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Irreducible Characters of the Generalized Symmetric Group

Huimin Gao, Naihuan Jing

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Abstract. The paper studies how to compute irreducible characters of the generalized symmetric group $C_k \wr S_n$ by iterative algorithms. After proving the Ariki-Koike version of the Murnaghan-Nakayama rule by vertex algebraic method, we formulate a new iterative formula for characters of the generalized symmetric group. As applications, we find a numerical relation between the character values of $C_k \wr S_n$ and modular characters of S_{kn}.

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1. Introduction

The character theory of the symmetric group S_n plays an important role in representation theory [8]. It is well known that the irreducible characters χ^λ of S_n are indexed by partitions λ of n, and explicit character values are given by the celebrated Frobenius formula, which expresses the character values as the transition coefficients between the Schur symmetric functions and the power-sum symmetric functions [8, 15]. The character formula can also be formulated using a vertex algebraic method [9], where the irreducible character values of S_n are expressed by matrix coefficients of Schur vertex operators [10] and products of Heisenberg operators.

The theory of irreducible characters of the wreath product $G \wr S_n$, where G is any finite group, was given by Specht [22]. Under the Frobenius-type charactersitic, the irreducible characters $\chi^{\boldsymbol{\lambda}}$ of $G \wr S_n$ correspond to the wreath product Schur symmetric functions [23] labelled by partitions $\boldsymbol{\lambda}$ colored by the irreducible characters of G. The character values are then given by the transition coefficients between the wreath product Schur symmetric functions and power-sum symmetric functions of partitions colored by the conjugacy classes of G.

Zelevinsky also studied the representations of $G \wr S_n$ using the language of PSH-algebras [25]. The Grothendieck group of the category of finite-dimensional complex representations possesses an additional structure of Hopf algebra obeying positivity and self-adjointness axioms.

In [5, 7] Frenkel, Wang and the second named author reformulated the Specht character theory using the vertex algebraic method in the context of the McKay

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correspondence. The character values are given by matrix coefficients of vertex operators of the form

$$\Big\langle \prod p_{\rho_i}(c), \prod X_\gamma(\lambda_i) \Big\rangle,$$

where $X_\gamma(\lambda_i)$ are the components of some vertex operators indexed by irreducible character λ_i of G and $p_{\rho_i}(c)$ are power-sum symmetric functions colored by the conjugacy classes of G.

It is generally believed that the character theory of *the generalized symmetric group* $C_p \wr S_n$, the wreath product of the cyclic group C_p and S_n, is closely related to the modular character theory of the symmetric group S_n over characteristic p (cf. [13]). All Weyl groups of classical type are essentially generalized symmetric groups.

If $k = 1$, the generalized symmetric group reduces to S_n; if $k = 2$, it is specially called the hyperoctahedral group, which is isomorphic to the Weyl group in type B. A character identity relating irreducible character values of the hyperocta-hedral group $C_2 \wr S_n$ and those of the symmetric group S_{2n} was recently found [14, 1]. It is natural to study how to effectively compute character values of the generalized symmetric group $C_k \wr S_n$. Pfeiffer has given programs to compute the character tables of the Weyl groups in GAP [21].

The classical Murnaghan-Nakayama rule is a combinatorial rule for computing the irreducible character χ^λ on conjugacy class ρ of the symmetric group S_n [18, 19]. Recall that a *partition* λ of n is a decreasing sequence of positive integers λ_i whose total sum is n, denoted by $\lambda \vdash n$. The number of λ_i is the length of λ, denoted by $l(\lambda)$. If the parts λ_i are not ordered, then we call λ a *composition* of n.

There are several generalizations of the classical Murnaghan-Nakayama rule. Osima [20] had given a Murnaghan-Nakayama rule in terms of skew representations of the generalized symmetric groups, and Stembridge [24] rediscovered the rule and studied associated combinatorial formulas. This version of the Murnaghan-Nakayama rule requires information of restricting an irreducible representation of $C_k \wr S_n$ to $C_k \wr S_{n-m}$, so it needs extra work for computer programming. Finally Ariki and Koike [2] found the Murnaghan-Nakayama rule for the generalized symmetric group that naturally generalizes the classical one and independent of a priori information of the restriction functor. There are also generalizations of the Murnaghan-Nakayama rule for other related structures [3, 4, 6, 12, 16].

In this paper, we study the Murnaghan-Nakayama rule for the generalized symmetric groups $C_k \wr S_n$ aiming to compute the characters effectively. We first reformulate the Ariki-Koike version of the Murnaghan-Nakayama rule using the vertex algebraic method. Our treatment relies upon the vertex operator realization of Schur functions [10, 11] to derive the iterative formula for the irreducible characters, which is perhaps a quicker derivation of the rule. Another benefit of the vertex algebraic method is that it also gives a *new iterative rule* or the dual rule for the generalized symmetric group by reducing the partition of the irreducible representation by rows.

Our reformulation of the Murnaghan-Nakayama rule is the following result, which is obtained by splitting the operators corresponding to conjugacy classes. This version gives an iterative formula to compute the irreducible characters that may simplify computational complexity.

Theorem 1.1. *We consider colored partitions* $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$ *and* $\boldsymbol{\rho} = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(k-1)})$ *of* n*, where* $\rho^{(s)} = (\rho_1^{(s)}, \cdots, \rho_m^{(s)}, \cdots, \rho_{l(s)}^{(s)})$*. For any fixed* $s \in I$ *and* m*, the value of the irreducible character* $\chi^{\boldsymbol{\lambda}}$ *of* $C_k \wr S_n$ *at the class* $\boldsymbol{\rho}$ *is given by*

$$\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} = \sum_{j=0}^{k-1} \sum_{\boldsymbol{\xi_j}} (-1)^{\mathrm{ht}(\boldsymbol{\lambda_j})} \omega^{-sj} \chi_{\boldsymbol{\rho} \backslash \rho_m^{(s)}}^{\boldsymbol{\lambda} \backslash \boldsymbol{\xi_j}}, \tag{1.1}$$

where $\boldsymbol{\xi_j}$ *runs through all colored* $\rho_m^{(s)}$*-rim hooks contained in* $\boldsymbol{\lambda}$ *that are supported at the j-th constituent.*

Moreover, we can also break down the irreducible characters into lower rank ones using a dual procedure, which leads to a new iterative rule for the generalized symmetric group. Our rule at $k = 1$ seems to be a new combinatorial formula for the irreducible character values of the symmetric group S_n as well.

Theorem 1.2. (New iterative rule) *Given colored partitions* $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(k-1)})$ *and* $\boldsymbol{\rho} = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(k-1)})$ *of* n*. For any fixed* $j \in I$*, the irreducible character value of* $C_k \wr S_n$ *is given by*

$$\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} = \sum_{\substack{\boldsymbol{\mu} \lhd \boldsymbol{\rho}, \|\boldsymbol{\mu}\| \geq \lambda_1^{(j)} \\ \boldsymbol{\tau} \vdash \|\boldsymbol{\mu}\| - \lambda_1^{(j)}}} \omega^{(\eta(\boldsymbol{\tau}) - \eta(\boldsymbol{\mu}))j} \frac{(-1)^{l(\boldsymbol{\tau})}}{k^{l(\boldsymbol{\tau})} z_{\tau}} \chi_{(\boldsymbol{\rho} \backslash \boldsymbol{\mu}) \cup \boldsymbol{\tau}}^{\boldsymbol{\lambda} \backslash \lambda_1^{(j)}} \tag{1.2}$$

summed over colored partitions $\boldsymbol{\mu}$ *and* $\boldsymbol{\tau}$ *such that* $\boldsymbol{\mu} \lhd \boldsymbol{\rho}$ *with weight bigger than* $\lambda_1^{(j)}$ *and* $\boldsymbol{\tau}$ *of weight* $\|\boldsymbol{\mu}\| - \lambda_1^{(j)}$*. Here* z_τ *is defined below* (2.10)*,* τ *is the rearrangement of* $\boldsymbol{\tau}$*, and* $\eta(\boldsymbol{\tau})$*,* $\eta(\boldsymbol{\mu})$ *are the weighted lengths* $\sum_i il(\tau^{(i)})$*,* $\sum_i il(\mu^{(i)})$ *respectively.*

We point it out that the dual Murnaghan-Nakayama rule for the generalized symmetric group gives a different iteration procedure. The former one given by Ariki and Koike simplifies the indexing colored-partition by removing all rim-hooks in each constituent, while our new rule simply removes the largest part in a fixed constituent. This reveals a different relation between representations of the generalized symmetric groups.

We also show how to use the Sagemath program [26] to implement the algorithm. In theory, any irreducible character value of the generalized symmetric group can be iteratively computed according to our program pending on computer's CPU. The source code for this program is included in the appendix. As applications, we also find a numerical relation between the irreducible character values of $C_k \wr S_n$ and those of modular S_{kn} mod k.

The paper is organized into three parts. In section 2, we recall the basic notions about generalized symmetric groups $C_k \wr S_n$ and the construction of the irreducible character value [5]. In section 3, we derive the analog of the Murnaghan-Nakayama rule and another iterative formula for the generalized symmetric groups, using the technique of vertex operators. The dual Maunaghan-Nakayama rule for the symmetric group is given as a consequence. In section 4, we discuss some examples and properties of irreducible character values based on our results. In the end, we formulate a numerical relation between the irreducible character values of $C_k \wr S_n$ and those of modular S_{kn}. The character tables of $C_3 \wr S_1$, $C_3 \wr S_2$, $C_3 \wr S_3$ are attached in the end.

2. Irreducible characters of $C_k \wr S_n$

The generalized symmetric group $C_k \wr S_n$ is the wreath product of the cyclic group C_k of order k with the symmetric group S_n of n elements, i.e. the semidirect product $C_k^n \rtimes S_n$, where the symmetric group S_n acts on the direct product $C_k^n = C_k \times \cdots \times C_k$ by permutating the factors. The multiplication is given by

$$(g;\sigma)\cdot(h;\tau) = (g\sigma(h);\sigma\tau), \tag{2.1}$$

where $g,h \in C_k^n$ and $\sigma,\tau \in S_n$.

