$ the "partial chamber"

$$(\mathfrak{a}^*_\Theta)^+ = \{\beta \in \mathfrak{a}^*_\Theta : \forall \alpha \in \Sigma \setminus \Theta,\ \langle \alpha, \beta \rangle > 0\},$$

which is the interior (in $\mathfrak{a}^*_\Theta$) of the convex cone $\mathrm{cl}\,(\mathfrak{a}^*_\Theta)^+$ spanned by $\Phi \setminus \Phi_\Theta$.
Now assume without loss of generality that $A^+ \cap \mathrm{int} S \neq \emptyset$ where $A^+ = \exp \mathfrak{a}^+$ (see [14, Lemma 3.2]). This assumption implies that origin x_0 of $\mathbb{F} = G/MAN$ belongs to the core C_0 of the invariant control set C. The convex cone

$$\Gamma_N = \{H \in \mathfrak{a} : \exists n \in N, \exists t > 0,\ e^{tH} n \in \mathrm{int} S\},$$

was considered in [20, Section 4]. The following inclusion was proved there.

Proposition 2.2. *Let $\mathbb{F}_\Theta$, $\Theta \subset \Sigma$ be the flag type of S. Then*

$$\Gamma_N \subset \Gamma_\Theta = \mathrm{cl}\Big(\bigcup_{w \in \mathcal{W}_\Theta} w\mathfrak{a}^+ \Big). \tag{1}$$

Moreover, $\Gamma_N \cap w\mathfrak{a}^+ \neq \emptyset$ for every $w \in \mathcal{W}_\Theta$.

Remark 2.3. In [20], it is proved that if the inclusion fails, then there are $w \notin \mathcal{W}_\Theta$, $H \in w\mathfrak{a}^+$ and $n \in N$ such that $g = e^H n \in \mathrm{int}\, S$. But this implies that x_0 is a fixed point of type w of g. Since $w \notin \mathcal{W}_\Theta$ this contradicts the fact that Θ is the flag type of S.

From the inclusion (1) we can prove the following estimate: for elements in $\mathrm{int}\, S$ fixing a point $x_0 \in \mathbb{F}$.

Lemma 2.4. *Suppose that $g \in \mathrm{int}\, S$ is such that $gx_0 = x_0$, and take $\lambda \in (\mathfrak{a}^*_\Theta)^+$, where Θ is the flag type of S. Then $\rho_\lambda(g, x_0) \geq 1$.*

Proof. If $gx_0 = x_0$, then we can write $g = man \in P = MAN$, in which case $\rho_\lambda (g, x_0) = e^{\lambda(H)}$ where $a = e^H$. We claim that $H \in \Gamma_\Theta$. In fact, since the elements of finite order are dense in compact groups, we can perturb g inside $\mathrm{int}\, S \cap P$ and assume that m has finite order. Then, since N normalizes A, it holds that $g^j = a^j \overline{n} = \mathrm{e}^{jH} \overline{n} \in AN$ for some $j \geq 1$. Therefore, $jH \in \Gamma_N \subset \Gamma_\Theta$, implying that $H \in \Gamma_\Theta$.

On the other hand, by the definition of the fundamental weights (see Appendix A), for any $\mu \in \Phi \setminus \Phi_\Theta$, $\alpha \in \Theta$, and $H' \in \mathfrak{a}$, it holds that $\mu(r_\alpha(H')) = \mu(H')$. Hence,

$$\mu(w(H')) = \mu(H'), \qquad \forall w \in \mathcal{W}_\Theta, \mu \in \Phi \setminus \Phi_\Theta, H' \in \mathfrak{a}.$$

Since $H \in \Gamma_\Theta$, there exists $H' \in \mathrm{cl}(\mathfrak{a}^+)$ and $w \in \mathcal{W}_\Theta$ satisfying $H = wH'$, which implies that

$$\mu(H) = \mu(wH') = \mu(H') \geq 0, \qquad \forall \mu \in \Phi \setminus \Phi_\Theta.$$

Finally, the fact that $\mathrm{cl}\,(\mathfrak{a}_\Theta^*)^+$ is generated by $\Phi \setminus \Phi_\Theta$ allows us to conclude that $\lambda(H) \geq 0$, showing that $\rho_\lambda\,(g, x_0) = \mathrm{e}^{\lambda(H)} \geq 1$ as claimed. ■

Combining this lemma with Lemma 2.1, we get at once the following estimate of cocycles.

Proposition 2.5. *Let S be a semigroup whose flag type is $\mathbb{F}_\Theta$, $\Theta \subset \Sigma$. Denote by C its invariant control set in the maximal flag manifold $\mathbb{F}$. Take $\lambda \in (\mathfrak{a}_\Theta^*)^+$ and $x \in C_0$. Then there exists $c > 0$ such that $\rho_\lambda\,(g, x) > c$ for all $g \in S$.*

This proposition will be applied later to get the lower bound estimate of Theorem 3.1. For this application, it is required only the following specialization to real rank one groups.

Corollary 2.6. *Suppose that $\mathfrak{g}$ (and G) has real rank one and let $S \subset G$ be a proper semigroup with $\mathrm{int}\,S \neq \emptyset$. Denote by α (and eventually 2α) the positive root. Then $\inf_{g \in S} \rho_\alpha\,(g, x) > 0$ if $x \in C_0$.*

Proof. In the rank one case, there is only the maximal flag manifold $\mathbb{F}$, and if S is proper, then its flag type is $\mathbb{F} = \mathbb{F}_\emptyset$ itself. The subspace $\mathfrak{a}_\Theta^*$ is one-dimensional, and $(\mathfrak{a}_\Theta^*)^+$ is the ray containing α. Hence α falls in the condition of the above proposition so that $\rho_\alpha\,(g, x)$ is lower bounded. ■

The following example illustrates the corollary with a concrete semigroup in $\mathrm{Sl}\,(2, \mathbb{R})$.

