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Admissible Systems and Graded Hermitian Superspaces

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Abstract. We introduce the notion of admissible systems for involutions on complex contragredient Lie superalgebras, and classify the involutions with admissible systems by circlings on extended Dynkin diagrams. We prove the graded Iwasawa decomposition of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ consisisting of the contragredient Lie superalgebra $\mathfrak{g}$ and the fixed points of an involution. We also show the representability in the category of complex superspaces of the corresponding real symmetric superspace.

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1. Introduction

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ be a complex contragredient Lie superalgebra which is not a Lie algebra, so it is one of $A(m,n), B(m,n), C(n), D(m,n), D(2,1;\alpha), F(4), G(3)$ [9]. Let $\operatorname{aut}(\mathfrak{g})$ and $\overline{\operatorname{aut}}(\mathfrak{g})$ respectively denote the set of bijective $\mathbb{C}$-linear and $\mathbb{C}$-antilinear maps which preserve the Lie bracket. We write

$$\operatorname{inv}(\mathfrak{g}), \operatorname{aut}_{2,4}(\mathfrak{g}) \subset \operatorname{aut}(\mathfrak{g}) \ , \ \overline{\operatorname{inv}}(\mathfrak{g}), \overline{\operatorname{aut}}_{2,4}(\mathfrak{g}) \subset \overline{\operatorname{aut}}(\mathfrak{g}),$$

where inv denotes involutions (i.e. order 2), and subscript $(2,4)$ denotes order 2 on $\mathfrak{g}_{\bar{0}}$ and order 4 on $\mathfrak{g}_{\bar{1}}$. There are bijective correspondences

$$\{\sigma \in \overline{\operatorname{inv}}(\mathfrak{g})\} \longleftrightarrow \{\text{standard real forms } \mathfrak{g}_{\mathbb{R}}\} \longleftrightarrow \{\theta \in \operatorname{aut}_{2,4}(\mathfrak{g})\}. \qquad (1)$$

In (1) as well as future discussions, we do not distinguish $\sigma, \mathfrak{g}_{\mathbb{R}}, \theta$ with their images under conjugations by $\mathfrak{g}$-automorphisms. The first correspondence of (1) is simply $\mathfrak{g}^\sigma = \mathfrak{g}_{\mathbb{R}}$, where $\mathfrak{g}^\sigma \subset \mathfrak{g}$ denotes the fixed points of σ. We say that $\mathfrak{g}_{\mathbb{R}}$ is a standard real form if it is a real subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} + i\mathfrak{g}_{\mathbb{R}}$ is a direct sum of real vector spaces. Often this is merely called a real form, but here we add the word *standard* to distinguish from the *graded* real forms discussed below (see [7] for details on standard and graded real forms). There exists a non-degenerate invariant bilinear form B on $\mathfrak{g}$ which is symmetric on $\mathfrak{g}_{\bar{0}}$ and skew-symmetric on $\mathfrak{g}_{\bar{1}}$. The correspondence between $\mathfrak{g}_{\mathbb{R}}$ and θ is as follows: θ stabilizes $\mathfrak{g}_{\mathbb{R}}$, and furthermore $\mathfrak{g}_{\mathbb{R},\bar{1}}$ and each simple ideal of $\mathfrak{g}_{\mathbb{R},\bar{0}}$ has $B(\cdot, \theta\cdot)$ or $-B(\cdot, \theta\cdot)$ as inner product (see [2, Thm.1.1] and [5, p.72]). We say that θ is a Cartan automorphism of $\mathfrak{g}_{\mathbb{R}}$, because it generalizes the Cartan involutions of real semisimple Lie algebras.

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The following correspondences are interesting parallel to (1),

$$\{\sigma \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})\} \longleftrightarrow \{\text{graded real forms } \mathfrak{g}_{\mathbb{R}}\} \longleftrightarrow \{\theta \in \mathrm{inv}(\mathfrak{g})\}. \tag{2}$$

In (2), the correspondence between $\mathfrak{g}_{\mathbb{R}}$ and σ is the following: the graded real form is obtained, via the functor of points in an appropriate category, as the fixed points of the conjugate linear natural transformation coming from σ. The correspondence between σ and θ is more explicit and comes from their direct realizations: see [7, Sec.5], where the theory of graded real forms is introduced and developed in full depth.

We call (1) the standard setting, and (2) the graded setting.

Let $\mathfrak{g}_{\mathbb{R}}$ and θ come from (1) or (2). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{g}^{\theta}$ are the fixed points of θ with center $\mathfrak{z}(\mathfrak{k})$, and $\mathfrak{p}$ is the sum of remaining eigenspaces of θ. It is natural to study and classify the irreducible factors of the adjoint $\mathfrak{k}$-representation on $\mathfrak{p}$, the dimension of $\mathfrak{z}(\mathfrak{k})$, and the existence of complex structures on $\mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_{\mathbb{R}}$. Admissible positive systems of a simple Lie algebra $\mathfrak{g}$ with an involution are essential to construct infinite dimensional representations of the real Lie group associated with the given involution [8]. They are realized in the space of holomorphic sections on the real hermitian symmetric space $G_{\mathbb{R}}/K_{\mathbb{R}}$, where $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$ is the real form of $\mathfrak{g}$ associated with the involution and $\mathfrak{k}_{\mathbb{R}} = \mathrm{Lie}(K_{\mathbb{R}})$ is its maximal compact subalgebra. These questions for Lie algebras are well studied and classified. For the standard real forms (1) of contragredient Lie superalgebras, we obtain analogous results in [3] (see also [11, 9]). The main purpose of this article is to study these problems for the graded real forms (2).

We make the following convention throughout the rest of this article: $\sigma, \mathfrak{g}_{\mathbb{R}}, \theta$ refer to the graded setting (2), $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ are the (± 1)-eigenspaces of $\theta \in \mathrm{inv}(\mathfrak{g})$, $\mathfrak{k}$ and $\mathfrak{g}$ have the same rank.