Since $C_k = \langle c\rangle$ has exactly k conjugacy classes $\{1\},\{c\},\cdots,\{c^{k-1}\}$, the order of the centralizer of each conjugacy class is k. Its irreducible characters are given by γ_i, $i = 0,\dots,k-1$, where $\gamma_i(c^j) = \omega^{ij}$, and $\omega = e^{2\pi i/k}$ is the k-th primitive root of unity. The space of class functions on C_k is given by

$$R(C_k) = \bigoplus_{i=0}^{k-1} \mathbb{C}\gamma_i.$$

The conjugacy classes of $C_k \wr S_n$ are parametrized by colored partitions. The color set is $I = \{0,1,\cdots,k-1\}$, the indexing set of conjugacy classes. The conjugacy class of an element $x = (g;\sigma) \in C_k \wr S_n$ corresponds to *the I-colored partition* $\boldsymbol{\lambda} = (\lambda^{(i)})$, which consists of k partitions $\lambda^{(i)}$ such that $\|\boldsymbol{\lambda}\| = \sum_{i=0}^{k-1}|\lambda^{(i)}| = n$, the ith partition $\lambda^{(i)}$ given by

$$\lambda^{(i)} = (1^{m_1(c^i)}2^{m_2(c^i)}\cdots),$$

where $m_j(c^i)$ equals to the number of j-cycles in σ whose cycle product (of the elements $g_a \in C_k$, where $a \in$ the cycle) is c^i. We also say that the colored partition $\boldsymbol{\lambda} = (\lambda^{(i)})$ is supported at its constituent subpartitions $\lambda^{(i)}$. In particular, a colored partition $\boldsymbol{\lambda}$ supported at only one color has one nontrivial constituent subpartition, and all other constituents are empty.

Given a partition $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_l)$, one can paint the parts with the colors $\{0,1,\dots,k-1\}$, to get $k^{l(\lambda)}$ colored partitions $\boldsymbol{\lambda}$, which are referred as the colored partitions arising from λ. On the other hand, all these $k^{l(\lambda)}$ colored partitions become to the partition λ by erasing their colors. If $\boldsymbol{\lambda} = (\lambda^{(i)})$ is a colored partition, we define the *weighted length* $\eta(\boldsymbol{\lambda})$ as $\sum_i il(\lambda^{(i)})$.

We also call $\boldsymbol{\lambda}$ the type of the conjugacy class. It is known that two elements are conjugate in $C_k \wr S_n$ if and only if they have the same type (cf [15]).

A *hook* is a special partition of the form $(a,1^b)$, where $a-1$ is the length of its arm and b is the length of its leg. The hook length is defined as $a+b$. For a general partition λ, its hook length at $x = (i,j) \in \lambda$, of the Young diagram, is defined by

$$h(x) = \lambda_i + \lambda'_j - i - j + 1,$$

and the hook length of λ is defined as the product of $h(x)$ for $x \in \lambda$, where λ' is the conjugate of λ.

Example 2.1.

In $C_3 \wr S_7$, $x = ((1,1,1,\omega^1,\omega^1,\omega^2,1);(1,2,3)(4,5)(6,7))$

and $y = ((1,\omega^2,\omega^1,1,1,1,\omega^1);(1,4,5)(2,6)(3,7))$

are conjugate, since they have the same type $((3),(\phi),(2,2))$, where (3) has color 0 and $(2,2)$ has color 2. ■

Irreducible characters of the generalized symmetric group $C_k \wr S_n$ were determined by Specht [22]. Frenkel, Wang, and the second author have given a vertex algebraic method to express the irreducible character values by matrix coefficients of certain vertex operators [5], which we now recall in the following.

Let $\{a_n(\gamma_i)|n \in \mathbb{Z}\backslash\{0\},\ i \in I\} \cup \{c\}$ be the set of generators of the Heisenberg algebra $\mathcal{H}$ with defining relations

$$[a_n(\gamma_i), a_m(\gamma_j)] = n\delta_{n,-m}\delta_{i,j}c, \tag{2.2}$$

$$[a_n(\gamma_i), c] = 0. \tag{2.3}$$

Consider the Fock space $V = \mathrm{Sym}(a_{-n}(\gamma_i)'s)$, the polynomial algebra on the variables $a_{-n}(\gamma_i)$, $n \in \mathbb{N}$ and $i \in I$, where $\mathbb{N}$ is the set of positive integers. The algebra $\mathcal{H}$ acts on V by the following rule:

$$a_{-n}(\gamma_i).v = a_{-n}(\gamma_i)v, \tag{2.4}$$

$$c.v = v, \tag{2.5}$$

$$\begin{aligned} &a_n(\gamma_i).a_{-n_1}(\alpha_1)a_{-n_2}(\alpha_2)\cdots a_{-n_k}(\alpha_k) \\ &= \sum_{j=1}^{k} \delta_{n,n_j}\langle\gamma_i,\alpha_j\rangle a_{-n_1}(\alpha_1)a_{-n_2}(\alpha_2)\cdots a_{-n_k}(\alpha_k) \end{aligned} \tag{2.6}$$

where $\alpha_j \in R(C_k)$, $\langle\gamma_i,\gamma_j\rangle = \delta_{i,j}$ and $\langle\gamma_i,\alpha_j\rangle$ is extended the whole space by sesque-linearity, where the second component is complex linear.

For convenience, we denote $a_n(\gamma_i) := a_{i,n}$, where the first index i stands for γ_i. The space V is equipped with the sesquelinear form inner given by

$$\langle 1,1\rangle = 1, \quad a^*_{i,n} = a_{i,-n}, \tag{2.7}$$

where $a^*_{i,n}$ denotes the adjoint of $a_{i,n}$. We denote by $A_i(z)$ and $A^*_i(z)$ the generating series:

$$A_i(z) = \sum_{n\in\mathbb{N}} a_{i,-n}z^n, \quad A^*_i(z) = \sum_{n\in\mathbb{N}} a_{i,n}z^{-n}. \tag{2.8}$$

Moreover, for a partition $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_l)$, we define

$$a_{i,-\lambda} = a_{i,-\lambda_1}a_{i,-\lambda_2}\cdots a_{i,-\lambda_l}, \tag{2.9}$$

which form an orthogonal basis:

$$\langle a_{i,-\lambda}, a_{j,-\mu}\rangle = \delta_{i,j}\delta_{\lambda\mu}z_\lambda, \tag{2.10}$$

where $z_\lambda = \prod_i i^{m_i}m_i!$ for $(1^{m_1}2^{m_2}\cdots)$, the exponential notation of λ.

In fact, $\mathcal{H}$ has another set of generators consisting of the Fourier transform of the first set: $\{\widetilde{a}_{i,n}|n \in \mathbb{Z}\backslash\{0\}, i \in I\} \cup \{c\}$, where

$$\widetilde{a}_{i,n} = \sum_{j=0}^{k-1} \omega^{-ij} a_{j,n}. \tag{2.11}$$

Under the inverse Fourier transform, we have that

$$a_{i,n} = \frac{1}{k} \sum_{j=0}^{k-1} \omega^{ij} \widetilde{a}_{j,n}. \tag{2.12}$$

Similarly, for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ we denote

$$\widetilde{a}_{i,-\lambda} = \widetilde{a}_{i,-\lambda_1} \widetilde{a}_{i,-\lambda_2} \cdots \widetilde{a}_{i,-\lambda_l}. \tag{2.13}$$

For a colored partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$, let

$$\widetilde{a}_{-\boldsymbol{\lambda}} = \widetilde{a}_{0,-\lambda^{(0)}} \widetilde{a}_{1,-\lambda^{(1)}} \cdots \widetilde{a}_{k-1,-\lambda^{(k-1)}}. \tag{2.14}$$

Then $\widetilde{a}_{-\boldsymbol{\lambda}}$ form another basis of $\mathcal{H}$.

We introduce the vertex operator $X_i(z)$ and its adjoint $X_i^*(z)$ as the linear maps: $V \longrightarrow V[[z, z^{-1}]]$ given by

$$X_i(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} a_{i,-n} z^n\Big) \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} a_{i,n} z^{-n}\Big) = \sum_{n\in\mathbb{Z}} X_{i,n} z^n, \tag{2.15}$$

$$X_i^*(z) = \exp\Big(-\sum_{n=1}^{\infty} \frac{1}{n} a_{i,-n} z^n\Big) \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} a_{i,n} z^{-n}\Big) = \sum_{n\in\mathbb{Z}} X_{i,n}^* z^{-n}. \tag{2.16}$$

In fact, for a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, it is known that

$$s_{i,\lambda} = X_{i,\lambda_1} X_{i,\lambda_2} \cdots X_{i,\lambda_l}.1 \tag{2.17}$$

is the Schur function associated to the parition λ in the variables $a_{i,-n}$ (as a power-sum symmetric function) [10]. Then we have

$$\langle s_{i,\lambda}, s_{i,\mu} \rangle = \delta_{\lambda\mu}. \tag{2.18}$$

For a colored partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$, we denote

$$s_{\boldsymbol{\lambda}} = s_{0,\lambda^{(0)}} s_{1,\lambda^{(1)}} \cdots s_{k-1,\lambda^{(k-1)}}. \tag{2.19}$$

We recall the following vertex algebraic formulation of Specht characters:

Theorem 2.1. [5] *For given colored partitions* $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$ *and* $\boldsymbol{\rho} = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(k-1)})$ *of* n*, the matrix coefficient*

$$\langle s_{\boldsymbol{\lambda}}, \widetilde{a}_{-\boldsymbol{\rho}} \rangle$$

is equal to the value of the irreducible character $\chi^{\boldsymbol{\lambda}}$ *at the conjugacy class of type* $\boldsymbol{\rho}$ *in* $C_k \wr S_n$*.*

3. Two iterative formulas

Using techniques of vertex operators, we have the following commutation relations for the vertex operators (cf. [11]):

$$zX_i(z)X_i(w) + wX_i(w)X_i(z) = 0, \tag{3.1}$$

$$A_i^*(z)X_i(w) = X_i(w)A_i^*(z) + \frac{w}{z-w}X_i(w), \tag{3.2}$$

$$X_i^*(z)A_i(w) = A_i(w)X_i^*(z) + \frac{w}{z-w}X_i^*(z). \tag{3.3}$$

Taking the coefficients of $z^m w^n$, we derive the following commutation relations among the components.