Example 2.7. For $\mathrm{Sl}\,(2, \mathbb{R})$ the only flag manifold is the projective line $\mathbb{P}^1$. The cocycle ρ_λ over $\mathbb{P}^1$ is defined by the relation

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 1,$$

and hence, $\rho_\lambda\,(g, [z]) = \|gz\| \,/\, \|z\|$, $0 \neq z \in \mathbb{R}^2$. Consider the cone

$$W = \{(a, b) \in \mathbb{R}^2 : a \geq 0,\ |b| \leq a\},$$

and define the semigroup $S_W = \{g \in \mathrm{Sl}\,(2, \mathbb{R}) : gW \subset W\}$. The core of the invariant control set for the action of S_W in $\mathbb{P}^1$ is

$$C_0 = \{[(a, b)] \in \mathbb{P}^1 : (a, b) \in \mathrm{int} W\}.$$

Let $g \in S_W$ satisfy $g\,(1, 0) = (a, b) \in W$. Then, for

$$h = \begin{pmatrix} 1 & 0 \\ b/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},$$

it holds that $g\,(1, 0) = h\,(1, 0)$.

Therefore, $h^{-1}g\,(1,0) = (1,0)$ implying that $h^{-1}g$ is upper triangular, and hence, g has the form

$$g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

with $\mu > 0$ and $|y| \leq 1$. Since $g\,(1,-1) \in W$ and $g\,(1,1) \in W$, it holds that

$$1 - x \geq 0, \quad |\mu y\,(1-x) - \mu^{-1}| \leq \mu\,(1-x)\,,$$

$$1 + x \geq 0 \quad \text{and} \quad |\mu y\,(1-x) + \mu^{-1}| \leq \mu\,(1+x)\,.$$

Hence $$|x| \leq 1 \quad \text{and} \quad \mu^{-1} - \mu|y|\,(1-x) \leq \mu\,(1-x)\,,$$

that is, $\mu^{-1} \leq \mu\,(1-x)\,(1+|y|)$. But $|y| \leq 1$ and $1 - x \leq 2$ so that $1 \leq 4\mu^2$ and $\mu \geq 1/2$. Since $g\,(1,0) = \mu\,(1,y)$, we get the lower bound

$$\|g\,(1,0)\| = \mu\sqrt{1+y^2} \geq \frac{1}{2}.$$

Now if $z = [(a,b)] \in C_0$, there exists $h \in S_W$ satisfying $z = hz_0$, where $z_0 = [(1,0)]$. Consequently, if $g \in S_W$, then $gh \in S_W$ so that

$$\rho_\lambda\,(g,z) = \frac{\|ghz_0\|}{\|hz_0\|} \geq \frac{\|ghz_0\|}{\|h\|} \geq \frac{1}{2\,\|h\|}\,,$$

showing that $c = 1/2\,\|h\|$ is the desired lower bound for $z = hz_0 \in C_0$. Notice that the lower bound $c = 1/2\,\|h\|$ depends on z through $\|h\|$ so that it is not uniform in z. Actually, a uniform lower bound (in C_0) cannot be obtained. In fact

$$h_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

belongs to S_W for all $t \in \mathbb{R}$. If $z = [(a,b)] \in C_0$ then

$$\rho_\lambda\,(h_t,z) = \frac{(a^2+b^2)\cosh 2t + 2ab\sinh 2t}{a^2+b^2}$$

which for fixed t converges to e^{-2t} as (a,b) approaches $(1,-1)$. This shows that a uniform lower bound does not necessarily exist.

The previous example shows that the lower bound c in Proposition 2.5 could depend on x and may not be uniform. Despite this fact, the next proposition shows that a uniform lower bound can be found for some coset Sh.

Proposition 2.8. *Under the assumptions of Proposition* 2.5, *there are* $h \in S$ *and* $c > 0$ *such that* $\rho_\lambda\,(g,y) > c$ *for all* $g \in Sh$ *and* $y \in C$.

Proof. Let $x_0 \in C_0$ and consider $g_0 \in \operatorname{int} S$ such that $g_0 x_0 = x_0$. Let $U \subset \operatorname{int} S$ be a compact neighborhood of g_0 and write $W = U\xi_0$, where $\xi_0 := \pi(x_0)$, with $\pi : \mathbb{F} \to \mathbb{F}_\Theta$ the canonical projection. Note that W is a compact neighborhood of ξ_0 in $\mathbb{F}_\Theta$, since

$$\xi_0 = \pi(x_0) = \pi(g_0 x_0) = g_0\pi(x_0) = g_0\xi_0 \in W.$$

Moreover, $$x_0 \in C_0 \quad \Longrightarrow \quad \xi_0 \in (C_\Theta)_0 \quad \Longrightarrow \quad W \subset (C_\Theta)_0,$$

where the last inclusion follows by the S-invariance of $(C_\Theta)_0$ inside C_Θ (see Appendix A, Section A.1).