Let $\mathfrak{h}$ be a common Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{g}$, with root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\Delta} \mathfrak{g}_{\alpha}$. Such common Cartan subalgebra exists because of our equal rank hypothesis. We have the disjoint union $\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$ of $\mathfrak{k}$-roots and $\mathfrak{p}$-roots. In addition if Δ^+ is a positive system, the positive and negative root spaces lead to $\mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^-$. We say that Δ^+ is *admissible* if

$$[\mathfrak{k}, \mathfrak{p}^{\pm}] \subset \mathfrak{p}^{\pm} \ , \ [\mathfrak{p}^{\pm}, \mathfrak{p}^{\pm}] = 0. \tag{3}$$

We shall study the existence of admissible systems, and classify all such cases. To do this, we next recall the extended Dynkin diagrams and circlings, which are effective tools to represent $\mathfrak{g}$-involutions.

Let D be an extended Dynkin diagram of $\mathfrak{g}$. The vertices and edges of D are drawn according to [9, p.54-55]. We write $\mathrm{D} = \mathrm{D}_{\bar{0}} \cup \mathrm{D}_{\bar{1}}$, where $\mathrm{D}_{\bar{0}}$ are the white vertices, and $\mathrm{D}_{\bar{1}}$ are the dark (black or grey) vertices. The vertices represent a simple system and its lowest root, so D depends on the choice of simple system. There is a distinguished choice such that [1, Thm.1.1]

$$\begin{aligned} &\mathrm{D}_{\bar{0}} = \text{Dynkin diagram of } [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}], \\ &\mathrm{D}_{\bar{1}} = \text{lowest (with respect to } \mathrm{D}_{\bar{0}}) \text{ weights of adjoint } \mathfrak{g}_{\bar{0}}\text{-action on } \mathfrak{g}_{\bar{1}}. \end{aligned} \tag{4}$$

We shall always let D denote this diagram. The vertices of D represent linearly dependent roots, and there are unique positive integers $\{a_{\alpha}\}_{\mathrm{D}}$ which are relatively prime such that $\sum_{\mathrm{D}} a_\alpha \alpha = 0$. For each connected component C of $\mathrm{D}_{\bar{0}}$, there are unique positive integers $\{b_\alpha\}_C$ such that $\sum_C b_\alpha \alpha$ is the highest weight of C. Figure 1 provides the list of all D together with $\{(a_\alpha, b_\alpha)\}_{\mathrm{D}_{\bar{0}}}$ and $\{a_\alpha\}_{\mathrm{D}_{\bar{1}}}$. See Remark 2.3 for $\mathfrak{sl}(n,n)$.

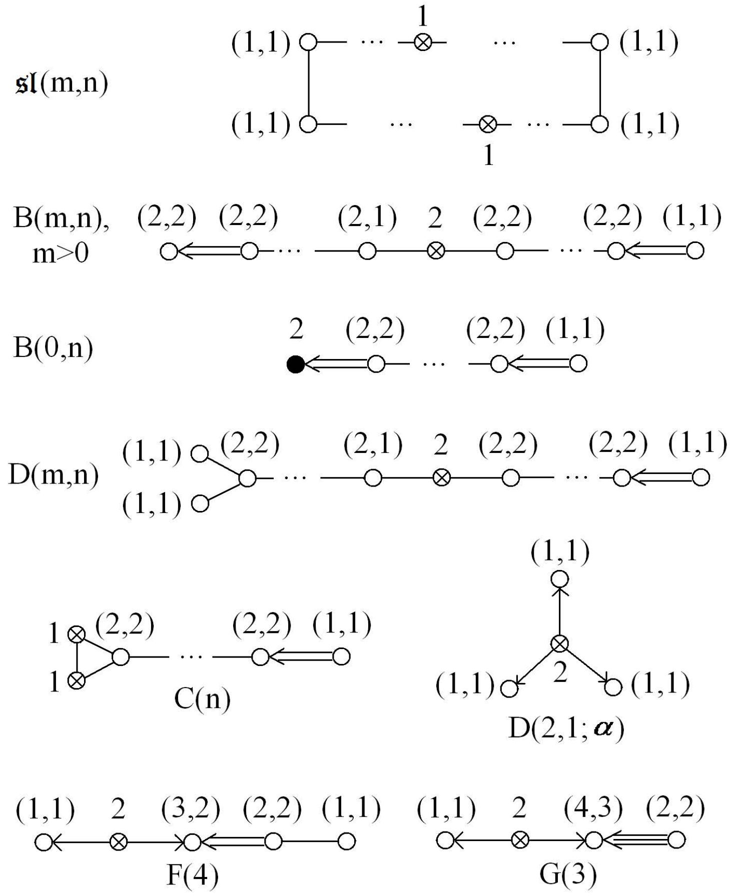

Figure 1: Extended Dynkin diagrams D with $\{(a_\alpha, b_\alpha)\}_{\mathrm{D}_{\bar{0}}}$ and $\{a_\alpha\}_{\mathrm{D}_{\bar{1}}}$.

A circling on D is a choice to circle some vertices of D. We say that a $\mathfrak{g}$-involution θ is represented by a circling if there are roots represented by D such that

$$\alpha \in \mathrm{D} \begin{cases} \text{not circled} \Longleftrightarrow \mathfrak{g}_\alpha \subset \mathfrak{k}, \\ \text{circled} \Longleftrightarrow \mathfrak{g}_\alpha \subset \mathfrak{p}. \end{cases} \tag{5}$$

Not every circling represents an involution, and conversely an involution can be represented by several circlings due to the choice of simple systems [1, Thm.1.3,1.4].

Theorem 1.1. *The $\mathfrak{g}$-involutions with admissible systems are classified by* (6), *and they are represented by the circlings in Figure* 2*. In each case, $\mathfrak{p}^\pm$ are irreducible $\mathfrak{k}$-modules.*

Theorem 1.1 excludes the trivial case of $\mathfrak{g} = C(n)$ with $\theta = 1$ on $\mathfrak{g}_{\bar{0}}$ and $\theta = -1$ on $\mathfrak{g}_{\bar{1}}$, where $\mathfrak{k} = \mathfrak{g}_{\bar{0}} = C_{n-1} + \mathbb{C}$.