Proposition 3.1. *For $m, n \in \mathbb{N}, i \in I$, we have*

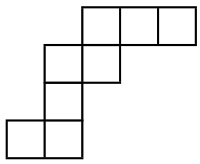
$$X_{i,m}X_{i,n} = -X_{i,n-1}X_{i,m+1}, \tag{3.4}$$

$$X_{i,m}^*.1 = 0, \quad a_{i,-m}^*.1 = 0, \tag{3.5}$$

$$a_{i,-m}^*X_{i,n} = X_{i,n}a_{i,-m}^* + X_{i,n-m}, \tag{3.6}$$

$$X_{i,m}^*a_{i,-n} = a_{i,-n}X_{i,m}^* + X_{i,m-n}^*. \tag{3.7}$$

Recall that for two partitions $\mu \subset \lambda$, if the skew partition $\lambda - \mu$ is connected and contains no 2×2 boxes, we call $\lambda-\mu$ a *rim hook* or *border strip*. The height of $\lambda-\mu$ is defined to be one less than the number of rows occupied, denoted by $\mathrm{ht}(\lambda-\mu)$ or $\mathrm{ht}(\lambda/\mu)$. For example, take $\lambda = (5,3,2,2)$, $\mu = (2,1,1)$, then $\lambda - \mu$ can be depicted as



which is a 8-rim hook of λ with height 3. Note that a hook is a special case of the rim hook.

If ξ is a rim hook contained in a Young diagram λ, then $\lambda\backslash\xi$ denotes the *subdiagram* of λ obtained by removing ξ. It is clear that $\lambda\backslash\xi$ is also a Young diagram, so we also use it refer to the associated partition. In the above example, $\xi = (5,3,2,2)-(2,1,1)$ is the rim hook, and the difference partition is

$$\lambda \setminus \xi = (2,1,1). \tag{3.8}$$

A colored partition $\boldsymbol{\lambda} = (\lambda^{(i)})$ is contained in a colored partition $\boldsymbol{\mu} = (\mu^{(i)})$, denoted by $\boldsymbol{\lambda} \subset \boldsymbol{\mu}$, if each constituent partition $\lambda^{(i)} \subset \mu^{(i)}$. The difference $\boldsymbol{\xi} = \boldsymbol{\lambda} - \boldsymbol{\mu}$ is called a rim-hook if one of the constituents $\lambda^{(i)} - \mu^{(i)}$ is a rim-hook and all other constituents are identical, i.e., the colored rim-hook $\boldsymbol{\xi}$ is only supported at the ith constituent.

For an integral l-tuple $\mu = (\mu_1, \ldots, \mu_l)$, we define the lowering operator T_n for $1 \le n \le l-1$ by

$$T_n: \quad (\mu_1, \cdots \mu_n, \mu_{n+1}, \cdots \mu_l) \mapsto (\mu_1, \cdots \mu_{n+1} - 1, \mu_n + 1, \cdots \mu_l).$$

Now we give a characterization of m-rim hook η by using lowering operators. Let $\xi = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_i - m, \lambda_{i+1}, \cdots, \lambda_l)$ be the integral l-tuple obtained by the action (3.6) of vertex operators.

Lemma 3.2. *Any m-rim hook η can be written as a difference of a partition λ and* $\mathrm{ht}(\eta)$ *applications of T_i to the integral l-tuple* $(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - m, \lambda_{i+1}, \dots, \lambda_l)$.

Proof. Assume that a m-rim hook η begins at the i-th row of λ. If its height is 0, i.e., it only occupies the i-th row of λ. Then $m \le \lambda_i - \lambda_{i+1}$, or $\lambda_i - m \ge \lambda_{i+1}$, so ξ is already a partition and $\eta = \lambda - \xi$. Suppose now $\mathrm{ht}(\eta) = 1$ and that it occupies the i-th and $(i+1)$-th rows of λ. By definition of a rim hook,

$$\lambda_i - \lambda_{i+1} + 2 \le m \le \lambda_i - \lambda_{i+2} + 1,$$

so $\lambda_{i+2} - 1 \le \lambda_i - m \le \lambda_{i+1} - 2$. Hence $\eta = \lambda - T_i(\xi)$ and $T_i(\xi)$ is a partition. In general, if the height of an m-rim hook is $r \ge 1$, one has

$$\begin{aligned} \sum_{j=0}^{r-1} (\lambda_{i+j} - \lambda_{i+j+1} + 1) + 1 &\le m \\ &\le \sum_{j=0}^{r-1} (\lambda_{i+j} - \lambda_{i+j+1} + 1) + (\lambda_{i+r} - \lambda_{i+r+1}). \end{aligned} \tag{3.9}$$

Then $\lambda_{i+r+1} - r \le \lambda_i - m \le \lambda_{i+r} - (r+1)$, and we obtain $\eta = \lambda - T_{i+r-1} \cdots T_i(\xi)$. ∎

Lemma 3.3. *For each $j \in I$, any partition $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_l)$, and any $m \in \mathbb{N}$, we have*

$$a^*_{j,-m} s_{j,\lambda} = \sum_{\mu} (-1)^{\mathrm{ht}(\lambda/\mu)} s_{j,\mu}, \tag{3.10}$$

summed over all partitions μ such that $\lambda - \mu$ is an m-rim hook.

Proof. Repeatedly using (3.6) we have that

$$a^*_{j,-m} s_{j,\lambda} = \sum_{i=1}^{l} X_{j,\lambda_1} \cdots X_{j,\lambda_i - m} \cdots X_{j,\lambda_l}.1. \tag{3.11}$$

The vector $X_{j,\lambda_1} \cdots X_{j,\lambda_i - m} \cdots X_{j,\lambda_l}.1$ is the Schur function associated with integral l-tuple $(\lambda_1, \dots, \lambda_{i-1}, \lambda_i - m, \lambda_{i+1} \dots, \lambda_l)$. If $\lambda_i - m < \lambda_{i+1}$, we need to straighten out the integral l-tuple to partition. We use (3.4) to rewrite

$$X_{j,\lambda_1} \cdots X_{j,\lambda_i - m} \cdots X_{j,\lambda_l}.1 = -X_{j,\lambda_1} \cdots X_{j,\lambda_{i+1} - 1} X_{j,\lambda_i - m + 1} \cdots X_{j,\lambda_l}.1.$$

If $\lambda_i - m + 1 = \lambda_{i+1}$, then the term vanishes. If $\lambda_i - m + 1 < \lambda_{i+2}$, then we repeat the swapping $(\lambda_i - m + 1, \lambda_{i+2}) \to (\lambda_{i+2} - 1, \lambda_{i+2} - m + 2)$ until we have a partition. Suppose after r steps, we reach the partition

$$\xi = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_{i+1} - 1, \cdots, \lambda_{i+r} - 1, \lambda_i - m + r, \lambda_{i+r+1}, \cdots, \lambda_l).$$

Then the skew partition

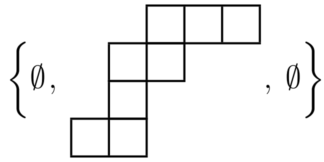
$$\lambda - \xi = (0^{i-1}, \lambda_i - \lambda_{i+1} + 1, \dots, \lambda_{i+r} - \lambda_i + m - r, \lambda_{i+r+1}, \dots, \lambda_l)$$

is a m-rim hook and $s_{j,\lambda} = (-1)^r s_{j,\xi}$.

Conversely, suppose ξ is a partition such that $\lambda - \xi$ is a horizontal m-strip. By Lemma 3.2 the term s_ξ appears in the expansion of $a^*_{j,-m} s_{j,\lambda}$. Therefore (3.10) is proved. ∎

Example 3.1. $$a^*_{1,-4}s_{1,(5,3,2)} = s_{1,(1,3,2)} + s_{1,(5,-1,2)} + s_{1,(5,3,-2)}$$
$$= -s_{1,(2,2,2)} - s_{1,(5,1,0)}\,.$$ ■

Similarly, a colored partition $\boldsymbol{\mu} \subset \boldsymbol{\lambda}$ such that $\boldsymbol{\lambda} - \boldsymbol{\mu}$ is a *colored rim hook* if and only if one of its constituents is a rim hook and the others are ϕ. The height of $\boldsymbol{\lambda} - \boldsymbol{\mu}$ is exactly the height of its nonempty constituent, denoted by $\mathrm{ht}(\boldsymbol{\lambda} - \boldsymbol{\mu})$ or $\mathrm{ht}(\boldsymbol{\lambda}/\boldsymbol{\mu})$. For example, if $\boldsymbol{\lambda} = ((2,1),(5,3,2,2),(4))$, and $\boldsymbol{\mu} = ((2,1),(2,1,1),(4))$, then $\boldsymbol{\lambda} - \boldsymbol{\mu}$ is an 8-rim hook colored at the second constituent with height 3.



If $\boldsymbol{\xi}$ is a colored rim hook contained in a colored Young diagram $\boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \setminus \boldsymbol{\xi}$ denotes the *colored subdiagram* of $\boldsymbol{\lambda}$ obtained by removing $\boldsymbol{\xi}$. Similarly we also use $\boldsymbol{\lambda} \setminus \boldsymbol{\xi}$ to denote the associated colored partition.