By Theorem 1.1, there exists $n_0 \in \mathbb{N}$ such that $h := g_0^{n_0}$ satisfies $hC_\Theta \subset W$ and hence $\pi^{-1}(W)$ is a compact neighborhood of x_0 satisfying,

$$hC = h\pi^{-1}(C_\Theta) = \pi^{-1}(hC_\Theta) \subset \pi^{-1}(W) \subset \pi^{-1}(C_\Theta) = C,$$

where the last equality follows from Proposition A.1. Since $\lambda \in (\mathfrak{a}_\Theta^*)^+$, we get by Lemma A.3 and Proposition 2.5 that

$$\exists c_1 > 0, \quad \rho_\lambda(g, ux_0) = \rho_\lambda(g, x_0) > c_1, \quad \text{for all} \quad g \in S, u \in K_\Theta.$$

Therefore, for any $g' \in U$ and $u \in K_\Theta$, we get

$$\rho_\lambda\left(gg', ux_0\right) = \rho_\lambda\left(g, g'ux_0\right)\rho_\lambda\left(g', ux_0\right),$$

or equivalently, $$\rho_\lambda\left(g, g'ux_0\right) = \rho_\lambda\left(gg', ux_0\right)\rho_\lambda\left(g', ux_0\right)^{-1}.$$

The first factor on the right-hand side of the previous equality is estimated by $\rho_\lambda\left(gg', ux_0\right) > c_1$, since $gg' \in \operatorname{int} S$. The second factor satisfies

$$\rho_\lambda\left(g', ux_0\right)^{-1} = \rho_\lambda\left(g', x_0\right)^{-1} \geq M^{-1}, \quad \text{where} \quad M := \max_{g' \in U} \rho_\lambda\left(g', x_0\right).$$

Since $\pi^{-1}(W) = U\pi^{-1}(\xi_0) = UK_\Theta x_0$, we conclude that

$$\rho_\lambda\left(g, y\right) > \frac{c_1}{M} > 0 \qquad y \in \pi^{-1}(W),\ g \in S.$$

Therefore, by considering $R = \min_{y\in\mathbb{F}} \rho_\lambda\left(h, y\right) > 0$, we conclude that

$$\rho_\lambda\left(gh, y\right) = \rho_\lambda\left(g, hy\right)\rho_\lambda\left(h, y\right) > \frac{c_1 R}{M} > 0,$$

proving the result. ■

3. The main result

In this section we prove our main result, namely, we prove the following:

Theorem 3.1. *Let $S \subset G$ be a semigroup with $\operatorname{int} S \neq \emptyset$ and write $\mathbb{F}_{\Theta(S)}$, $\Theta(S) \subset \Sigma$, for its flag type. Denote by C the invariant control set of S in the maximal flag manifold $\mathbb{F}$. If $x_0 \in C_0$, then*

$$\inf_{g\in S} \rho_\alpha\left(g, x_0\right) > 0$$

for any $\alpha \in \Sigma \setminus \Theta(S)$. Conversely, $\inf_{g\in S} \rho_\alpha\left(g, x_0\right) = 0$ if $\alpha \in \Theta(S)$.

The proof of our main result is divided into the next two sections.

3.1. Nonexistence of lower bound for $\alpha \in \Theta(S)$

We start by showing that the cocycle ρ_α, for roots α inside the type flag of S, does not admit a lower bound, that is,

$$\inf_{g\in S} \rho_\alpha\left(g, x_0\right) = 0,$$

if $\alpha \in \Theta(S)$ and $x_0 \in C_0$.

This is an immediate consequence of Proposition 2.2. In fact, let us assume w.l.o.g. that $x_0 \in C_0$ is the origin of $\mathbb{F}$ and assume further that $A^+ \cap \operatorname{int} S \neq \emptyset$. By Proposition 2.2 it holds that $\Gamma_N \cap r_\alpha \mathfrak{a}^+ \neq \emptyset$ and hence, there exists $H \in r_\alpha \mathfrak{a}^+$, $t > 0$, and $n \in N$ such that $g = \mathrm{e}^{tH} n \in \operatorname{int} S$ (see Remark 2.3).

Since $\alpha(H) < 0$, we conclude that $\rho_\alpha(g, x_0) = \mathrm{e}^{t\alpha(H)} < 1$, implying that

$$\rho_\alpha\left(g^k, x_0\right) = \rho_\alpha(g, x_0)^k \to 0 \qquad \text{as} \qquad k \to +\infty.$$

Therefore, $\rho_\alpha(g, x_0)$ admits no positive lower bound in S when $\alpha \in \Theta(S)$.

3.2. Simple root outside $\Theta(S)$

We apply now the results from Section 3 to get a lower bound for $\rho_\alpha(g, x)$, $g \in S$, $x \in C_0$ when $\alpha \in \Sigma \setminus \Theta(S)$. The main idea is to reduce our problem to a rank one group associated with α and apply Corollary 2.6. For this purpose we exploit the fibration $\mathbb{F} \to \mathbb{F}_\alpha$ over the flag manifold defined by α.

Let $\mathfrak{g}(\alpha)$ be the real rank one subalgebra generated by $\mathfrak{g}_{\pm\alpha}$ and write $G(\alpha)$ for the connected subgroup having Lie algebra $\mathfrak{g}(\alpha)$. This subalgebra is the direct sum of $\mathfrak{a}(\alpha) = \operatorname{span}\{H_\alpha\}$ and the root spaces $\mathfrak{g}_{\pm\alpha}$ and $\mathfrak{g}_{\pm 2\alpha}$ (eventually 2α is not a root, in which case $\mathfrak{g}_{\pm 2\alpha} = \{0\}$). Consider the Iwasawa decomposition $\mathfrak{g}(\alpha) = \mathfrak{k}(\alpha) \oplus \mathfrak{a}(\alpha) \oplus \mathfrak{n}(\alpha)$ with $\mathfrak{k}(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{k}$ and $\mathfrak{n}(\alpha) = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. It defines the global decomposition $G(\alpha) = K(\alpha)A(\alpha)N(\alpha)$ by taking exponentials of the Lie algebra components. The restriction of α to $\mathfrak{a}(\alpha)$ is a root of $\mathfrak{g}(\alpha)$, which is positive for the choice of positive roots in $\mathfrak{a}(\alpha)^*$ that yields this Iwasawa decomposition.