For complex simple Lie algebras, the existence of admissible systems for θ is equivalent to the presence of a 1-dimensional center $\mathfrak{z}(\mathfrak{k})$ in the fixed point set $\mathfrak{k}$ of θ. The following corollary provides an analogous result for contragredient Lie superalgebras.

$$\begin{array}{cll} & \mathfrak{g} & \mathfrak{k} \\ \hline \text{(a)} & \mathfrak{sl}(m,n) & \mathfrak{s}(\mathfrak{gl}(p,q)+\mathfrak{gl}(m-p,n-q)) \\ \text{(b)} & B(m,n) & B(m-1,n)+\mathbb{C} \\ \text{(c)} & D(m,n) & D(m-1,n)+\mathbb{C} \\ \text{(d)} & C(n) & A(n-2,0)+\mathbb{C} \\ \text{(e)} & D(m,n) & A(m-1,n-1)+\mathbb{C} \\ \text{(f)} & D(2,1;\alpha) & A(1,0)+\mathbb{C} \\ \text{(g)} & F(4) & C(3)+\mathbb{C} \end{array} \tag{6}$$

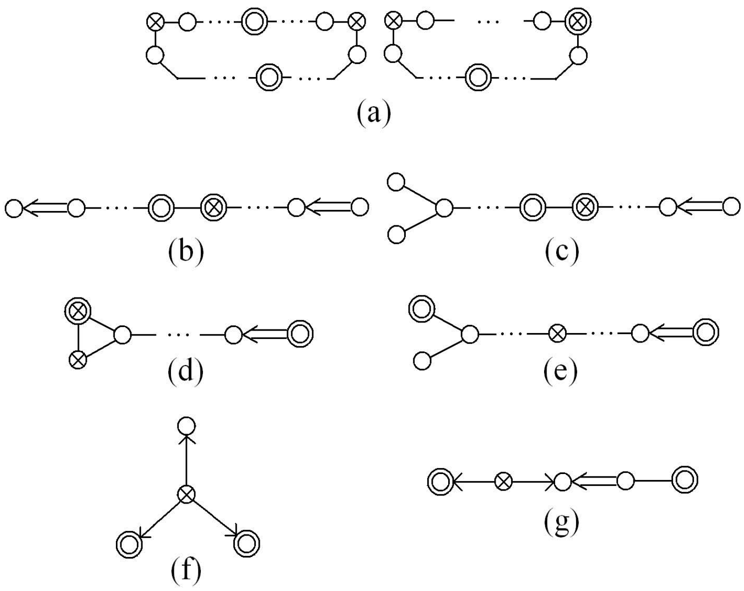

Figure 2: Circlings of involutions with admissible systems.

Corollary 1.2. *For a contragredient Lie superalgebra* $\mathfrak{g} \neq \mathfrak{sl}(n,n)$*, we have*

$$\dim \mathfrak{z}(\mathfrak{k}) = \begin{cases} 1 & \text{if } \theta \text{ has admissible systems,} \\ 0 & \text{if } \theta \text{ has no admissible system.} \end{cases}$$

For $\mathfrak{g} = \mathfrak{sl}(n,n)$, $\dim \mathfrak{z}(\mathfrak{k})$ does not determine the existence of admissible systems. In the admissible situation where $\mathfrak{k} = \mathfrak{s}(\mathfrak{gl}(p,q)+\mathfrak{gl}(n-p,n-q))$, we have $\dim \mathfrak{z}(\mathfrak{k}) = 1, 2$ depending on $p \neq q$ or $p = q$. But $\dim \mathfrak{z}(\mathfrak{k}) = 1$ also occurs in the non-admissible situation where θ permutes the two simple ideals $\mathfrak{sl}(n,\mathbb{C})^2$ of $\mathfrak{g}_{\bar{0}}$, as $\theta I = I$ leads to $I \in \mathfrak{z}(\mathfrak{k})$.

Recall that in the graded setting (2), θ corresponds to a graded real form $\mathfrak{g}^\sigma$. Here $\mathfrak{g}^\sigma$ is given as a functor and it is not retrieved as the fixed points of σ; in general, it is not representable in the category of superspaces. We can nevertheless prove an analogue of the Iwasawa decomposition (see Proposition 3.1)

$$\mathfrak{g}(A) = \mathfrak{g}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A) \text{ for all } A \in (\text{salg})^{\text{gr}}_{\mathbb{C}},$$

where $(\text{salg})^{\text{gr}}_{\mathbb{C}}$ is the category of commutative superalgebras with graded involutions. This is the graded version of the result proven in the case of hermitian forms of Lie superalgebras in [3], then later on generalized in [12, Thm.5.3].

As in the standard setting [3, Thm.1.4], if θ is admissible, then there exists an invariant complex structure on the functor $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma$, leading to its representability as complex superspace as follows.

Theorem 1.3. *Let* $\mathfrak{b}$ *be the Borel subalgebra with respect to an admissible positive system for* $\mathfrak{g}$*. There exists a parabolic subalgebra* $\mathfrak{q} \supset \mathfrak{b}$ *such that* $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma \cong \mathfrak{g}/\mathfrak{q}$*. Hence,* $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma$ *acquires an invariant complex structure and it is representable as complex superspace.*

In [7] we have applied Chevalley's recipe (see also [6]) to obtain the global counterparts $G^\sigma \supset K^\sigma$ of the graded real forms $\mathfrak{g}^\sigma$ and $\mathfrak{k}^\sigma$. As expected, we then have an injective natural transformation from $G^\sigma(A)/K^\sigma(A)$ into a suitable complex superscheme corresponding to the infinitesimal isomorphism $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma \cong \mathfrak{g}/\mathfrak{q}$.

