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Multiplicative Lie Algebra Structures on the Semi-Direct Product of Groups

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Abstract. We give a method to determine the multiplicative Lie algebra structures on the semi-direct product of groups under certain conditions. Consequently, we see that every multiplicative Lie algebra structure on the semi-direct product of two groups is completely determined by multiplicative Lie algebra structures on them.

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1. Introduction

A multiplicative Lie algebra structure on a group G is a function that satisfies identities similar to the universal identities of the commutator function. We know that if G is a non-cyclic group, then there are always at least two distinct multiplicative Lie algebra structures on G. Also, if $G = \mathbb{Z}_p^n$, where p is a prime, then distinct multiplicative Lie algebra structures on G can be determined by the classification problem of n-dimensional Lie algebras over the field $\mathbb{Z}_p$. Thus, the following are interesting problems:

Problem 1. *How many distinct (up to isomorphism) multiplicative Lie algebra structures exist on a group G?*

Problem 2. *Let H be a subgroup of G with a multiplicative Lie algebra structure $\star$ on H. Can we define a multiplicative Lie algebra structures $\tilde{\star}$ on G with the help of $\star$?*

In 2019, Walls ([4]) investigated the construction of a multiplicative Lie algebra structure on G (for details, one can see Theorem 3.7 of [4]). In [3], Pandey and Upadhyay discussed Problem 1 and gave a precise characterization of the group homomorphisms from the exterior square $G \wedge G$ to G which determine a multiplicative Lie algebra structure on G. They also found the number of distinct (up to isomorphism) multiplicative Lie algebra structures on some classes of finite groups like D_n, Q_n, etc.

The main aim of this paper is to determine the multiplicative Lie algebra structures on a semi-direct product of groups motivated by Problem 2. More precisely, let H be an abelian group with trivial multiplicative Lie algebra structure and K be a

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multiplicative Lie algebra. Then with the help of multiplicative Lie algebra structure on K, we define multiplicative Lie algebra structures on the semi-direct product G of H by K such that H is an ideal of G. In particular, if $G = H \times K$ and $(|H|, |K|) = 1$, then we see that every multiplicative Lie algebra structure on G is completely determined by multiplicative Lie algebra structure on K. This method will help to determine all distinct multiplicative Lie algebra structures on a given group.

Now, we give a few definitions and results which are useful for the article.

Definition 1.1. [1] A *multiplicative Lie algebra* is a triple $(K, \cdot, \star)$, where K is a set, $\cdot$ and $\star$ are two binary operations on K such that $(K, \cdot)$ is a group (need not be abelian) and for all $x, y, z \in K$, the following identities hold:

1. $x \star x = 1$,
2. $x \star (y \cdot z) = (x \star y) \cdot {}^{y}(x \star z)$,
3. $(x \cdot y) \star z = {}^{x}(y \star z) \cdot (x \star z)$,
4. $((x \star y) \star {}^{y}z)((y \star z) \star {}^{z}x)((z \star x) \star {}^{x}y) = 1$,
5. ${}^{z}(x \star y) = ({}^{z}x \star {}^{z}y)$,

where ${}^{x}y$ denotes xyx^{-1}. We say that $\star$ is a multiplicative Lie algebra structure on the group K. ■

Definition 1.2. A short exact sequence

$$1 \longrightarrow H \xrightarrow{\alpha} G \xrightarrow{\beta} K \longrightarrow 1$$

of multiplicative Lie algebras is called an *extension* of H by K, where α and β are multiplicative Lie algebra homomorphisms. A map $t : K \to G$ is called a section of the extension if $\beta \circ t = I_K$ and $t(1) = 1$.

Note that α is an injective map, so without loss of generality, we can assume that α is an inclusion map and H is an ideal of G. Now onwards, we denote α by i. ■

Remark 1.3. [2] Let H be an abelian group and $\mathrm{End}(H)$ be the set of all group endomorphisms on H. Then $(\mathrm{End}(H), \cdot, \star)$ is a multiplicative Lie algebra, where $(F_1 \cdot F_2)(h) = F_1(h)F_2(h)$ and $(F_1 \star F_2)(h) = F_1(F_2(h))F_2(F_1(h^{-1}))$.

2. Multiplicative Lie algebra structures on semi-direct product

Consider an extension $1 \longrightarrow H \xrightarrow{i} G \xrightarrow{\beta} K \longrightarrow 1$ of H by K, where H is an abelian group with trivial multiplicative Lie algebra structure and K is a group with multiplicative Lie algebra structure $\star$. Let $t : K \to G$ be a section. Then it is easy to see that every element of the group G can be uniquely expressed as $ht(x)$, for some $h \in H$ and $x \in K$, that is, $G = Ht(K)$ and $H \cap t(K) = \{1\}$. Now by Remark 4.14 ([2]), the group operation " $\cdot$ " and the multiplicative Lie algebra structure $\star$ on G are given by

$$ht(x) \cdot kt(y) = h\sigma_x^t(k)f^t(x, y)t(xy),$$

$$ht(x) \star kt(y) = hk\Gamma_x^t(k)\sigma_{(x \star y)}^t(h^{-1}k^{-1}\Gamma_y^t(h^{-1}))h^t(x, y)t(x \star y),$$

where $\sigma_x^t(k) = t(x)kt(x)^{-1}$, $\Gamma_x^t(k) = t(x) \star k$ are group homomorphisms on H and $f^t, h^t : K \times K \longrightarrow H$ are maps satisfying the following identities

1. $f^t(1,x) = f^t(x,1) = 1$ and $f^t(x,y)f^t(xy,z) = \sigma_x^t(f^t(y,z))f^t(x,yz)$;
2. $h^t(x,1) = h^t(1,x) = h^t(x,x) = 1$.

