

Poincaré Inequalities on Carnot Groups and Spectral Gap of Schrödinger Operators

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Abstract. We give a sufficient condition under which the global Poincaré inequality on Carnot groups holds true for a large family of probability measures absolutely continuous with respect to the Lebesgue measure. Additionally, we show that the global Poincaré inequality holds true on any Carnot group for a certain choice of a probability measure adapted to the structure of each Carnot group, and whose formula is explicitly given. Consequently, we extend the results of a previous work by the authors [*q*-Poincaré inequalities on Carnot groups with a filiform Lie algebra, Potential Analysis 60/3 (2024) 1067–1092] targeted on filiform Carnot groups to any Carnot group. As a result, the Schrödinger operators associated with the density of the considered probability measure have a spectral gap.

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Key Words: Poincaré inequalities, Carnot groups, sub-gradient, spectral gap.

1. Introduction

1.1. Hörmander sum of squares and Poincaré inequalities

The main question that we address in this work is the determination of the sufficient conditions that the density of a probability measure μ on a Carnot group \mathbb{G} must satisfy in order to formulate the so-called “*global Poincaré*” or *q*-Poincaré inequality in this setting, that is an inequality of the form (3). Another aspect of this investigation is whether we can construct a homogeneous quasi-norm giving rise to the global Poincaré inequality on any stratified group. The validity of the Poincaré inequality will, in turn, allow to decide whether a certain type of Schrödinger operator on \mathbb{G} has a spectral gap.

The Schrödinger-type operators we are referring to are defined via a system of vector fields $\mathbb{X} = \{X_1, \dots, X_n\}$ on \mathbb{R}^n that satisfies Hörmander’s condition, that is operators that take the form

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$$-\sum_{i=1}^n X_i^2 + V(x), \quad x \in \mathbb{R}^n, \quad (1)$$

where V is a suitable potential.

In the present paper the operator $\sum_{i=1}^n X_i^2$ is viewed as the sub-Laplacian of a certain Carnot group. However, not all operators of this form – that is those with vector fields satisfying Hörmander’s condition – are necessarily sub-Laplacians of a Carnot group, although they can always be “approximated” by sub-Laplacians on a Carnot group of higher topological dimension $N > n$. The latter is, roughly speaking, the main idea of the celebrated lifting theorem by Rothschild and Stein [40].

We recall that if $\mathbb{G} \equiv \mathbb{R}^n$ is a group with a Lie algebra that can be generated by vector fields X_j , $1 < j \leq n_1 \leq n$, using a finite number of repeated commutators, then the negative operator of the form

$$\Delta_{\mathbb{G}} = \sum_{i=1}^{n_1} X_i^2,$$

is called the *sub-Laplacian* on \mathbb{G} , where n_1 is the dimension of the first stratum of the Carnot group. In this case, the system \mathbb{X} is the system of generators of the Lie algebra of \mathbb{G} . Such operators, thanks to Hörmander’s celebrated result [29], are the most typical examples of operators that are not elliptic but hypoelliptic. We recall that when the group structure is not available (i.e. when elements of \mathbb{X} are just vector fields on \mathbb{R}^n), the system \mathbb{X} that satisfies Hörmander’s condition consists of (possibly degenerate, non-commutative) vector fields on \mathbb{R}^n such that

$$\text{rank}(\text{Lie}\{\mathbb{X}\}(x)) = n, \quad x \in \mathbb{R}^n,$$

where $\text{Lie}\{\mathbb{X}\}$ is the smallest Lie algebra that can be generated by the system \mathbb{X} using repeated commutators.

The first pioneering result about Poincaré inequalities involving Hörmander’s vector fields was proved by Jerison in [33]. For the reader’s convenience, we recall the statement of the aforementioned inequality in Theorem 1.1 below.

Theorem 1.1. *Let \mathbb{G} be any nilpotent Lie group with the Haar measure dx . For any $p \in [1, \infty)$ there exists a constant $P_0(r) = P_0(r, p)$ such that, for every $x \in \mathbb{G}$ and $f \in C^\infty(B_r(x))$,*

$$\int_{B_r(x)} |f(y) - f_{B_r(x)}|^p dy \leq P_0(r) \int_{B_r(x)} |\nabla_{\mathbb{G}} f(y)|^p dy, \quad (2)$$

where $f_{B_r(x)} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y) dy$ and $B_r(x) := \{y \in \mathbb{G} : d(x, y) \leq r\}$ is the ball of radius r centered at x with respect to the Carnot-Carathéodory distance d .

We shall refer to (2) as the *local* Poincaré inequality, since it holds locally on a ball of radius r .

The Poincaré inequalities involving Hörmander’s vector fields that we are interested in in this paper are the *global* Poincaré inequalities of the form (2), where $B_r(x)$ and the Haar measure on \mathbb{G} (which in this case coincides with the Lebesgue measure) are replaced by the whole \mathbb{G} and by a suitable probability measure on \mathbb{G} , respectively.

To be precise, for a suitable probability measure such global Poincaré inequality takes the form

$$\mu(|f - \mu(f)|^q) \leq C\mu(|\nabla_{\mathbb{G}}f|^q), \quad q \geq 1, \quad (3)$$

for all functions f for which the right hand side is well defined. Here and later on μ will be a probability measure on a Carnot group \mathbb{G} , $\mu(f)$ will denote the integral $f_{\mu} = \int_{\mathbb{G}} f d\mu$, and $\nabla_{\mathbb{G}}$ will be the vector-valued operator $\nabla_{\mathbb{G}} = (X_1, \dots, X_{n_1})$ which is the sub-gradient on the group \mathbb{G} .

The interest for these inequalities is strongly motivated by the fact that when $q = 2$ then (3) can be written as

$$\mu(|f - \mu(f)|^2) \leq C\mu(f(\mathcal{L}f)), \quad (4)$$

for \mathcal{L} as in (7), which is referred to as the *spectral gap inequality*. The significance of this estimates lies in the fact that, whenever it holds, it implies the existence of a gap at the bottom of the spectrum of the operator \mathcal{L} . Finally, we remark that estimates of the form (4) go back to Henri Poincaré and imply the exponential convergence of the associated semigroup $P_t \equiv e^{t\mathcal{L}}$ to the invariant measure μ , see [11], [25], [38].

1.2. Operators with discrete spectra on Carnot groups and the U -bounds method

As previously mentioned there is a link between functional inequalities and spectral properties of Schrödinger operators. In the work [18], using a probabilistic approach, Driver and Melcher show that the heat kernel measure in the Heisenberg group satisfies a spectral gap inequality. Non-probabilistic approaches to the problem were considered in [20] and [35]. As for the investigation of the spectra of other classes of operators in relation to functional inequalities, we refer to the recent works of Cipriani [15] and Wang [42].

Another approach to prove coercive inequalities related to spectral problems was developed by Hebisch and Zegarlinski in [26]. Their method, called the *U -bounds method*, works on a general metric space M equipped with the (non-)commuting vector fields $\{X_1, \dots, X_m\}$. More specifically, the *U -bounds method* consists in proving some inequalities, also called *U -bounds*, allowing to derive Poincaré and other inequalities with respect to suitable probability measures.

More precisely, in the Carnot group setting, a *U -bound* is expressed as

$$\int |f|^q g(d) d\mu \leq A_q \int |\nabla_{\mathbb{G}}f|^q d\mu + B_q \int |f|^q d\mu, \quad (5)$$

where g is a positive unbounded function on $[0, +\infty)$ (that goes to infinity as d goes to infinity), $d\mu = e^{-U(d)} dx$ is a probability measure defined via a suitable unbounded function U , and dx denotes the Haar-measure on the group.

In [26] it is shown that, under suitable conditions on μ , one can pass from the *U -bounds* inequalities (5) to the global (that is with respect to a probability measure) q -Poincaré inequalities. This method was successfully applied in the case of step 2 groups in [26], [30], [6],[32], [5] and [14], and on Carnot groups of higher step in [7]. In particular, this is also the approach we shall use in the current paper.