The sections in this article are arranged as follows. In Section 2, we prove Theorem 1.1 and Corollary 1.2. In Section 3, we prove Proposition 3.1, which provides an Iwasawa decomposition for the graded real form. In Section 4, we apply the Iwasawa decomposition to prove Theorem 1.3.

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2. Admissible systems

In this section, we prove Theorem 1.1. Let $\mathfrak{g}$ be a contragredient Lie superalgebra, let $\theta \in \mathrm{inv}(\mathfrak{g})$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its (± 1)-eigenspaces. We assume that $\mathfrak{k}$ and $\mathfrak{g}$ have a common Cartan subalgebra $\mathfrak{h}$, with root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\Delta} \mathfrak{g}_\alpha .$$

Let Δ^+ be a positive system, and $\Delta^- = -\Delta^+$. We write $\Delta = \Delta_\mathfrak{k} \cup \Delta_\mathfrak{p}$ for the $\mathfrak{k}$ and $\mathfrak{p}$-roots, $\Delta_\mathfrak{p}^\pm = \Delta_\mathfrak{p} \cap \Delta^\pm$, and

$$\mathfrak{k} = \mathfrak{h} + \sum_{\Delta_\mathfrak{k}} \mathfrak{g}_\alpha \ , \ \mathfrak{p}^\pm = \sum_{\Delta_\mathfrak{p}^\pm} \mathfrak{g}_\alpha \ , \ \mathfrak{p} = \mathfrak{p}^+ + \mathfrak{p}^- . \tag{7}$$

The admissible condition (3) is equivalent to

$$(\Delta_\mathfrak{k} + \Delta_\mathfrak{p}^\pm) \cap \Delta \subset \Delta_\mathfrak{p}^\pm \ , \ (\Delta_\mathfrak{p}^\pm + \Delta_\mathfrak{p}^\pm) \cap \Delta = \emptyset . \tag{8}$$

Recall that D is the distinguished extended Dynkin diagram (4), with integers $\{(a_\alpha, b_\alpha)\}_{\mathrm{D}_{\bar{0}}}$ and $\{a_\alpha\}_{\mathrm{D}_{\bar{1}}}$ given in Figure 1. Its vertices represent a simple system Π and its lowest root φ. A circling c on D is said to represent θ if D represents some $\Pi \cup \{\varphi\}$ such that (5) holds. Let $c(\mathrm{D}) \subset \mathrm{D}$ denote the vertices circled by c. Then c represents a $\mathfrak{g}$-involution if and only if $\sum_{c(\mathrm{D})} a_\alpha$ is even [1, Thm.1.3]. All the circlings in Figure 2 satisfy this condition, for instance Figure 2(a) satisfies $\sum_{c(\mathrm{D})} a_\alpha = 1 + 1 = 2$, and Figure 2(b,c) satisfy $\sum_{c(\mathrm{D})} a_\alpha = 2 + 2 = 4$. So all the circlings in Figure 2 represent $\mathfrak{g}$-involutions, and one sees that they represent (6).

Proposition 2.1. *Every $\mathfrak{g}$-involution in (6) has admissible systems, and $\mathfrak{p}^\pm$ are irreducible $\mathfrak{k}$-representations.*

Proof. We tentatively replace $\mathfrak{g} = \mathfrak{sl}(n,n)$ of (6)(a) by $\mathfrak{g} = \mathfrak{gl}(n,n)$ (see Remark 2.3 for the reason). The involutions in (6) are represented by the circlings in Figure 2. We start with Figure 2(a,d,e,f,g). In each case there are two circled vertices $\{\beta, \varphi\}$, where $a_\beta = a_\varphi = 1$. Let $\Pi = \mathrm{D}\backslash\{\varphi\}$. Then Π is a simple system of $\mathfrak{g}$ with lowest root φ. Let Δ^+ be the positive system generated by Π. Every member of Δ has the form $\sum_\Pi r_\alpha \alpha$, and $r_\beta = -1, 0, 1$. In particular,

$$\Delta_{\mathfrak{k}} = \Big\{ \sum_\Pi r_\alpha \alpha \in \Delta \ ; \ r_\beta = 0 \Big\} \ , \ \Delta_{\mathfrak{p}}^\pm = \Big\{ \sum_\Pi r_\alpha \alpha \in \Delta \ ; \ r_\beta = \pm 1 \Big\}. \tag{9}$$

Then (9) implies (8), so Δ^+ is admissible. Since Π is a connected subdiagram of D, it implies that $\mathfrak{p}^\pm$ is irreducible. Namely, each member of $\Delta_{\mathfrak{p}}^+$ can be expressed as $\beta + \alpha_1 + ... + \alpha_n$ such that $\alpha_i \in \Pi\backslash\{\beta\}$ and $\beta + \alpha_1 + ... + \alpha_i \in \Delta_{\mathfrak{p}}^+$ for all i. This proves the proposition for (6)(a,d,e,f,g).

If $\theta \in \mathrm{inv}(\mathfrak{sl}(n,n))$ is represented by Figure 2(a), then it acts trivially on the Cartan subalgebra of $\mathfrak{sl}(n,n)$. We extend it to $\widetilde{\theta} \in \mathrm{inv}(\mathfrak{gl}(n,n))$ by acting trivially on the Cartan subalgebra of $\mathfrak{gl}(n,n)$. By the above discussions, $\widetilde{\theta}$ has admissible systems, hence θ also has admissible systems.