Now we are ready to give a vertex algebraic proof of the Murnaghan-Nakayama rule for the generalized symmetric group $C_k \wr S_n$. The formula (3.12) was first proved by group theoretical method in [2].

Theorem 3.4. *We consider again colored partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$ and $\boldsymbol{\rho} = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(k-1)})$ of n, where $\rho^{(s)} = (\rho^{(s)}_1, \cdots, \rho^{(s)}_m, \cdots, \rho^{(s)}_{l(s)})$. For any fixed $s \in I$ and m, the value of the irreducible character $\chi^{\boldsymbol{\lambda}}$ of $C_k \wr S_n$ at the class $\boldsymbol{\rho}$ is given by*

$$\chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}} = \sum_{j=0}^{k-1} \sum_{\boldsymbol{\xi_j}} (-1)^{\mathrm{ht}(\boldsymbol{\xi_j})} \omega^{-sj} \chi^{\boldsymbol{\lambda} \setminus \boldsymbol{\xi_j}}_{\boldsymbol{\rho} \setminus \rho^{(s)}_m}, \tag{3.12}$$

where $\boldsymbol{\xi_j}$ runs through all colored $\rho^{(s)}_m$-rim hooks contained in $\boldsymbol{\lambda}$ that are supported at the j-th constituent.

Proof. Fixing $j \in I$, it follows from (2.6) that

$$a^*_{j,-\rho^{(s)}_m} s_{i,\lambda^{(i)}} = 0, \quad \text{if } i \neq j\,.$$

Then nonzero contributions only come from

$$a^*_{j,-\rho^{(s)}_m} s_{\boldsymbol{\lambda}} = s_{0,\lambda^{(0)}} \cdots (a^*_{j,-\rho^{(s)}_m} s_{j,\lambda^{(j)}}) \cdots s_{k-1,\lambda^{(k-1)}}\,.$$

By Lemma 3.3, $a^*_{j,-\rho^{(s)}_m} s_{\boldsymbol{\lambda}}$ is expanded into an alternating sum of $s_{\boldsymbol{\lambda} \setminus \boldsymbol{\xi_j}}$ such that $\boldsymbol{\xi_j}$ is a colored $\rho^{(s)}_m$-rim hook supported at the j-th constituent. By Theorem 2.1 and converting back to the basis $\widetilde{a}_{s,-\rho^{(s)}_m}$, using

$$\widetilde{a}_{s,-\rho^{(s)}_m} = \sum_{j=0}^{k-1} \omega^{-sj} a_{j,-\rho^{(s)}_m},$$

we have shown the theorem. ■

Example 3.2. In the group $C_3 \wr S_{12}$, we have

$$\chi^{((4,1),(3,1,1),(2))}_{((5,2),(3),(1,1))} = -\chi^{(\emptyset,(3,1,1),(2))}_{((2),(3),(1,1))} + \chi^{((4,1),\emptyset,(2))}_{((2),(3),(1,1))}.$$

Similarly in $C_3 \wr S_{14}$, one has

$$\chi^{((3,2),(4,2,1),(2))}_{((2,1),(4,3),(3,1))} = -\chi^{((1),(4,2,1),(2))}_{((2,1),(3),(3,1))} - \omega^2 \chi^{((3,2),(1,1,1),(2))}_{((2,1),(3),(3,1))},$$

where ω is the 3-rd primitive root of unity. ■

Similar to the symmetric group, we have the following result by applying Theorem 3.4. Recall that a *colored hook* is a special colored partition if and only if one of its constituent is a hook and the others are $\emptyset$. The length of arm and leg of a colored hook are exactly those of its nonempty constituent respectively. Obviously a colored hook is a special colored rim hook.

Proposition 3.5. *Let $\boldsymbol{\kappa}_i$ be a colored partition supported only at the i-th constituent (n) and ω the k-th primitive root of unity. Then the irreducible character $\chi^{\boldsymbol{\lambda}}$ of $C_k \wr S_n$ satisfies*

$$\chi^{\boldsymbol{\lambda}}(\boldsymbol{\kappa}_i) = (-1)^r \omega^{-ij} \quad or \quad 0 \tag{3.13}$$

according as $\boldsymbol{\lambda}$ is a colored hook or not, where j is the index of its non-empty constituent and r is its length of leg.

Now we study how to decompose the irreducible characters using the dual vertex operator in the other direction.

For each partition $\rho = (\rho_1, \rho_2, \cdots, \rho_l)$, a *subpartition* denoted by $\mu \lhd \rho$ is $\mu = \emptyset$ or $\mu = (\rho_{i_1}, \rho_{i_2}, \cdots, \rho_{i_s})$, for some $1 \leqslant i_1 < \cdots < i_s \leqslant l$. Obviously, there are $2^{l(\rho)}$ subpartitions μ such that $\mu \lhd \rho$ and $\rho\backslash\mu$ is also a partition. For example, if $\rho = (4,2,2,1)$, and $\mu = (2,1)$, then $\mu \lhd \rho$ and $\rho\backslash\mu = (4,2)$. Moreover, $\mu \cup \tau$ of partitions μ and τ, is defined as the union of parts of μ and τ in the descending order.

Lemma 3.6. *For any partition $\rho = (\rho_1, \rho_2, \cdots, \rho_l)$, $m \in \mathbb{N}$, and $i, j \in I$, one has that*

$$X^*_{j,m} \widetilde{a}_{i,-\rho} = \sum_{\mu \lhd \rho} \omega^{-ij\cdot(l(\mu))} \widetilde{a}_{i,-\rho\backslash\mu} X^*_{j,m-|\mu|},$$

summed over all subpartitions of ρ.

Proof. For each $1 \leq s \leq l$, it follows from the linearity of $\widetilde{a}_{i,-\rho_s}$ and the action of (3.7) that

$$X^*_{j,m} \widetilde{a}_{i,-\rho_s} = \widetilde{a}_{i,-\rho_s} X^*_{j,m} + \omega^{-ij} X^*_{j,m-\rho_s}. \tag{3.14}$$

In the case of length 2 partition, the expression becomes

$$\widetilde{a}_{i,-\rho} X^*_{j,m} + \omega^{-ij} \widetilde{a}_{i,-\rho_1} X^*_{j,m-\rho_2} + \omega^{-ij} \widetilde{a}_{i,-\rho_2} X^*_{j,m-\rho_1} + \omega^{-ij\cdot 2} X^*_{j,m-|\rho|}.$$

Using induction on $l(\rho)$, we get the lemma. ■

For two colored partitions $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$, if for each $i \in I$, such that $\mu^{(i)} \lhd \rho^{(i)}$, we write $\boldsymbol{\mu} \lhd \boldsymbol{\rho}$. The set minus $\boldsymbol{\rho}\backslash\boldsymbol{\mu}$ (removing the empty rows) is also a colored partition. For example, if $\boldsymbol{\rho} = ((5,3,1),(4,2,2,1),(4))$, and $\boldsymbol{\mu} = ((3),(2,1),(4))$, then $\boldsymbol{\mu} \lhd \boldsymbol{\rho}$ and $\boldsymbol{\rho}\backslash\boldsymbol{\mu} = ((5,1),(4,2),\emptyset)$.

A colored partition $\boldsymbol{\rho}$ naturally gives rise to a partition by forgetting the colors, we call this partition the *rearrangement* of $\boldsymbol{\rho}$. For example, take $\boldsymbol{\rho} = ((5,1),\emptyset,(4,2))$, it can be rearranged into a regular partition $(5,4,2,1)$. We also define the length of a colored partition $l(\boldsymbol{\rho})$ as the sum of $l(\rho^{(i)})$. Recall that the weighted length $\eta(\boldsymbol{\rho})$ is defined as $\sum_i il(\rho^{(i)})$. The juxtaposition $\boldsymbol{\mu} \cup \boldsymbol{\tau}$ of two colored partitions is defined as $(\mu^{(i)} \cup \tau^{(i)})$.

Now we are ready to give our second main result–the dual Murnaghan-Nakayama rule for the generalized symmetric group.

Theorem 3.7. *We consider again colored partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$ and $\boldsymbol{\rho} = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(k-1)})$ of n. For any fixed $j \in I$, the irreducible character value of $C_k \wr S_n$ is given by*

$$\chi^{\boldsymbol{\lambda}}_{\boldsymbol{\rho}} = \sum_{\substack{\boldsymbol{\mu} \lhd \boldsymbol{\rho}, \|\boldsymbol{\mu}\| \geq \lambda_1^{(j)} \\ \boldsymbol{\tau} \vdash \|\boldsymbol{\mu}\| - \lambda_1^{(j)}}} \omega^{(\eta(\boldsymbol{\tau}) - \eta(\boldsymbol{\mu}))j} \frac{(-1)^{l(\boldsymbol{\tau})}}{k^{l(\boldsymbol{\tau})} z_\tau} \chi^{\boldsymbol{\lambda}\backslash\lambda_1^{(j)}}_{(\boldsymbol{\rho}\backslash\boldsymbol{\mu})\cup\boldsymbol{\tau}} \tag{3.15}$$

where the sum is over colored partitions $\boldsymbol{\mu}$ and $\boldsymbol{\tau}$ such that $\boldsymbol{\mu} \lhd \boldsymbol{\rho}$, with weight bigger than $\lambda_1^{(j)}$, and $\boldsymbol{\tau}$ of weight $\|\boldsymbol{\mu}\| - \lambda_1^{(j)}$. z_τ is defined in (2.10), *τ is the rearrangement of $\boldsymbol{\tau}$, and $\eta(\boldsymbol{\tau})$, $\eta(\boldsymbol{\mu})$ are the weighted lengths $\sum_i il(\tau^{(i)})$, $\sum_i il(\mu^{(i)})$ respectively.*