The type of relation between spectral analysis and coercive inequalities that we are going to investigate here is as follows:

Consider a probability space (M, μ) and a positive self-adjoint operator \mathcal{L} on $\mathcal{D}(\mathcal{L}) \subset L^2(\mu)$; then the operator \mathcal{L} has a spectral gap if and only if there exists a constant $C > 0$ so that

$$\mu(|f - \mu(f)|^2) \leq C\mathcal{E}(f, f), \quad (6)$$

where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the Dirichlet form associated to \mathcal{L} , that is the closure of the form

$$\mathcal{E}(f, g) = \mu(f\mathcal{L}g), \quad f, g \in \mathcal{D}(\mathcal{L}).$$

Now, for U as in (5), that is U being an unbounded function such that $d\mu = e^{-U(d)}dx$ is a probability measure, by taking

$$\mathcal{L} = -\Delta_{\mathbb{G}} + \nabla_{\mathbb{G}}U \cdot \nabla_{\mathbb{G}}, \quad (7)$$

we get

$$\mathcal{E}(f, f) = \mu(f\mathcal{L}f) = \mu(|\nabla_{\mathbb{G}}f|^2).$$

At the same time, we know that if an operator \mathcal{L} satisfies the relation

$$(\mathcal{L}f, f)_{L^2(\mu)} = \int_{\mathbb{G}} |\nabla_{\mathbb{G}}f(x)|^2 d\mu(x),$$

then it is an operator of the form (1) which is positive and self-adjoint on $L^2(\mu)$. Therefore, by the previous analysis, (6) becomes

$$\mu(|f - \mu(f)|^2) \leq C\mu(|\nabla_{\mathbb{G}}f|^2), \quad f \in \mathcal{D}(\mathcal{L}), \quad (8)$$

for $q = 2$, where inequality (8) is equivalent to the existence of a spectral gap for the operator \mathcal{L} in (7).

Let us stress that the *U-bounds* method was introduced to deal with the validity of coercive inequalities in non doubling measure spaces, which is also the setting considered in this paper. In the non-doubling setting, a global $L^2 - L^2$ Poincaré inequality on connected non-compact Lie groups was proved in [10] in the possibly non-unimodular case, and in [41] in the unimodular case. In both [10] and [41] the result is based on the existence of a Lyapunov function – an approach different from the one we use here. As for Poincaré inequalities in doubling metric measure spaces, several results have been obtained so far, and we refer the interested reader to [3], [21] and references therein.

1.3. The class of probability measures for which the spectral gap inequality holds true

The quadratic form bounds in [26], that is the *U-bounds* in (5) for $q = 2$, resemble the ones in the works of Rosen [39] and Adams [1] in the Euclidean setting. In the case of the Heisenberg group (see [26]), the potential U in the probability measure is taken with respect to the Carnot-Carathéodory distance. In this consideration, several coercive inequalities, including the Log-Sobolev inequality, are proven. However, in the same work, the authors prove that replacing the Carnot-Carathéodory distance by any other smooth homogeneous quasi-norm forces the Log-Sobolev inequality to fail.

In the case of the spectral gap inequality that is of interest to us, the potential is given with respect to a homogeneous quasi-norm on the group. The control over the

potential that is present in the U -bounds is granted by the lower bound of the length of the sub-gradient of the quasi-norm. In general, it is the control over the potential U that is needed for such types of inequalities; see also [Corollary 4.2.4 [30]] where it was proved that, as in the Euclidean setting, a Schrödinger operator with potential V as in (7) has a discrete spectrum if V grows to infinity in all directions. We wish to mention that results on the discreteness of the spectrum of operators as in (7) in the case of H-type groups can be found in [31], while for similar results when \mathbb{G} is a Métivier group we refer to in [9]. Let us say that in the aforementioned papers the discreteness of the spectrum is obtained by considering a potential defined in terms of a norm N on \mathbb{G} being either the Carnot-Charathéodory norm or the Kaplan norm, while in [9] N is the Kaplan norm.

1.4. Main result and spectral gap

The main results of this work are contained in Theorems 1.2 and 1.4 below. Particularly, in Theorem 1.2 we give sufficient conditions on the probability measure so that global Poincaré inequalities with respect to that measure hold true, while in Theorem 1.4 we show that global Poincaré inequalities on a Carnot group \mathbb{G} always hold for a particular choice of a quasi-norm on \mathbb{G} .

Theorem 1.2. *Let \mathbb{G} be a Carnot group of step r on \mathbb{R}^n , let N be a homogeneous quasi-norm on \mathbb{G} smooth away from the origin, and let μ_p be defined as in (13). If there exists an index $j_0 \in \{1, \dots, n_1\}$, and a positive integer $\gamma \geq 2$, such that*

$$|\nabla_{\mathbb{G}}N(x)| \gtrsim \frac{|x_{j_0}|^{\gamma-1}}{N^{\gamma-1}(x)}, \quad \forall x \in \mathbb{G}, \tag{9}$$

then, for all $p \geq 2\gamma$, and for q being the conjugate exponent of p , there exists $c_0 \in (0, \infty)$ such that

$$\mu_p(|f - f_{\mu_p}|^q) \leq c_0 \mu_p(|\nabla_{\mathbb{G}}f|^q) \tag{10}$$

for all functions f which are Lipschitz (with respect to the Carnot-Charathéodory distance).

A direct consequence of this result is the spectral gap for a suitable operator \mathcal{L}_p .

Corollary 1.3. *Under the hypotheses of Theorem 1.2 the positive self-adjoint operator*

$$\mathcal{L}_p := -\Delta_{\mathbb{G}} + apN^{p-1}\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}},$$

on $L^2(\mu_p)$ has a spectral gap.

Theorem 1.4. *Let \mathbb{G} be a Carnot group on \mathbb{R}^n . Then for N, γ as in (36), μ_p as in (13) with $p \geq 2\gamma$, and for q being the conjugate exponent of p , there exists $c_0 \in (0, \infty)$ such that*

$$\mu_p(|f - f_{\mu_p}|^q) \leq c_0 \mu_p(|\nabla_{\mathbb{G}}f|^q) \tag{11}$$

for all functions f which are Lipschitz (with respect to the Carnot-Charathéodory distance). Consequently, also the corresponding operator \mathcal{L}_p , as in Corollary 1.3, has a spectral gap.

The general result in Theorem 1.4 is a combination of Theorem 1.2 together with a result by Helffer and Nourrigat in [28], where the authors show that there always

exists an admissible change of coordinates in \mathbb{G} so that the vector fields that generate the corresponding Lie algebra admit a particular form. After reducing to this form, one can show that the lower bound (9) is satisfied, allowing global Poincaré inequalities to hold as an application of Theorem 1.2.

1.5. Organisation of the paper

The paper is organised as follows: in Section 2 we provide the reader with the necessary notions around Carnot groups. In Section 3 we prove Theorem 1.2 about the existence of sufficient conditions for the global Poincaré/spectral gap inequalities to hold. Those sufficient conditions are related to the group structure and/or to the choice of the homogeneous quasi-norm, that is, in other words, to the choice of the probability measure. Consequently, the spectral analysis for the self-adjoint operators \mathcal{L}_p follows as an immediate application of Corollary 1.3. The analysis in this section includes that in [14] as a special case. In particular, from Theorem 1.2 it follows that the global Poincaré inequality holds true for every member of the family of the Carnot groups studied in [14] whose Lie algebra is of filiform type. Finally, in Section 4 we give other examples of groups where the sufficient conditions described earlier are satisfied. These include many important examples of Carnot groups that are frequently studied in the literature. Most importantly, in Section 4 we prove our main general result, i.e., the existence of a quasi-norm on any Carnot group \mathbb{G} so that the global Poincaré inequality holds true.

2. Preliminaries

In this section we shall recall some properties of the Lie algebra of a Carnot group \mathbb{G} . The homogeneous structure of Carnot groups plays a fundamental role in our analysis, therefore we start by recalling the definition of homogeneous Lie groups, homogeneous functions, and homogeneous differential operators.

Definition 2.1. (Homogeneous Lie group on \mathbb{R}^n) Let $\mathbb{G} = (\mathbb{R}^n, \circ)$ be a Lie group on \mathbb{R}^n . We say that \mathbb{G} is a *homogeneous* Lie group (on \mathbb{R}^n) if there exists an n -tuple of positive integers $\sigma = (\sigma_1, \dots, \sigma_n)$, with $1 = \sigma_1 \leq \dots \leq \sigma_n$, such that the dilation

$$\delta_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \delta_\lambda(x_1, \dots, x_n) := (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n)$$

is an automorphism of the group \mathbb{G} for any $\lambda > 0$. We shall denote by $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ the datum of a homogeneous Lie group on \mathbb{R}^n , where \circ is the composition law and $\{\delta_\lambda\}_{\lambda>0}$ is the dilation group.

Definition 2.2. (\mathbb{G} -length of a multi-index) Given a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, we define the δ_λ -length of α as

$$|\alpha|_{\mathbb{G}} := \langle \alpha, \sigma \rangle = \sum_{i=1}^n \alpha_i \sigma_i,$$

where σ_i are as in Definition 2.1.