In fact, we have a group homomorphism $\sigma^t : K \to \mathrm{Aut}(H)$defined by $\sigma^t(x) = \sigma_x^t$ and a map $\Gamma^t : K \to \mathrm{End}(H)$ defined by $\Gamma^t(x) = \Gamma_x^t$.

Proposition 2.1. *The maps σ^t and Γ^t are independent of the choice of section t.*

Proof. Let s and t be two sections of K in G. Then there exists a map $g : K \to H$ with $g(1) = 1$ such that $s(x) = g(x)t(x)$ for every $x \in K$. Now,

$$\sigma_x^s(h) = s(x)hs(x)^{-1} = g(x)t(x)ht(x)^{-1}g(x)^{-1} = g(x)\sigma_x^t(h)g(x)^{-1} = \sigma_x^t(h)$$

(since H is abelian). This shows that the group homomorphism $\sigma^t : K \to \mathrm{Aut}(H)$ is independent on the choice of section t.

Also, $\Gamma_x^s(h) = s(x) \star h = (g(x)t(x)) \star h = {}^{g(x)}(t(x) \star h)(g(x) \star h) = \Gamma_x^t(h)$ (since H is abelian with trivial multiplicative Lie algebra structure). This shows that the map $\Gamma^t : K \to \mathrm{End}(H)$ is independent on the choice of section t. ■

So, now onwards we denote σ^t and Γ^t by σ and Γ, respectively. Suppose t is a group homomorphism, that is, $G \cong H \rtimes_\sigma K$. Then $f^t(x,y) = 1$, for all $x, y \in K$. Hence, we have $ht(x) \cdot kt(y) = h\sigma_x(k)t(xy)$ and

$$ht(x) \star kt(y) = hk\Gamma_x(k)\sigma_{(x \star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)t(x \star y).$$

Proposition 2.2. *If t is a splitting, then we have $\Gamma_{xy}(h) = \Gamma_x(h)\sigma_x(\Gamma_y(h))$ and $\Gamma_{x \star y}(\sigma_y(h)) = \Gamma_x(\Gamma_y(h))\Gamma_{xyx^{-1}}(\Gamma_x(h^{-1}))$, for all $x, y \in K$ and $h \in H$.*

Proof. Since t is a group homomorphism, we have

$$\begin{aligned} \Gamma_{xy}(h) &= t(xy) \star h = (t(x)t(y)) \star h = {}^{t(x)}(t(y) \star h)(t(x) \star h) \\ &= \Gamma_x(h)^{t(x)}(\Gamma_y(h)) = \Gamma_x(h)\sigma_x(\Gamma_y(h)). \end{aligned}$$

Now,
$$\begin{aligned} \Gamma_{x \star y}(h) &= t(x \star y) \star h = (h^t(x,y)^{-1}(t(x) \star t(y))) \star h \\ &= {}^{h^t(x,y)^{-1}}((t(x) \star t(y)) \star h)(h^t(x,y)^{-1} \star h) = (t(x) \star t(y)) \star h. \end{aligned}$$

Since $((t(x) \star t(y)) \star {}^{t(y)}h)((t(y) \star h) \star {}^h t(x))((h \star t(x)) \star {}^{t(x)} t(y)) = 1$, we have

$$\begin{aligned} &((t(x) \star t(y)) \star \sigma_y(h))(\Gamma_y(h) \star {}^h t(x))(\Gamma_x(h^{-1}) \star t(xyx^{-1})) = 1 \\ &\implies (h^t(x,y)t(x \star y) \star \sigma_y(h))^h(\Gamma_y(h) \star t(x))(\Gamma_x(h^{-1}) \star t(xyx^{-1})) = 1 \\ &\implies \Gamma_{(x \star y)}(\sigma_y(h))(\Gamma_y(h) \star t(x))\Gamma_{xyx^{-1}}(\Gamma_x(h)) = 1 \\ &\implies \Gamma_{(x \star y)}(\sigma_y(h))\Gamma_x(\Gamma_y(h^{-1}))\Gamma_{xyx^{-1}}(\Gamma_x(h)) = 1 \\ &\implies \Gamma_{(x \star y)}(\sigma_y(h)) = \Gamma_x(\Gamma_y(h))\Gamma_{xyx^{-1}}(\Gamma_x(h^{-1})). \end{aligned}$$
■

Lemma 2.3. *If K is an abelian group, then $\sigma_x \circ \Gamma_z = \Gamma_z \circ \sigma_x$, for all $x, z \in K$.*

Proof. Let $h \in H$. Then

$$(\sigma_x \circ \Gamma_z)(h) = \sigma_x(t(z) \star h) = t(x)(t(z) \star h)t(x)^{-1} = {}^{t(x)}(t(z) \star h). \tag{i}$$