Proof. Applying Lemma 3.6 repeatedly, one has

$$X^*_{j,m} \widetilde{a}_{-\boldsymbol{\rho}} = \sum_{\boldsymbol{\mu} \lhd \boldsymbol{\rho}} \omega^{-\eta(\boldsymbol{\mu})j} \widetilde{a}_{-\boldsymbol{\rho}\backslash\boldsymbol{\mu}} X^*_{j,m-\|\boldsymbol{\mu}\|}.1, \tag{3.16}$$

summed over all colored partitions $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} \lhd \boldsymbol{\rho}$. By (3.5), nonzero items survive only when $m - \|\boldsymbol{\mu}\| \leq 0$. Taking the coefficient of z^n in $X_j(z)^*$, we have

$$X^*_{j,n}.1 = \sum_{\rho \vdash n} \frac{(-1)^{l(\rho)}}{z_\rho} a_{j,-\rho} \quad for \quad n \geq 0. \tag{3.17}$$

By (2.9) and (2.11), for a partition $\tau = (\tau_1, \tau_2, \cdots, \tau_{l(\tau)})$ we have

$$\begin{aligned} a_{j,-\tau} &= \frac{1}{k^{l(\tau)}} \prod_{s=1}^{l(\tau)} \left(\sum_{i_s=0}^{k-1} \omega^{j i_s} \widetilde{a}_{i_s,-\tau_s} \right) \\ &= \frac{1}{k^{l(\tau)}} \sum_{(i_1,\cdots i_{l(\tau)}) \in I^{l(\tau)}} \omega^{|i|j} \widetilde{a}_{i_1,-\tau_1} \cdots \widetilde{a}_{i_{l(\tau)},-\tau_{l(\tau)}}, \end{aligned} \tag{3.18}$$

where $(i_1, \cdots i_{l(\tau)})$ runs through the Cartesian product $I^{l(\tau)}$ and $|i| = i_1 + \cdots + i_{l(\tau)}$. We rewrite the $k^{l(\tau)}$ summands by reordering factors according to their colors, where the pairs $(i_1,\tau_1),(i_2,\tau_2),\ldots,(i_{l(\tau)},\tau_{l(\tau)})$ are rearranged according to their colors. Then the composition $(\tau_1, \tau_2, \ldots, \tau_{l(\tau)})$ becomes a colored partition $\boldsymbol{\tau}$ and the summation of its colors $|i| = i_1 + \cdots + i_{l(\tau)}$ becomes $\eta(\boldsymbol{\tau})$. Therefore

$$a_{j,-\tau} = \frac{1}{k^{l(\tau)}} \sum_{\boldsymbol{\tau}} \omega^{j\eta(\boldsymbol{\tau})} \tilde{a}_{-\boldsymbol{\tau}}, \tag{3.19}$$

where the sum runs through all colored partitions arising from τ. Plugging (3.19) into (3.18) and using (3.16) and (3.17), we have derived the recurrence formula. ∎

We remark that the dual Murnaghan-Nakayama rule of $C_k \wr S_n$ offers a different iterative route from the Ariki-Koike version by directly simplifying the indexing colored-partition of the irreducible representation by removing the largest row in a fixed constituent, while the usual Murnaghan-Nakayama rule is iterated by removing all possible rim-hooks from the indexing partition in all constituents.
When $k = 1$, we have the following dual Murnaghan-Nakayama rule for the symmetric group, which seems new to the authors' knowledge.

Theorem 3.8. *The irreducible character value of S_n is given by*

$$\chi^{\lambda}_{\rho} = \sum_{\substack{\mu \lhd \rho, |\mu| \geq \lambda_1 \\ \tau \vdash |\mu| - \lambda_1}} \frac{(-1)^{l(\tau)}}{z_\tau} \chi^{\lambda \backslash \lambda_1}_{(\rho \backslash \mu) \cup \tau}, \tag{3.20}$$

summed over partitions μ and τ such that μ is a subpartition of ρ with weight bigger than λ_1, and τ of weight $|\mu| - \lambda_1$.

Example 3.3. Using the recurrence relation, we see that in $C_2 \wr S_4$

$$\chi^{((1),(3))}_{((2,1),(1))} = -\frac{1}{2}\chi^{((1),\emptyset)}_{((1),\emptyset)} + \frac{1}{2}\chi^{((1),\emptyset)}_{(\emptyset,(1))}.$$

Similarly in $C_3 \wr S_8$, we have

$$\begin{aligned}
\chi^{((2),(5),(1))}_{((3,1),(2),(2))} = \;& \frac{8}{9}\chi^{((2),\emptyset,(1))}_{((3),\emptyset,\emptyset)} - \frac{1}{9}\omega\chi^{((2),\emptyset,(1))}_{(\emptyset,(3),\emptyset)} - \frac{1}{9}\omega^2\chi^{((2),\emptyset,(1))}_{(\emptyset,\emptyset,(3))} \\
& - \frac{1}{9}\chi^{((2),\emptyset,(1))}_{((2,1),\emptyset,\emptyset)} - \frac{5}{18}\omega^2\chi^{((2),\emptyset,(1))}_{(\emptyset,(2,1),\emptyset)} - \frac{5}{18}\omega\chi^{((2),\emptyset,(1))}_{(\emptyset,\emptyset,(2,1))} \\
& + \frac{1}{18}\omega\chi^{((2),\emptyset,(1))}_{((2),(1),\emptyset)} + \frac{1}{18}\omega^2\chi^{((2),\emptyset,(1))}_{((2),\emptyset,(1))} - \frac{5}{18}\chi^{((2),\emptyset,(1))}_{(\emptyset,(2),(1))} \\
& + \frac{5}{9}\omega\chi^{((2),\emptyset,(1))}_{((1),(2),\emptyset)} + \frac{5}{9}\omega^2\chi^{((2),\emptyset,(1))}_{((1),\emptyset,(2))} - \frac{5}{18}\chi^{((2),\emptyset,(1))}_{(\emptyset,(1),(2))} \\
& + \frac{4}{81}\chi^{((2),\emptyset,(1))}_{((1,1,1),\emptyset,\emptyset)} - \frac{1}{162}\chi^{((2),\emptyset,(1))}_{(\emptyset,(1,1,1),\emptyset)} - \frac{1}{162}\chi^{((2),\emptyset,(1))}_{(\emptyset,\emptyset,(1,1,1))} \\
& + \frac{5}{54}\omega\chi^{((2),\emptyset,(1))}_{((1,1),(1),\emptyset)} + \frac{5}{54}\omega^2\chi^{((2),\emptyset,(1))}_{((1,1),\emptyset,(1))} - \frac{1}{54}\omega\chi^{((2),\emptyset,(1))}_{(\emptyset,(1,1),(1))} \\
& - \frac{1}{54}\omega^2\chi^{((2),\emptyset,(1))}_{(\emptyset,(1),(1,1))} + \frac{1}{27}\omega^2\chi^{((2),\emptyset,(1))}_{((1),(1,1),\emptyset)} + \frac{1}{27}\omega\chi^{((2),\emptyset,(1))}_{((1),\emptyset,(1,1))} \\
& + \frac{2}{27}\chi^{((2),\emptyset,(1))}_{((1),(1),(1))},
\end{aligned}$$

where ω is the 3-rd primitive root of unity. In the expansion of $\chi^{((2),\emptyset,(1))}_{(\emptyset,(2,1),\emptyset)}$, there are only two terms: $\boldsymbol{\mu} = ((3,1),\emptyset,(2))$, $\boldsymbol{\tau} = (\emptyset,(1),\emptyset)$ and $\boldsymbol{\mu'} = ((3,1),(2),(2))$, $\boldsymbol{\tau'} = (\emptyset,(2,1),\emptyset)$. Therefore its coefficient is a sum of two terms. ■

The following is a special case of a direct consequence of Theorem 3.7.

Proposition 3.9. *Let $\boldsymbol{\kappa}_i$ be a colored partition supported only at the ith constituent (n). Then for any colored partition $\boldsymbol{\rho} = (\rho^{(0)},\rho^{(1)},\cdots,\rho^{(k-1)})$ of n, we have that*

$$\chi^{\boldsymbol{\kappa}_i}_{\boldsymbol{\rho}} = \omega^{-\eta(\boldsymbol{\rho})i}, \tag{3.21}$$

where $\eta(\boldsymbol{\rho})$ is the weight length $\sum_i il(\rho^{(i)})$.

4. Further examples

For completeness, we first calculate the degree of any irreducible character of $C_k \wr S_n$ [7]. Recall that the inner product in V is given by:

$$\langle a_{-\boldsymbol{\lambda}}, a_{-\boldsymbol{\mu}}\rangle = \prod_{i=0}^{k-1}\langle a_{-\lambda^{(i)}}, a_{-\mu^{(i)}}\rangle = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}\prod_{i=0}^{k-1} z_{\lambda^{(i)}}. \tag{4.1}$$

Under this inner product the wreath Schur functions are orthonormal:

$$\langle s_{\boldsymbol{\lambda}}, s_{\boldsymbol{\mu}}\rangle = \prod_{i=0}^{k-1}\langle s_{\lambda^{(i)}}, s_{\mu^{(i)}}\rangle = \delta_{\boldsymbol{\lambda}\boldsymbol{\mu}}. \tag{4.2}$$

Then we have

$$\langle s_{\boldsymbol{\lambda}}, a_{-\boldsymbol{\mu}}\rangle = \prod_{i=0}^{k-1}\langle s_{\lambda^{(i)}}, a_{-\mu^{(i)}}\rangle. \tag{4.3}$$

The following result is well-known for the wreath products and is included here using the vertex operator approach.

Theorem 4.1. [20] *For any colored partition $\boldsymbol{\lambda} = (\lambda^{(0)},\lambda^{(1)},\cdots,\lambda^{(k-1)})$ of n, the degree of the irreducible character $\chi^{\boldsymbol{\lambda}}$ of $C_k \wr S_n$ is given by*

$$\chi^{\boldsymbol{\lambda}}(\mathbf{1}) = \frac{n!}{\prod_{i=0}^{k-1} h(\lambda^{(i)})}, \tag{4.4}$$

where $\mathbf{1} = ((1^n),\emptyset,\cdots,\emptyset)$ and $h(\lambda^{(i)})$ is the hook length of $\lambda^{(i)}$.