Definition 2.3. (δ_λ -homogeneous function of degree m) Let a be a real function on $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$, we say that a is a δ_λ -homogeneous function of degree $m \in \mathbb{R}$ if $a \not\equiv 0$ and for any $x \in \mathbb{G} \setminus 0$ and $\lambda > 0$, we have $a(\delta_\lambda(x)) = \lambda^m a(x)$.

Definition 2.4. (\mathbb{G} -degree of a polynomial function) Given a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$, and a polynomial function $p : \mathbb{G} \rightarrow \mathbb{R}$ defined (as a finite sum) as follows

$$p(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}, \quad c_{\alpha} \in \mathbb{R},$$

we shall call the \mathbb{G} -degree of p the quantity

$$\text{deg}_{\mathbb{G}}(p) := \max\{|\alpha|_{\mathbb{G}} : c_{\alpha} \neq 0\}.$$

Proposition 2.5 (Smooth δ_λ -homogeneous functions). *Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ and $a \in C^\infty(\mathbb{G}; \mathbb{R})$. Then a is δ_λ -homogeneous of degree m if and only if it is a polynomial function. As a consequence, the set of the degrees of the smooth (non-vanishing) δ_λ -homogeneous functions is*

$$\mathcal{A} = \{|\alpha|_{\mathbb{G}} : \alpha \in (\mathbb{N} \cup \{0\})^n\}, \quad |\alpha|_{\mathbb{G}} := \sum_{i=1}^n \sigma_i \alpha_i,$$

with $|\alpha|_{\mathbb{G}} = 0$ if and only if a is constant.

Remark 2.6. If a function a is smooth and δ_λ -homogeneous of degree m , then $m \geq 0$. Additionally, given a multi-index α , one has

$$D^\alpha a(x) = \begin{cases} 0 & , \forall \alpha \text{ such that } |\alpha|_{\mathbb{G}} > m, \\ \delta_\lambda\text{-homogeneous of degree } m - |\alpha|_{\mathbb{G}} & , \forall \alpha \text{ such that } |\alpha|_{\mathbb{G}} \leq m. \end{cases}$$

Definition 2.7 (δ_λ -homogeneous vector field of degree m). Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$. Then, given a vector field $X \in \mathfrak{g} = \text{Lie}(\mathbb{G})$, we say that X is δ_λ -homogeneous of degree m if, for any $\varphi \in C^\infty(\mathbb{G})$, $x \in \mathbb{G} \setminus \{0\}$ and $\lambda > 0$,

$$X(\varphi(\delta_\lambda(x))) = \lambda^m (X\varphi)(\delta_\lambda(x)).$$

Proposition 2.8 (Smooth δ_λ -homogeneous vector fields). *Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ and $X \in \mathfrak{g} = \text{Lie}(\mathbb{G})$, that is*

$$X = \sum_{j=1}^n a_j(x) \partial_{x_j}.$$

Then X is δ_λ -homogeneous of degree $k \in \mathbb{R}$ if and only if for every $j = 1, \dots, n$, a_j is a polynomial function δ_λ -homogeneous of degree $\sigma_j - k$ (unless $a_j \equiv 0$). Hence the degree of δ_λ -homogeneity of X belongs to every set

$$\mathcal{A}_j = \{\sigma_j - |\alpha|_{\mathbb{G}} : \alpha \in (\mathbb{N} \cup \{0\})^n\},$$

whenever j is such that a_j is not identically 0. In other words, for any fixed $j = 1, \dots, n$, $k \leq \sigma_j$ and $k = \sigma_j - |\alpha|_{\mathbb{G}}$, for some $\alpha \in (\mathbb{N} \cup \{0\})^n$.

For more details on homogeneous Lie groups and for the proof of the above propositions, we refer the interested reader to [4], [16],[22] and [23].

We shall now proceed with the formal definition of Carnot groups. We use the following notation: For V, W vector spaces such that $V, W \subseteq \mathfrak{g}$, we denote by

$$[V, W] := \{[v, w] : v \in V, w \in W\},$$

where $[\cdot, \cdot]$ is the Lie bracket.

Definition 2.9. Let $\mathbb{G} = (\mathbb{R}^n, \circ)$ be a Lie group on \mathbb{R}^n , and let \mathfrak{g} be the Lie algebra of \mathbb{G} . Then \mathbb{G} is called a stratified group, or *Carnot group*, if \mathfrak{g} admits a vector space decomposition (stratification) of the form

$$\mathfrak{g} = \bigoplus_{j=1}^r V_j, \quad \text{such that} \quad \begin{cases} [V_1, V_{i-1}] = V_i, & 2 \leq i \leq r, \\ [V_1, V_r] = \{0\}, \end{cases} \quad (12)$$

where the positive integer r is called the *step* of \mathbb{G} .

Remark 2.10. Carnot groups are naturally homogeneous Lie groups. Therefore, each Carnot group \mathbb{G} can be equipped with a mapping δ_λ , $\lambda > 0$, as in Definition 2.1 that is an automorphism of the group \mathbb{G} . The diffeomorphic map $\exp_{\mathbb{G}} : \mathfrak{g} \rightarrow \mathbb{G}$ ensures that δ_λ is also an automorphism of \mathfrak{g} , where \mathfrak{g} is the Lie algebra of \mathbb{G} . Note that the stratification of a Lie algebra \mathfrak{g} is not unique. However, the mapping δ_λ as in Definition 2.1 does not depend on the choice of the stratification; cf. [Proposition 2.2.8 [4]]. We can then denote by $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ the Carnot group equipped with the natural dilations $\{\delta_\lambda\}$. ■

We shall conclude this section by recalling the definition of sub-Laplacian and sub-gradient on a Carnot group \mathbb{G} .

Definition 2.11. Let \mathbb{G} be a Carnot group, and let \mathfrak{g} be the corresponding Lie algebra. If X_j , $1 \leq j \leq n_1$, are the canonical left (right) invariant vector fields that generate \mathfrak{g} , then the second order differential operator

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{n_1} X_j^2,$$

is called the *canonical left (right) invariant sub-Laplacian* on \mathbb{G} , while the vector valued operator

$$\nabla_{\mathbb{G}} = (X_1, \dots, X_{n_1}),$$

is called the *canonical left (right) invariant \mathbb{G} -gradient*.

Definition 2.12. We call *homogeneous (quasi-)norm* on the Carnot group \mathbb{G} , every continuous, with respect to the Euclidean topology, mapping $N : \mathbb{G} \rightarrow [0, \infty)$

such that $N(x) = 0$ if and only if $x = 0$,

and $N(\delta_\lambda(x)) = \lambda N(x)$, for every $\lambda > 0$, $x \in \mathbb{G}$.

3. Sufficient conditions for global Poincaré inequalities: proof of Theorem 1.2

In this section, we establish *sufficient* (though not necessary) conditions for global q -Poincaré inequalities to hold for some probability measures on Carnot groups, by proving Theorem 1.2. The probability measures suitable for our purposes – denoted by μ_p and defined below – have densities (with respect to the Haar measure) depending on a parameter p and on a fixed homogeneous quasi-norm N on \mathbb{G} .

As for the necessity of our conditions, it is not known yet if they are also necessary to obtain Poincaré inequalities with respect to probability measures of the form considered here.

Given a homogeneous quasi-norm N , we define the probability measure μ_p as

$$\mu_p := \frac{1}{Z} e^{-aN^p} dx, \tag{13}$$

where Z is a normalization constant, $a \in (0, \infty)$, and $p \in (0, \infty)$. As we will explain later, all our results will hold for a large class of perturbations of such measures.

Here and later on we use a convention $a \gtrsim b$ (resp. $a \lesssim b$) to say that there exists a constant $C \in (0, \infty)$ independent of a, b so that $a \geq Cb$ (resp. $a \leq Cb$). Also, for $p, q \in [1, \infty]$ we say that q is the conjugate exponent of p iff $\frac{1}{p} + \frac{1}{q} = 1$.

We remark that the requirement that f is Lipschitz, with respect to the Carnot-Carathéodory distance, guarantees that the sub-gradient $\nabla_{\mathbb{G}} f$ exists a.e., see Theorem 2.5 in [36]. Moreover, by Theorem 4.2.4 in [27] such functions are dense in $L^q(\mu)$, for $q \in [1, \infty)$.