Take the flag manifold $\mathbb{F}_\alpha = G/P_\alpha$ and let $\pi : \mathbb{F} \to \mathbb{F}_\alpha$ be the canonical projection. If x_0 is the origin of $\mathbb{F} = G/P$, then the origin of $\mathbb{F}_\alpha$ is $x_\alpha = \pi(x_0)$. The fiber $F(\alpha) = \pi^{-1}\{x_\alpha\}$ is invariant by $G(\alpha)$ and the action of this group on $F(\alpha)$ is transitive. The isotropy subalgebra at x_0 for the action of $G(\alpha)$ in $F(\alpha)$ is the parabolic subalgebra $\mathfrak{q}(\alpha)$ of $\mathfrak{g}(\alpha)$ given by $\mathfrak{p}(\alpha) = \mathfrak{m}(\alpha) \oplus \mathfrak{a}(\alpha) \oplus \mathfrak{n}(\alpha)$. Hence $F(\alpha)$ becomes identified with the maximal flag manifold of $G(\alpha)$, namely, $F(\alpha) = G(\alpha)/P(\alpha)$. For this identification, x_0 is the origin of $F(\alpha)$. Moreover, since $G(\alpha)$ is a real rank one semisimple Lie group, the flag manifold $F(\alpha)$ is diffeomorphic to a sphere (see [8, Chapter VII, Section 7]).

Moreover, the restriction of α is a root in $\mathfrak{a}(\alpha)^*$, and it defines, by the Iwasawa decomposition of $G(\alpha)$ a cocycle on $G(\alpha) \times F(\alpha)$ (see Appendix A, Section A.3, for details of the definition of cocycles on the maximal flag manifold of a semisimple Lie group). This cocycle is denoted the same way by $\rho_\alpha(g, x)$ since it is the restriction to the fiber $F(\alpha)$ of the cocycle over $G \times \mathbb{F}$ as follows by the inclusions $K(\alpha) \subset K$, $A(\alpha) \subset A$ and $N(\alpha) \subset N$. Therefore, if $T \subset G(\alpha)$ is a semigroup with nonempty interior and its associated invariant control set $C(T)$ in $F(\alpha)$ satisfies $x_0 \in C(T)_0$, then $\rho_\alpha(g, x_0)$ is bounded below in T by Corollary 2.6. This fact will be applied soon to get a lower bound for $\rho_\alpha(g, x_0)$ in the semigroup $S \subset G$.

As another ingredient for the proof of the required estimate, we consider the Langlands decomposition determined by α, and given by

$$P_\alpha = MG(\alpha) A_\alpha N_\alpha,$$

where $A_\alpha = \exp \mathfrak{a}_\alpha$, $N_\alpha = \exp \mathfrak{n}_\alpha$ and $\mathfrak{n}_\alpha = \sum_\beta \mathfrak{g}_\beta$ with the sum extended to the positive roots $\beta \neq \alpha, 2\alpha$ (see Appendix A for more details.)

Going back to the semigroup S, suppose without loss of generality that x_0 belongs to the core C_0 of the invariant control set of S in $\mathbb{F}$. A semigroup $T \subset G(\alpha)$ is defined from S by the following steps:

1. The projection $C_\alpha = \pi(C)$ is the invariant control set of S in $\mathbb{F}_\alpha$ whose core $(C_\alpha)_0$ contains $\pi(C_0)$.
2. Let x_α be the origin of $\mathbb{F}_\alpha$. Then $x_\alpha = \pi(x_0) \in \pi(C_0) \subset (C_\alpha)_0$ implying that the semigroup $S_\alpha = S \cap P_\alpha$ has a nonempty interior in P_α.
3. Taking into account the decomposition $P_\alpha = MG(\alpha) A_\alpha N_\alpha$ define
$$\Gamma = \{g \in MG(\alpha) : \exists b \in A_\alpha N_\alpha,\ gb \in S\}.$$
Since $MG(\alpha)$ normalizes $A_\alpha N_\alpha$, it follows that Γ is a subsemigroup. Moreover, the fact that S_α has a nonempty interior in P_α implies that Γ has a nonempty interior in $MG(\alpha)$.
4. Now define also $T = \Gamma \cap G(\alpha)$, which is a subsemigroup of $G(\alpha)$ with nonempty interior. In fact, take $mg \in \mathrm{int}\Gamma$, with $m \in M$ and $g \in G(\alpha)$ such that m has finite order, say $m^k = 1$. This is possible because M is compact so that the set of its elements of finite order is dense. Then $(mg)^k = m^k g_1 = g_1 \in G(\alpha)$ because M normalizes $G(\alpha)$. Thus $g_1 \in \mathrm{int}(\Gamma \cap G(\alpha))$, that is, $\mathrm{int}T \neq \emptyset$.

The next lemma tells about the invariant control sets of Γ and T in the fiber $F(\alpha)$.