Proof. By definition, $\chi^{\boldsymbol{\lambda}}(\mathbf{1}) = \langle s_{\boldsymbol{\lambda}}, (\widetilde{a}_{0,-1})^n\rangle = \langle s_{\boldsymbol{\lambda}}, (\sum_{j=0}^{k-1} a_{j,-1})^n\rangle$. The non-zero items are only

$$\langle s_{\boldsymbol{\lambda}}, \prod_{j=0}^{k-1} a_{j,-1}^{|\lambda^{(j)}|}\rangle = \prod_{j=0}^{k-1}\langle s_{\lambda^{(j)}}, a_{j,-1}^{|\lambda^{(j)}|}\rangle = \prod_{j=0}^{k-1}\frac{|\lambda^{(j)}|!}{h(\lambda^{(j)})},$$

where the second equality has used the degree formula of irreducible characters in $S_{\lambda^{(j)}}$. Since the coefficient of $\prod_{j=0}^{k-1} a_{j,-1}^{|\lambda^{(j)}|}$ in $(\sum_{j=0}^{k-1} a_{j,-1})^n$ equals to

$$C_n^{|\lambda^{(0)}|}C_{n-|\lambda^{(0)}|}^{|\lambda^{(1)}|}\cdots = \frac{n!}{\prod_{j=0}^{k-1}|\lambda^{(j)}|!},$$

and the theorem is proved. ■

Corollary 4.2. *Given colored partitions $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(k-1)})$ of n, let $\boldsymbol{\kappa}^j$ be a colored partition supported only at the j-th constituent (1^n), and ω be the k-th primitive root of unity. Then we have*

$$\chi^{\boldsymbol{\lambda}}_{\boldsymbol{\kappa}^j} = \frac{\omega^{-\deg(\boldsymbol{\lambda})j} n!}{\prod_{i=0}^{k-1} h(\lambda^{(i)})}, \tag{4.5}$$

where $\deg(\boldsymbol{\lambda})$ *is defined as the sum of* $i|\lambda^{(i)}|$.

In the dual case, one has the following result.

Proposition 4.3. *Given a colored partition $\boldsymbol{\rho} = (\rho^{(0)}, \rho^{(1)}, \cdots, \rho^{(k-1)})$ of n, the irreducible character value is*

$$\chi^{\boldsymbol{\kappa}^j}_{\boldsymbol{\rho}} = \omega^{-\eta(\boldsymbol{\rho})j}(-1)^{l(\boldsymbol{\rho})+n}, \tag{4.6}$$

where $\boldsymbol{\kappa}^j$ is a colored partition supported only at the j-th constituent (1^n), $\eta(\boldsymbol{\rho})$ is the weighted length $\sum_i il(\rho^{(i)})$.

Proof. In view of (4.3), the only nonzero factor of $\langle s_{\boldsymbol{\kappa}^j}, \widetilde{a}_{\boldsymbol{\rho}}\rangle$ is

$$\langle s_{j,(1^n)}, \prod_{i=0}^{k-1} \omega^{-ij|\rho^{(i)}|} a_{j,-\rho^{(i)}}\rangle$$

since $|\lambda^{(j)}|$ exhausts n. Note that $\prod_{i=0}^{k-1} a_{j,-\rho^{(i)}}$ can be written as $a_{j,-\rho}$, where ρ is the rearrangement of $\boldsymbol{\rho}$.

For the character value $\chi^{(1^n)}_{\rho}$, we apply Theorem 3.8 with induction on n. First $\chi^{(1)}_{(1)} = 1$ is clear. The inductive hypothesis says that

$$\chi^{(1^{n-1})}_{(\rho\backslash\mu)\cup\tau} = (-1)^{l(\rho)-l(\mu)+l(\tau)+n-1}.$$

Note that
$$\sum_{\lambda\vdash n} \frac{1}{z_\lambda} = 1.$$

Then
$$\chi^{(1^n)}_{\rho} = (-1)^{l(\rho)+n-1} \sum_{\mu\lhd\rho, |\mu|\geq 1} (-1)^{l(\mu)},$$

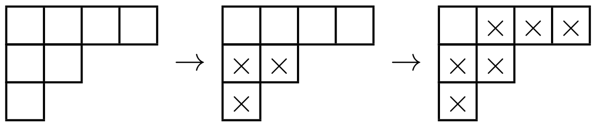
where the sum is over all nonempty subpartitions $\mu \lhd \rho$, which is an alternating sum of binomial coefficients:

$$\sum_{k=1}^{l(\rho)} (-1)^k \binom{l(\rho)}{k} = -C^0_{l(\rho)} = -1,$$

Therefore $\chi^{(1^n)}_{\rho} = (-1)^{l(\rho)+n}$. Obviously $l(\boldsymbol{\rho}) = l(\rho)$, and $\prod_{i=0}^{k-1} \omega^{-ij|\rho^{(i)}|}$ equals to $\omega^{-\eta(\boldsymbol{\rho})j}$, subsequently we have shown $\chi^{\boldsymbol{\kappa}^j}_{\boldsymbol{\rho}}$ is given as stated. ∎

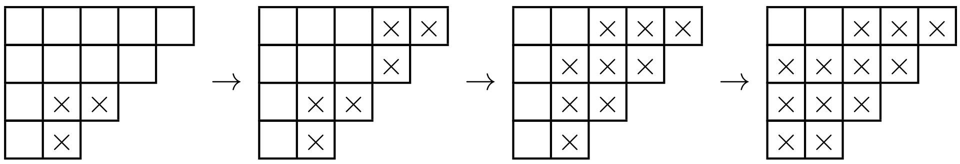
Next we move on to study a relationship between irreducible character values of $C_k \wr S_n$ and those of S_{kn} mod k. Note that k does not have to be a prime.

Recall that the k-*core* γ_λ of partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ is defined by removing all k-rim hooks of λ. For example, let $\lambda = (4, 2, 1)$, $k = 3$,

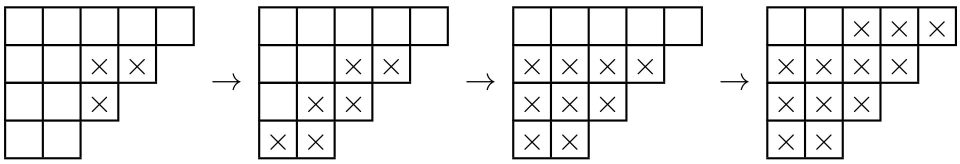


then the 3-core of λ is (1).

Let $T = T(\lambda)$ be the set of paths from partition λ to its core γ_λ, where each path is a procedure of removing k-rim hooks. For each path $t \in T$, denote by $\mathrm{ht}(t)$ the sum of heights of k-rim hooks in path t. It is known that $\sigma_\lambda := (-1)^{\mathrm{ht}(t)}$ is well defined and independent of the paths in $T(\lambda)$ [17, Prop. 2.2]. For example, the 3-core of $\lambda = (5, 4, 3, 2)$ is (2). Consider two paths $t_1, t_2 \in T$ as follows. t_1 has $\mathrm{ht}(t_1) = 5$:



while t_2 has $\mathrm{ht}(t_2) = 3$:



so σ_λ equals to -1.

The k-*quotient* $\boldsymbol{\beta}_\lambda$ of λ can be constructed as follows. Choose m as a multiple of k such that $m \geq l(\lambda)$, let $\delta_m = (m-1, m-2, \cdots, 1, 0)$, $\xi = \lambda + \delta_m$, which is a strict partition. For each $0 \leq r \leq k-1$, suppose there are m_r parts ξ_i that are congruent to r mod k. Each of these $\xi_i\,(1 \leq i \leq m_r)$ is divided by k and reordered as

$$\xi_i = k \cdot \xi_s^{(r)} + r$$

such that $\xi_1^{(r)} > \xi_2^{(r)} > \cdots > \xi_{m_r}^{(r)} \geq 0$. So for each $0 \leq r \leq k-1$, we obtain a partition $\lambda^{(r)} = (\lambda_1^{(r)}, \lambda_2^{(r)}, \cdots, \lambda_{m_r}^{(r)})$ by letting $\lambda_s^{(r)} = \xi_s^{(r)} - m_r + s$ $(1 \leq s \leq m_r)$. Now the colored partition $(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(k-1)}) = \boldsymbol{\beta}_\lambda$ is the k-quotient of λ.

For example, given $\lambda = (4, 2, 1)$ and $k = 3$ as above. Take $m = 3$, then $\xi = (6, 3, 1)$, $m_0 = 2, m_1 = 1$ and $m_2 = 0$. We have

$$\lambda_1^{(0)} = 2 - 2 + 1 = 1, \quad \lambda_2^{(0)} = 1 - 2 + 2 = 1; \quad \lambda_1^{(1)} = 0 - 1 + 1 = 0,$$

thus the 3-quotient of λ is $((1, 1), \emptyset, \emptyset)$.

The above procedure shows that any partition λ determines its k-core γ_λ and k-quotient $\boldsymbol{\beta}_\lambda$ uniquely. Moreover,

$$|\lambda| = k \cdot \|\boldsymbol{\beta}_\lambda\| + |\gamma_\lambda|.$$

In fact, λ also can be determined uniquely by γ_λ and β_λ conversely (cf. [15, I.I. Ex. 8]).