Remark 3.1 (Regarding the condition (9)). Observe that condition (9) implies that

$$\text{Ker}(|\nabla_{\mathbb{G}} N|) := \{x \in \mathbb{R}^n : |\nabla_{\mathbb{G}} N(x)| = 0\} \subseteq \{x \in \mathbb{R}^n : x_{j_0} = 0\}.$$

In other words, a necessary (but not sufficient) condition to have (9) is that $\text{Ker}(|\nabla_{\mathbb{G}} N|)$ is contained in the hyperplane $\{x \in \mathbb{R}^n : x_{j_0} = 0\}$. ■

For the proof of Theorem 1.2 the following two auxiliary lemmas are necessary.

Lemma 3.2. *Let \mathbb{G} be a Carnot group on \mathbb{R}^n , and let N be a homogeneous quasi-norm on \mathbb{G} smooth away from the origin. Then, for all $x \in \mathbb{G} \setminus \{0\}$, we have*

$$|\nabla_{\mathbb{G}} N(x)| \lesssim 1, \tag{14}$$

$$|\Delta_{\mathbb{G}} N(x)| \lesssim \frac{1}{N(x)}. \tag{15}$$

Proof. The proof follows from the homogeneity properties of N and the fact that $|\nabla_{\mathbb{G}} N|^2$ and $\Delta_{\mathbb{G}} N$ are homogeneous functions of degree 0 and -1 , respectively. ■

Lemma 3.3. *Let $p \geq 2\gamma$ and let q be its conjugate exponent. Let N be a homogeneous quasi-norm smooth away from the origin satisfying (9) for some $j_0 \in \{1, \dots, n_1\}$, and let $\mu_p := e^{-aN^p} dx$. Then there exist $A \in (0, \infty)$ and $B \in [0, \infty)$ such that*

$$\mu_p(|f|^q N^{p-2\gamma} |x_{j_0}|^{2\gamma}) \leq A\mu_p(|\nabla_{\mathbb{G}} f|^q) + B\mu_p(|f|^q), \tag{16}$$

for every function f which is Lipschitz (with respect to the Carnot-Carathéodory distance).

Proof. We shall prove the result for $f \geq 0$ such that $f \in C_0^\infty(\mathbb{G})$, which, by approximation, gives the result when f is non-negative and Lipschitz (this can be seen by using the fact that the density of the probability measure has strong decay properties, so $C_0^\infty(\mathbb{G})$ is dense in $L^p(\mu)$ and $\text{Lip}(\mathbb{G}) \subset L^p(\mu)$).

The result for a general f will then follow from this case by the fact that a.e. one has $|\nabla_{\mathbb{G}}|f|| \leq |\nabla_{\mathbb{G}}f|$. By the Leibniz rule we have that

$$e^{-aN^p} \nabla_{\mathbb{G}} f = \nabla_{\mathbb{G}}(e^{-aN^p} f) + apN^{p-1} e^{-aN^p} f(\nabla_{\mathbb{G}} N),$$

therefore, taking the inner product of the vectors $e^{-aN^p} \nabla_{\mathbb{G}} f$ and $\nabla_{\mathbb{G}} N$ we have

$$\begin{aligned} & \int \nabla_{\mathbb{G}} N(x) \cdot (\nabla_{\mathbb{G}} f(x)) e^{-aN(x)^p} dx \\ &= \int \nabla_{\mathbb{G}} N(x) \cdot \nabla_{\mathbb{G}}(e^{-aN(x)^p} f(x)) dx + ap \int e^{-aN(x)^p} f(x) N^{p-1}(x) |\nabla_{\mathbb{G}} N(x)|^2 dx. \end{aligned} \quad (17)$$

We then apply Cauchy-Schwarz inequality and (14) to the left-hand side of (17), integrate by parts the first term on the right-hand side of (17), and apply (9) on the second term of the right hand side of it. This gives, for $C_2 \geq 1$ and $C_1 > 0$ such that $|\nabla_{\mathbb{G}} N(x)| \leq C_1$ (see (14)), that

$$C_1 \int |\nabla_{\mathbb{G}} f(x)| e^{-aN(x)^p} dx \geq \int |\nabla_{\mathbb{G}} N(x)| |\nabla_{\mathbb{G}} f(x)| e^{-aN(x)^p} dx \quad (18)$$

$$\geq - \int (\Delta_{\mathbb{G}} N(x)) f(x) e^{-aN(x)^p} dx \quad (19)$$

$$+ ap \int e^{-aN(x)^p} f(x) N^{p-1}(x) |\nabla_{\mathbb{G}} N(x)|^2 dx \quad (20)$$

$$\geq -C_2 \int |\Delta_{\mathbb{G}} N(x)| f(x) e^{-aN(x)^p} dx \quad (21)$$

$$+ ap C_0 \int N(x)^{p-2\gamma+1} |x_{j_0}|^{2\gamma-2} f(x) e^{-aN(x)^p} dx. \quad (22)$$

Then, moving the term containing $\Delta_{\mathbb{G}} N$ on the right hand side of (22) to the left hand side of (22), we get

$$\begin{aligned} & ap C_0 \int N(x)^{p-2\gamma+1} |x_{j_0}|^{2\gamma-2} f(x) e^{-aN(x)^p} dx \\ & \leq C_1 \int |\nabla_{\mathbb{G}} f(x)| e^{-aN(x)^p} dx + C_2 \int |\Delta_{\mathbb{G}} N(x)| f(x) e^{-aN(x)^p} dx, \end{aligned}$$

where j_0 is the index appearing in (9) and $C_0, C_1, C_2 \in (0, \infty)$ are some constants independent of the function f . Since, by homogeneity, we have $|\Delta_{\mathbb{G}} N(x)| \lesssim 1/N(x)$ and $|x_{j_0}|^{2\gamma-2} \gtrsim |x_{j_0}|^{2\gamma-1}/N(x)$, we get that

$$\begin{aligned} & ap C_0 \int N(x)^{p-2\gamma} |x_{j_0}|^{2\gamma-1} f(x) e^{-aN(x)^p} dx \\ & \leq C_1 \int |\nabla_{\mathbb{G}} f(x)| e^{-aN(x)^p} dx + C_2 \int \frac{1}{N(x)} f(x) e^{-aN(x)^p} dx, \end{aligned}$$

possibly with new constants that we keep denoting here and later on by C_0, C_1, C_2 . Note that the previous inequality can be rewritten in short notation as

$$ap C_0 \mu_p(f N^{p-2\gamma} |x_{j_0}|^{2\gamma-1}) \leq C_1 \mu_p(|\nabla_{\mathbb{G}} f|) + C_2 \mu_p\left(\frac{f}{N}\right).$$

Now (using suitable approximation), we replace f with $f \cdot |x_{j_0}|$ in the inequality above and get

$$\begin{aligned} & ap C_0 \int N(x)^{p-2\gamma} |x_{j_0}|^{2\gamma} f(x) e^{-aN(x)^p} dx \\ & \leq C_1 \int |\nabla_{\mathbb{G}}(f(x)|x_{j_0}|)| e^{-aN(x)^p} dx + C_2 \int \frac{|x_{j_0}|}{N(x)} f(x) e^{-aN(x)^p} dx. \end{aligned}$$

Using that $|\nabla_{\mathbb{G}}(f(x)|x_{j_0}|)| \leq |\nabla_{\mathbb{G}}f(x)||x_{j_0}| + f(x)$ (since $|\nabla|x_{j_0}|| = 1$ and $f \geq 0$), and that $\frac{|x_{j_0}|}{N(x)} \lesssim 1$, it follows that

$$\begin{aligned} & ap C_0 \int N(x)^{p-2\gamma} |x_{j_0}|^{2\gamma} f(x) e^{-aN(x)^p} dx \\ & \leq C_1 \int |\nabla_{\mathbb{G}}f(x)||x_{j_0}| e^{-aN(x)^p} dx + C_2 \int f(x) e^{-aN(x)^p} dx. \end{aligned}$$

Replacing f with f^q in the previous estimate, we get

$$\begin{aligned} & ap C_0 \int N(x)^{p-2\gamma} |x_{j_0}|^{2\gamma} f(x)^q e^{-aN(x)^p} dx \\ & \leq C_1 \int q f^{q-1} |\nabla_{\mathbb{G}}f(x)||x_{j_0}| e^{-aN(x)^p} dx + C_2 \int f(x)^q e^{-aN(x)^p} dx. \end{aligned}$$

An application of Young's inequality, for any $\varepsilon > 0$, yields

$$q f^{q-1} |\nabla_{\mathbb{G}}f||x_{j_0}| \leq \frac{1}{\varepsilon^{q-1}} |\nabla_{\mathbb{G}}f|^q + \frac{q}{\varepsilon} |x_{j_0}|^p f^q.$$

Therefore, since $|x_{j_0}|^p \leq N^{p-2\gamma}(x) \cdot |x_{j_0}|^{2\gamma}$, we have

$$\left(ap C_0 - C_1 \frac{q}{\varepsilon} \right) \mu_p(f^q N^{p-2\gamma} |x_{j_0}|^{2\gamma}) \leq \frac{C_1}{\varepsilon^{q-1}} \mu_p(|\nabla_{\mathbb{G}}f|^q) + C_2 \mu_p(f^q).$$

Finally, choosing $\varepsilon > 0$ sufficiently small in such a way that the constant on the left hand side is positive, we conclude (16) for non-negative compactly supported functions, which implies the result for suitable general f . ■

Remark 3.4. Note that in Theorem 1.2 and in the lemmas above we require the quasi-norm N to be smooth away from the origin. This assumption is used to avoid technical problems when applying the vector fields to N . However, in the case when N is not smooth on some hyperplane, one can split the domain into connected components where N is differentiable. In such cases, working locally inside each connected component, and using a suitable approximation argument like in [14], one can extend the result to the whole domain.