On the other hand,

$$\begin{aligned}(\Gamma_z \circ \sigma_x)(h) &= \Gamma_z(t(x)ht(x)^{-1}) = t(z) \star t(x)ht(x)^{-1} = t(z) \star {}^{t(x)}h \\ &= {}^{t(x)}({}^{t(x)^{-1}}t(z) \star h) = {}^{t(x)}(t(x^{-1}zx) \star h) = {}^{t(x)}(t(z) \star h).\end{aligned} \tag{ii}$$

By equations (i) and (ii), we have$\Gamma_z \circ \sigma_x = \sigma_x \circ \Gamma_z$. That is, Γ_z and σ_x commutes with each other. ∎

Now we discuss the properties of multiplicative Lie algebra structure $\star$ on G. Consider the expression

$$\begin{aligned}&(ht(x) \cdot kt(y)) \star lt(z) = (h\sigma_x(k)t(xy)) \star lt(z) \\ &= hl\sigma_x(k)\Gamma_{xy}(l)\sigma_{(xy\star z)}(h^{-1}l^{-1}\sigma_x(k^{-1})\Gamma_z(h^{-1}\sigma_x(k^{-1})))h^t(xy, z)t(xy \star z).\end{aligned} \tag{1}$$

On the other hand

$$\begin{aligned}(ht(x) \cdot kt(y)) \star lt(z) &= {}^{ht(x)}(kt(y) \star lt(z)) \cdot (ht(x) \star lt(z)) \\ &= ht(x)kl\Gamma_y(l)\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y, z)t(y \star z)t(x)^{-1}h^{-1} \cdot hl\Gamma_x(l) \\ &\quad \cdot \sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x, z)t(x \star z) \\ &= h\sigma_x(kl\Gamma_y(l)\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y, z))t({}^x(y \star z)) \cdot l\Gamma_x(l) \\ &\quad \cdot \sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x, z)t(x \star z) \\ &= h\sigma_x(kl\Gamma_y(l)\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y, z))\sigma_{{}^x(y\star z)}(l\Gamma_x(l) \\ &\quad \cdot \sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x, z))t({}^x(y \star z))t(x \star z)).\end{aligned} \tag{2}$$

From equations (1) and (2), we have

$$\begin{aligned}&l\Gamma_x(l)\sigma_{(xy\star z)}(\sigma_x(k^{-1})\Gamma_z(\sigma_x(k^{-1})))h^t(xy, z) \\ &\qquad = \sigma_x(l\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y, z))\sigma_{{}^x(y\star z)}(l\Gamma_x(l)h^t(x, z)).\end{aligned} \tag{3}$$

Now consider the expression

$$\begin{aligned}&ht(x) \star (kt(y) \cdot lt(z)) = ht(x) \star (k\sigma_y(l)t(yz)) \\ &\quad = hk\sigma_y(l)\Gamma_x(k\sigma_y(l))\sigma_{(x\star yz)}(h^{-1}k^{-1}\sigma_y(l^{-1})\Gamma_{yz}(h^{-1}))h^t(x, yz)t(x \star yz).\end{aligned} \tag{4}$$

On the other hand

$$\begin{aligned}ht(x) \star (kt(y) \cdot lt(z)) &= (ht(x) \star kt(y)) \cdot {}^{kt(y)}(ht(x) \star lt(z)) \\ &= hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x, y)t(x \star y) \cdot kt(y)hl\Gamma_x(l) \\ &\quad \cdot \sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x, z)t(x \star z)t(y)^{-1}k^{-1} \\ &= hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x, y)t(x \star y) \cdot k\sigma_y(hl\Gamma_x(l) \\ &\quad \cdot \sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x, z))t(y)t(x \star z)t(y)^{-1}k^{-1} \\ &= hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x, y)\sigma_{x\star y}(k\sigma_y(hl\Gamma_x(l) \\ &\quad \cdot \sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x, z)))\sigma_{(x\star yz)}(k^{-1})t((x \star y)^y(x \star z)).\end{aligned} \tag{5}$$

From equations (4) and (5), we have

$$\begin{aligned}&\Gamma_x(\sigma_y(l))\sigma_y(l)\sigma_{(x\star yz)}(h^{-1}\sigma_y(l^{-1})\Gamma_{yz}(h^{-1}))h^t(x,yz)\\&=\sigma_{(x\star y)}(h^{-1}\Gamma_y(h^{-1})\sigma_y(hl\Gamma_x(l)\sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x,z)))h^t(x,y).\end{aligned} \tag{6}$$