It remains to consider Figure 2(b,c). We use the root notations for the vertices as in [1, Fig.3]. So the grey vertex is $\delta_n - \epsilon_1$, the circled white vertex is $\epsilon_1 - \epsilon_2$. For $B(m,n)$, we have

$$\Delta_{\mathfrak{p}} = \{x \pm \epsilon_1 \ ; \ x = 0, \pm\epsilon_i (i \neq 1), \pm\delta_j\}. \tag{10}$$

Pick a positive system $\Delta_{\mathfrak{k}}^+$ of $\Delta_{\mathfrak{k}}$. Define

$$\Delta_{\mathfrak{p}}^+ = \{x + \epsilon_1 \in \Delta \ ; \ x \in \Delta_{\mathfrak{k}} \cup \{0\}\} \ , \ \Delta^+ = \Delta_{\mathfrak{k}}^+ \cup \Delta_{\mathfrak{p}}^+. \tag{11}$$

Then Δ^+ is a positive system of $\mathfrak{g}$ which satisfies (8), so it is admissible. By (10) and (11), each $y + \epsilon_1 \in \Delta_{\mathfrak{p}}^+$ is obtained from $x + \epsilon_1 \in \Delta_{\mathfrak{p}}^+$ by adding $y - x \in \Delta_{\mathfrak{k}}$, so $\mathfrak{p}^+$ is irreducible. This proves the proposition for $B(m,n)$.

The root system of $D(m,n)$ is obtained by removing δ_i, ϵ_j from $B(m,n)$. Therefore, by removing such elements from (10) and (11), the same arguments work for $D(m,n)$. ■

Let c be a circling on D. For a circled white vertex α, we define a new circling $F_\alpha(c)$ by reversing the circling of the vertices adjacent to α (except for longer root joint to α by a double edge); see [1, (6.2)] for details. Let $\theta_a, \theta_b \in \mathrm{inv}(\mathfrak{g})$ be represented by circlings c_a, c_b respectively. If there is a sequence of circlings

$$c_a = c_1 \to c_2 \to ... \to c_n = c_b \tag{12}$$

such that each $c_i \to c_{i+1}$ is given either by some F_α discussed above or a diagram symmetry on D, then θ_a, θ_b are conjugated by a $\mathfrak{g}$-automorphism [1, Thm.1.4].

We next prove the converse of Proposition 2.1.

Proposition 2.2. *Every* $\mathfrak{g}$*-involution with admissible systems belongs to* (6)*.*

Proof. For $\mathfrak{g} = \mathfrak{sl}(m,n)$, all the involutions such that $\mathfrak{g}$ and $\mathfrak{k}$ have the same rank are already given by (6)(a). So we consider the remaining contragredient Lie superalgebras.

Given an involution θ, we let $\theta_{\bar{0}}$ and $\theta_{\bar{1}}$ be its restrictions to $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ respectively. We now consider all the cases where $\theta_{\bar{0}}$ has admissible systems, because this is necessary for θ to have admissible systems. Recall that $c(\mathrm{D})$ denotes the vertices circled by a circling c. It suffices to consider all the circlings c such that:

(a) $\sum_{c(\mathrm{D})} a_\alpha$ is even,
(b) at most one vertex β is circled in each connected component of $\mathrm{D}_{\bar{0}}$, and $b_\beta = 1$. (13)

Condition (13)(a) is needed so that the circling represents a $\mathfrak{g}$-involution [1, Thm.1.3]. Condition (13)(b) says that $\theta_{\bar{0}}$ has admissible systems, and is a consequence of Kac's theorem [10, Ch.8].

Two circlings are said to be *associated* if they are the same on $\mathrm{D}_{\bar{0}}$, and opposite on $\mathrm{D}_{\bar{1}}$. We denote a pair of associated circlings by c, c'. If they represent involutions θ, θ', then $\theta_{\bar{0}} = \theta'_{\bar{0}}$ and $\theta_{\bar{1}} = -\theta'_{\bar{1}}$. By checking through the diagrams in Figure 1, we see that all circlings which satisfy (13) either occur in Figure 2, or are associated to circlings that occur in Figure 2, or are circlings for $C(n)$ where the two grey vertices are circled. The last situation represents $\theta_{\bar{0}} = 1$ and $\theta_{\bar{1}} = -1$ on $C(n)$, which we have said to ignore.

For each circling c in Figure 2(d,e,f,g), it is not hard to check case by case that we can transform c' to c by a sequence described in (12). For example in $D(2,1;\alpha)$, if the grey vertex γ is circled, then the circling on γ can disappear immediately by applying F_α where α is a circled vertex. Hence for Figure 2(d,e,f,g), θ, θ' are conjugated to each other by $\mathfrak{g}$-automorphisms.

It remains to show that for all c in Figure 2(b,c), the involution represented by c' has no admissible system. Similar to Proposition 2.1, we use the root notations of [1, Fig.3]. There is only one circled vertex $c'(\mathrm{D}) = \{\epsilon_1 - \epsilon_2\} \subset \Delta_{\mathfrak{p}}$. One checks that

$$x_1 = \delta_n - \epsilon_1 \ , \ x_2 = -\epsilon_1 - \delta_n \ , \ x_3 = \epsilon_2 + \epsilon_3 \ , \ x_4 = \epsilon_2 - \epsilon_3$$

are all $\mathfrak{k}$-roots. Then $-\epsilon_1 + \epsilon_2 = (\epsilon_1 - \epsilon_2) + x_1 + x_2 + x_3 + x_4$, $(\epsilon_1 - \epsilon_2) + x_1 + ... + x_i \in \Delta_{\mathfrak{p}}$ for all i. Hence the root spaces of $\epsilon_1 - \epsilon_2$ and $-\epsilon_1 + \epsilon_2$ are in the same $\mathfrak{k}$-module, namely there is no admissible system. ■

Proof of Theorem 1.1. By Propositions 2.1 and 2.2, all the involutions with admissible systems are given by (6). Proposition 2.1 also says that in these cases, $\mathfrak{p}^{\pm}$ are irreducible. This proves Theorem 1.1.

Proof of Corollary 1.2. Let $\mathfrak{g} = \mathfrak{sl}(m,n)$, where $m \neq n$. Then, in (6)(a), $\mathfrak{k} = \mathfrak{s}(\mathfrak{gl}(p,q) + \mathfrak{gl}(m-p, n-q))$ has a 1-dimensional center. The remaining involutions are the ones which permute $\mathrm{D}_{\bar{0}}$ nontrivially [1, Thm.1.3], and in these cases $\dim \mathfrak{z}(\mathfrak{k}) = 0$.