For convenience, we define the *standard form* of an irreducible character value of $C_k \wr S_n$. According to Theorem 3.4, each $\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}$ can be written as

$$c_0 + c_1\omega + \cdots + c_{k-1}\omega^{k-1}, c_i \in \mathbb{Z}$$

since the coefficients of the recursive formula lie in $\mathbb{Z}[\omega]$. However, the k-tuple $(c_0, c_1, \cdots, c_{k-1})$ is not unique due to $1 + \omega + \cdots + \omega^{k-1} = 0$. Indeed,

$$c_0 + c_1\omega + \cdots + c_{k-1}\omega^{k-1} = c_0' + c_1'\omega + \cdots + c_{k-1}'\omega^{k-1}$$

if and only if $c_0 - c_0' = c_1 - c_1' = \cdots = c_{k-1} - c_{k-1}'$. For example, the values of $\chi_{(\emptyset,(1),(1,1))}^{(\emptyset,(2),(1))}$ can be

$$\vdots$$
$$2 + \omega + 3\omega^2$$
$$1 + 2\omega^2$$
$$-\omega + \omega^2$$
$$\vdots$$

The standard form is the *unique* form such that the sum of c_i lies in the color set $I = \{0, 1, \ldots, k-1\}$. We also denote by $d(\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}})$ the sum of c_i in the standard form. Then $-\omega + \omega^2$ is the standard form in the above example, and $d(\chi_{(\emptyset,(1),(1,1))}^{(\emptyset,(2),(1))}) = 0$.

Theorem 4.4. (Relation between $C_k \wr S_n$ and S_{kn}) *Given colored partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ of n. Let λ be the partition of kn with the k-core $\emptyset$ and k-quotient $\boldsymbol{\lambda}$, and let ρ be the rearrangement of $k\boldsymbol{\rho}$. Then*

$$d(\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}) = \sigma_\lambda(\chi_\rho^\lambda \mod k).$$

Proof. Let P be the set of paths from $\boldsymbol{\lambda}$ to $\emptyset$ that remove colored $|\rho_1^{(0)}|$-rim hooks, colored $|\rho_2^{(0)}|$-rim hooks,..., colored $|\rho_{l(k-1)-1}^{(k-1)}|$-rim hooks, colored $|\rho_{l(k-1)}^{(k-1)}|$-rim hooks. By Theorem 3.4 it follows that the irreducible character value $\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}$ of $C_k \wr S_n$ can be written as

$$\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}} = \sum_{p \in P} \sigma_p \omega(p),$$

where the sum is over all paths $p \in P$ and $\sigma_p := (-1)^{\mathrm{ht}(p)}$ with $\mathrm{ht}(p)$ being the sum of all colored rim-hook heights in path p. Note that $\omega(p)$ equals to some power of ω depending on p. Suppose the standard form of $\chi_{\boldsymbol{\rho}}^{\boldsymbol{\lambda}}$ is of the form

$$\sum_{i=0}^{k-1} c_i \omega^i.$$

On the other hand, let $\tilde{P}$ be the set of paths from λ to $\emptyset$ that remove $k|\rho_1|$-rim hooks, $k|\rho_2|$-rim hooks,..., and $k|\rho_{l(\lambda)}|$-rim hooks. By the ordinary Murnaghan-Nakayama rule of S_{kn}, we have that

$$\chi_\rho^\lambda = \sum_{\tilde{p} \in \tilde{P}} \sigma_{\tilde{p}},$$

where $\sigma_{\tilde{p}} = (-1)^{\mathrm{ht}(\tilde{p})}$ is defined similarly as σ_p.

It is known that σ_p and $\sigma_{\tilde{p}}$ are related by the fundamental relation [20]

$$\sigma_p = \sigma_\lambda \sigma_{\tilde{p}}, \tag{4.7}$$

and there is a bijection between the sets P and $\tilde{P}$. This implies that $\sum_i c_i$ in the standard form equals to $\sigma_\lambda \cdot \chi_\rho^\lambda \mod k$, which finishes the proof. ■

Finally we discuss how to implement our iterative formulas using the SageMath program.

An algorithm design for Theorem 3.4 to display all coefficients of the MN rule is as follows:

Step 1: Given two colored partitions of length k and weight n, named $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$;

Step 2: Assume the largest part (of weight m) in $\boldsymbol{\rho}$ be removed (to reduce the times of iterations), list all colored partitions of weight $n-m$ included in $\boldsymbol{\lambda}$;

Step 3: Filter out colored partitions $\boldsymbol{\lambda_j}$ such that skew partition $\boldsymbol{\lambda} - \boldsymbol{\lambda_j}$ is a rim hook, and compute the corresponding coefficient.

An algorithm outline of listing all coefficients for Theorem 3.7 goes as follows:

Step 1: Given two colored partitions of length k and weight n, named $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$;

Step 2: Identify and remove the largest part in $\boldsymbol{\lambda}$ (of weight s), filter out sub-partitions $\boldsymbol{\mu}$ of $\boldsymbol{\rho}$ with weight less than or equal to $n-s$, and corresponding partitions τ of size $n-s-|\boldsymbol{\mu}|$, then compute the corresponding coefficients;

Step 3: List all possible colored partitions $\boldsymbol{\tau}$ corresponding τ, also compute each of coefficient.

To end this section, we list the character tables of $C_3 \wr S_1$, $C_3 \wr S_2$, $C_3 \wr S_3$, we denote by ω the 3-rd primitive root of unity.

Table 1: $C_3 \wr S_1$

$\gamma \backslash c$	c_1	c_2	c_3
γ_1	1	1	1
γ_2	ω	ω^2	1
γ_3	ω^2	ω	1

In Table 1:

$\gamma_1 = c_3 = ((1), \emptyset, \emptyset)$; $\gamma_2 = c_2 = (\emptyset, (1), \emptyset)$; $\gamma_3 = c_1 = (\emptyset, \emptyset, (1))$.

In Table 2: (see next page)

$\gamma_1 = c_9 = ((2), \emptyset, \emptyset)$; $\gamma_2 = c_8 = ((1,1), \emptyset, \emptyset)$; $\gamma_3 = c_7 = ((1), (1), \emptyset)$;
$\gamma_4 = c_6 = ((1), \emptyset, (1))$; $\gamma_5 = c_5 = (\emptyset, (2), \emptyset)$; $\gamma_6 = c_4 = (\emptyset, (1,1), \emptyset)$;
$\gamma_7 = c_3 = (\emptyset, (1), (1))$; $\gamma_8 = c_2 = (\emptyset, \emptyset, (2))$; $\gamma_9 = c_1 = (\emptyset, \emptyset, (1,1))$.

In Table 3: (see next but one page)

$\gamma_1 = c_{22} = ((3), \emptyset, \emptyset)$; $\gamma_2 = c_{21} = ((2,1), \emptyset, \emptyset)$; $\gamma_3 = c_{20} = ((1,1,1), \emptyset, \emptyset)$;
$\gamma_4 = c_{19} = ((2), (1), \emptyset)$; $\gamma_5 = c_{18} = ((1,1), (1), \emptyset)$; $\gamma_6 = c_{17} = ((2), \emptyset, (1))$;

Table 2: $C_3 \wr S_2$

$\gamma\backslash c$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
γ_1	1	1	1	1	1	1	1	1	1
γ_2	1	-1	1	1	-1	1	1	1	-1
γ_3	2ω	0	-1	$2\omega^2$	0	$-\omega^2$	$-\omega$	2	0
γ_4	$2\omega^2$	0	-1	2ω	0	$-\omega$	$-\omega^2$	2	0
γ_5	ω^2	ω	1	ω	ω^2	ω	ω^2	1	1
γ_6	ω^2	$-\omega$	1	ω	$-\omega^2$	ω	ω^2	1	-1
γ_7	2	0	-1	2	0	-1	-1	2	0
γ_8	ω	ω^2	1	ω^2	ω	ω^2	ω	1	1
γ_9	ω	$-\omega^2$	1	ω^2	$-\omega$	ω^2	ω	1	-1

Note that Table 3 is presented on the next page.

Table 4: S_9 mod 3

$\lambda\backslash c$	(9)	$(6,3)$	$(3,3,3)$
$(7,1,1)$	1	1	1
$(4,1,1,1,1,1)$	2	0	2
$(1,1,1,1,1,1,1,1,1)$	1	2	1
$(4,3,2)$	0	2	0
$(2,2,2,1,1,1)$	0	2	0
$(4,4,1)$	0	1	0
$(3,2,1,1,1,1)$	0	1	0
$(5,2,2)$	0	1	0
$(2,2,2,2,1)$	0	1	0
$(3,3,3)$	0	0	0
$(6,2,1)$	0	2	0
$(3,2,2,2)$	0	2	0
$(8,1)$	2	2	2
$(5,1,1,1,1)$	1	0	1
$(2,1,1,1,1,1,1,1)$	2	1	2
$(5,4)$	0	2	0
$(3,3,1,1,1)$	0	2	0
$(6,3)$	0	1	0
$(3,3,2,1)$	0	1	0
(9)	1	1	1
$(6,1,1,1)$	2	0	2
$(3,1,1,1,1,1,1)$	1	2	1