Consider the expressions,

$$\begin{aligned}(ht(x)\star kt(y))\star^{kt(y)}lt(z)&=(ht(x)\star kt(y))\star(kt(y)lt(z)t(y)^{-1}k^{-1})\\&=hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)t(x\star y)\star k\sigma_y(l)\sigma_{{}^yz}(k^{-1})t({}^yz)\\&=hk^2\Gamma_x(k)\sigma_y(l)\sigma_{{}^yz}(k^{-1})\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)\Gamma_{(x\star y)}(k\sigma_y(l)\sigma_{{}^yz}(k^{-1}))\\&\quad\cdot\sigma_{((x\star y)\star{}^yz)}(h^{-1}k^{-2}\Gamma_x(k^{-1})\sigma_y(l^{-1})\sigma_{{}^yz}(k)\sigma_{(x\star y)}(hk\Gamma_y(h))h^t(x,y)^{-1}\\&\quad\cdot\Gamma_{{}^yz}(h^{-1}k^{-1}\Gamma_x(k^{-1})\sigma_{(x\star y)}(hk\Gamma_y(h))h^t(x,y)^{-1}))\\&\quad\cdot h^t(x\star y,{}^yz)t((x\star y)\star{}^yz).\end{aligned}$$

Thus, we have

$$\begin{aligned}&(ht(x)\star kt(y))\star^{kt(y)}lt(z)\\&=hk^2\Gamma_x(k)\sigma_y(l)\sigma_{{}^yz}(k^{-1})\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)\Gamma_{(x\star y)}(k\sigma_y(l)\sigma_{{}^yz}(k^{-1}))\\&\quad\cdot\sigma_{((x\star y)\star{}^yz)}(h^{-1}k^{-2}\Gamma_x(k^{-1})\sigma_y(l^{-1})\sigma_{{}^yz}(k)\sigma_{(x\star y)}(hk\Gamma_y(h))\\&\quad\cdot h^t(x,y)^{-1}\Gamma_{{}^yz}(h^{-1}k^{-1}\Gamma_x(k^{-1})\sigma_{(x\star y)}(hk\Gamma_y(h))h^t(x,y)^{-1}))\\&\quad\cdot h^t(x\star y,{}^yz)t((x\star y)\star{}^yz).\end{aligned}$$

Similarly, we can calculate

$$(kt(y)\star lt(z))\star^{lt(z)}ht(x)\qquad\text{and}\qquad(lt(z)\star ht(x))\star^{ht(x)}kt(y).$$

Since

$$((ht(x)\star kt(y))\star^{kt(y)}lt(z))((kt(y)\star lt(z))\star^{lt(z)}ht(x))((lt(z)\star ht(x))\star^{ht(x)}kt(y))=1,$$

we have the following equation $\qquad ABC=1, \qquad$ (7)

where

$$\begin{aligned}A&=hk^2\Gamma_x(k)\sigma_y(l)\sigma_{{}^yz}(k^{-1})\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)\Gamma_{(x\star y)}(k\sigma_y(l)\sigma_{{}^yz}(k^{-1}))\\&\quad\cdot\sigma_{((x\star y)\star{}^yz)}(h^{-1}k^{-1}\Gamma_x(k^{-1})\sigma_y(l^{-1})\sigma_{{}^yz}(k)\sigma_{(x\star y)}(hk\Gamma_y(h))\\&\quad\cdot h^t(x,y)^{-1}\Gamma_{{}^yz}(h^{-1}k^{-1}\Gamma_x(k^{-1})\sigma_{(x\star y)}(hk\Gamma_y(h))h^t(x,y)^{-1}))\\&\quad\cdot h^t(x\star y,{}^yz),\\B&=\sigma_{((x\star y)\star{}^yz)}(l^2\Gamma_y(l)\sigma_z(h)\sigma_{{}^zx}(l^{-1})\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y,z)\Gamma_{(y\star z)}(l\sigma_z(h)\sigma_{{}^zx}(l^{-1}))\\&\quad\cdot\sigma_{((y\star z)\star{}^zx)}(k^{-1}l^{-1}\Gamma_y(l^{-1})\sigma_z(h^{-1})\sigma_{{}^zx}(l)\sigma_{(y\star z)}(kl\Gamma_z(k))h^t(y,z)^{-1}\\&\quad\cdot\Gamma_{{}^zx}(k^{-1}l^{-1}\Gamma_y(l^{-1})\sigma_{(y\star z)}(kl\Gamma_z(k))h^t(y,z)^{-1}))\\&\quad\cdot h^t(y\star z,{}^zx)),\\C&=\sigma_{((y\star z)\star{}^zx)((z\star x)\star{}^xy)}(h^2\Gamma_z(h)\sigma_x(k)\sigma_{{}^xy}(h^{-1})\sigma_{(z\star x)}(l^{-1}h^{-1}\Gamma_x(l^{-1}))\\&\quad\cdot h^t(z,x)\Gamma_{(z\star x)}(h\sigma_x(k)\sigma_{{}^xy}(h^{-1}))\\&\quad\cdot\sigma_{((z\star x)\star{}^xy)}(l^{-1}h^{-2}\Gamma_z(h^{-1})\sigma_x(k^{-1})\sigma_{{}^xy}(h)\sigma_{(z\star x)}(lh\Gamma_x(l))h^t(z,x)^{-1}\\&\quad\cdot\Gamma_{{}^xy}(l^{-1}h^{-1}\Gamma_z(h^{-1})\sigma_{(z\star x)}(lh\Gamma_x(l))h^t(z,x)^{-1}))\\&\quad\cdot h^t(z\star x,{}^xy)).\end{aligned}$$