With the previous lemma at our disposal we can now prove Theorem 1.2.

Proof of Theorem 1.2. It is easy to check that for all $m \in \mathbb{R}$, a.e. we have

$$\mu_p(|f - f_{\mu_p}|^q) \leq 2^q \mu_p(|f - m|^q).$$

So it is enough to prove the estimate for $\mu_p(|f - m|^q)$ with a suitable choice of m . Then, with some $R > 0$ and $L > 0$ to be chosen later, we have

$$\begin{aligned} \mu_p(|f - m|^q) &= \mu_p(|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \geq R\}}) + \mu_p(|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \leq L\}}) \\ &\quad + \mu_p(|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \geq L\}}) \\ &= I + II + III. \end{aligned} \quad (23)$$

For the term I , by Lemma 3.3, we have

$$\begin{aligned} I &= \mu_p(|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \geq R\}}) \leq \frac{1}{R} \mu_p(|f - m|^q N^{p-2\gamma} |x_{j_0}|^{2\gamma}) \\ &\leq \frac{A}{R} \mu_p(|\nabla_{\mathbb{G}} f|^q) + \frac{B}{R} \mu_p(|f - m|^q). \end{aligned}$$

To estimate II we first observe that, given any homogeneous quasi-norm N , there exists $C \in (0, \infty)$ such that the Carnot-Carathéodory distance $d(x) = d(x, 0)$ satisfies

$$C^{-1}N(x) \leq d(x) \leq CN(x), \quad \forall x \in \mathbb{G}.$$

Therefore, given $L > 0$, there exists $L_1, L_2 > 0$ such that for the Carnot-Carathéodory ball B_{L_1} of radius L_1 centered at the origin, we have

$$\{N \leq L\} \subset B_{L_1} \subset \{N \leq L_2\}. \quad (24)$$

Choosing $m = \frac{1}{|B_{L_1}|} \int_{B_{L_1}} f(x) dx$, we get

$$\begin{aligned} II &= \mu_p(|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \leq L\}}) \\ &\leq \mu_p(|f - m|^q \mathbb{1}_{\{N \leq L\}}) = \frac{1}{Z} \int_{\{N \leq L\}} |f(x) - m|^q e^{-aN(x)^p} dx \\ &\leq \frac{1}{Z} \int_{\{N \leq L\}} |f(x) - m|^q dx \leq \frac{1}{Z} \int_{B_{L_1}} |f(x) - m|^q dx \\ &\leq \frac{P_0(L_1)}{Z} \int_{B_{L_1}} |\nabla_{\mathbb{G}} f(x)|^q dx \leq \frac{P_0(L_1)e^{aL_1^p}}{Z} \int_{\{N \leq L_2\}} |\nabla_{\mathbb{G}} f(x)|^q e^{-aN(x)^p} dx \\ &\leq P_0(L_1)e^{aL_1^p} \mu_p(|\nabla_{\mathbb{G}} f(x)|^q), \end{aligned}$$

where in the fourth line we applied the Poincaré inequality on balls (see [33]).

We are now left with the estimate of term III . To this end define a set

$$A_{L,R} := \{x \in \mathbb{G} : |x_{j_0}|^{2\gamma} \leq R, N(x) \geq L\}.$$

We claim that for all $x \in A_{L,R}$ there exist a positive constant $c' = c'(R) < 1$ sufficiently small and a horizontal curve $\gamma_x : [0, t] \mapsto G$ such that $\gamma_x(0) = e$, $(x \circ \gamma_x(t))_{j_0} > c'R^{\frac{1}{2\gamma}}$, and

$$R \leq N(x \circ \gamma_x(s)) < N(x), \quad \forall s \in (0, t]. \quad (25)$$

Let us clarify that the reason for the subscript x in γ_x is to stress that the choice of the curve with the above properties with respect to x depends on x , so x is not the starting point of γ_x .

Next, we give the proof of our claim. In the following we shall say that a geodesic $\gamma : [0, t] \rightarrow \mathbb{G}$ connects x and y if $\gamma(0) = e$, hence $x = x \circ \gamma(0)$, and $y = x \circ \gamma(t)$. Since N is smooth and there exists $r = r(x) > 0$ such that $x \in \partial B_r$, we have $N(x) > N(y)$ for all $y \in B_r$. Then, we choose $y \in B_r$ and take $\gamma_{1,x} : [0, t] \mapsto \mathbb{G}$ the horizontal geodesic connecting x and y , that is such that $x \circ \gamma_{1,x}(0) = x$ and $x \circ \gamma_{1,x}(t) = y$. Note that $N(x \circ \gamma_{1,x}(s)) < N(x)$ for all $s \in (0, t]$. Due to our choice of y , which satisfies $N(y) < N(x)$, there exists $c = c(r, R)$, with $|c| < 1$ sufficiently small, such that

$$N(y \circ (0, \dots, cR^{\frac{1}{2\gamma}}, \dots, 0)) = N(y_1, \dots, y_{j_0} + cR^{\frac{1}{2\gamma}}, y_{j_0+1}, \dots, y_n) < N(x)$$

and
$$y_{j_0} + cR^{\frac{1}{2\gamma}} > c'R^{\frac{1}{2\gamma}}$$

for some $0 < c' < 1$ small. Now we call $\gamma_{2,x}$ the horizontal geodesic connecting y and $y \circ (0, \dots, cR^{\frac{1}{2\gamma}}, \dots, 0)$, that is such that $\gamma_{2,x}(0) = e$, $y \circ \gamma_{2,x}(0) = y$ and $y \circ \gamma_{2,x}(t) = y \circ (0, \dots, cR^{\frac{1}{2\gamma}}, \dots, 0)$, and $\gamma_x : [0, t] \mapsto \mathbb{G}$ the union of $\gamma_{1,x}$ and $\gamma_{2,x}$, more precisely $\gamma_x(s) := \gamma_{1,x}(s) \circ \gamma_{2,x}(s)$, $s \in [0, t]$. Since γ_x connects x and $x \circ h(x) = y(x) \circ (0, \dots, cR^{\frac{1}{2\gamma}}, \dots, 0)$, where $h(x)$ is a suitable point which depends on x and satisfying the previous identity, we conclude that γ_x satisfies (25) and hence the claim.