Lemma 3.2. *Let C be the S-invariant control set in $\mathbb{F}$. Then, $F(\alpha) \cap C$ is the unique invariant control set for Γ. Moreover, x_0 belongs to the core $C(T)_0$ of the unique invariant control $C(T)$ of T in $F(\alpha)$.*

Proof. By general facts of semigroup actions on fiber bundles, C is the union of invariant control sets on the fibers of $\pi : \mathbb{F} \to \mathbb{F}_\alpha$ (see [3, Theorem 4.4]). In particular, $F(\alpha) \cap C$ is the invariant control set of the subsemigroup S_α leaving invariant the fiber $F(\alpha)$. Now in the decomposition $P_\alpha = MG(\alpha) A_\alpha N_\alpha$ if $n \in A_\alpha N_\alpha$ then its restriction to $F(\alpha)$ is the identity map, since $A_\alpha N_\alpha$ is a normal subgroup of P_α and $A_\alpha N_\alpha \subset AN$ implies $nx_0 = x_0$. Therefore, the orbits of Γ in $F(\alpha)$ are equal to the orbits of S_α, implying that $F(\alpha) \cap C$ is the invariant control set of Γ as well. Concerning the invariant control set $C(T)$ of T in $F(\alpha)$ we have $C(T) \subset F(\alpha) \cap C$ because $T \subset \Gamma$. Since $x_0 \in C_0$, there exists $g \in (\mathrm{int}\, S) \cap P$ such that x_0 is an attractor fixed point of g (see Appendix A, Section A.2). Therefore, $g^k z \to x_0$, as $k \to \infty$, for all $z \in F(\alpha) \cap C$. In view of the Langlands decomposition $P_\alpha = MG(\alpha) A_\alpha N_\alpha$ and taking into account that $P \subset P_\alpha$, write $g = mhn$ with $m \in M$, $h \in G(\alpha)$ and $n \in A_\alpha N_\alpha$. We have that $g_1 = mh$ belongs to $\mathrm{int}\Gamma$ and x_0 is an attractor fixed point of g_1 in $F(\alpha)$ because $A_\alpha N_\alpha$ acts trivially in the fiber $F(\alpha)$. By assuming that m has finite order, which is possible by the compactness of M and the fact that $g_1 \in \mathrm{int}\Gamma$, we can conclude that $g_2 = g_1^{k_0} \in \mathrm{int}T$ for some integer k_0.

Finally, for any $z \in C(T) \subset C$, we have $g_2^k z \to x_0$ so that $x_0 \in C(T)$. Actually x_0 belongs to the core $C(T)_0$ of $C(T)$ because $g_2 \in \mathrm{int}T$ and $g_2 x_0 = x_0$, concluding the proof. ■

Having these constructions at hand, we can get the estimate that allows us to apply Lemma 2.1.

Lemma 3.3. *Let α be a simple root outside $\Theta(S)$. Then for any $x \in C_0$ there exists $d > 0$ such that $\rho_\alpha(g,x) > d$ for every $g \in S_x^\circ$.*

Proof. Assume without loss of generality that x is the origin x_0 of $\mathbb{F}$ so that we can apply the above comments, including Lemma 3.2. Let $d > 0$ be a lower bound for $\rho_\alpha(g,x_0)$ with $g \in T$ which exists by Corollary 2.6.

We are required to get such a lower bound for g in $S_{x_0}^\circ = (\operatorname{int} S) \cap P$. We can write $g \in S_{x_0}^\circ$ as

$$g = m(hn)h_1 n_1,$$

with $m \in M$, $h \in A(\alpha)$, $n \in N(\alpha)$, $h_1 \in A_\alpha$ and $n_1 \in N_\alpha$ (see 3). We have

$$\rho_\alpha(g,x_0) = \rho_\alpha(hn, h_1 n_1 x_0)\,\rho_\alpha(h_1 n_1, x_0) = \rho_\alpha(h, x_0),$$

because $m \in M$, $n, n_1 \in N$, and $\alpha(\log h_1) = 0$. The value of the cocycle does not depend on $m \in M$ so that it can be changed slightly, keeping g within $\operatorname{int} S \cap P$, and assume that $m^k = 1$ for some integer k. In this case, if $g_1 = m(hn)$, then $g_1^k = h^k n'$ with $n' \in N(\alpha)$ because m normalizes this subgroup and commutes with h. Thus $g_1^k \in \operatorname{int} T$ so that

$$d < \rho_\alpha\left(g_1^k, x_0\right) = \rho_\alpha\left(h^k, x_0\right) = \rho_\alpha(h, x_0)^k = \rho_\alpha(g, x_0)^k.$$

Therefore we have proved that for any $g \in S_{x_0}^\circ = (\operatorname{int} S) \cap P$ there exists an integer k such that $\rho_\alpha(g,x_0)^k > d$. Hence if $\rho_\alpha(g,x_0) < 1$, then

$$\rho_\alpha(g,x_0) > \rho_\alpha(g,x_0)^k > d,$$

showing that $\min\{d,1\} > 0$ is a lower bound for $\rho_\alpha(g,x_0)$ with g running through $(\operatorname{int} S) \cap P = S_{x_0}^\circ$. ■

Combining the above lemma with Lemma 2.1, we get at once the following statement, which concludes the proof of Theorem 3.1.

Proposition 3.4. *For any $\alpha \in \Sigma \setminus \Theta(S)$ and $x_0 \in C_0$, there exists $c > 0$ such that for every $g \in S$ it holds that $\rho_\alpha(g,x) > c$.*

A. Preliminaries

A.1. Semisimple theory

Here we introduce the main results and notations related to semisimple theory. For more on the subject, the reader can consult [6, 7, 8, 9, 21]. Let G be a connected, semisimple, non-compact Lie group G with a finite center and associated Lie algebra $\mathfrak{g}$. Choose a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$. It induces an inner product $B_\theta(X,Y) = -C(X,\theta(Y))$, where $C(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y))$ is the Cartan-Killing form. Since $\theta^2 = \mathrm{id}$, we get that θ is self-adjoint with respect to B_θ, and hence, we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, where $\mathfrak{k}$ is the 1-eigenspace and $\mathfrak{s}$ the (-1)-eigenspace of θ. We also have an associated Cartan decomposition $G = KS$ of the group with $K = \exp \mathfrak{k}$ and $S = \exp \mathfrak{s}$. The derivations $\operatorname{ad}(X)$, $X \in \mathfrak{k}$, are skew-symmetric w.r.t. B_θ and hence the automorphisms $\operatorname{Ad}(k)$, $k \in K$, are isometries.