Now consider the expression

$$
\begin{aligned}
{}^{lt(z)}(ht(x)\star kt(y)) &= lt(z)hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)t(x\star y)t(z)^{-1}l^{-1}\\
&= l\sigma_z(hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y))\sigma_{{}^z(x\star y)}(l^{-1})t({}^z(x\star y)). \qquad (8)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
{}^{lt(z)}ht(x)\star {}^{lt(z)}kt(y) &= (lt(z)ht(x)t(z)^{-1}l^{-1})\star(lt(z)kt(y)t(z)^{-1}l^{-1})\\
&= l\sigma_z(h)\sigma_{{}^zx}(l^{-1})t({}^zx)\star l\sigma_z(k)\sigma_{{}^zy}(l^{-1})t({}^zy)\\
&= l^2\sigma_z(hk)\sigma_{{}^zx}(l^{-1})\sigma_{{}^zy}(l^{-1})\Gamma_{{}^zx}(l\sigma_z(k)\sigma_{{}^zy}(l^{-1}))\\
&\quad\cdot\sigma_{({}^zx\star{}^zy)}(l^{-2}\sigma_z(h^{-1}k^{-1})\sigma_{{}^zx}(l)\sigma_{{}^zy}(l)\Gamma_{{}^zy}(l^{-1}\sigma_z(h^{-1})\sigma_{{}^zx}(l)))\\
&\quad\cdot h^t({}^zx,{}^zy)t({}^zx\star{}^zy). \qquad (9)
\end{aligned}
$$

From equations (8) and (9), we have

$$
\begin{aligned}
&\sigma_{{}^zx}(l)\sigma_{{}^zy}(l)\sigma_z(\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y))\\
&= l\Gamma_{{}^zx}(l\sigma_z(k)\sigma_{{}^zy}(l^{-1}))\sigma_{({}^zx\star{}^zy)}(l^{-1}\sigma_z(h^{-1}k^{-1})\sigma_{{}^zx}(l)\sigma_{{}^zy}(l)\Gamma_{{}^zy}(l^{-1}\sigma_z(h^{-1})\sigma_{{}^zx}(l)))\\
&\qquad\cdot h^t({}^zx,{}^zy). \qquad (10)
\end{aligned}
$$

From the above discussion, we have the following theorem:

Theorem 2.4. *Let $G = H \rtimes_\sigma K$, where H is an abelian group with trivial multiplicative Lie algebra structure and K is a group. Suppose $\star$ is a multiplicative Lie algebra structure on K, and there are maps $\Gamma : K \to \mathrm{End}(H)$ and $h^t : K\times K \to H$ that satisfies the following conditions for all $x, y, z \in K$ and $h, k, l \in H$:*

1. $h^t(x,1) = h^t(1,x) = h^t(x,x) = 1$;
2. $\Gamma_{xy}(h) = \Gamma_x(h)\sigma_x(\Gamma_y(h))$ *and* $\Gamma_{x\star y}(\sigma_y(h)) = \Gamma_x(\Gamma_y(h))\Gamma_{xyx^{-1}}(\Gamma_x(h^{-1}))$;
3. $l\Gamma_x(l)\sigma_{(xy\star z)}(\sigma_x(k^{-1})\Gamma_z(\sigma_x(k^{-1})))h^t(xy,z)$
   $= \sigma_x(l\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y,z))\sigma_{{}^x(y\star z)}(l\Gamma_x(l)h^t(x,z))$;
4. $\Gamma_x(\sigma_y(l))\sigma_y(l)\sigma_{(x\star yz)}(h^{-1}\sigma_y(l^{-1})\Gamma_{yz}(h^{-1}))h^t(x,yz)$
   $= \sigma_{(x\star y)}(h^{-1}\Gamma_y(h^{-1})\sigma_y(hl\Gamma_x(l)\sigma_{(x\star z)}(h^{-1}l^{-1}\Gamma_z(h^{-1}))h^t(x,z)))h^t(x,y)$;
5. $ABC = 1$, *where*

$$
\begin{aligned}
A &= hk^2\Gamma_x(k)\sigma_y(l)\sigma_{{}^yz}(k^{-1})\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y)\Gamma_{(x\star y)}(k\sigma_y(l)\sigma_{{}^yz}(k^{-1}))\\
&\quad\cdot\sigma_{((x\star y)\star{}^yz)}(h^{-1}k^{-1}\Gamma_x(k^{-1})\sigma_y(l^{-1})\sigma_{{}^yz}(k)\sigma_{(x\star y)}(hk\Gamma_y(h))\\
&\quad\cdot h^t(x,y)^{-1}\Gamma_{{}^yz}(h^{-1}k^{-1}\Gamma_x(k^{-1})\sigma_{(x\star y)}(hk\Gamma_y(h))h^t(x,y)^{-1}))\\
&\quad\cdot h^t(x\star y,{}^yz),\\
B &= \sigma_{((x\star y)\star{}^yz)}(l^2\Gamma_y(l)\sigma_z(h)\sigma_{{}^zx}(l^{-1})\sigma_{(y\star z)}(k^{-1}l^{-1}\Gamma_z(k^{-1}))h^t(y,z)\\
&\qquad\Gamma_{(y\star z)}(l\sigma_z(h)\sigma_{{}^zx}(l^{-1}))\\
&\quad\cdot\sigma_{((y\star z)\star{}^zx)}(k^{-1}l^{-1}\Gamma_y(l^{-1})\sigma_z(h^{-1})\sigma_{{}^zx}(l)\sigma_{(y\star z)}(kl\Gamma_z(k))h^t(y,z)^{-1}\\
&\quad\cdot\Gamma_{{}^zx}(k^{-1}l^{-1}\Gamma_y(l^{-1})\sigma_{(y\star z)}(kl\Gamma_z(k))h^t(y,z)^{-1}))\\
&\quad\cdot h^t(y\star z,{}^zx)),
\end{aligned}
$$