Next we consider $\mathfrak{g} \neq \mathfrak{sl}(m,n)$. Here $\dim \mathfrak{z}(\mathfrak{k}) \in \{0,1\}$ [4, Prop.2.6], and furthermore $\dim \mathfrak{z}(\mathfrak{k}) = 1$ occurs exactly when $(\mathfrak{g},\mathfrak{k})$ belongs to (6) [4, Thm.1.1]. This proves Corollary 1.2.

Remark 2.3. For $\mathfrak{g} = \mathfrak{sl}(n,n)$ the root spaces have dimensions 2, and the vertices of the extended Dynkin diagram D do not represent the roots well enough. For instance with $\mathfrak{g} = \mathfrak{sl}(2,2)$, in the first diagram of Figure 1, D has four vertices denoted by the standard notation

$$\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_1, \delta_1 - \delta_2, \delta_2 - \epsilon_1.$$

The two grey vertices are $\gamma_1 = \epsilon_2 - \delta_1$ and $\gamma_2 = \delta_2 - \epsilon_1$. The super-trace function is

$$\mathrm{str} = \epsilon_1 + \epsilon_2 - \delta_1 - \delta_2 = \gamma_1 - \gamma_2.$$

The super-trace vanishes on the Cartan subalgebra $\mathfrak{h}$, so γ_1 and γ_2 are the same element in $\mathfrak{h}^*$. So the root space $\mathfrak{g}_{\gamma_1} = \mathfrak{g}_{\gamma_2}$ is 2-dimensional.

To overcome this problem, we let each vertex of D represent a 1-dimensional subspace of $\mathfrak{g}_\alpha$ instead of α. With notation as above, we let γ_1 and γ_2 represent the 1-dimensional spaces $\mathbb{C}E_{23}$ and $\mathbb{C}E_{41}$ respectively. In this way, D represents $\mathfrak{sl}(n,n)$.

If we need to work with the roots, such as in Proposition 2.1, we can also consider $\mathfrak{g} = \mathfrak{gl}(n,n)$. Then root spaces of $\mathfrak{g}$ are 1-dimensional, the vertices of D represent the roots of $\mathfrak{g}$ nicely, and the admissible properties of $\mathfrak{gl}(n,n)$ and $\mathfrak{sl}(n,n)$ are the same.

3. Graded Iwasawa decomposition

A *graded real structure* on a complex Lie superalgebra $\mathfrak{g}$ is an antilinear automorphism $\sigma \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$ of order 2 (resp. 4) on the even (resp. odd) part of $\mathfrak{g}$. The corresponding *graded real form* is the functor $\mathfrak{g}^\sigma : (\mathrm{salg})_{\mathbb{C}}^{\mathrm{gr}} \longrightarrow (\mathrm{smod})_{\mathbb{C}}$ from the category of complex Lie superalgebras with graded real structures to the category of complex supermodules, given on the objects by:

$$\mathfrak{g}^\sigma(A) = \{u \in \mathfrak{g}(A) \; ; \; \sigma_A(u) = u\} = \{a \otimes x + \widetilde{a} \otimes \sigma(x) \in \mathfrak{g}(A)\}$$

where σ_A is the involution of $\mathfrak{g}(A) := A_{\bar{0}} \otimes \mathfrak{g}_{\bar{0}} \oplus A_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}$ induced by σ, and $\widetilde{a}$ is the image of $a \in A$ under the graded conjugation $\widetilde{\ }$. Hereafter, by *graded conjugation* on $A \in (\mathrm{salg})_{\mathbb{C}}$ we mean any $\mathbb{C}$-antilinear automorphism $\widetilde{\ } : A \longrightarrow A$ – as a superalgebra – whose order is 2, resp. 4, on the even, resp. odd, part of A; this graded conjugation is what gives A a “graded real structure”. Then the pairs $\big(A\,,\widetilde{\ }\big)$ are the objects of a new category – with obvious morphisms – that we denote by $(\mathrm{salg})_{\mathbb{C}}^{\mathrm{gr}}$ (see [7, Sec.2] for more details).

We remark that $\mathfrak{g}(A)$ is a $\mathbb{Z}_2$-graded Lie algebra, not a Lie superalgebra.

Let $\mathfrak{g}$ be a complex contragredient Lie superalgebra, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{g} = \mathfrak{h} + \sum_\Delta \mathfrak{g}_\alpha$ the root space decomposition. Let $\Pi \subset \Delta$ be a simple system, and it leads to the sets of positive and negative roots $\Delta^\pm$. Write $\Pi = \Pi_{\bar{0}} \cup \Pi_{\bar{1}}$ for the even and odd roots.

We have a one to one correspondence between the antilinear automorphisms $\sigma \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$ and the linear involutions $\theta \in \mathrm{inv}(\mathfrak{g})$ given by $\sigma = \omega \circ \theta$, where $\omega \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$ satisfies

$$\omega|_{\mathfrak{h}} = -\mathrm{id}, \qquad \omega(x_{\pm\alpha}) = -x_{\mp\alpha}, \qquad \omega(x_{\pm\beta}) = \pm x_{\mp\beta} \tag{14}$$

for some $\{x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}, x_{\pm\beta} \in \mathfrak{g}_{\pm\beta} \ ; \ \alpha \in \Pi_{\bar{0}}, \beta \in \Pi_{\bar{1}}\}$. See also Section 5, Props. 5.1, 5.8 of [7]. In particular, when $\mathfrak{g}^\theta$ and $\mathfrak{g}$ have the same rank, (14) implies that $\sigma(\mathfrak{g}_\alpha) \subset \mathfrak{g}_{-\alpha}$, for all $\alpha \in \Delta$.