Note that, by choosing L much larger than R , then for all $x \in A_{L,R}$ we can take $r(x) \sim R$ in the argument above so that $d(x, x \circ h) = d(h) \lesssim R$. Hence, using the properties of the curve γ_x and the notation $g = f - m$, we get

$$\begin{aligned} III &= \mu_p(|g|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \geq L\}}) \lesssim \int_{A_{L,R}} |g(x)|^q d\mu_p(x) \\ &\lesssim \int_{A_{L,R}} |g(x) - g(x \circ h)|^q d\mu_p(x) + \int_{A_{L,R}} |g(x \circ h)|^q d\mu_p(x). \end{aligned} \tag{26}$$

Let us now define $h := h(x) = \gamma_x(t)$. Then by Hölder’s inequality and the fact that d is the control distance, i.e., we have $d(x, x \circ h) = d(h) \leq t$ (since $d(h) := d(e, h)$, by the definition of d , is the shortest “time” needed to go from e to h via a horizontal path), we obtain

$$\begin{aligned} \int_{A_{L,R}} |g(x) - g(x \circ h)|^q d\mu_p(x) &= \int_{A_{L,R}} \left| \int_0^t \frac{d}{ds} g(x \circ \gamma_x(s)) ds \right|^q d\mu_p(x) \\ &\leq \int_{A_{L,R}} t^q \int_0^t |\nabla_{\mathbb{G}} g(x \circ \gamma_x(s))|^q ds d\mu_p(x) \\ &\leq \int_{A_{L,R}} d(h)^q \int_0^t |\nabla_{\mathbb{G}} g(x \circ \gamma_x(s))|^q ds d\mu_p(x) \\ &\lesssim R^q \int_{A_{L,R}} \int_0^t |\nabla_{\mathbb{G}} g(x \circ \gamma_x(s))|^q ds d\mu_p(x) \\ &\stackrel{(25)}{\lesssim} R^q \int_{A_{L,R}} \int_0^{cR} |\nabla_{\mathbb{G}} g(x \circ \gamma_x(s))|^q ds d\mu_p(x \circ \gamma_x(s)) \\ &\lesssim R^{q+1} \int_{\mathbb{G}} |\nabla_{\mathbb{G}} g(x)|^q d\mu_p(x) \\ &= CR^{q+1} \mu_p(|\nabla_{\mathbb{G}} g(x)|^q). \end{aligned} \tag{27}$$

To estimate the second term in (26) we use (25) and the estimate $(x \circ h)_{j_0} = (y(x) \circ (0, \dots, cR^{\frac{1}{2\gamma}}, \dots, 0)) \gtrsim R^{\frac{1}{2\gamma}}$ (which implies $N(x \circ h) \gtrsim R$), so that an application of Lemma 3.3 yields

$$\int_{A_{L,R}} |g(x \circ h)|^q d\mu_p(x) \lesssim \frac{1}{R^p} \int_{A_{L,R}} |g(x \circ h)|^q |(x \circ h)_{j_0}|^{2\gamma} N(x \circ h)^{p-2\gamma} d\mu_p(x \circ h) \quad (28)$$

$$\lesssim \frac{1}{R^p} \int_{\mathbb{G}} |g(x)|^q |(x)_{j_0}|^{2\gamma} N(x)^{p-2\gamma} d\mu_p(x) \quad (29)$$

$$\leq \frac{A}{R^p} \mu_p(|\nabla_{\mathbb{G}} f|^q) + \frac{B}{R^p} \mu_p(|f - m|^q), \quad (30)$$

which gives $III \leq \left(\frac{A}{R^p} + CR^{q+1}\right) \mu_p(|\nabla_{\mathbb{G}} f|^q) + \frac{B}{R^p} \mu_p(|f - m|^q)$ (31)

Finally, putting together the estimates for the three terms I, II and III , we get

$$\left(1 - \frac{B}{R^p} - \frac{B}{R}\right) \mu_p(|f - m|^q) \leq \left(\frac{A}{R} + P_0(L_1)e^{aL_2^p} + \frac{A}{R^p} + CR^{q+1}\right) \mu_p(|\nabla_g f|^q),$$

hence, by choosing R and L sufficiently large, with $L > R$, we get the result. This concludes the proof. ■

Proof of Corollary 1.3. The proof is an immediate consequence of Theorem 1.2 combined with the fact that, whenever (10) holds true for some $q > 1$, then it holds true also for $q' > q$ under the same unchanged probability measure; see Proposition 2.1.11 in [30]. ■

Remark 3.5. We want to point out that the choice of the homogeneous quasi-norm N on the Carnot group provides a radical difference on the spectrum of the operators of the form (7). To be more precise, for a probability measure on an H -type group of the form $Z^{-1}e^{-aN^p}$, $p \in (1, 2)$, the corresponding operator (7) has empty essential spectrum when N is the Kaplan norm, and does not even have a spectral gap when N is the Carnot-Carathéodory distance; see Remark 4.5.4 in [30].

As a corollary of Theorem 1.2 one has that the global Poincaré inequalities of Theorem 1.2 are satisfied by a family of measures whose potential is a perturbation of the one appearing in the density of μ_p .

Corollary 3.6. *Let $d\mu_W := \tilde{Z}^{-1}e^{-W}d\mu_p$ be a probability measure with a potential W which is Lipschitz (with respect to the Carnot-Carathéodory distance) satisfying*

$$|\nabla_{\mathbb{G}} W(x)|^q \leq \delta N(x)^{p-2\gamma} |x_{j_0}|^{2\gamma} + \gamma_{\delta}, \quad \text{for almost all } x \in \mathbb{G}, \quad (32)$$

for some $0 < \delta \ll 1$ and $\gamma_{\delta} \in (0, \infty)$. Then the measure μ_W satisfies inequality (16) for $p \geq 2\gamma$ and q such that $\frac{1}{q} + \frac{1}{p} = 1$. Moreover, if there exists $\tilde{C} > 0$ such that $W \leq \tilde{C}N$ then μ_W satisfies the global Poincaré inequality.

Proof. Let us start by replacing f by $f e^{-\frac{W}{q}}$ in the inequality (16). This gives

$$\mu_p \left(e^{-W} |f|^q N^{p-2\gamma} |x_{j_0}|^{2\gamma} \right) \leq C \mu_p \left(|\nabla_{\mathbb{G}} (e^{-\frac{W}{q}} f)|^q \right) + D \mu_p \left(|e^{-\frac{W}{q}} f|^q \right). \quad (33)$$

Now, since $|\nabla_{\mathbb{G}}(e^{-\frac{W}{q}} f)|^q = \left| \left(\nabla_{\mathbb{G}} e^{-\frac{W}{q}} \right) f + e^{-\frac{W}{q}} \nabla_{\mathbb{G}} f \right|^q$

$$\leq \left(\frac{|\nabla_{\mathbb{G}} W|}{q} |e^{-\frac{W}{q}} f| + e^{-\frac{W}{q}} |\nabla_{\mathbb{G}} f| \right)^q$$

$$\leq C(q) (|\nabla_{\mathbb{G}} W|^q e^{-W} |f|^q + e^{-W} |\nabla_{\mathbb{G}} f|^q), \tag{34}$$

substituting the latter in (33), and using (32) we get

$$\begin{aligned} \mu_W (|f|^q N^{p-2\gamma} |x_{j_0}|^{2\gamma}) &\leq C'(q) \mu_W (|\nabla_{\mathbb{G}} W|^q |f|^q) + C'(q) \mu_W (|\nabla_{\mathbb{G}} f|^q) + D \mu_W (|f|^q) \\ &\leq \delta C'(q) \mu_W (N^{p-2\gamma} |x_{j_0}|^{2\gamma} |f|^q) + \gamma_{\delta} C'(q) \mu_W (|f|^q) \\ &\quad + C'(q) \mu_W (|\nabla_{\mathbb{G}} f|^q) + D \mu_W (|f|^q). \end{aligned}$$

Therefore, if δ is such that $1 - \delta C'(q) > 0$, the last inequality gives the result.

Now to prove the global Poincaré for the measure μ_W , we have to assume that W is such that $W \lesssim N$. For $L > 1$ and $R > 0$ as in the proof of Theorem 1.2, we decompose the quantity $\mu_W (|f - m|^q)$ as follows

$$\begin{aligned} \mu_W (|f - m|^q) &= \mu_W (|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \geq R\}}) \\ &\quad + \mu_W (|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \leq L\}}) \\ &\quad + \mu_W (|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \geq L\}}). \end{aligned} \tag{35}$$

For the first and the third term of (35) one can proceed as in Theorem 1.2, while for the second term, arguing as in Theorem 1.2, we arrive at

$$\begin{aligned} \mu_W (|f - m|^q \mathbb{1}_{\{|x_{j_0}|^{2\gamma} N^{p-2\gamma} \leq R\}} \mathbb{1}_{\{N \leq L\}}) &\leq \frac{P_0(L_1)}{\tilde{Z}} \int_{\{N \leq L_2\}} |\nabla_{\mathbb{G}} f|^q dx \\ &\leq \frac{P_0(L_1)}{\tilde{Z}} e^{aL_2^p + \tilde{C}L_2} \mu_W (|\nabla_{\mathbb{G}} f|^q), \end{aligned}$$

where $L_2 > 0$ is the one appearing in (24). In the last inequality we have used the fact that in $\{N \leq L_2\}$ we have $W \leq \tilde{C}N \leq \tilde{C}L_2$. The proof is now complete. \blacksquare

4. Examples and general results

In this section we (I) describe classes of Carnot groups for which the sufficient condition of Theorem 1.2, expressed by (9), is satisfied, and (II) construct a quasi-norm on a Carnot group \mathbb{G} (with a formula depending on \mathbb{G}) which gives rise to the corresponding global Poincaré inequalities. This guarantees the existence of the spectral gap for the corresponding self-adjoint operator \mathcal{L}_p as in Corollary 1.3.