$$C = \sigma_{((y\star z)\star^z x)((z\star x)\star^x y)}(h^2\Gamma_z(h)\sigma_x(k)\sigma_{^xy}(h^{-1})\sigma_{(z\star x)}(l^{-1}h^{-1}\Gamma_x(l^{-1}))$$
$$\cdot h^t(z,x)\Gamma_{(z\star x)}(h\sigma_x(k)\sigma_{^xy}(h^{-1}))$$
$$\cdot \sigma_{((z\star x)\star^x y)}(l^{-1}h^{-2}\Gamma_z(h^{-1})\sigma_x(k^{-1})\sigma_{^xy}(h)\sigma_{(z\star x)}(lh\Gamma_x(l))h^t(z,x)^{-1}$$
$$\cdot \Gamma_{^xy}(l^{-1}h^{-1}\Gamma_z(h^{-1})\sigma_{(z\star x)}(lh\Gamma_x(l))h^t(z,x)^{-1}))$$
$$\cdot h^t(z\star x, {^xy})).$$

6. $\sigma_{^zx}(l)\sigma_{^zy}(l)\sigma_z(\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y))$
$= l\Gamma_{^zx}(l\sigma_z(k)\sigma_{^zy}(l^{-1}))\sigma_{(^zx\star^zy)}(l^{-1}\sigma_z(h^{-1}k^{-1})\sigma_{^zx}(l)\sigma_{^zy}(l)\Gamma_{^zy}(l^{-1}\sigma_z(h^{-1})\sigma_{^zx}(l)))$
$\cdot h^t({^zx},{^zy})$.

Then we have a multiplicative Lie algebra structure $\tilde{\star}$ on G defined by

$$(h,x)\,\tilde{\star}\,(k,y) = (hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y), x\star y),$$

for all $(h,x),(k,y)\in G$.

In this case, we say $\tilde{\star}$ is induced by $\star$, and maps Γ and h^t.

Conversely, let $\tilde{\star}$ be a multiplicative Lie algebra structure on G such that H is an ideal. Then there is a multiplicative Lie algebra structure $\star$ on K, and maps $\Gamma : K \to \mathrm{End}(H)$ and $h^t : K\times K \to H$ that satisfies conditions from (1) to (6) given above such that

$(h,x)\,\tilde{\star}\,(k,y) = (hk\Gamma_x(k)\sigma_{(x\star y)}(h^{-1}k^{-1}\Gamma_y(h^{-1}))h^t(x,y), x\star y)$ *for all* $(h,x),(k,y)\in G$.

In this case, we say $\tilde{\star}$ is determined by $\star$, and maps Γ and h^t.

The above theorem is very technical and hard to apply in general. So, now onwards, we restrict ourself to the case of direct product of groups.

Remark 2.5. In particular, suppose $G = H\times K$ (that is, $\sigma = I_H$) and there are maps $\Gamma : K\to \mathrm{End}(H)$ and $h^t : K\times K\to H$ that satisfies the following conditions for all $x,y,z\in K$ and $h,k,l\in H$:

1. $h^t(x,1) = h^t(1,x) = h^t(x,x) = 1$;
2. $\Gamma_{xy}(h) = \Gamma_x(h)\Gamma_y(h)$ and $\Gamma_{x\star y}(h) = \Gamma_x(\Gamma_y(h))\Gamma_y(\Gamma_x(h^{-1}))$, that is, Γ is a multiplicative Lie algebra homomorphism;
3. $h^t(xy,z) = h^t(x,z)h^t(y,z)$;
4. $h^t(x,yz) = h^t(x,y)h^t(x,z)$;
5. $\Gamma_z(h^t(x,y)^{-1})\Gamma_x(h^t(y,z)^{-1})\Gamma_y(h^t(z,x)^{-1})h^t(x\star y,{^yz})h^t(y\star z,{^zx})h^t(z\star x,{^xy}) = 1$;
6. $h^t({^zx},{^zy}) = h^t(x,y)$.