Let us consider a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$, and for each $\alpha \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \ : \ \mathrm{ad}(H)X = \alpha(H)X, \ \forall H \in \mathfrak{a}\}.$$

The derivations $\mathrm{ad}(H)$, $H \in \mathfrak{a}$, commute and are self-adjoint w.r.t. the inner product B_θ. Hence, they can be diagonalized simultaneously, and the nontrivial $\mathfrak{g}_\alpha$ are the associated eigenspaces. The set

$$\Pi := \{\alpha \in \mathfrak{a}^* \backslash \{0\} : \ \mathfrak{g}_\alpha \neq 0\}$$

is called the *set of roots of* $\mathfrak{g}$. The associated spaces $\mathfrak{g}_\alpha$, $\alpha \in \Pi$, are called *root spaces.* The set of *split-regular elements of* $\mathfrak{a}$[1] is given by

$$\{H \in \mathfrak{a} \ : \ \alpha(H) \neq 0, \ \forall \alpha \in \Pi\}.$$

The connected components of this set are the *Weyl chambers.* Choosing an (arbitrary) Weyl chamber $\mathfrak{a}^+$, we define

$$\alpha > 0 \qquad \Leftrightarrow \qquad \alpha|_{\mathfrak{a}^+} > 0, \quad \alpha \in \Pi,$$

and hence, the *sets of positive and negative roots*, respectively, are

$$\Pi^+ := \{\alpha \in \Pi \ : \ \alpha > 0\} \quad \text{and} \quad \Pi^- := -\Pi^+,$$

and $\Pi = \Pi^+ \cup \Pi^-$ is a disjoint union. We now define the nilpotent subalgebras $\mathfrak{n} := \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- := \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha$, which yields the *Iwasawa decomposition* of the Lie algebra:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

Associated with each of the previous subalgebras we have corresponding connected Lie subgroups of G, denoted by K, A, N, and N^-, respectively. They induce the Iwasawa decomposition $G = KAN$ of the group G. The *Weyl group* $\mathcal{W}$ of G is the quotient M^*/M, where M^* and M are the normalizer and the centralizer of $\mathfrak{a}$ in K, respectively, i.e.,

$$M^* = \{k \in K : \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\} \text{ and } M = \{k \in K : \mathrm{Ad}(k)H = H, \ \forall H \in \mathfrak{a}\}.$$

Alternatively, the Weyl group is the group generated by the orthogonal reflections at the hyperplanes $\ker \alpha$, $\alpha \in \Pi$. The Weyl group acts simply transitively on the Weyl chambers. The *set of simple roots* $\Sigma \subset \Pi^+$ are the positive roots that cannot be written as linear combinations of other positive roots. It forms a basis of $\mathfrak{a}^*$. Moreover, the reflections at $\ker \alpha$, $\alpha \in \Sigma$, generate the Weyl group. There exists a unique element $w_0 \in \mathcal{W}$ of order 2 that takes Π^+ to Π^-, called the *principal involution.* For any $\alpha \in \mathfrak{a}^*$, let $H_\alpha \in \mathfrak{a}$ denote the coroot of α, defined by $B_\theta(H_\alpha, H) = \alpha(H)$, $H \in \mathfrak{a}$. By writing $\Sigma = \{\alpha_1, \ldots, \alpha_l\}$ for the simple roots, the set of the *fundamental weights* is the set $\Phi = \{\mu_1, \ldots, \mu_l\} \subset \mathfrak{a}^*$ of linear functions satisfying

$$\frac{2\langle \alpha_i, \mu_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \delta_{ij}, \qquad \text{where} \qquad \langle \alpha, \beta \rangle := B_\theta(H_\alpha, H_\beta).$$

In particular, the fundamental weights are nonnegative on the positive Weyl chamber $\mathfrak{a}^+$.

[1] More generally, any element of the form $\mathrm{Ad}(k)H$ for $H \in \mathfrak{a}^+$ and $k \in K$ is called a split-regular element of $\mathfrak{g}$.

Let $\Theta \subset \Sigma$ be an arbitrary subset and $\langle\Theta\rangle$ for the set of roots that are linear combinations (over $\mathbb{Z}$) of elements in Θ. Moreover, we put

$$\mathfrak{a}(\Theta) := \mathrm{span}\{H_\alpha \ : \ \alpha \in \Theta\}.$$

The subalgebra $\mathfrak{g}(\Theta)$ generated by $\mathfrak{a}(\Theta) \oplus \sum_{\alpha\in\langle\Theta\rangle} \mathfrak{g}_\alpha$ is a semisimple Lie algebra. We put $\mathfrak{k}(\Theta) := \mathfrak{k} \cap \mathfrak{g}(\Theta)$ and $\mathfrak{n}(\Theta) := \mathfrak{n} \cap \mathfrak{g}(\Theta)$. Then $\mathfrak{g}(\Theta)$ is the Lie algebra of a semisimple Lie group $G(\Theta) \subset G$, and $\mathfrak{g}(\Theta) = \mathfrak{k}(\Theta) \oplus \mathfrak{a}(\Theta) \oplus \mathfrak{n}(\Theta)$ is an Iwasawa decomposition of $\mathfrak{g}(\Theta)$, while Θ is the corresponding set of simple roots. We write $K(\Theta)$ for the connected Lie subgroup with Lie algebra $\mathfrak{k}(\Theta)$ and $A(\Theta) = \exp \mathfrak{a}(\Theta)$, $N(\Theta) = \exp \mathfrak{n}(\Theta)$ (which are also connected subgroups). Then $G(\Theta) = K(\Theta)A(\Theta)N(\Theta)$ is an Iwasawa decomposition of $G(\Theta)$. Let