4.1. Homogeneous quasi-norms smooth away from the origin

Before starting with the investigation of the validity of (9), we briefly discuss here some homogeneous quasi-norms we will be using in the current section. The works [19], [37], [34] and [2], contain a recent exposition of examples of homogeneous quasi-norms on Carnot groups.

Let us remark that the quasi-norm we have used so far is smooth away from the origin. For convenience we shall simply call such quasi-norms *smooth norms*, where the smoothness property shall be regarded as smoothness everywhere except for the origin.

A natural example of such quasi-norm on a Carnot group on \mathbb{R}^n equipped with the dilation $\delta_\lambda(x) = (\lambda^{\sigma_1}x_1, \dots, \lambda^{\sigma_n}x_n)$ is given by the formula:

$$N(x) = \left(\sum_{j=1}^n a_j |x_j|^{2\beta_j} \right)^{1/\gamma}, \quad (36)$$

where a_j , for all $j = 1, \dots, n$, is a positive real number, and β_j , for all $j = 1, \dots, n$, is such that $\beta_j \in \mathbb{N}$ and $2\sigma_j\beta_j = \gamma \geq 2\sigma_n$. Homogeneous quasi-norms of the form (36) are indeed natural extensions of the Euclidean norm on \mathbb{R}^n to the dilated structure of the Carnot group. A particular case of a quasi-norm of the above form that one often encounters in the literature is the one with $\beta_j = \sigma_n!/\sigma_j$ and $\gamma = 2\sigma_n!$. Given a Carnot group on \mathbb{R}^n as above, one can use quasi-norms of the form (36) on groups of variables to give rise to other homogeneous quasi-norms. Explicitly, we can define the sum

$$\tilde{N}(x) := \left(\sum_{j=1}^m N_j(x_{j_1}, x_{j_2}, \dots, x_{j_{k_j}}) \right)^{1/\alpha}, \quad (37)$$

where $k_j \leq n$, $j_1, \dots, j_{k_j} \in \{1, \dots, n\}$, $m \in \mathbb{N}$, and N_j are quasi-norms of the form (36) defined on a subspace of \mathbb{R}^n ¹. In (37) we have chosen $\alpha \in \mathbb{N}$, and thus the homogeneity requires that the α_j 's are such that $\alpha_j/\gamma_j = \alpha$, where γ_j is the exponent in N_j as in (36).

In Carnot groups \mathbb{G} of step two, the general formula (37) boils down to the quasi-norm given in (38) below, which can be viewed as a generalization of the *Kaplan norm* on H-type groups (see [30]). Precisely, if \mathbb{G} is a group on \mathbb{R}^{m+n} , with m being the number of generators, then denoting by $(x, t) \in \mathbb{R}^m \times \mathbb{R}^n$ an element of the group, one has that a class of smooth quasi-norms on \mathbb{G} is given by

$$N_\alpha(x, t) = \left(\|x\|^{4\alpha} + \sum_{j=1}^n c_j t_j^{2\alpha} \right)^{\frac{1}{4\alpha}}, \quad (38)$$

where α is a positive integer, c_j , for every $j = 1, \dots, n$, is a positive real number, and $\|x\| := (\sum_{k=1}^m x_k^2)^{1/2}$.

Let us also remark that all the quasi-norms listed above are as in the hypothesis of Theorem 1.2.

Considering the norms defined above, we will now focus on examples of groups where these norms allow to recover global Poincaré inequalities. We stress once more that spectral gaps for suitable corresponding operators follow from the 2-Poincaré inequality.

In what follows we will first show that on step two Carnot groups, by choosing N as in (38), we have that our sufficient condition (9) holds true, and, consequently, so does also the corresponding global Poincaré inequality for the suitable probability measure.

¹ To be precise, the quasi-norms N_j above are

$$N_j(x_{j_1}, x_{j_2}, \dots, x_{j_{k_j}}) = \left(\sum_{i=1}^{k_j} a_{ji} |x_{j_i}|^{2\beta_i} \right)^{1/\gamma_j},$$

where the parameters are as in (36), that is such that $\beta_i/\sigma_{j_i} = \gamma_j \geq 2\sigma_{j_{k_j}}$ for every $i = 1, \dots, k_j$, while a_{ji} are positive real numbers.

Next, for a wide class of Carnot groups of arbitrary step, we will show that condition (9) is satisfied if N is as in (36) or, more generally, as in (37). This, once again, will imply the validity of global Poincaré inequalities, and, as before, of the spectral gaps for operators as in (7), or, more precisely, as in Corollary 1.3.

4.2. Carnot groups of step 2

We start with the investigation of step 2 Carnot groups. Below we shall use the notation $(\mathbb{R}^{m+n}, \circ)$ for an $N = m + n$ -dimensional Carnot group of step 2 with m generators and composition law \circ . A point in $\mathbb{G} = (\mathbb{R}^{m+n}, \circ)$ will be denoted by (x, t) , with $x \in \mathbb{R}^m$ and $t \in \mathbb{R}^n$.

We recall that any N -dimensional Carnot group of step 2 and m generators is naturally isomorphic to a step two Carnot group $(\mathbb{R}^{m+n}, \circ')$, where $n = N - m$, the composition law is

$$(x, t) \circ' (\xi, \tau) = (x + \xi, t_1 + \tau_1 + \frac{1}{2} \langle B^{(1)}x, \xi \rangle, \dots, t_n + \tau_n + \frac{1}{2} \langle B^{(n)}x, \xi \rangle), \tag{39}$$

and $B^{(j)}$, for all $j = 1, \dots, n$, is an $m \times m$ skew-symmetric matrix. Therefore, without loss of generality, hereafter we consider two step Carnot groups with a composition law of the form \circ' defined through some skew-symmetric matrices $B^{(j)}$'s. Note that, for a Carnot group of step 2 with m generators on \mathbb{R}^{m+n} , (39) gives that

$$X_j = \partial_{x_j} + \frac{1}{2} \sum_{k=1}^n \sum_{i=1}^m B_{ij}^{(k)} x_i \partial_{t_k}, \quad \text{for all } j = 1, \dots, m. \tag{40}$$

Formula (40) will be very useful to prove the following proposition.

Proposition 4.1. *Let $\mathbb{G} = (\mathbb{R}^{n_1+n}, \circ')$ be a Carnot group of step 2 and n_1 generators, and let N_α be a smooth quasi-norm on \mathbb{G} as in (38), where α can be any positive integer. Then, for every $x \in \mathbb{G} \setminus \{0\}$ there exists a constant $C > 0$ such that*

$$|\nabla N_\alpha(x)| \geq C \frac{\|(x_1, \dots, x_{n_1})\|^{4\alpha-1}}{N_\alpha(x)^{4\alpha-1}}, \tag{41}$$

where $\|\cdot\|$ stands for the Euclidean norm on \mathbb{R}^{n_1} .

Proof. Recall that, for a Carnot group \mathbb{G} of step 2, the generators X_1, \dots, X_{n_1} , are of the form (40). Therefore, since

$$\begin{aligned} |X_j N_\alpha(x)|^2 &= \left(\frac{8\|x\|^{4\alpha-2} x_j + \sum_{k=1}^n \sum_{i=1}^{n_1} B_{ij}^{(k)} x_i c_k t_k}{4N_\alpha(x)^{4\alpha-1}} \right)^2 \\ &= \frac{64\|x\|^{8\alpha-4} x_j^2 + (\sum_{k=1}^n \sum_{i=1}^{n_1} B_{ij}^{(k)} x_i c_k t_k)^2 + 2 \sum_{k=1}^n \sum_{i=1}^{n_1} B_{ij}^{(k)} x_i x_j c_k t_k}{16N_\alpha(x)^{8\alpha-2}}, \end{aligned}$$

and since $\sum_{i,j=1}^{n_1} B_{ij}^{(k)} x_i x_j = 0$ by the skew-symmetry of $B^{(k)}$, we have

$$|\nabla N_\alpha(x)|^2 \geq \frac{4\|x\|^{8\alpha-2}}{N_\alpha(x)^{8\alpha-2}},$$

and the proof is complete. ■

Remark 4.2. Thanks to Proposition 4.1 and Theorem 1.2 we have that global q -Poincaré inequalities hold true on step 2 Carnot groups for probability measures whose density is of the form $e^{-aN_\alpha^p}$, with N_α as in (38) and q conjugate exponent of p . Moreover, the 2-Poincaré inequality yields the validity of the spectral gap for the operator $\mathcal{L}_p := -\Delta_{\mathbb{G}} + pN_\alpha^{p-1}\nabla_{\mathbb{G}}N \cdot \nabla_{\mathbb{G}}$, for p as in μ_p . ■

Summarizing, we have proved the following theorem.