Then we have a multiplicative Lie algebra structure $\tilde{\star}$ on G defined by

$$(h,x)\,\tilde{\star}\,(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1})h^t(x,y), x\star y), \quad \text{for all } (h,x),(k,y)\in G.$$

Conversely, let $\tilde{\star}$ be a multiplicative Lie algebra structure on G such that H is an ideal. Then there is a multiplicative Lie algebra structure $\star$ on K, and maps $\Gamma : K\to\mathrm{End}(H)$ and $h^t : K\times K\to H$ that satisfies conditions from (1) to (6) given above such that

$$(h,x)\,\tilde{\star}\,(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1})h^t(x,y), x\star y), \quad \text{for all } (h,x),(k,y)\in G.$$

Remark 2.6. Let $G = H \times K$, where H is an abelian group with trivial multiplicative Lie algebra structure and K is a finite group generated by two elements a and b such that $(|H|, |a|) = 1$. It is easy to verify that there is no non-trivial alternating map from $K \times K$ to H. Therefore, every multiplicative Lie algebra structure $\tilde{\star}$ on G with respect to which H is an ideal is determined by a multiplicative Lie algebra structure $\star$ on K and a multiplicative Lie algebra homomorphism $\Gamma : K \to \mathrm{End}(H)$, and it is defined by

$$(h, x) \,\tilde{\star}\, (k, y) = (\Gamma_x(k)\Gamma_y(h^{-1}), x \star y), \quad \text{for all } x, y \in K \text{ and } h, k \in H.$$

Proposition 2.7. *Let $G = H \times K$, where H is an abelian group of order m with trivial multiplicative Lie algebra structure and K is a group of order n such that $(m, n) = 1$. Then every multiplicative Lie algebra structure $\tilde{\star}$ on G is determined by a multiplicative Lie algebra structure $\star$ on K and a multiplicative Lie algebra homomorphism $\Gamma : K \to \mathrm{End}(H)$, and it is defined as*

$$(h, x) \,\tilde{\star}\, (k, y) = (\Gamma_x(k)\Gamma_y(h^{-1}), x \star y), \quad \textit{for all } x, y \in K \textit{ and } h, k \in H.$$

Proof. Let $\tilde{\star}$ be a multiplicative Lie algebra structure on G. By Remark 2.5, it is sufficient to show that H is an ideal of G and there is only trivial bilinear map h^t from $K \times K$ to H.

Let $a \in H$ and $g \in G$. Then $1 = a^m \,\tilde{\star}\, g = (a \,\tilde{\star}\, g)^m$. Suppose $a \,\tilde{\star}\, g = hy$, where $h \in H$ and $y \in K$. Since $H \subseteq Z(G), 1 = (a \,\tilde{\star}\, g)^m = h^m y^m = y^m$. This implies that $y = 1$. Hence, $a \,\tilde{\star}\, g \in H, \forall g \in G$.

Suppose $x, y \in K$ and order of y is k. Then $h^t(x, y^k) = 0 = h^t(x, y)^k$. Since $(m, k) = 1$, we have $h^t(x, y) = 0$. Now, it is easy to see that $h^t(a, b) = 0$ for all $a, b \in K$. ∎

Example 2.8. Let D_p be the dihedral group of order $2p$ with multiplicative Lie algebra structure $\star$, where p is a prime number. Suppose $G = \mathbb{Z}_p \times D_p$. Then by Remark 2.5, we have a multiplicative Lie algebra structure $\tilde{\star}$ on G induced by $\star$, and maps Γ and h^t, and it is defined as

$$(h, x) \,\tilde{\star}\, (k, y) = (\Gamma_x(k)\Gamma_y(h^{-1})h^t(x, y), x \star y).$$

It is clear that h is a bilinear map. Let $x, y \in D_p$ with order of y is 2. Then $h^t(x, y^2) = 0 = h^t(x, y)^2$. Since $\mathbb{Z}_p$ has no element of order 2, we have $h^t(x, y) = 0$. Now, it is easy to see that $h^t(a, b) = 0$ for all $a, b \in D_p$.
Since there is only trivial homomorphism $D_p \to \mathrm{End}(\mathbb{Z}_p) \cong \mathbb{Z}_p$, Γ is trivial. Hence, $(h, x)\tilde{\star}(k, y) = (0, x \star y)$. Since D_p has only two multiplicative Lie algebra structures, G has also two multiplicative Lie algebra structures for which $\mathbb{Z}_p$ is ideal. ∎

We already know that the symmetric group S_3 has two distinct multiplicative Lie algebra structure [3]. The following example gives another method to compute the same with the help of Theorem 2.4.

Example 2.9. Let $G = \mathbb{Z}_3 \rtimes_\sigma \mathbb{Z}_2$ and $\tilde{\star}$ be a non-trivial multiplicative Lie algebra structure on G, where $\sigma : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3)$ is non-trivial group homomorphism. Since $\mathbb{Z}_3$ is the only proper normal subgroup of G, $G \,\tilde{\star}\, G = \mathbb{Z}_3$.