$$\mathfrak{a}_\Theta := \{H \in \mathfrak{a} \ : \ \alpha(H) = 0, \ \forall \alpha \in \Theta\},$$

be the orthogonal complement of $\mathfrak{a}(\Theta)$ in $\mathfrak{a}$. The subset Θ singles out a subgroup $\mathcal{W}_\Theta$ of $\mathcal{W}$ consisting of those elements that act trivially on $\mathfrak{a}_\Theta$. Alternatively, $\mathcal{W}_\Theta$ can be defined as the subgroup generated by the reflections at the hyperplanes $\ker \alpha$, $\alpha \in \Theta$. Then $\mathcal{W}_\Theta$ is isomorphic to the Weyl group $\mathcal{W}(\Theta)$ of $G(\Theta)$. We let Z_Θ denote the centralizer of $\mathfrak{a}_\Theta$ in G and $K_\Theta = Z_\Theta \cap K$. The group Z_Θ is a reductible Lie group and admits an Iwasawa decomposition that is given by $Z_\Theta = K_\Theta AN(\Theta)$.

The *parabolic subalgebra of type* $\Theta \subset \Sigma$ is defined by $\mathfrak{p}_\Theta := \mathfrak{n}^-(\Theta) \oplus \mathfrak{p}$, where $\mathfrak{p} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, $\mathfrak{n}^-(\Theta) = \mathfrak{n}^- \cap \mathfrak{g}(\Theta)$, and $\mathfrak{m}$ is the Lie algebra of M, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, or, respectively, the part of the common 0-eigenspace of the maps $\mathrm{ad}(H)$, $H \in \mathfrak{a}$, contained in $\mathfrak{k}$. The associated *parabolic subgroup* P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G. Then $\mathfrak{p}_\Theta$ is the Lie algebra of P_Θ. The empty set $\Theta = \emptyset$ yields the minimal parabolic subalgebra $\mathfrak{p}_\emptyset = \mathfrak{p}$. An Iwasawa decomposition of the associated subgroup P is $P = MAN^+$. The parabolic subgroup P_Θ also decomposes as

$$P_\Theta = K_\Theta AN = MG(\Theta)A_\Theta N_\Theta, \tag{2}$$

where $A_\Theta = \exp \mathfrak{a}_\Theta$ and $N_\Theta = \exp \mathfrak{n}_\Theta$, with $\mathfrak{n}_\Theta = \sum_{\alpha\in\Pi^+\setminus\langle\Theta\rangle} \mathfrak{g}_\alpha$ the nilradical of $\mathfrak{p}_\Theta$. The first equality of (2) is known as *Langlands decomposition*, and the second one follows from the fact that $K_\Theta = MK(\Theta)$, $A = A_\Theta A(\Theta) = A(\Theta)A_\Theta$, $N = N(\Theta)N_\Theta = N_\Theta N(\Theta)$ and $A_\Theta N(\Theta) = N(\Theta)A(\Theta)$. In particular, any $g \in P$ can be written as

$$g = m(hn)h_1 n_1, \tag{3}$$

with $m \in M$, $h \in A(\Theta)$, $n \in N(\Theta)$, $h_1 \in A_\Theta$ and $n_1 \in N_\Theta$.

The *flag manifold of type* Θ is the $\mathrm{Ad}(G)$-orbit $\mathbb{F}_\Theta := \mathrm{Ad}(G)\mathfrak{p}_\Theta$ with base point $x_\Theta := \mathfrak{p}_\Theta$ in the Grassmann manifold of $(\dim \mathfrak{p}_\Theta)$-dimensional subspaces of $\mathfrak{g}$. $\mathbb{F}_\Theta$ is compact because the natural K-action on $\mathbb{F}_\Theta$ is transitive; in fact, it holds that $\mathbb{F}_\Theta = K/K_\Theta$. In the case $\Theta = \emptyset$, we also write $x_0 = x_\emptyset$ and $\mathbb{F} = \mathbb{F}_\emptyset$ for the maximal flag manifold. Since the isotropy group of x_Θ is the subgroup P_Θ, $\mathbb{F}_\Theta$ can be identified with the homogeneous space G/P_Θ via $\mathrm{Ad}(g)\mathfrak{p}_\Theta \mapsto gP_\Theta$. If $\Theta_1 \subset \Theta_2$, then $P_{\Theta_1} \subset P_{\Theta_2}$ and the projection $\pi^{\Theta_1}_{\Theta_2} : \mathbb{F}_{\Theta_1} \to \mathbb{F}_{\Theta_2}$, $gP_{\Theta_1} \mapsto gP_{\Theta_2}$, is a well-defined fibration. In case $\Theta_1 = \emptyset$, we just write π_{Θ_2} for this map. The choice of the previous subalgebras and subgroups is not unique. In fact, by conjugation with any $k \in K$, one obtains a new set of subalgebras and subgroups. For instance, $\mathrm{Ad}(k)\mathfrak{a}$ is another maximal abelian subspace of $\mathfrak{s}$, and $\mathrm{Ad}(k)\mathfrak{a}^+$ is an associated positive

Weyl chamber. Also $\mathfrak{g} = \mathfrak{k} \oplus \mathrm{Ad}(k)\mathfrak{a} \oplus \mathrm{Ad}(k)\mathfrak{n}$ is another Iwasawa decomposition. We will use these conjugated settings frequently.