Table 3: $C_3 \wr S_3$

$\gamma \backslash c$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}	c_{21}	c_{22}
γ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
γ_2	2	0	-1	2	0	2	0	2	0	-1	2	0	2	2	0	2	0	2	0	2	0	-1
γ_3	1	-1	1	1	-1	1	-1	1	-1	1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1
γ_4	3ω	ω	0	ω-1	ω^2	ω^2-1	ω	$3\omega^2$	ω^2	0	$\omega - \omega^2$	1	0	$\omega^2 - \omega$	1	1-ω^2	ω	1-ω	ω^2	3	1	0
γ_5	3ω	-ω	0	ω-1	-ω^2	ω^2-1	-ω	$3\omega^2$	-ω^2	0	$\omega - \omega^2$	-1	0	$\omega^2 - \omega$	-1	1-ω^2	-ω	1-ω	-ω^2	3	-1	0
γ_6	$3\omega^2$	ω^2	0	ω^2-1	ω	ω-1	ω^2	3ω	ω	0	$\omega^2 - \omega$	1	0	$\omega - \omega^2$	1	1-ω	ω^2	1-ω^2	ω	3	1	0
γ_7	$3\omega^2$	-ω^2	0	ω^2-1	-ω	ω-1	-ω^2	3ω	-ω	0	$\omega^2 - \omega$	-1	0	$\omega - \omega^2$	-1	1-ω	-ω^2	1-ω^2	-ω	3	-1	0
γ_8	$3\omega^2$	ω	0	1-ω	ω	1-ω^2	ω^2	3ω	ω^2	0	ω-1	ω	0	ω^2-1	ω^2	$\omega - \omega^2$	1	$\omega^2 - \omega$	1	3	1	0
γ_9	$3\omega^2$	-ω	0	1-ω	-ω	1-ω^2	-ω^2	3ω	-ω^2	0	ω-1	-ω	0	ω^2-1	-ω^2	$\omega - \omega^2$	-1	$\omega^2 - \omega$	-1	3	-1	0
γ_{10}	6	0	0	0	0	0	0	6	0	0	0	0	-3	0	0	0	0	0	0	6	0	0
γ_{11}	3ω	ω^2	0	1-ω^2	ω^2	1-ω	ω	$3\omega^2$	ω	0	ω^2-1	ω^2	0	ω-1	ω	$\omega^2 - \omega$	1	$\omega - \omega^2$	1	3	1	0
γ_{12}	3ω	-ω^2	0	1-ω^2	-ω^2	1-ω	-ω	$3\omega^2$	-ω	0	ω^2-1	-ω^2	0	ω-1	-ω	$\omega^2 - \omega$	-1	$\omega - \omega^2$	-1	3	-1	0
γ_{13}	1	ω^2	ω	ω	1	ω^2	1	1	ω	ω^2	ω^2	ω	1	ω	ω^2	ω	ω	ω^2	ω^2	1	1	1
γ_{14}	2	0	-ω	2ω	0	$2\omega^2$	0	2	0	-ω^2	$2\omega^2$	0	2	2ω	0	2ω	0	$2\omega^2$	0	2	0	-1
γ_{15}	1	-ω^2	ω	ω	-1	ω^2	-1	1	-ω	ω^2	ω^2	-ω	1	ω	-ω^2	ω	-ω	ω^2	-ω^2	1	-1	1
γ_{16}	3ω	1	0	$\omega^2 - \omega$	ω^2	$\omega - \omega^2$	ω	$3\omega^2$	1	0	1-ω	ω	0	1-ω^2	ω^2	ω-1	ω^2	ω^2-1	ω	3	1	0
γ_{17}	3ω	-1	0	$\omega^2 - \omega$	-ω^2	$\omega - \omega^2$	-ω	$3\omega^2$	-1	0	1-ω	-ω	0	1-ω^2	-ω^2	ω-1	-ω^2	ω^2-1	-ω	3	-1	0
γ_{18}	$3\omega^2$	1	0	$\omega - \omega^2$	ω	$\omega^2 - \omega$	ω^2	3ω	1	0	1-ω^2	ω^2	0	1-ω	ω	ω^2-1	ω	ω-1	ω^2	3	1	0
γ_{19}	$3\omega^2$	-1	0	$\omega - \omega^2$	-ω	$\omega^2 - \omega$	-ω^2	3ω	-1	0	1-ω^2	-ω^2	0	1-ω	-ω	ω^2-1	-ω	ω-1	-ω^2	3	-1	0
γ_{20}	1	ω	ω^2	ω^2	1	ω	1	1	ω^2	ω	ω	ω^2	1	ω^2	ω	ω^2	ω^2	ω	ω	1	1	1
γ_{21}	2	0	-ω^2	$2\omega^2$	0	2ω	0	2	0	-ω	2ω	0	2	$2\omega^2$	0	$2\omega^2$	0	2ω	0	2	0	-1
γ_{22}	1	-ω	ω^2	ω^2	-1	ω	-1	1	-ω^2	ω	ω	-ω^2	1	ω^2	-ω	ω^2	-ω^2	ω	-ω	1	-1	1

$\gamma_7 = c_{16} = ((1,1),\emptyset,(1))$; $\gamma_8 = c_{15} = ((1),(2),\emptyset)$; $\gamma_9 = c_{14} = ((1),(1,1),\emptyset)$;
$\gamma_{10} = c_{13} = ((1),(1),(1))$; $\gamma_{11} = c_{12} = ((1),\emptyset,(2))$; $\gamma_{12} = c_{11} = ((1),\emptyset,(1,1))$;
$\gamma_{13} = c_{10} = (\emptyset,(3),\emptyset)$; $\gamma_{14} = c_9 = (\emptyset,(2,1),\emptyset)$; $\gamma_{15} = c_8 = (\emptyset,(1,1,1),\emptyset)$;
$\gamma_{16} = c_7 = (\emptyset,(2),(1))$; $\gamma_{17} = c_6 = (\emptyset,(1,1),(1))$; $\gamma_{18} = c_5 = (\emptyset,(1),(2))$;
$\gamma_{19} = c_4 = (\emptyset,(1),(1,1))$; $\gamma_{20} = c_3 = (\emptyset,\emptyset,(3))$; $\gamma_{21} = c_2 = (\emptyset,\emptyset,(2,1))$;
$\gamma_{22} = c_1 = (\emptyset,\emptyset,(1,1,1))$.

Moreover, in consequence of Theorem 4.4 one obtains Table 4, where λ corresponds to $\gamma_1 \sim \gamma_{22}$. They are consistent with the elements in the character table of $S_9 \bmod 3$.

Appendix

A. The source code for Theorem 3.4

```
#  Take k=3, n=5
P=PartitionTuples(3,5)
import random
lam=random.choice(P)
rho=random.choice(P)
print(lam,rho)
#  Set environment
m=max(max(rho))
j=rho.index(max(rho))
r=Zmod(3)
w=SR.var('w')
import numpy
def ht(nums):
  return numpy.count_nonzero(nums)-1
#  Filter out lam satisfying conditions
for k in range(3):
  if lam[k].size()>=m:
  tem=Partitions(lam[k].size()-m,outer=lam[k]).list()

  for i in tem:
    if SkewPartition([lam[k],i]).is_ribbon():
      coff =
      (-1)^ht(SkewPartition([lam[k],i]).row_lengths())*w^r(-j*k)
      l=list(lam)
      l[k]=i
      print(coff,"new lam:",PartitionTuple(l))
```

B. Theorem 3.7

```
# Take k=3, n=5
P=PartitionTuples(3,5)
import random
```

```
lam=random.choice(P)
rho=random.choice(P)
print(lam,rho)
```

Set environment

```
s=max(max(lam))
j=lam.index(max(lam))
r=Zmod(3)
w=SR.var('w')

def sub(nums):
  res = [[]]
  for num in nums:
  res += [ i + [num] for i in res]
  return res

import itertools
array1 = sub(rho[0])
array2 = sub(rho[1])
array3 = sub(rho[2])
combs = itertools.product(array1,array2,array3)
```

Simulate the path

```
import copy
class Solution:
  def subsets(self, nums):
    result = []
    path = []
    self.backtracking(nums, 0, path, result)
    return result

  def backtracking(self, nums, startIndex, path, result):
    result.append(path[:])
    for i in range(startIndex, len(nums)):
      path.append(nums[i])
      self.backtracking(nums, i + 1, path, result)
      path.pop()
```

Filter out rho satisfying conditions

```
for comb in combs:
  if sum(comb[0])+sum(comb[1])+sum(comb[2])<=5-s:
    box0 = comb[0]
    box1 = comb[1]
    box2 = comb[2]
    taus=
    Partitions(5-s-sum(comb[0])-sum(comb[1])-sum(comb[2])).list()
    solu=Solution()
    result =

    for tau in taus:
```

```
d=Partition(tau).centralizer_size()
c=(-1/3)^len(tau)/d*w^r(-(len(rho[1])-
len(comb[1]))*1*j-(len(rho[2])-len(comb[2]))*2*j)
all_index_3 = list(range(len(tau)))
all_index_1 = solu.subsets(all_index_3)

for i_i in range(len(all_index_1)):
  index_2_3 = list(set(all_index_3)-set(all_index_1[i_i]))
  all_index_2_3 = solu.subsets(index_2_3)
  box0_i_index = copy.deepcopy(all_index_2_3)
  box1_i_index = copy.deepcopy(all_index_2_3)
  box2_i_index = copy.deepcopy(all_index_2_3)

  for i_i_i in range(len(all_index_2_3)):
    box0_i_index[i_i_i]
      = list(set(all_index_3)-set(index_2_3))
    box1_i_index[i_i_i]
      = list(set(index_2_3)-set(all_index_2_3[i_i_i]))

    for k_0 in range(len(box0_i_index[i_i_i])):
    box0_i_index[i_i_i][k_0] = tau[box0_i_index[i_i_i][k_0]]
    for k_1 in range(len(box1_i_index[i_i_i])):
    box1_i_index[i_i_i][k_1] = tau[box1_i_index[i_i_i][k_1]]
    for k_2 in range(len(box2_i_index[i_i_i])):
    box2_i_index[i_i_i][k_2] = tau[box2_i_index[i_i_i][k_2]]

    coff =c*( w^r((len(box0_i_index[i_i_i]) * 0
      + len(box1_i_index[i_i_i]) * 1
      + len(box2_i_index[i_i_i]) * 2)*j))
    x = box0 + box0_i_index[i_i_i]
    y = box1 + box1_i_index[i_i_i]
    z = box2 + box2_i_index[i_i_i]
    tmp = list()
    tmp.append(x)
    tmp.append(y)
    tmp.append(z)
    result["rho"] = tmp
    print(coff,"new rho:",result["rho"])
```

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Huimin Gao, School of Mathematics, Southern University of Science and Technology, Shenzhen, China; 12431006@mail.sustech.edu.cn

Naihuan Jing, Department of Mathematics, North Carolina State University, Raleigh, NC 27695, U.S.A.; jing@ncsu.edu.

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