Now, by Theorem 2.4, $\tilde{\star}$ determined by a multiplicative Lie algebra structure $\star$ on $\mathbb{Z}_2$, and maps Γ and h^t. Since $\mathbb{Z}_2$ has only trivial multiplicative Lie algebra structure,

$$(h,x)\,\tilde{\star}\,(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1})h^t(x,y), 0).$$

It is easy to see that $h^t(x,y) = 0$ for all $x, y \in \mathbb{Z}_2$. Hence,

$$(h,x)\,\tilde{\star}\,(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1}), 0).$$

Also, it is easy to see that there is a non zero map $\Gamma : \mathbb{Z}_2 \to \mathrm{End}(\mathbb{Z}_3)$ which satisfies $\Gamma_{xy}(h) = \Gamma_x(h)\sigma_x(\Gamma_y(h))$ for every $h \in \mathbb{Z}_3$. ■

Example 2.10. Consider the group $G = \mathbb{Z}_p \times D_n$, $(p, 2n) = 1$. Then by Proposition 2.7, every multiplicative Lie algebra structure $\tilde{\star}$ on G is determined by a multiplicative Lie algebra structure $\star$ on D_n and map Γ. Since there is only trivial homomorphism $D_n \to \mathrm{End}(\mathbb{Z}_p) \cong \mathbb{Z}_p$, Γ is trivial. Therefore,

$$(h,x)\,\tilde{\star}\,(k,y) = (0, x \star y).$$

Now, by Theorem 2.5 of [3], D_n has $\tau(n)$ multiplicative Lie algebra structures. So, G has also $\tau(n)$ multiplicative Lie algebra structures for which $\mathbb{Z}_p$ is ideal. ■

Example 2.11. Let Q_n be the dicyclic group of order $4n$. Suppose $G = \mathbb{Z}_p \times Q_n$, $(p, 4n) = 1$. Then by Proposition 2.7, every multiplicative Lie algebra structure $\tilde{\star}$ on G is determined by a multiplicative Lie algebra structure $\star$ on Q_n and map Γ. Therefore, $(h,x)\,\tilde{\star}\,(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1}), x \star y)$.
Since there is only trivial homomorphism $Q_n \to \mathrm{End}(\mathbb{Z}_p) \cong \mathbb{Z}_p$, Γ is trivial. Hence, $(h,x)\,\tilde{\star}\,(k,y) = (0, x \star y)$. Now, by Theorem 2.5 of [3], Q_n has $\tau(n)$ multiplicative Lie algebra structures. So, G has also $\tau(n)$ multiplicative Lie algebra structures for which $\mathbb{Z}_p$ is ideal. ■

Example 2.12. Let $D_4 = \langle a, b \mid a^2 = 1 = b^4 = 1, ab = b^{-1}a \rangle$ be the dihedral group of order 8 with multiplicative Lie algebra structure $\star$. Suppose $G = \mathbb{Z}_4 \times D_4$. Let $\Gamma : D_4 \to \mathrm{End}(\mathbb{Z}_4) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$ and $h^t : D_4 \times D_4 \to \mathbb{Z}_4$ be maps satisfying all conditions given in Remark 2.5. Hence, we have a multiplicative Lie algebra structure $\tilde{\star}$ on G defined as

$$(h,x)\,\tilde{\star}\,(k,y) = (\Gamma_x(k)\Gamma_y(h^{-1})h^t(x,y), x \star y).$$

Suppose $x, y \in D_4$, where order of y is 2. Then $h^t(x, y^2) = \bar{0} = h^t(x,y)^2$, order of $h^t(x,y)$ is either 1 or 2. Hence, there are only two bilinear maps, one is trivial and the other one is defined by $h^t(a,b) = \bar{2}$. Also, there are four group homomorphism Γ from D_4 to $\mathrm{End}(\mathbb{Z}_4) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$ defined by

1. $\Gamma_a = \tilde{0}$ and $\Gamma_b = \tilde{0}$;
2. $\Gamma_a = \tilde{2}$ and $\Gamma_b = \tilde{0}$;
3. $\Gamma_a = \tilde{0}$ and $\Gamma_b = \tilde{2}$;
4. $\Gamma_a = \tilde{2}$ and $\Gamma_b = \tilde{2}$.

We know that D_4 has three distinct multiplicative Lie algebra structures defined as $a \star b = 1$, $a \star b = b$ and $a \star b = b^2 = [a, b]$ (Theorem 2.5, [3]).

Case I: For $a \star b = 1$, it is easy to see that every pair of (Γ, h^t) satisfies all the conditions given in Remark 2.5, where $\Gamma : D_4 \to \mathrm{End}(\mathbb{Z}_4)$ is a group homomorphism and $h^t : D_4 \times D_4 \to \mathbb{Z}_4$ is a bilinear map.

If $\tilde{\star}$ is non trivial, then $G \,\tilde{\star}\, G \cong \mathbb{Z}_2$.

Case II: For $a \star b = b$, there are two multiplicative Lie algebra homomorphism Γ from D_4 to $\mathrm{End}(\mathbb{Z}_4) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$ given by

1. $\Gamma_a = \tilde{0}$ and $\Gamma_b = \tilde{0}$;
2. $\Gamma_a = \tilde{2}$ and $\Gamma_b = \tilde{0}$.

Also, these two multiplicative Lie algebra homomorphisms satisfy all the conditions given in Remark 2.5 with every bilinear map $h^t : D_4 \times D_4 \to \mathbb{Z}_4$.

In this case, $G \,\tilde{\star}\, G \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_4$.

Case III: Similarly, for $a \star b = b^2$, it is easy to see that every pair of (Γ, h^t) satisfies all the conditions given in Remark 2.5, where $\Gamma : D_4 \to \mathrm{End}(\mathbb{Z}_4)$ is a group homomorphism and $h^t : K \times K \to H$ is a bilinear map.

In this case, $G \,\tilde{\star}\, G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2$. ■

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