Proposition 3.1. (Graded Iwasawa Decomposition). *Suppose that $\mathfrak{g}$ and $\mathfrak{k}$ have the same rank, and let $\sigma \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$. Then*

$$\mathfrak{g}(A) = \mathfrak{g}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A) \qquad \textit{for all} \quad A \in (\mathrm{salg})^{\mathrm{gr}}_{\mathbb{C}}, \tag{15}$$

where $\mathfrak{n}^+$ is the nilpotent Lie subalgebra of $\mathfrak{g}$ associated with our choice of $\mathfrak{h}$ and of a set of positive roots.

Proof. Let $\mathfrak{g}'(A) = \mathfrak{g}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A)$, $A \in (\mathrm{salg})^{\mathrm{gr}}_{\mathbb{C}}$. We have immediately that $\mathfrak{g}'(A) \subset \mathfrak{g}(A)$. Now we show the other inclusion. Since

$$\mathfrak{g}(A) = \mathfrak{n}^-(A) \oplus \mathfrak{h}(A) \oplus \mathfrak{n}^+(A), \quad \text{with } \mathfrak{n}^\pm = \sum_{\Delta^\pm} \mathfrak{g}_\alpha,$$

we have $\mathfrak{n}^+(A) \subset \mathfrak{g}'(A)$. Furthermore, since $\mathfrak{h}(A) = \mathfrak{h}^\sigma(A) + i\mathfrak{h}^\sigma(A)$, we also have $\mathfrak{h}(A) \subset \mathfrak{g}'(A)$. So we are left to show $\mathfrak{g}_{-\alpha}(A) \subset \mathfrak{g}'(A)$, $\alpha \in \Delta^+$. Since $\mathfrak{h}$ is a common Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$, here $\mathfrak{h}^\sigma$ is a compact Cartan subalgebra of $\mathfrak{g}^\sigma$. Hence the root vectors of $\mathfrak{g}^\sigma(A)$ are of the form

$$ax_{-\alpha} + \widetilde{a}\sigma(x_{-\alpha}) \in \mathfrak{g}^\sigma(A) \ , \ a \in A$$

where $a, \widetilde{a}$ are non-zero (here we omit $\otimes$ for clarity). Since $\sigma(x_{-\alpha}) \in \mathfrak{g}_\alpha$, we have:

$$ax_{-\alpha} \in \mathfrak{g}^\sigma(A) \oplus \mathfrak{n}^+(A) \ , \ a \in A$$

so that $\mathfrak{g}_{-\alpha}(A) \subset \mathfrak{g}'(A)$. ■

Remark 3.2. In general, the functor $\mathfrak{g}^\sigma$ is not representable in the category of superspaces, i.e. there is no superspace V such that $V(A) = \mathfrak{g}^\sigma(A)$ for all $A \in (\mathrm{salg})^{\mathrm{gr}}_{\mathbb{C}}$. Hence the equation (15) has to be interpreted accordingly.

4. Graded Hermitian symmetric spaces

Let $\mathfrak{g}$ be a contragredient Lie superalgebra, and $\mathfrak{h}$ a Cartan subalgebra. Let $\theta \in \mathrm{inv}(\mathfrak{g})$ be admissible, and we write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}^+ + \mathfrak{p}^-$ and $\Delta = \Delta_{\mathfrak{k}} \cup \Delta^+_{\mathfrak{p}} \cup \Delta^-_{\mathfrak{p}}$ with the same meanings as (7), where $\mathfrak{k}$ and $\mathfrak{g}$ have the same rank. Let $\sigma \in \overline{\mathrm{aut}}_{2,4}(\mathfrak{g})$ be associated to θ, i.e. $\sigma = \omega \circ \theta$ as in (14). Since θ commutes with ω, we have that $(\mathfrak{k}, \sigma|_{\mathfrak{k}})$ is a graded real structure on $\mathfrak{k}$. We have the following.

Proposition 4.1. *Let $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Then*

$$\mathfrak{g}^\sigma(A)/\mathfrak{k}^\sigma(A) = \mathfrak{g}(A)/\mathfrak{q}(A) \ \textit{for all } A \in (\mathrm{salg})^{\mathrm{gr}}_{\mathbb{C}}.$$

Proof. By Proposition 3.1, $\mathfrak{g}(A) = \mathfrak{g}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A)$, so we can write

$$\mathfrak{g}(A)/(\mathfrak{k}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A)) = \mathfrak{g}^\sigma(A)/\mathfrak{k}^\sigma(A). \tag{16}$$

Notice that
$$\mathfrak{k}^\sigma(A) = \mathfrak{h}^\sigma(A) \oplus \Big(\sum_{\Delta_\mathfrak{k}} \mathfrak{g}_\alpha\Big)^\sigma(A). \tag{17}$$

So
$$\mathfrak{k}^\sigma(A) + i\mathfrak{h}^\sigma(A) = \mathfrak{h}(A) + \Big(\sum_{\Delta_\mathfrak{k}} \mathfrak{g}_\alpha\Big)^\sigma(A).$$

Consider
$$a x_{-\alpha} + \widetilde{a}\sigma(x_{-\alpha}) \in \mathfrak{k}^\sigma(A), \qquad \alpha \in \Delta_\mathfrak{k}^+, \qquad a \in A.$$

Since $\sigma = \omega \circ \theta$, we have $\sigma(x_{-\alpha}) \in \mathfrak{g}_\alpha$. Hence

$$a x_{-\alpha} \in \mathfrak{k}^\sigma(A) + \mathfrak{n}^+(A), \qquad \alpha \in \Delta_\mathfrak{k}^+, \qquad a \in A.$$