Theorem 4.3. *Let \mathbb{G} be a Carnot group of step 2, α a positive integer, and N_α a homogeneous quasi-norm on \mathbb{G} as in (38). Then*

$$\mu_p(|f - f_{\mu_p}|^q) \leq \mu_p(|\nabla f|^q)$$

for every $p \geq 8\alpha$, and with q being the conjugate exponent of p .

Example 4.4 (\mathbb{H} -type groups.). Note that, for a Carnot group of step 2 of \mathbb{H} -type on \mathbb{R}^{m+n} , the quasi-norm N_α in (38) with $\alpha = 1$ and $c_j = \frac{1}{16}$, for all $j = 1, \dots, n$, coincides with the so called Kaplan norm

$$N_1(x) = (\|x\|^4 + \frac{1}{16}|z|^2)^{1/4}, \quad x = (w, z) \in \mathbb{R}^n \times \mathbb{R}^m.$$

In [30] the author showed that, on \mathbb{H} -type groups, $|\nabla N_1(x)| = \frac{\|x\|}{N(x)}$, and that global q -Poincaré inequalities with respect to $d\mu_p = \frac{1}{Z}e^{-aN_\alpha^p(x)}dx$, with $a > 0$, $p \geq 2$, and q conjugate exponent of p , hold true. We remark that the previous identity also gives

$$|\nabla N_1(x)| = \frac{\|x\|}{N(x)} = \frac{\|x\|^3}{\|x\|^2 N(x)} \geq \frac{\|x\|^3}{N(x)^3},$$

which is condition (9) in Theorem 1.2 giving the validity of global Poincaré inequalities.

Moreover, here Theorem 4.3 applies and generalizes, in some sense, the result in [30], allowing to conclude global Poincaré inequalities for the class of probability measures defined as $d\mu_p = \frac{1}{Z}e^{-aN_\alpha^p(x)}dx$, for $a > 0$ and α being any positive integer.

Example 4.5 (The anisotropic Heisenberg group.). We conclude this part dedicated to step 2 groups by considering a group treated in [5] to which our results apply. The group under consideration is the so called anisotropic Heisenberg group $\mathbb{H}_{2n}(\frac{1}{2}, 1)$. Since $\mathbb{H}_{2n}(\frac{1}{2}, 1)$ is a Carnot group of step 2, we get the validity of global q -Poincaré inequalities for any smooth homogeneous quasi-norm as in (38) by Theorem 4.3. However, the validity of such inequalities was first proved in [5] by using probability measures whose density depends on a specific smooth quasi-norm N , that is, for N being the fundamental solution for the sub-Laplacian. The quasi-norm in [5] is

$$N(x) = \frac{(B^2 + t^2)^{1/4n}(AB + t^2 + A\sqrt{A^2 + B^2})^{1/2-1/4n}}{(B + \sqrt{B^2 + t^2})^{1/2}},$$

where

$$A = \frac{x_1^2}{2} + \frac{x_{n+1}^2}{2} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq n+1}}^{2n} x_j^2, \quad \text{and} \quad B = \frac{x_1^2}{4} + \frac{x_{n+1}^2}{4} + \frac{1}{2} \sum_{\substack{j=1 \\ j \neq n+1}}^{2n} x_j^2.$$

Note that the quasi-norm N is smooth away from 0.

Moreover, in [5] the authors proved that, for all $x \neq 0_{\mathbb{G}}$,

$$|\nabla N(x)| \geq C \frac{\|x\|^2}{N^2} \geq C \frac{\|x\|^{\gamma-1}}{N^{\gamma-1}}, \quad \forall \gamma \geq 3,$$

where $\|x\|$ is the Euclidean norm of $x = (x_1, \dots, x_{2n})$, therefore the sufficient condition (9) is satisfied and Theorem 1.2 applies for all $\gamma \geq 3$.

Summarizing, on the anisotropic Heisenberg group one can apply both Theorem 1.2 (with N as in [5]) and Theorem 4.3, and get, in the first case, the same result as in [5], while, in the second case, Poincaré inequalities with respect to different probability measures defined through N_α as in (38).

Let us finally remark that in [5] the authors had to deal with the non trivial problem of finding a fundamental solution of the sub-Laplacian in order to find the suitable probability measure to prove the inequalities. Theorem 4.3, instead, is direct, and gives already a class of measures for which the inequalities are true.

4.3. Carnot groups of step $r \geq 2$

Besides Carnot groups of step 2, there are other groups to which our result applies, that is groups such that the sufficient condition for the Poincaré inequality is verified by any homogeneous quasi-norm on the group being smooth away from the origin and of the form (36) or (37). Such groups include those described in the following result.

Lemma 4.6. *Let \mathbb{G} be a Carnot group of step r on \mathbb{R}^n , and let $\{X_j\}_{1 \leq j \leq n_1}$ be the generators of the first stratum V_1 . If there exists $j_0 \in \{1, \dots, n_1\}$ such that*

$$X_{j_0} = \partial_{x_{j_0}}, \tag{42}$$

then (9) holds with $j = j_0$ and with N as in (36) or (37).

Proof. We give the proof for N as in (36), since for quasi-norms of the form (37) one can proceed similarly.

Note that, due to the form of X_{j_0} and of the quasi-norm N in (36),

$$|X_{j_0} N(x)| = \left| \frac{2\beta_{j_0} a_{j_0} x_{j_0}^{2\beta_{j_0}-1}}{\gamma N^{\gamma-1}} \right| = C \frac{|x_{j_0}|^{\gamma-1}}{N^{\gamma-1}}.$$

Therefore, since $|\nabla_{\mathbb{G}} N(x)| \geq |X_{j_0} N(x)|$ for all $x \in \mathbb{G}$, the previous estimate amounts to (9) with $j = j_0$.

For combinations of smooth quasi-norms as in (37), by repeating the same considerations as above we trivially get the same result. This concludes the proof. ■

By using Lemma 4.6, one obtains the following corollary of Theorem 1.2.

Proposition 4.7. *Let \mathbb{G} be a Carnot group of step r on \mathbb{R}^n and let $\{X_j\}_{1 \leq j \leq n_1}$ be the generators of the first stratum V_1 . If there exists $j_0 \in \{1, \dots, n_1\}$ such that (42) holds, then the hypotheses of Theorem 1.2 are satisfied for N as in (36) or (37).*

Among the groups to which Lemma 4.6 applies, and therefore satisfying conditions (42) and (9), for N as in (36) or (37), we have:

- Carnot groups of "Engel type", see e.g. [12], [13], on \mathbb{R}^n (i.e. Carnot groups of filiform-type), for $n \geq 4$, with polynomial coordinates² (see [14]);
- The Cartan group (see [17]);
- *Kolmogorov*-type groups (see [4]);
- Carnot groups arising from some Sub-Laplacians, like, for instance, the ones arising from the lifting of Bony-type Sub-Laplacians and those related to some Sub-Laplacian arising in control theory (see Section 4.3 in [4] for details);
- Sums of Carnot groups of the previous type.

A combination of Lemma 4.6, Proposition 4.7 and Theorem 1.2, leads to the proof of Theorem 1.4. For completeness, we give the proof in the following.

Proof of Theorem 1.4. By the result of Helffer and Nourrigat in [28, p. 99], see also [24, Theorem 2.2], there exists an admissible change of coordinates such that one of the generating vector fields, say X_1 after suitably relabeling the generating vector fields, is sent into the vector field $Y_1 = \partial_{x_1}$, while X_2, \dots, X_{n_1} are sent into suitable vector fields Y_2, \dots, Y_{n_1} . Since $\{Y_j\}_{j=1, \dots, n_1}$ are generating vector fields of a stratified Lie algebra (isomorphic to the starting one) satisfying (42) with $j_0 = 1$, by Proposition 4.7 we conclude that, possibly after a change of coordinates, Theorem 1.2 holds with N as in (36), which concludes the proof. ■

Remark 4.8. We note that Theorem 1.4 holds true also for μ_p corresponding to N given by (37). In this case we have $p \geq 8\alpha$.

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² Here we use polynomial coordinates arising from a strong Malcev basis of the Lie algebra of the group; see [16] for details about such coordinates.

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