A.2. Semigroup actions on flag manifolds

Here we highlight the main results involving the actions of semigroups on the flag manifolds. The results presented here can be found in the references [14, 15, 16, 20].

Let $S \subset G$ be a semigroup with nonempty interior. The semigroup S acts on the flag manifolds $\mathbb{F}_\Theta$ of G by left translation. This action is S is not transitive in $\mathbb{F}_\Theta$ unless $S = G$.

A *control set* for the S-action on $\mathbb{F}_\Theta$ is a subset $D \subset \mathbb{F}_\Theta$ that is maximal w.r.t. set inclusion, satisfying:

1. $\mathrm{int} D \neq \emptyset$;
2. $D \subset \mathrm{cl}(Sx)$ for all $x \in D$.

We say that D is S-invariant, provided that $SD \subset D$.

As was shown in [14, Theorem 3.1], there exists exactly one invariant control set $C_\Theta \subset \mathbb{F}_\Theta$, with $C_\Theta \neq \mathbb{F}_\Theta$ if S is a proper semigroup. The *core* of C_Θ is the open, dense, and S-invariant subset given by

$$(C_\Theta)_0 := \{x \in C_\Theta : x \in \mathrm{int}\, Sx\}.$$

Moroever, for any $\Theta \subset \Sigma$, it holds that

$$\pi_\Theta(C) = C_\Theta \quad \text{and} \quad \pi_\Theta(C)_0 \subset (C_\Theta)_0.$$

Let $H \in \mathfrak{a}^+$ and $k \in K$, and consider a split-regular element $Z = \mathrm{Ad}(k)H$. The action of Z on the flag manifold $\mathbb{F}_\Theta$ admits as fixed points the elements of the form $\mathrm{fix}_\Theta(Z, w) = kwx_\Theta$ for $w \in \mathcal{W}$. Those points are isolated and have stable and unstable manifolds given by $\mathrm{st}_\Theta(Z, w) := kN^- wx_\Theta$ and $\mathrm{un}_\Theta(Z, w) = kNwx_\Theta$, respectively. In particular, there is a unique attractor fixed point $\mathrm{at}_\Theta(Z) = kx_\Theta$, whose stable manifold $\mathrm{st}_\Theta(Z) = \mathrm{st}_\Theta(Z, 1) = kN^- x_\Theta$ is open and dense, and a unique repeller fixed point $\mathrm{rp}_\Theta = kw_0 x_\Theta$, whose unstable manifold

$$\mathrm{un}_\Theta(Z) = \mathrm{st}_\Theta(Z, w_0) = kN^+ w_0 x_\Theta$$

is also open and dense. In particular, the core of the invariant control set C_Θ is characterized by attractor points cite[Theorem 3.4]smt as

$$(C_\Theta)_0 = \{\mathrm{at}_\Theta(Z) : Z \text{ is split-regular and } \mathrm{e}^Z \in \mathrm{int}\, S\}.$$

The geometry of the invariant control sets associated with a semigroup in G is described by the following results, proved in [20].

Proposition A.1. *There exists $\Theta \subset \Sigma$ such that $\pi_\Theta^{-1}(C_\Theta) = C$ is the invariant control set in the maximal flag manifold $\mathbb{F}$. Among the subsets Θ satisfying this property, there exists a unique maximal one (w.r.t. set inclusion).*

The maximal subset of the previous result is called the *flag type* of S and is denoted by $\Theta(S)$. Alternatively, we call the flag type of S the corresponding flag manifold $\mathbb{F}_{\Theta(S)}$ (see [16, 15, 20, 17] for further discussions about the flag type of a semigroup and its applications).

Another characterization of the flag type is given by the following result.

Proposition A.2. *The flag type $\Theta(S)$ is the minimal subset (w.r.t. set inclusion) satisfying: If $Z \in \mathfrak{g}$ is split-regular, then*

$$\mathrm{e}^Z \in \operatorname{int} S \qquad \Longrightarrow \qquad C_\Theta \subset \mathrm{st}_\Theta(Z).$$

A.3. Cocycles on flag manifolds

A cocycle ρ on the maximal flag manifold $\mathbb{F}$ is a function $\rho : G \times \mathbb{F} \to \mathbb{R}$, satisfying

$$\rho(gh, x) = \rho(g, hx)\rho(h, x) \quad \forall g, h \in G, x \in \mathbb{F}.$$

A natural cocycle on $\mathbb{F}$ appears naturally from an Iwasawa decomposition. In fact, let us start with the map $\mathsf{a} : G \times K \to \mathfrak{a}$ defined by the Iwasawa decomposition

$$gk = u \exp(\mathsf{a}(g, k))n, \quad u \in K, n \in N.$$

Since the second component of a is invariant by M, the map a factors to the flag $\mathbb{F}$. Since a satisfies

$$\mathsf{a}(gh, x) = \mathsf{a}(g, hx) + \mathsf{a}(h, x),$$

it holds that, for any $\lambda \in \mathfrak{a}$, the map

$$\rho_\lambda : G \times \mathbb{F} \to \mathbb{R}, \quad \rho_\lambda(g, x) = \mathrm{e}^{\lambda \mathsf{a}(g,x)},$$

is a cocycle[2]. The next result [2, Lemma 6.1] shows that a cocycle can factor to the partial flag manifolds under some conditions on the functional λ.

Lemma A.3. *If $\lambda \in \mathfrak{a}$ satisfies $\lambda(\mathfrak{a}(\Theta)) = 0$, then*

$$\rho_\lambda (g, x) = \rho_\lambda (gu, x), \quad \forall u \in K_\Theta.$$

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