So we have

$$\mathfrak{k}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A) = \mathfrak{h}(A) \oplus \sum_{\Delta_\mathfrak{k}^+} \mathfrak{g}_{-\alpha}(A) \oplus \mathfrak{n}^+(A) = \mathfrak{k}(A) + \mathfrak{p}^+(A), \tag{18}$$

where we recall $\mathfrak{n}^+ = \mathfrak{p}^+ \oplus \sum_{\Delta_\mathfrak{k}^+} \mathfrak{g}_\alpha$. Let $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$. Then

$$\begin{aligned} \mathfrak{g}(A)/\mathfrak{q}(A) &= \mathfrak{g}(A)/(\mathfrak{k}^\sigma(A) \oplus i\mathfrak{h}^\sigma(A) \oplus \mathfrak{n}^+(A)) && \text{by (18)} \\ &= \mathfrak{g}^\sigma(A)/\mathfrak{k}^\sigma(A). && \text{by (16)} \end{aligned}$$

This proves the proposition. ■

We now want to address the question of representability. We say that a functor

$$F : (\text{salg})_\mathbb{C}^{\text{gr}} \longrightarrow (\text{sets})$$

is *representable* by a superspace if there exists a superspace

$$X : (\text{salg})_\mathbb{C} \longrightarrow (\text{sets}),$$

such that $X \circ \mathcal{F} = F$, where $\mathcal{F}$ is the forgetful functor $\mathcal{F} : (\text{salg})_\mathbb{C}^{\text{gr}} \longrightarrow (\text{salg})_\mathbb{C}$.
Based on our previous discussion, we have the following.

Corollary 4.2. *The functor* $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma : (\text{salg})_\mathbb{C}^{\text{gr}} \longrightarrow (\text{smod})_\mathbb{C}$ *defined on the objects as* $(\mathfrak{g}^\sigma/\mathfrak{k}^\sigma)(A) := \mathfrak{g}^\sigma(A)/\mathfrak{k}^\sigma(A)$ *is representable in the category of superspaces and it is represented by the superspace* $\mathfrak{p}^-$.

Proof of Theorem 1.3. This follows immediately from Proposition 4.1 and Corollary 4.2.

Corollary 4.2 gives also a complex structure on the superspace $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma$, since it gives its identification with $\mathfrak{p}^-$; hence we call $\mathfrak{g}^\sigma/\mathfrak{k}^\sigma$ an *infinitesimal graded hermitian symmetric space.*

We now want to give a global version of Proposition 4.1. Let $P^\pm$, K, Q be the complex subsupergroups of G with $\mathfrak{p}^\pm = \text{Lie}(P^\pm)$, $\mathfrak{k} = \text{Lie}(K)$, $\mathfrak{q} = \text{Lie}(Q)$. We may also write $Q = KP^+$. This is immediate because of the existence of admissible systems, so that $\mathfrak{p}^\pm$, $\mathfrak{q} = \mathfrak{k} + \mathfrak{p}^+$ are Lie subsuperalgebras.

Proposition 4.3. *Let the notation be as above. Then, we have an injective natural transformation*

$$\phi_A : G^\sigma(A)/K^\sigma(A) \longrightarrow G(A)/Q(A), \qquad A \in (\mathrm{salg})_{\mathbb{C}}^{\mathrm{gr}}.$$

Proof. We have a well defined natural transformation

$$\psi_A : G^\sigma(A) \longrightarrow G(A)/Q(A).$$

Notice that $\psi_A(u) \in Q(A)$ if and only if $u \in K^\sigma(A)$. ∎

Remark 4.4. After a suitable sheafification of functors in the Zariski site, when $A = A_0$, Proposition 4.3 gives the classical open embedding of the ordinary hermitian symmetric space G_0^σ/K_0^σ into G_0/K_0, the index 0 denoting the ordinary group scheme underlying a supergroup scheme. Hence, through this embedding, G_0^σ/K_0^σ becomes equipped with an invariant complex structure.

This embedding is rarely an isomorphism, for the following reason. It relies on the Iwasawa decomposition (15), which, at the Lie algebra setting, is

$$\mathfrak{g} = \mathfrak{g}^\sigma \oplus i\mathfrak{h}^\sigma \oplus \mathfrak{n}^+.$$

This means that at the Lie group level, $G^\sigma \exp(i\mathfrak{h}^\sigma)N^+$ is an open subset of G. This open subset is the entire G if and only if G^σ is compact. Real contragredient Lie superalgebras with compact even parts occur only in specific cases of types A and C.

References

[1] M. K. Chuah: *Finite order automorphisms on contragredient Lie superalgebras*, J. Algebra 351 (2012) 138–159.

[2] M. K. Chuah: *Cartan automorphisms and Vogan superdiagrams*, Math. Zeitschrift 273 (2013) 793–800.

[3] M. K. Chuah, R. Fioresi: *Hermitian real forms of contragredient Lie superalgebras*, J. Algebra 437 (2015) 161–176.

[4] M. K. Chuah, M. Zhang: *Symplectic symmetric pairs of contragredient Lie superalgebras*, Ann. Mat. Pura Appl. (4) 200/3 (2021) 1195–1215.

[5] M. K. Chuah, M. Zhang: *Outer automorphism groups of real forms of contragredient Lie superalgebras*, J. Algebra 619 (2023) 60–98.

[6] R. Fioresi, F. Gavarini: *Chevalley Supergroups*, Memoirs of the AMS, Vol. 215, No. 1014, American Mathematical Society, Providence (2012).

[7] R. Fioresi, F. Gavarini: *Real forms of complex Lie superalgebras and supergroups*, Comm. Math. Phys. 397/2 (2023) 937–965.

[8] Harish-Chandra: *Representations of semi-simple Lie groups IV*, Amer. J. Math. 77 (1955) 743–777.

[9] V. G. Kac: *Lie superalgebras*, Adv. Math. 26 (1977) 8–96.

[10] V. G. Kac: *Infinite Dimensional Lie Algebras*, 3rd. ed., Cambridge University Press, Cambridge (1990).

[11] V. Serganova: *Classification of real simple Lie superalgebras and symmetric superspaces*, Funct. Anal. Appl. 17 (1983) 200–207.

[12] A. Sherman: *Iwasawa decomposition for Lie superalgebras*, J. Lie Theory 32 (2022) 973–996.

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